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PRÉFACE.

In the past twenty years (since 1938) there has been developed, as a sequel to the theory of approximation by polynomials and by other rational functions of one complex variable, a theory of approximation by functions analytic and bounded in a given region. This new theory thus studies approximation by functions which may be regarded as the most useful non-trivial functions, analytic in a given region which is not merely the plane with one or more points deleted. This new theory has application to the study of approximation by polynomials and by more general rational functions, but applies also to topics in numerical analysis, and indeed is of significance whenever a sequence of functions analytic in a region D converges in a subregion of D.

The purpose of the present essay is to set forth both in broad outline and in detail some of the principal results of the new theory, including some previously unpublished methods and results, and to indicate promising directions for future research. This essay can be read independently of any other treatment of approximation, although naturally occasional proofs are merely sketched or omitted. There is included much of the pertinent theory of approximation by polynomials and other rational functions, although the new theory is not intended in any way to supersede the old.

While the theory here described, including the related theory of approximation by rational functions, has been unfolding in recent years, it has been the present writer's privilege to be personally associated with other workers in the field, notably H. G. Russell, J. L. Doob, J. H. Curtiss, W. E. Sewell, Y. C. Shen, E. N. Nilson, A. Spitzbart, H. M. Elliott, P. Davis, J. P. Evans and A. Sinclair.

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J. L. WALSH.

INTRODUCTION.

APPROXIMATION TO $f(z)$ ON A CLOSED SET E BY FUNCTIONS ANALYTIC AND BOUNDED IN A REGION D CONTAINING E.

Beyond the study of the possibility of uniform approximation to real functions by polynomials or by trigonometric polynomials (Weierstrass), there has been developed in the past half-century a theory relating order (degree) of approximation by polynomials or by trigonometric polynomials of given degree to the continuity properties of the functions approximated (de la Vallée Poussin, Lebesgue, D. Jackson, S. Bernstein, Montel).

An analogue of this theory has later been developed, study of approximation on a closed point set E in the plane of the complex variable z, to a function $f(z)$ given on E, by polynomials or more general rational functions of z; here the main problem is as before to relate degree of approximation on E on the one hand to continuity properties (including analyticity, existence of derivatives on the boundary of E, Lipschitz conditions on such derivatives, etc.) of $f(z)$ on E on the other hand (1). Analyticity of $f(z)$ on E is related to geometric degree of approximation, weaker continuity properties to weaker degree of approximation.

(1) We shall not need to use general results on the *possibility* of uniform approximation by bounded analytic functions, but mention by way of background the following [1935, chap. II, theorem 15; chap. I, theorem 8]: *Let the closed point set E be bounded by a finite number of mutually disjoint Jordan curves and let $f(z)$ be analytic in the interior points of* E *and continuous on* E, *or let* E *be an arbitrary closed set and $f(z)$ analytic on* E. *Let points z_k be given, at least one in each of the regions into which* E *separates the plane. Then on* E *the function $f(z)$ can be approximated as closely as desired by a rational function whose poles lie in the z_k.*

Dates in square brackets refer to works mentioned in the Bibliography.

Similar problems arise in the plane of the complex variable if E lies interior to a region D and $f(z)$ is approximated on E by functions $\varphi_n(z)$ required merely to be analytic and bounded : $|\varphi_n(z)| \leqq M_n$ in D. Thus we relate degree of approximation to $f(z)$ on E expressed in terms of asymptotic properties of M_n on the one hand to continuity properties of $f(z)$ on E on the other hand. This problem (in the continued notation already introduced) is the primary topic of the present essay; the theory here set forth essentially includes many phases of the problem of approximation even by polynomials and rational functions. The theory is by no means complete in the sense that no further open questions exist; nevertheless the main outline of a complete theory now seems to be taking shape, and appropriate indications for continued research seem clear.

Our study is divided into several parts: Problem A deals with a function $f(z)$ analytic on E, and with geometric degree of convergence as measured in terms of M_n; Problem α deals with weaker properties than analyticity (e. g. existence of derivatives and Lipschitz conditions) of $f(z)$ on E and slower than geometric degree of convergence; Problem β deals with such weaker properties of $f(z)$ not on E but on a closed set E_1 containing E and contained in D, where $f(z)$ is analytic on E but not throughout E_1, and degree of convergence on E is geometric but expressed with various refinements depending upon the properties of $f(z)$ on E_1. This general topic of approximation by bounded analytic functions has been treated in a number of separate papers (*see* Bibliography), but no combined exposition has hitherto been available, even in outline.

Chapter I deals with Problem A, chapters II and III with Problems α and β, in each case giving the main features of the theory with some detailed proofs. Chapter IV is devoted to a summary of further related results, mainly without proofs.

CHAPTER I.

Problem A : $f(z)$ analytic on E.

1.1. E the unit disc, D a concentric disc. — The Taylor development is both a principal tool in the study of analytic functions and a model for other series expansions, especially those defined by interpolation. So we present first a relatively simple geometric situation [1946], for the purpose of indicating to the reader our general problems and methods without topological complications.

For convenient reference we state some well known properties of the Taylor development

$$(1.1.1) \qquad f(z) \equiv a_0 + a_1 z + a_2 z^2 + \ldots$$

of a function $f(z)$ analytic in the disc $|z| < \rho$, but analytic throughout no larger concentric disc (1). Then we have (Cauchy-Hadamard)

$$(1.1.2) \qquad \lim_{n \to \infty} \sup |a_n|^{\frac{1}{n}} = \frac{1}{\rho},$$

whence if we set

$$(1.1.3) \qquad S_n(z) \equiv a_0 + a_1 z + \ldots + a_n z^n,$$

we have also

$$(1.1.4) \quad \lim_{n \to +\infty} \sup [\max |f(z) - S_n(z)|, \text{ for } |z| \leq r]^{\frac{1}{n}} = \frac{r}{\rho} \qquad (r < \rho),$$

$$(1.1.5) \quad \lim_{n \to \infty} \sup [\max |S_n(z)|, \text{ for } |z| \leq r]^{\frac{1}{n}} = \frac{r}{\rho} \qquad (r \geq \rho).$$

