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Volumes engendrés

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VOLUMES ENGENDRÉS

(Mêmes figures que pour les surfaces). — Fin (voir p. 361)

PAR M. HUET,

ancien professeur de mathématiques spéciales à l'École de Sorrèze,
régent de mathématiques spéciales au collége de Pamiers

1° *Triangle.*

$$\text{Vol}\,T = \text{vol}\,CBEF - 2\,\text{vol}\,ABF = \pi\overline{BF}^2.CB - \frac{2}{3}\pi\overline{BF}^2.AF =$$

$$= \pi\overline{BF}^2\left(CB - \frac{2}{3}AF\right) = \frac{2}{3}CB.\overline{BF}^2;$$

or $$AB = R\sqrt{3},\ BF = \frac{3}{2}R\,;$$

donc $$\mathrm{vol}\,T = \frac{3}{2}\pi R^3\sqrt{3}.$$

2° *Carré.*

$$\mathrm{Vol}\,C = 2\,\mathrm{vol}\,CAB = 2\,(\mathrm{vol}\,CAEB - \mathrm{vol}\,ABE) =$$

$$= \frac{2}{3}\pi AE(\overline{CA}^2 + \overline{BE}^2 + CA.BE - \overline{BE}^2) = \frac{2}{3}\pi AE.CA(CA + BE) =$$

$$= \frac{2}{3}\pi AE.\frac{3}{2}CA^2 = \frac{1}{2}\pi\overline{CA}^3;$$

or $$CA = 2R\,;$$

donc $$\mathrm{vol}\,C = 4\pi R^3.$$

3° *Pentagone.*

$$\mathrm{Vol}\,P = 2\mathrm{vol}\,ABCD = 2\mathrm{vol}\,DCAG + 2(\mathrm{vol}\,CBHG - \mathrm{vol}\,ABH) =$$

$$= 2\mathrm{vol}\,T + \mathrm{vol}\,T').$$

$$\mathrm{Vol}\,T = \pi AG.\overline{CG}^2;\quad \mathrm{vol}\,T' = \mathrm{vol}\,t - \mathrm{vol}\,t'\,;$$

or $\mathrm{vol}\,t = \frac{1}{3}\pi GH(\overline{CG}^2 + \overline{BH}^2 + CG.BH)$, et $\mathrm{vol}\,t' = \frac{1}{3}\pi AH.\overline{BH}^2$.

Cela posé, $AC = D = \frac{R}{2}\sqrt{10 + 2\sqrt{5}}$; $AB = \frac{R}{2}\sqrt{10 - 2\sqrt{5}}$.

$AG = \frac{AB}{2} = \frac{R}{4}\sqrt{10 - 2\sqrt{5}}$; $CG = \frac{R}{4}(5 + \sqrt{5})$; $AH = \frac{D}{2} =$
$\frac{R}{4}\sqrt{10 + 2\sqrt{5}}$; $GH = \frac{D}{2} - AG = \frac{R}{4}\left(\sqrt{10+2\sqrt{5}} - \sqrt{10-2\sqrt{5}}\right)$

$$= \frac{R}{4}\sqrt{20 - 8\sqrt{5}} = \frac{R}{2}\sqrt{5 - 2\sqrt{5}}\,;$$

donc $$\mathrm{vol}\,T = \pi\frac{R}{4}\sqrt{10 - 2\sqrt{5}}.\frac{R^2}{16}(30 + 10\sqrt{5}) =$$

$$= \frac{1}{4}.\frac{5}{8}\pi R^3\sqrt{10 - 2\sqrt{5}} \times (3 + \sqrt{5}) = \frac{1}{4}.\frac{5}{8}\pi R^3\sqrt{80 + 32\sqrt{5}} =$$

$$= \frac{5}{8}\pi R^3\sqrt{5 + 2\sqrt{5}}.$$

$$\text{Vol}\,t = \frac{1}{3}\pi\frac{R}{2}\sqrt{5-2\sqrt{5}}\left\{\frac{R^2}{16}(30+10\sqrt{5})+\frac{R^2}{16}(30-10\sqrt{5})+\right.$$

$$\left.+\frac{20R^2}{16}\right\} = \frac{5}{6}\pi R^3\sqrt{5-2\sqrt{5}}.$$

$$\text{Vol}\,t' = \frac{1}{3}\pi\frac{R}{4}\sqrt{10+2\sqrt{5}}\,.\,\frac{R^2}{16}(30-10\sqrt{5}) =$$

$$=\frac{5}{3}\cdot\frac{1}{2}\cdot\frac{1}{16}\pi R^3\sqrt{80-32\sqrt{5}} = \frac{5}{3}\cdot\frac{1}{8}\pi R^3\sqrt{5-2\sqrt{5}}.$$

Par suite, $\text{vol}\,T' = \frac{5}{3}\cdot\frac{1}{8}\pi R^3\left\{4\sqrt{5-2\sqrt{5}}-\sqrt{5-2\sqrt{5}}\right\} =$

$$=\frac{5}{8}\pi R^3\sqrt{5-2\sqrt{5}}\,;$$

et partant,

$$\text{vol}\,P = 2(\text{vol}T+\text{vol}T') = \frac{5}{4}\pi R^3\sqrt{5+2\sqrt{5}}+\frac{5}{4}\pi R^3\sqrt{5-2\sqrt{5}} =$$

$$=\frac{5}{4}\pi R^3\left(\sqrt{5+2\sqrt{5}}+\sqrt{5-2\sqrt{5}}\right) = \frac{5}{4}\pi R^3\sqrt{10+2\sqrt{5}}.$$

4° *Hexagone.*

$$\text{Vol}\,H = 2\,\text{vol}\,ABCD = 2\,(\text{vol}\,ADCG - \text{vol}\,ABG) =$$

$$= \frac{2}{3}\pi AG\left(\overline{DA}^2+\overline{CG}^2+DA.CG-\overline{BG}^2\right) =$$

$$=\frac{2}{3}\pi AG\left\{\overline{DA}^2+(CG+BG)(CG-BG)+DA\,.\,CG\right\} =$$

$$=\frac{2}{3}\pi AG.DA(DA+CB+CG) = \frac{2}{3}\pi AG.2R(4R+BG) = 6\pi R^2.AG\,;$$

or $\quad AG = \frac{R}{2}\sqrt{3}, \quad$ donc $\text{vol}\,H = 3\pi R^3\sqrt{3}.$

5° *Octogone.*

$$\text{Vol}\,O = 2\,\text{vol}\,ABCDE = 2\,(\text{vol}\,ABDE + \text{vol}\,BCD) =$$

$$= 2\,(\text{vol}\,T + \text{vol}\,T').$$

$$\text{Vol}T = \text{vol}AFDE - \text{vol}ABF = \frac{1}{3}\pi AF\left(\overline{AE}^2+\overline{DF}^2+AE.DF-\overline{BF}^2\right)$$

$$= \frac{1}{3}\pi AF \left\{ \overline{AE}^2 + (DF+BF)(DF-BF) + AE \cdot DF \right\} =$$

$$= \frac{1}{3}\pi FG.DB(AE+DB+DF) = \frac{2}{3}\pi AF.CG\left(3CG+\frac{3}{2}DB\right) =$$

$$= \pi AF \cdot CG\,(2CG+DB).$$

$$\mathrm{Vol}\,T' = \mathrm{vol}\,DCGF - \mathrm{vol}\,BFGC =$$

$$= \frac{1}{3}\pi FG\left(\overline{DF}^2 + \overline{CG}^2 + DF \cdot CG - \overline{BF}^2 - \overline{CG}^2 - BF \cdot CG\right) =$$

$$= \frac{1}{3}\pi FG\left\{(DF+BF)(DF-BF) + CG\,(DF-BF)\right\} =$$

$$= \frac{1}{3}\pi FG \cdot DB\,(AE+CG) = \pi FG \cdot DB \cdot CG.$$

Or, $\quad CG = R\,;\ DB = R\sqrt{2}\,;\quad AF = \frac{DB}{2} = \frac{R\sqrt{2}}{2}\,;$

$$FG = R - AF = R - \frac{R\sqrt{2}}{2} = \frac{R}{2}(2-\sqrt{2})\,;$$

donc

$$\mathrm{vol}T = \pi.\frac{R\sqrt{2}}{2}.R(2R+R\sqrt{2}) = \frac{\pi R^3}{2}(2+2\sqrt{2}) = \pi R^3(1+\sqrt{2}).$$

$$\mathrm{Vol}T' = \pi\,.\,\frac{R}{2}(2-\sqrt{2})\,.\,R\sqrt{2}\,.\,R = \frac{\pi R^3}{2}\,(2\sqrt{2}-2) =$$

$$= \pi R^3\,(\sqrt{2}-1)\,;$$

par suite $\qquad \mathrm{vol}\,T + \mathrm{vol}\,T' = 2\pi R^3\sqrt{2}\,,$

et partant $\qquad \mathrm{vol}\,O = 4\pi R^3\sqrt{2}.$

6° *Décagone*.