The fact that the first member of (1.1.4) is not greater than the second member is an immediate consequence of (1.1.2); if the first member of (1.1.4) is less than the second member, we have for n sufficiently large and for $|z| = r$

$$|f(z) - S_n(z)| \leq \frac{r_1^n}{\rho^n} \qquad (r_1 < r),$$

$$|S_n(z) - S_{n-1}(z)| = |a_n| r^n \leq \frac{r_1^n}{\rho^n} + \frac{r_1^{n-1}}{\rho^{n-1}} = \frac{A r_1^n}{\rho^n},$$

in contradiction to (1.1.2). Equation (1.1.5) may be established similarly. These equations are to be used in proving:

THEOREM 1.1.1. — *Let the function $f(z)$ be analytic in the disc $|z| < \rho\ (> 1)$ but not continuable so as to be analytic throughout*

(1) A function is *analytic* on a point set if it is single valued there and if in some neighborhood of each point of the set it can be represented by a convergent power series.

any larger concentric disc. For each $M (> 0)$ *let* $\varphi_M(z)$ *denote the (or a) function analytic and in modulus not greater than* M *in the disc* $D : |z| \leq R (> \rho)$ *for which*

$$(1.1.6) \qquad m_M = [\max |f(z) - \varphi_M(z)|, \ z \text{ on } E], \qquad E : \ |z| \leq 1,$$

is least. Then we have

$$(1.1.7) \qquad \lim_{M \to \infty} \sup m_M^{\frac{1}{\log M}} = \exp\left[\frac{-\log \rho}{\log R - \log \rho}\right].$$

The existence of the extremal function $\varphi_M(z)$ follows readily by the use of the Montel theory of normal families. The uniqueness has been established in this case by Agmon (unpublished). To study the asymptotic relationship between M and m_M we use the $S_n(z)$ as comparison approximating functions.

If $\rho_1 (< \rho)$ is arbitrary, we have from (1.1.4)

$$(1.1.8) \qquad |f(z) - S_n(z)| \leq \frac{A_1}{\rho_1^n} \qquad (|z| \leq 1),$$

and from (1.1.5)

$$(1.1.9) \qquad |S_n(z)| \leq \frac{A_2 R^n}{\rho_1^n} \qquad (|z| \leq R);$$

here and in the sequel the numbers A with or without subscripts usually represent constants independent of n and z, constants which may vary from one formula to another. The functions $S_n(z)$ form a sequence whereas the functions $\varphi_M(z)$ depend on a continuous parameter; in order to use the $S_n(z)$ for comparison we now relate M and n by the inequalities

$$(1.1.10) \qquad \frac{A_2 R^n}{\rho_1^n} \leq M < \frac{A_2 R^{n+1}}{\rho_1^{n+1}},$$

so it follows from (1.1.9) that for M sufficiently large $S_n(z)$ is one of the competing functions in the class whose extremal function is $\varphi_M(z)$. Then we have by (1.1.6) and (1.1.8) and by the first of inequalities (1.1.10)

$$(1.1.11) \qquad m_M \leq \frac{A_1}{\rho_1^n} \leq \frac{A_1 M}{A_2 R^n}.$$

From the second of inequalities (1.1.10) we may write

$$\frac{\log M - \log A_2}{\log R - \log \rho_1} < n + 1,$$

so the extreme inequality of (1.1.11) implies

$$m_M \leqq A_3 M \exp\left[-\frac{(\log M - \log A_2)\log R}{\log R - \log \rho_1}\right],$$

for M sufficiently large. When $M \to \infty$ and then $\rho_1 \to \rho$ we now deduce

$$(1.1.12) \qquad \lim_{M\to\infty} \sup m_M^{\frac{1}{\log M}} \leqq \exp\left[\frac{-\log\rho}{\log R - \log\rho}\right].$$

To complete the proof of (1.1.7) we use Hadamard's three-circle theorem, to the effect that for an analytic function $\Phi(z)$, the function $\log[\max|\Phi(z)|$, for $|z| = r]$ is a convex function of $\log r$. It turns out that the strong inequality in (1.1.12) would imply the uniform convergence of a sequence of the functions $\varphi_M(z)$ throughout a region $|z| < r\,(>\rho)$, which contradicts the definition of ρ. We choose the specific values $M = e^n\,(n = 1, 2, \ldots)$, and denote the corresponding extremal functions $\varphi_M(z)$ by $\Phi_n(z)$ respectively. The (Fatou) boundary values of $\Phi_n(z)$ on the circumference $|z| = R$ exist, and for those values we have

$$(1.1.13) \qquad \begin{cases} |\Phi_n(z)| \leqq e^n, \qquad |\Phi_{n+1}(z)| \leqq e^{n+1}, \\ \qquad |\Phi_{n+1}(z) - \Phi_n(z)| \leqq 2e^{n+1}. \end{cases}$$

We assume the strong inequality in (1.1.12), whence for suitably chosen $R_1\,(\rho < R_1 < R)$ and for n sufficiently large

$$m_M^{\frac{1}{\log M}} \leqq \exp\left[\frac{-\log R_1}{\log R - \log R_1}\right],$$

so by (1.1.6) for $M = e^n$ and $M = e^{n+1}$ on the circle $|z| = 1$ we have

$$(1.1.14) \qquad |\Phi_{n+1}(z) - \Phi_n(z)| \leqq 2\exp\left[\frac{-n\log R_1}{\log R - \log R_1}\right].$$

The last inequality of (1.1.13), together with (1.1.14), yields by the three circle theorem (which indeed applies not only to the maximum of the modulus of an analytic function on three circumferences, but also to the superieur limit of the maximum modulus of a sequence on these circumferences)

$$\lim_{n\to\infty} \sup\,[\max|\Phi_{n+1}(z) - \Phi_n(z)|, \text{ for } |z| = r]^{\frac{1}{n}} \leqq \exp\left[\frac{\log r - \log R_1}{\log R - \log R_1}\right]$$
$$(1 \leqq r \leqq R);$$

this last member is less than unity for every $r < R_1$, hence is less than unity for some $r_0\,(\rho < r_0 < R_1)$. The sequence $\Phi_n(z)$ converges uniformly in $|z| \leq r_0$, and by (1.1.12) converges to $f(z)$ in $|z| \leq 1$, so $f(z)$ can be continued so as to be analytic throughout $|z| < r_0$ $(> \rho)$, contrary to hypothesis.