$$\mathrm{Vol}\,D = 2\,\mathrm{vol}\,ABCDEF = 2\,(\mathrm{vol}\,ABEF + \mathrm{vol}\,BCDE) =$$

$$= 2\,(\mathrm{vol}\,T + \mathrm{vol}\,T').$$

$$\mathrm{Vol}\,T = \mathrm{vol}\,AGEF - \mathrm{vol}\,ABG =$$

$$= \frac{1}{3}\pi AG\left(\overline{AF}^2 + \overline{EG}^2 + AF.\,EG - \overline{BG}^2\right) =$$

$$= \frac{1}{3}\pi AG\left\{\overline{AF}^2 + (EG+BG)\,(EG-BG) + AF.\,EG\right\} =$$

$$= \frac{1}{3}\pi AG.AF(AF + EB + EG) = \frac{2}{3}\pi AG.AO\left(3AO + \frac{3}{2}EB\right) =$$
$$= \pi AG\,(2AO + EB).$$

$$\mathrm{Vol}\,T = \frac{1}{3}\pi GK(\overline{EG}^2 + \overline{DK}^2 + EG.DK - \overline{BG}^2 - \overline{CK}^2 - BG.CK) =$$

$$= \frac{1}{3}\pi GK\left\{(EG+BG)(EG-BG)+(DK+CK)(DK-CK)+EG.DK-\right.$$
$$\left.-BG.CK\right\} = \frac{1}{3}\pi GK\,(AE.EB + AE.DC + EG.DK - BG.CK).$$

Cela posé, on a $OA - R$; $AE = 2R$;

$$PB = \frac{R}{2}\sqrt{10-2\sqrt{5}};\quad AG = \frac{PB}{2} = \frac{R}{4}\sqrt{10-2\sqrt{5}};$$

$$EY = \sqrt{R^2 - \overline{AG}^2} = \sqrt{R^2 - \frac{R^2}{16}(10-2\sqrt{5})} = \frac{R}{4}\sqrt{6+2\sqrt{5}}$$
$$= \frac{R}{4}(1+\sqrt{5})\,;\ EB = 2EY = \frac{R}{2}(1+\sqrt{5});\ DC = C = \frac{R}{2}(\sqrt{5}-1);$$

$$AK = r = \sqrt{R^2 - \frac{C^2}{4}} = \sqrt{R^2 - \frac{R^2}{16}(6 - 2\sqrt{5}} =$$
$$= \frac{R}{4}\sqrt{10+2\sqrt{5}};\ GK = AK - AG = \frac{R}{4}\left(\sqrt{10+2\sqrt{5}} -\right.$$
$$\left.-\sqrt{10-2\sqrt{5}}\right) = \frac{R}{4}\sqrt{20-8\sqrt{5}} = \frac{R}{2}\sqrt{5-2\sqrt{5}};$$

$$BG = \sqrt{\overline{AB}^2 - \overline{AG}^2} = \frac{R}{4}\sqrt{14-6\sqrt{5}} = \frac{R}{4}(3-\sqrt{5});$$

$$CK = R - \frac{C}{2} = \frac{R}{4}(5-\sqrt{5});\ EG = EB + BG = \frac{R}{4}(5+\sqrt{5})\,;$$
$$DK = C + CK = \frac{R}{4}(3+\sqrt{5}).$$

On a donc $\quad \mathrm{vol}T = \pi.\frac{R^2}{4}\sqrt{10-2\sqrt{5}}\left\{2R + \frac{R}{2}(1+\sqrt{5})\right\} =$

$$= \frac{\pi R^3}{8}(5+\sqrt{5})\sqrt{10-2\sqrt{5}} = \frac{\pi R^3}{8}\sqrt{200+40\sqrt{5}} =$$
$$= \frac{\pi R^3}{4}\sqrt{50+10\sqrt{5}}.$$

$$\mathrm{Vol\,T'} = \frac{1}{3}\pi . \frac{R}{2}\sqrt{5-2\sqrt{5}}\left\{R^2(1+\sqrt{5}) + R^2(\sqrt{5}-1) + \right.$$

$$\left. + \frac{R^2}{16}(5+\sqrt{5})(3+\sqrt{5}) - \frac{R^2}{16}(3-\sqrt{5})(5-\sqrt{5})\right\} =$$

$$= \frac{1}{2}.\frac{1}{3}.\frac{1}{16}\pi R^3\sqrt{5-2\sqrt{5}} . 48\sqrt{5} = \frac{1}{2}\pi R^3\sqrt{25-10\sqrt{5}};$$

partant,

$$\mathrm{vol\,T} + \mathrm{vol\,T'} = \frac{1}{4}\pi R^3\left\{\sqrt{50+10\sqrt{5}} + 2\sqrt{25-10\sqrt{5}}\right\}.$$

En élevant la parenthèse au carré et en extrayant la racine carrée, on obtient :

$$\mathrm{vol\,T} + \mathrm{vol\,T'} = \frac{1}{4}\pi R^3\sqrt{250-50\sqrt{5}} = \frac{5}{4}\pi R^3\sqrt{10-2\sqrt{5}};$$

et par suite, $\mathrm{vol\,D} = \frac{5}{2}\pi R^3\sqrt{10-2\sqrt{5}}$,

expression égale à la surface engendrée par le pentagone, multipliée par la moitié du rayon R ; résultat remarquable.

7° Dodécagone.

$$\mathrm{Vol\,D} = 2\,\mathrm{vol\,ABCDEFG} = 2\,(\mathrm{vol\,T} + \mathrm{vol\,T'} + \mathrm{vol\,T''}).$$

$$\mathrm{Vol\,T} = \mathrm{vol\,ABFG} = \mathrm{vol\,AHFC} - \mathrm{vol\,ABH} =$$

$$= \frac{1}{3}\pi AH\,(\overline{AC}^2 + \overline{HF}^2 + AG\,.\,HF - \overline{BH}^2) =$$

$$= \frac{1}{3}\pi AH\left\{\overline{AG}^2 + (HF+BH)(HF-BH) + AG\,.\,HF\right\} =$$

$$= \frac{1}{3}\pi AH\,.\,AG(AG+BF+HF) = \frac{2}{3}\pi AH\,.\,AO\,(3AO + \frac{3}{2}BF) =$$

$$= \pi AH\,.\,AO(2AO+BF).$$

$$\mathrm{Vol\,T'} = \mathrm{vol\,HFEK} - \mathrm{vol\,HBCK} =$$

$$= \frac{1}{3}\pi KH(\overline{HF}^2 + \overline{KE}^2 + HF.\,KE - \overline{BH}^2 - \overline{CK}^2 - BH\,.\,CK) =$$

$$= \frac{1}{3}\pi KH\left\{(HF+BH)(HF-BH) + (KE+CK)(KE-KC)\right. +$$

$$\left. + HF\,.\,KE - BH\,.\,CK\right\} =$$

$$= \frac{1}{3}\pi KH \left\{ AG \,.\, BF + AG \,.\, CE + HF \,.\, KE - BH \,.\, CK \right\}.$$

$$\text{Vol}\, T'' = \text{vol}\, KEDL - \text{vol}\, KCDL =$$

$$\frac{1}{3}\pi KL\left(\overline{KE}^2 + \overline{DL}^2 + EK\,.\,DL - \overline{DL}^2 - \overline{CK}^2 - DL\,.\,CK\right) =$$

$$= \frac{1}{3}\pi KL \left\{ KE(KE+DL) - (DL+CK)\,CK \right\}.$$

Or, on trouve facilement $OA = RB = CE = DL = R$; $AG = 2R$; $AH = OV = \frac{1}{2}R$; $FB = R\sqrt{3}$; $AB = C = = R\sqrt{2-\sqrt{3}}$; $BH = \sqrt{C^2 - \overline{AH}^2} = \frac{R}{2}\sqrt{7-4\sqrt{3}} = = \frac{R}{2}(2-\sqrt{3})$; $FH = FB + BH = R\sqrt{3} + \frac{R}{2}(2-\sqrt{3}) = \frac{R}{2}(2+\sqrt{3})$; $KE = \frac{3}{2}R$; $CK = EK - EC = \frac{1}{2}R$; $DY = \frac{1}{2}R$; $KL = CY = \sqrt{C^2 - \frac{R^2}{4}} = \frac{R}{2}(2-\sqrt{3})$; $HK = AL - AH - KL = \frac{R}{2} - \frac{1}{2}R\,(2-\sqrt{3}) = \frac{1}{2}R\,(\sqrt{3}-1)$.