As an immediate application of theorem 1.1.1 we prove :

THEOREM 1.1.2. — *Under the conditions of theorem* 1.1.1 *let* $\psi_M(z)$ *represent a set of arbitrary functions analytic in* D *and satisfying there the respective inequalities* $|\psi_M(z)| \leq M$. *If we set*

$$\mu_M = [\max |f(z) - \psi_M(z)|,\ z \text{ on } E],$$

we have

$$\lim_{M\to\infty} \sup \mu_M^{\frac{1}{\log M}} \geq \exp\left[\frac{-\log\rho}{\log R - \log\rho}\right]. \tag{1.1.15}$$

From the definition of the $\varphi_M(z)$ follows $\mu_M \geq m_M$, so (1.1.15) follows from (1.1.7).

We emphasize the fact that the conditions on the $\psi_M(z)$ are very light; the conclusion applies to very many sets $\psi_M(z)$ converging uniformly in a circle. Theorem 1.1.2 and its analogues (which for the most part are henceforth left to the reader for formulation) are among the most interesting results set forth in the present essay, due to the light hypothesis and the frequency with which the situation occurs in analysis. Although in theorems 1.1.1 and 1.1.2 we have required M to become continuously infinite, it is in fact sufficient if M becomes monotonically infinite through a sequence of values M_n such that the quotients $\dfrac{\log M_n}{\log M_{n+1}}$ approach unity; the original form of (1.1.7) follows if we set $\varphi_M(z) \equiv \varphi_{M_n}(z)$, $M_n \leq M < M_{n+1}$.

Further use of the three circle theorem (details are similar to those above) establishes :

COROLLARY 1.1.1. — *Let the functions* $\varphi_M(z)$ (*extremal or not*) *for every* M *sufficiently large be analytic and in modulus not greater than* M *in the region* $|z| < R\,(> 1)$, *let* $f(z)$ *be defined on* $E : |z| = 1$, *and let* m_M *be defined by* (1.1.6). *If we have*

$$\lim_{M\to\infty} \sup m_M^{\frac{1}{\log M}} \leq \exp\left[\frac{-\log\rho}{\log R - \log\rho}\right], \tag{1.1.16}$$

then we have

$$(1.1.17)\quad \left\{ \begin{array}{l} \lim_{M\to\infty} \sup [\max |f(z)-\varphi_M(z)|, \textit{for } |z|=r]^{\frac{1}{\log M}} \leqq \exp\left[\dfrac{\log r - \log\rho}{\log R - \log\rho}\right] \\ \qquad\qquad (1 \leqq r < \rho), \end{array}\right.$$

$$(1.1.18)\quad \left\{ \begin{array}{l} \lim_{M\to\infty} \sup [\max |\varphi_M(z)|, \textit{for } |z|=r]^{\frac{1}{\log M}} \leqq \exp\left[\dfrac{\log r - \log\rho}{\log R - \log\rho}\right] \\ \qquad\qquad (\rho \leqq r \leqq R); \end{array}\right.$$

consequently $f(z)$ *can be extended from* E *so as to be analytic throughout* $|z|<\rho$. *If it is known that* $f(z)$ *cannot be extended from* E *so as to be analytic throughout any* $|z|<\rho'$ $(\rho'>\rho)$, *then the equality signs hold in* (1.1.16), (1.1.17) *and* (1.1.18).

Many results in the sequel admit analogues of corollary 1.1.1, which refer to the convergence of a given set of functions which are not necessarily extremal; henceforth also these analogues, which are readily stated and proved, are ordinarily left to the reader.

1.2. Analyticity in an annulus. — In theorem 1.1.1 we have required both $f(z)$ and the $\varphi_M(z)$ to be analytic throughout suitable discs, even though the details of the conclusion are concerned primarily with the annulus $1<|z|<R$. A result related to theorem 1.1.1 can be established, as we now show, when $f(z)$ and the $\varphi_M(z)$ are analytic merely in suitable annuli.

A function $f(z)$ analytic in an annulus $1<|z|<\rho$ is expressed there by Cauchy's integral formula

$$(1.2.1)\quad f(z) \equiv f_1(z) + f_2(z) \qquad (1<|z|<\rho),$$

$$(1.2.2)\quad f_1(z) \equiv \frac{1}{2\pi i}\int_{|t|=\rho-\varepsilon} \frac{f(t)\,dt}{t-z} \qquad (|z|<\rho,\ 0<\varepsilon<\rho-1),$$

$$(1.2.3)\quad f_2(z) \equiv \frac{1}{2\pi i}\int_{|t|=1+\varepsilon} \frac{f(t)\,dt}{t-z} \qquad (|z|>1);$$

the function $f_1(z)$ defined by (1.2.2) is analytic throughout the disc $|z|<\rho$, where the integral is taken in the counterclockwise sense, $0<\varepsilon<\rho-|z|$; the function $f_2(z)$ defined by (1.2.3) is analytic throughout $|z|>1$, where the integral is taken in the clockwise sense, $\varepsilon<|z|-1$. We consider $f_1(z)$ and $f_2(z)$ as the *components* of $f(z)$. If $f(z)$ is continuous (i. e. in the two-dimensional sense) on $|z|=1$, so also is $f_2(z)$ as defined on $|z|=1$ by (1.2.1) or by continuity from (1.2.3); likewise if $f(z)$ is continuous on $|z|=\rho$, so also is $f_1(z)$. An analogue of theorem 1.1.1 now suggests itself :

THEOREM 1.2.1. — *Let the function $f^0(z)$ be analytic in the annulus $1<|z|<\rho$ but not continuable so as to be analytic throughout any annulus $1<|z|<\rho'$ $(\rho'>\rho)$, and let $f^0(z)$ be continuous on $|z|=1$. For each $M(>0)$ let $\varphi_M^0(z)$ denote the (or a) function analytic and in modulus not greater than M in the annulus $1<|z|<R\,(>\rho)$ for which*

$$(1.2.4)\quad m_M=\left[\max|f^0(z)-\varphi_M^0(z)|,\ z \text{ on E}\right],\qquad \text{E}:\ |z|=1,$$

is least. Then (1.1.7) is valid.

Set $f^0(z)\equiv f_1(z)+f_2(z)$ as in (1.2.1), where the components $f_1(z)$ and $f_2(z)$ are defined by the analogues of (1.2.2) and (1.2.3), and $f_2(z)$ is continuous on E, analytic in $|z|>1$. Then $f_1(z)$ is analytic in $|z|<\rho$, and by theorem 1.1.1 there exist functions $\theta_N(z)$ analytic with $|\theta_N(z)|\leq N$ in $|z|<R$ such that

$$(1.2.5)\quad \limsup_{N\to\infty}\left[\max|f_1(z)-\theta_N(z)|,\ z \text{ on E}\right]^{\frac{1}{\log N}}=\exp\left[\frac{-\log\rho}{\log R-\log\rho}\right].$$

We now set $\varphi_M(z)\equiv\theta_N(z)+f_2(z)$ in $1\leq|z|<R$, whence

$$f_1(z)-\theta_N(z)\equiv f^0(z)-\varphi_M(z)\quad\text{on E},$$

and for suitably chosen N_0 we have

$$|\varphi_M(z)|\leq M=N+N_0\qquad\text{in }\ 1\leq|z|<R,$$

where N_0 is independent of M and z. From (1.2.5) follows (1.1.12), with the notation (1.2.4), and the equality sign follows precisely as in the proof of theorem 1.1.1. Theorem 1.2.1 is established.

We remark however that proof of the existence of the extremal function $\varphi_M^0(z)$ minimizing (1.2.4) requires additional discussion. We need not assume $f^0(z)$ continuous on E, but do assume $f^0(z)$ bounded in a neighborhood of E, and in (1.2.4) we use the least upper bound and (Fatou) boundary values of $f^0(z)$ and $\varphi_M^0(z)$ almost everywhere on E. For fixed M suppose we have $\psi_n(z)$ analytic and $|\psi_n(z)| \leq M$ in $1 < |z| < R$, with

$$\mu_n = [\sup |f^0(z) - \psi_n(z)|, \ z \text{ on } E] \to m_M,$$

where m_M is the greatest lower bound of the expression

$$[\sup |f^0(z) - \psi(z)|, \ z \text{ on } E]$$

for all $\psi(z)$ analytic with $|\psi(z)| \leq M$ in $1 < |z| < R$. Let ρ_1 be fixed, $1 < \rho_1 < \rho$. For some A_1 we have

$$|f^0(z) - \psi_n(z)| \leq A_1 \quad \text{on} \quad |z| = \rho_1,$$

where A_1 is independent of n, and on $|z| = 1$ we have

$$|f^0(z) - \psi_n(z)| \leq \mu_n.$$

If $\psi_0(z)$ is a limit function of the sequence $\psi_n(z)$ in $1 < |z| < R$, we have

$$|\psi_0(z)| \leq M \quad \text{in} \quad 1 < |z| < R,$$

and in $1 < |z| < \rho_1$ the function $\log|f^0(z) - \psi_0(z)|$ is dominated by the harmonic function whose value on $|z| = \rho_1$ is $\log A_1$ and whose value on $|z| = 1$ is $\log m_M$. Consequently (1.2.4), involving boundary values of the functions on E, is valid with $\varphi_M^0(z) \equiv \psi_0(z)$.

A new result for an arbitrary annulus A can be found from theorem 1.2.1 by mapping A onto an annulus $1 < |z| < R$:

THEOREM 1.2.2. — *Let* A *be an annulus bounded by two Jordan curves* C_0 *and* C_1, *with* C_0 *interior to* C_1, *let* $u(z)$ *be harmonic in* A, *continuous in* $A + C_0 + C_1$, *and equal to zero and unity on* C_0 *and* C_1 *respectively. Let* C_σ *denote generically the Jordan curve* $u(z) = \sigma$ $(0 < \sigma < 1)$ *in* A, *and let* Γ_σ *denote the annulus bounded by* C_0 *and* C_σ.

Let $f(z)$ *be analytic throughout* Γ_ρ *but not continuable so as to be analytic throughout any* Γ_σ $(\sigma > \rho)$, *and let* $f(z)$ *be continuous on* C_0. *For each* $M (> 0)$ *let* $\varphi_M(z)$ *denote the (or a) function analytic and in modulus not greater than* M *in* A *such that*

$$m_M = [\max |f(z) - \varphi_M(z)|, \ z \text{ on } C_0] \tag{1.2.6}$$

is least. Then we have

$$\limsup_{M \to \infty} m^{\frac{1}{\log M}} = e^{\frac{\rho}{\rho - 1}} \tag{1.2.7}$$

If A is the annulus $1 < |z| < R$, we have $u(z) \equiv \frac{\log |z|}{\log R}$, and $u(z)$ is invariant under conformal mapping.

The analogue of corollary 1.1.1 is of course valid, and indeed is a consequence of corollary 1.1.1. We prefer to formulate for later use a more general result :

Corollary 1.2.2. — *Let* D *be a region containing a closed set* E, *and let* D — E *be connected. Let* $u(z)$ *exist, harmonic in* D — E, *continuous and equal to zero and unity on the boundaries of* E *and* D *respectively. Let* C_σ *denote generically the locus* $u(z) = \sigma$ $(0 \leqq \sigma \leqq 1)$ *in the closure of* D — E, *and let* Γ_σ *denote the set* $0 < u(z) < \sigma$ *in* D — E. *Let* $f(z)$ *be defined on* C_0, *and for each* $M (> 0)$ *let* $\varphi_M(z)$ *extremal or not be analytic with* $|\varphi_M(z)| \leqq M$ *in* D — E. *If the first member of* (1.2.7) *is not greater than the second member, and* m_M *is given by* (1.2.6), *we have*

$$(1.2.8)\quad \limsup_{M \to \infty} \left[\max |f(z) - \varphi_M(z)|,\ z \text{ on } C_\sigma\right]^{\frac{1}{\log M}} \leqq e^{\frac{\rho-\sigma}{\rho-1}} \qquad (0 \leqq \sigma < \rho),$$

$$(1.2.9)\quad \limsup_{M \to \infty} \left[\max |\varphi_M(z)|,\ z \text{ on } C_\sigma\right]^{\frac{1}{\log M}} \leqq e^{\frac{\rho-\sigma}{\rho-1}} \qquad (\rho \leqq \sigma \leqq 1);$$

consequently $f(z)$ *can be extended from* E *so as to be analytic throughout* Γ_ρ. *If it is known that* $f(z)$ *cannot be extended from* E *so as to be analytic throughout any* $\Gamma_{\rho'}$ $(\rho' > \rho)$, *then the equality signs hold in* (1.2.7), (1.2.8) *and* (1.2.9).