Donc $\text{vol}\, T = \pi\,.\,\frac{R^2}{2}(2R + R\sqrt{3}) = \frac{1}{2}\pi R^3(2+\sqrt{2})$;

$$\text{Vol}\, T' = \frac{1}{3}\pi\,.\,\frac{R}{2}\,(\sqrt{3}-1)\left\{2R^2\sqrt{3} + 2R^2 + \frac{3}{4}R^2(2+\sqrt{2}) - \right.$$

$$\left. - \frac{R^2}{4}(2-\sqrt{2})\right\} = \frac{1}{3}\,.\,\frac{1}{8}\pi R^3(\sqrt{3}-1)\,(12+12\sqrt{3}) =$$

$$= \frac{1}{2}\pi R^3(\sqrt{3}-1)\,(\sqrt{3}+1) = \pi R^3.$$

$$\text{Vol}\, T'' = \frac{1}{3}\pi\,.\,\frac{R}{2}(2-\sqrt{2})\left\{\frac{3}{2}R\left(\frac{3}{2}R+R\right) - \frac{1}{2}R\left(\frac{1}{2}R+R\right)\right\} =$$

$$= \frac{1}{3}\,.\,\frac{1}{8}\pi R^3(2-\sqrt{2})\,.\,12 = \frac{1}{2}\pi R^3(2-\sqrt{2}).$$

Partant, $\text{vol}\, T + \text{vol}\, T' + \text{vol}\, T'' = \frac{1}{2}\pi R^3 . 6 = 3\pi R^3$,

et par suite, $\text{vol}\, D = 6\pi R^3$,

résultat d'une simplicité remarquable.

Nous allons maintenant vérifier ces résultats par la méthode de Guldin. D'abord on trouve facilement pour expressions des surfaces des polygones réguliers en fonction de R :

$$\text{surf}\,T = \frac{3}{4}R^2\sqrt{3}\,;\ \ \text{surf}\,C = 2R^2\,;$$

$$\text{surf}\,P = \frac{5}{8}R^2\sqrt{10+2\sqrt{5}}\,;\ \ \text{surf}\,H = \frac{3}{2}R^2\sqrt{3}\,;$$

$$\text{surf}\,O = 2R^2\sqrt{2}\,;\ \ \text{surf}\,DC'C = \frac{5}{4}R^2\sqrt{5-2\sqrt{5}}\,;$$

$$\text{surf}\,Dod = 3R^2.$$

On a donc, en appliquant la formule $\text{vol}\,Pol = S\,.\,2\pi R$:

$$\text{vol}T = \frac{3}{4}R^2\sqrt{3}\,.\,2\pi R = \frac{3}{2}\pi R^3\sqrt{3}\,;\ \text{vol}\,C = 2R^2\,.\,2\pi R = 4\pi R^3;$$

$$\text{vol}P = \frac{5}{8}R^2\sqrt{10+2\sqrt{5}}\,.\,2\pi R = \frac{5}{4}\pi R^3\sqrt{10+2\sqrt{5}}\,;$$

$$\text{vol}\,H = \frac{3}{2}R^2\sqrt{3}\,.\,2\pi R = 3\pi R^3\sqrt{3}\,;$$

$$\text{vol}O = 2R^2\sqrt{2}\,.\,2\pi R = 4\pi R^3\sqrt{2}\,;$$

$$\text{vol}\,D = \frac{5}{4}R^2\sqrt{10+2\sqrt{5}}\,.\,2\pi R = \frac{5}{2}\pi R^3\sqrt{10-2\sqrt{5}}\,;$$

$$\text{vol}\,Dod = 3R^2\,.\,2\pi R = 6\pi R^3\,;$$

résultats identiques avec ceux que nous venons de trouver par la géométrie.

On pourrait encore, en suivant l'une des deux marches que nous venons d'employer, trouver les expressions des mêmes volumes, soit en fonction du côté c, soit en fonction de l'apothème r.

Par exemple, en fonction du côté, on trouverait :

$$\text{vol}T = \frac{1}{2}\pi c^3\,;\ \ \text{vol}C = \pi c^3\sqrt{2}\,;\ \ \text{vol}P = \frac{1}{4}\pi c^3(5+3\sqrt{5})\,;$$

$$\text{vol}H = 3\pi c^3\sqrt{3}\,;\ \text{vol}O = 2\pi c^3\sqrt{20+14\sqrt{5}}\,;$$

$$\text{vol}\,Dc'c = \frac{5}{2}\pi c^3\sqrt{50+22\sqrt{5}}\,;\ \text{vol}\,Dod = 3\pi c^3(3\sqrt{6}+5\sqrt{2}).$$