Corollary 1.2.2 remains valid if we allow M to become infinite merely through a monotonic sequence of values M_n such that the quotients $\frac{\log M_{n+1}}{\log M_n}$ approach unity.

The proof of corollary 1.2.2 can be given at once by use of the Nevanlinna two-constant theorem, a generalization of the three-circle theorem, which for the present purposes merely expresses the fact that a suitable upper bound for $\varphi_M(z)$ $(M = e^n)$ in D — E is readily calculated in terms of $e^{u(z)}$, thanks to our hypothesis concerning $\varphi_M(z)$ on C_0 and C_1. Indeed the two-constant theorem applies directly to the superior limit of the maximum modulus of a sequence of functions $\psi_n(z)$ analytic and bounded in A, on three level loci of $u(z)$;

we deduce

$$\limsup_{n\to\infty} [\max |\psi_n(z)|, z \text{ on } C_\sigma]^{\frac{1}{n}} \leqq e^{a+(b-a)\sigma},$$

where we assume

$$\limsup_{n\to\infty} [\max |\psi_n(z)|, z \text{ on } C_0]^{\frac{1}{n}} \leqq e^{a},$$

$$\limsup_{n\to\infty} [\max |\psi_n(z)|, z \text{ on } C_1]^{\frac{1}{n}} \leqq e^{b}.$$

The harmonic function $a + (b - a)\, u(z)$ dominates

$$\limsup_{n\to\infty} \log \left\{ [\max |\psi_n(z)|, z \text{ on } C_\sigma]^{\frac{1}{n}} \right\} \text{ for every } \sigma \qquad (0 \leqq \sigma \leqq 1).$$

Inequalities (1.2.8) and (1.2.9) thus proved [as in 1940] for $M = e^n$, hold also for functions $\Phi_M(z)$ defined as $\varphi_{e^n}(z)$ for $e^n \leqq M < e^{n+1}$, and combine with the hypothesis on the $\varphi_M(z)$ to prove the corollary,

The functions $\varphi_M(z)$ need not be continuous on C_0, but if not we use the (Fatou) boundary values on C_0 and the least upper bound in (1.2.6). An inequality similar to (1.1.14) shows that the sequence $\varphi_M(z)$ $(M = e^n)$ converges uniformly in a neighborhood of C_0, and the limit function $f(z)$ is bounded in such a neighborhood, and possesses the given $f(z)$ as boundary values almost everywhere on C_0.

To complete the analogy with the case that C_0 and C_1 are concentric circles, one may inquire whether in theorem 1.2.2 the $\varphi_M(z)$ can be chosen analytic throughout the closed interior of C_1 if $f(z)$ is given analytic throughout the interior of C_ρ. We prove

THEOREM 1.2.3 — *Under the conditions of theorem 1.2.2 let $f(z)$ be analytic throughout the interior of C_ρ; then the $\varphi_M(z)$ of theorem 1.2.2 can be chosen analytic (and in modulus not greater than M) throughout the closed interior of C_1.*

If $\varphi_M(z)$ is the function already defined and used in theorem 1.2.2, we consider its components $\varphi_{M1}(z)$ and $\varphi_{M2}(z)$ analytic respectively interior to C_1 and exterior to C_0; in A we have

$$\varphi_M(z) \equiv \varphi_{M1}(z) + \varphi_{M2}(z).$$

For z on C_0 we have $(0 < \varepsilon < \rho)$

$$(1.2.10) \qquad f(z) - \varphi_{M1}(z) \equiv \frac{1}{2\pi i}\int_{C_\varepsilon} \frac{[f(t) - \varphi_{M1}(t)]\,dt}{t - z}$$
$$\equiv \frac{1}{2\pi i}\int_{C_\varepsilon} \frac{[f(t) - \varphi_{M}(t)]\,dt}{t - z};$$

in fact, for z on C_0 the integral of $\frac{\varphi_{M2}(t)}{t-z}$ over C_ε equals the integral of $\frac{\varphi_{M2}(t)}{t-z}$ over a circle of variable radius r containing C_ε, and the latter (constant) integral approaches zero as $r \to \infty$ by virtue of $\varphi_{M2}(\infty) = 0$. We may now write from (1.2.10)

$$\left[\max |f(z) - \varphi_{M1}(z)|,\ z \text{ on } C_0\right] \leqq A_1\left[\max |f(z) - \varphi_{M}(z)|,\ z \text{ on } C_\varepsilon\right],$$

whence by (1.2.8)

$$(1.2.11) \qquad \limsup_{M \to \infty}\left[\max |f(z) - \varphi_{M1}(z)|,\ z \text{ on } C_0\right]^{\frac{1}{\log M}} \leqq e^{\frac{\rho - \varepsilon}{\rho - 1}},$$

and $\varepsilon \to 0$ yields the fact that the first member of (1.2.11) is not greater than the second member of (1.2.7).

To be sure, although the functions $\varphi_{M1}(z)$ are analytic interior to C_1, they have not been shown to satisfy $|\varphi_{M1}(z)| \leqq M$ there. Nevertheless we have for z in the neighborhood of C_1

$$\varphi_{M2}(z) = \frac{1}{2\pi i}\int_{C_\varepsilon} \frac{\varphi_M(t)\,dt}{t - z},$$

$$|\varphi_{M2}(z)| \leqq A_2 M, \qquad |\varphi_{M1}(z)| \leqq |\varphi_M(x)| + |\varphi_{M2}(z)| \leqq (1 + A_2)M.$$

Since $(1 + A_2)M$ is a bound for $|\varphi_{M1}(z)|$ in the neighborhood of C_1, it is also such a bound throughout the interior of C_1. We also have as $M \to \infty$

$$\frac{\log(1 + A_2)M}{\log M} = \frac{\log(1 + A_2) + \log M}{\log M} \to 1,$$

from which it follows that the first member of (1.2.11) is not greater than the second member of (1.2.7), where M now indicates a bound on the modulus of $\varphi_{M1}(z)$ interior to C_1. However, the strong inequality is not possible (compare corollary 1.2.2), so the proof of theorem 1.2.3 is complete.

A proposition weaker in some respects than theorems 1.2.2 and 1.2.3 and corollary 1.2.2 is still of interest :

Corollary 1.2.3. — *Let* C *be a Jordan curve, and let* $f(z)$ *be continuous on* C. *A necessary and sufficient condition that* $f(z)$ *be the boundary value of a function analytic in an annular region of which* C *is one of the two boundaries, is that there exist a sequence of functions* $f_n(z)$ *continuous on* C *and analytic in an annular region* A *of which* C *is one of the two boundaries, such that*

$$\limsup_{n\to\infty}[\max|f(z)-f_n(z)|,\ z \text{ on } C]^{\frac{1}{n}} < 1,$$

$$\limsup_{n\to\infty}[\max|f_n(z)|,\ z \text{ in } A]^{\frac{1}{n}} < \infty.$$

A necessary and sufficient condition that $f(z)$ *be analytic on* C *is that there exist a sequence of functions* $f_n(z)$ *analytic in an annular region* A *containing* C *(which separates the two boundary components of* A*) and satisfying these inequalities.*

If $f(z)$ is given, analytic respectively in an annular region of which C is one of the two bounding curves or in an annular region containing C, the annular region A may be chosen arbitrarily, satisfying the topological conditions mentioned.

The direct parts of corollary 1.2.3 may be proved by approximating separately the two components of the given $f(z)$, using theorem 1.2.3; the indirect parts may be proved by applying the two-constant theorem to the sequence $f_n(z)-f_{n-1}(z)$ in A; in the first case the application is made but once, using the annular region A; in the second case the application is made to the two annular regions into which C separates A.

The topological situation of theorem 1.2.2 can be generalized so that for instance C_0 is replaced by a Jordan arc.

Theorem 1.2.4. — *Let* D *be a simply connected region whose boundary contains more than one point, and let* E *be a continuum containing more than one point, interior to* D. *Let* A *denote the annular region* D — E, *and let* $u(z)$ *be harmonic interior to* A, *continuous in the closure* $\overline{A}$ *of* A, *equal to zero and unity on the* *boundaries of* E *and* D *respectively. Let* C_σ *denote generically the Jordan curve* $u(z) = \sigma$ $(0 < \sigma < 1)$ *in* $\overline{A}$, *and let* Γ_σ *denote the locus* $0 < u(z) < \sigma$ *in* A.

Let $f(z)$ *be analytic throughout* $\Gamma_\rho + E$, *but not continuable so as to be analytic throughout any* Γ_σ $(\sigma > \rho)$. *For each* $M(> 0)$ *let* $\varphi_M(z)$ *denote the (or a) function analytic and in modulus not greater than* M *in* D *such that*

$$(1.2.12) \qquad m_M = [\max |f(z) - \varphi_M(z)|, \ z \text{ on } E]$$

is least. Then we have

$$(1.2.13) \qquad \limsup_{M \to \infty} m_M^{\frac{1}{\log M}} = e^{\frac{\rho}{\rho - 1}}.$$

Let ε $(0 < \varepsilon < \rho)$ be arbitrary, and let us set

$$(1.2.14) \qquad m_M(\varepsilon) = [\max |f(z) - \varphi_{M\varepsilon}(z)|, \ z \text{ on } C_\varepsilon],$$

where $\varphi_{M\varepsilon}(z)$ denotes the (or a) function analytic and in modulus not greater than M in D such that (1.2.14) is least. Since E lies interior to C_ε we obviously have

$$(1.2.15) \qquad m_M \leq m_M(\varepsilon).$$

It is no loss of generality to assume D the interior of a Jordan curve C_1. The function harmonic in the annulus bounded by C_ε and C_1, continuous in the closure of the annulus and equal to zero and unity on C_ε and C_1 respectively is $u_\varepsilon(z) \equiv \frac{u(z) - \varepsilon}{1 - \varepsilon}$, which takes the value $\frac{\rho - \varepsilon}{1 - \varepsilon}$ on C_ρ. By theorem 1.2.3 we have

$$\limsup_{M \to \infty} [m_M(\varepsilon)]^{\frac{1}{\log M}} = e^{\frac{\rho - \varepsilon}{\rho - 1}};$$

use of (1.2.15) and approach of ε to zero implies that the first member of (1.2.13) is not greater than the second member. However, by corollary 1.2.2, the strong inequality is impossible, so theorem 1.2.4 follows.

1.3. Approximation by polynomials. — The discussion of Problem A (relation of regions of analyticity to geometric degree of convergence), culminating in theorems 1.2.2, 1.2.3 and 1.2.4, is entirely satisfactory so far as concerns approximation on a set E consisting of a single continuum by functions $\varphi_M(z)$ analytic and bounded in a simply connected region D containing E. The direct theorems [proof of the existence of the $\varphi_M(z)$] are based on the Taylor development, and the indirect theorems (proof of the analyticity properties of the approximated function when order of approximation is given) are based on the three-circle theorem. We now engage in the study of more complicated topological situations, notably point sets E consisting of several continua and regions D that are not simply connected. The two constant theorem (as in the proof of corollary 1.2.2) is adequate for the indirect theorems, but the Taylor development is not adequate for the direct theorems, and less familiar expansions are to be used.

A limiting case of a region D bounded by a continuum not a single point is that of the plane of finite points; boundedness of the approximating functions in D is then no longer feasible, but it is appropriate to study approximation by polynomials, as we now proceed to do [1].

The discussion of paragraph 1.2 is significant in the study of approximation by polynomials; we prove

THEOREM 1.3.1. — *Let* E *be a closed bounded set whose complement* K *is connected and possesses a Green's function* $g(z)$ *with pole at infinity. Let* E_R $(R > 1)$ *denote generically the locus* $g(z) = \log R$ *in* K. *If* $f(z)$ *is defined on* E, *and if* $\{p_n(z)\}$ *is a sequence of polynomials of respective degrees* n, *then the relation*

$$(1.3.1)\qquad \begin{cases} \limsup\limits_{n\to\infty} \mu_n^{\frac{1}{n}} = \dfrac{1}{\rho} \qquad (\rho > 1), \\ \mu_n = \max\left[\,|f(z) - p_n(z)|,\ z \text{ on } E\right], \end{cases}$$

implies that $f(z)$ *can be continued analytically from* E *so as to be analytic throughout the interior of* E_ρ. *Indeed, the sequence* $\{p_n(z)\}$ *converges uniformly on every closed set interior to* E_ρ.

[1] *A polynomial of degree n* is a function of the form $a_0 z^n + a_1 z^{n-1} + \ldots + a_n$ whether or not $a_0 = 0$. A *rational function of degree n* is a rational function having poles whose total order is not greater than *n*.

A point z is considered to be *interior* to E_ρ (which may consist of a finite number of Jordan curves, mutually exterior except that each of a finite number of points may belong to several such curves) if z is separated by E_ρ from the point at infinity.

An important tool is a lemma due to S. Bernstein in the case that E is a finite line segment; we use [1935, § 4.6] a method of proof first published by M. Riesz, independently found by M. Montel.

GENERALIZED BERNSTEIN LEMMA. — *Let* E *and* E_R *satisfy the conditions of theorem* 1.3.1. *If* $P_n(z)$ *is a polynomial of degree n which satisfies the inequality* $|P_n(z)| \leq M_0$ *on* E, *then we have*

$$|P_n(z)| \leq M_0 R^n \tag{1.3.2}$$

throughout the closed interior of E_R.

Green's function $g(z)$ is defined by the properties of being harmonic at every finite point of K, of being continuous with value zero on the boundary of K, and of having the form

$$g(z) \equiv \log|z| + g_1(z) \quad \text{for large } |z|,$$

where $g_1(z)$ is harmonic at infinity. If $h(z)$ is a function conjugate to $g(z)$ in K, the function $\frac{P_n(z)}{[e^{g(z)+ih(z)}]^n}$ is analytic although perhaps not single valued throughout K, and its modulus is single valued there. This function has a modulus which is continuous and not greater than M_0 on the boundary of K, hence which is not greater than M_0 throughout K. Then for z on E_R we have (1.3.2), so (1.3.2) is valid throughout the closed interior of E_R.

We are now in a position to prove theorem 1.3.1. Équation (1.3.1) yields by elementary algebraic inequalities

$$\limsup_{n \to \infty} [\max |p_n(z) - p_{n-1}(z)|, \ z \text{ on } E]^{\frac{1}{n}} \leq \frac{1}{\rho},$$

whence by the generalized Bernstein lemma,

$$\limsup_{n \to \infty} [\max |p_n(z) - p_{n-1}(z)|, \ z \text{ on } E_R]^{\frac{1}{n}} \leq \frac{R}{\rho} \qquad (R > 1).$$

The sequence $p_n(z)$ converges uniformly on every E_R $(1 < R < \rho)$, so theorem 1.3.1 follows. It is worth noting too that for an alternate proof we may choose a fixed R ($> \rho$), may write from (1.3.2)

$$\limsup_{n\to\infty}\left[\max|p_n(z)|,\ z \text{ on } E_R\right]^{\frac{1}{n}} \leq \frac{R}{\rho}$$

and apply corollary 1.2.2 with D the interior of E_R (which may consist of several mutually disjoint regions) and with $u(z) \equiv \frac{g(z)}{\log R}$.

Green's function, of fundamental importance in the (indirect) theorem 1.3.1 on approximation by polynomials, is also of fundamental importance in direct theorems.

Theorem 1.3.2. — *Let* B *denote a finite number of mutually exterior analytic Jordan curves,* K *the infinite region bounded by* B, *and* $g(z)$ *Green's function for* K *with pole at infinity. Suppose for large* $|z|$ *we have*

$$g(z) \equiv \log|z| + g_1(z), \qquad \textit{where} \quad g_1(\infty) = -g_0.$$

Then for z *in* K *we have*

$$g(z) + g_0 \equiv \int_B \varphi(s)\log r\, ds, \qquad \varphi(s) \equiv \frac{1}{2\pi}\frac{\partial g}{\partial \nu}, \tag{1.3.3}$$

where ν *denotes inner normal for* K *and* $r = |z - t|$, t *on* B; *we also have*

$$\int_B \varphi(s)\, ds = 1. \tag{1.3.4}$$

Consequently, if for $n = 1, 2, \ldots$ *the points* $\zeta_1^{(n)}, \zeta_2^{(n)}, \ldots, \zeta_n^{(n)}$ *are equally spaced on* B *with respect to the parameter* $\sigma_0 = \int \varphi(s)\, ds$, *we have*

$$\left\{\begin{array}{l} \lim\limits_{n\to\infty} |\omega_n(z)|^{\frac{1}{n}} = e^{g(z)+g_0}, \\ \omega_n(z) \equiv (z-\zeta_1^{(n)})(z-\zeta_2^{(n)})\ldots(z-\zeta_n^{(n)}), \end{array}\right. \tag{1.3.5}$$

uniformly on any closed bounded set in K.

Thanks to the special properties of $g(z)$ at infinity and the relation $g(z) = 0$ on B, Green's formula

$$g(z) \equiv \frac{1}{2\pi}\int_{B'}\left(\log r\frac{\partial g}{\partial \nu} - g\frac{\partial \log r}{\partial \nu}\right) ds,$$

where B′ is B together with a large circle whose center is z, reduces to (1.3.3), and (1.3.4) follows by $g(z) \equiv \log|z| + g_1(z)$. Equation (1.3.5) expresses the exponential of the Riemann sums for the integral in (1.3.3), where the points $t = \zeta_j^{(n)}$ of subdivision of B are equally spaced with respect to $\sigma_0 = \int \varphi(s)\,ds$, and uniformity of convergence results from equicontinuity. Equation (1.3.5) was first used by Hilbert in the case that B is a single curve, and by Faber in the more general case. The full use of (1.3.5) in relation to geometric degree of convergence by polynomials is due to Walsh and Russell [1934]:

THEOREM 1.3.3. — *Let* E *and* E_R *satisfy the conditions of theorem* 1.3.1. *Let* $f(z)$ *be analytic throughout the interior of* E_ρ *but not throughout the interior of any* $E_{\rho'}$ $(\rho' > \rho)$.

If $t_n(z)$ *denotes the polynomial of degree* n *of best approximation to* $f(z)$ *on* E *in the sense of Tchebycheff, then we have*

$$(1.3.6)\qquad \limsup_{n\to\infty}[\max|f(z) - t_n(z)|,\ z \text{ on } E]^{\frac{1}{n}} = \frac{1}{\rho}.$$

Of course $t_n(z)$ is the polynomial of degree n which minimizes the square bracket in (1.3.6); this polynomial is known to exist and be unique. By the method of proof of theorem 1.2.4, it is sufficient here to suppose E bounded by a finite number of mutually exterior analytic Jordan curves B. For z interior to $E_{\rho-\varepsilon}$ $(0 < 2\varepsilon < \rho)$ we use Hermite's interpolation formula

$$(1.3.7)\qquad f(z) - p_n(z) \equiv \frac{1}{2\pi i}\int_{E_{\rho-\varepsilon}} \frac{\omega_{n+1}(z) f(t)\,dt}{\omega_{n+1}(t)(t-z)},$$

where $\omega_{n+1}(z)$ is defined by (1.3.5) and $p_n(z)$ is the polynomial of degree n which interpolates to $f(z)$ in the points $\zeta_j^{(n+1)}$ on B. There follows

$$\begin{aligned}[\max|f(z) - t_n(z)|,\ z \text{ on } E] &\leqq [\max|f(z) - p_n(z)|,\ z \text{ on } E]\\ &\leqq [\max|f(z) - p_n(z)|,\ z \text{ on } E_{1+\varepsilon}],\end{aligned}$$

and the superior limit of the n^{th} root of this last member is by (1.3.5) not greater than $\frac{1+\varepsilon}{\rho-\varepsilon}$; thus $(\varepsilon \to 0)$ the first member of (1.3.6) is not greater than the second member; equality follows by theorem 1.3.1.

Any sequence of polynomials $t_n(z)$ of respective degrees n satisfying (1.3.6) is said to converge *maximally* to $f(z)$ on E. Theorem 1.3.3 is due to S. Bernstein if E is a line segment.

Theorems 1.3.1 and 1.3.3 form a satisfactory solution of Problem A for approximation by polynomials. They include the special case of Theorem 1.2.3 in which C_1 is a level locus of Green's function for the exterior of C_0 with pole at infinity. In order to prove a direct theorem concerning Problem A for approximation by bounded analytic functions in general regions, we establish such a theorem for approximation by rational functions, of which theorem 1.3.3 is a limiting case.

1.4. Approximation by rational functions; applications. — Theorems 1.3.1 and 1.3.2 are limiting cases of results to be proved by similar methods [1935].

THEOREM 1.4.1. — *Let* D *interior to* C_1 *be a region whose boundary consists of mutually disjoint analytic Jordan curves* B_1, $B_2, \ldots, B_\mu$; $C_1, C_2, \ldots, C_\nu$, *and let* $u(z)$ *be harmonic in* D, *equal to zero on* $B=\sum B_k$ *and equal to unity on* $C=\sum C_k$. *Then for* z *in* D *we have*

$$(1.4.1)\qquad \begin{cases} u(z)-1\equiv\displaystyle\int_B \varphi(s)\log r\,ds-\int_C \varphi(s)\log r\,ds, \\ \varphi(s)\equiv\dfrac{1}{2\pi}\left|\dfrac{\partial u}{\partial N}\right|, \end{cases}$$

where N *denotes inner normal for* D *and* $r=|z-t|$, t *on* B *or* C; *we set*

$$(1.4.2)\qquad \int_B \varphi(s)\,ds=\tau.$$

Consequently, if for $n=1, 2, \ldots$ *the points* $\beta_1^{(n)}, \beta_2^{(n)}, \ldots, \beta_n^{(n)}$ *are equally spaced on* B *with respect to the parameter* $\sigma_0=\int\varphi(s)\,ds$ *and the* $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_n^{(n)}$ *similarly spaced on* C, *we have*

$$(1.4.3)\qquad \lim_{n\to\infty}|\omega_n(z)|^{\frac{1}{n}}=e^{\frac{u(z)-1}{\tau}},$$

$$(1.4.4)\qquad \omega_n(z)\equiv\frac{(z-\beta_1^{(n)})\ldots(z-\beta_n^{(n)})}{(z-\alpha_1^{(n)})\ldots(z-\alpha_n^{(n)})},$$

uniformly on any closed set in D.

Equation (1.4.1) is merely Green's formula for the function $u(z)$ and the region D; the logarithm of the two members of equation (1.4.3) expresses the convergence of the Riemann sums for the integrals in (1.4.1), where the points $t = \beta_j^{(n)}$ and $\alpha_j^{(n)}$ subdividing B and C are chosen equally spaced with respect to the parameter

$$\sigma_0 = \int \varphi(s)\, ds;$$

uniformity of convergence follows from the uniform continuity of the harmonic functions involved.

For purposes of interpolation by rational functions it is convenient to modify (1.4.4) by setting

$$(1.4.5) \qquad \omega_n(z) \equiv \frac{(z-\beta_1^{(n)})\ldots(z-\beta_{n+1}^{(n)})}{(z-\alpha_1^{(n)})\ldots(z-\alpha_n^{(n)})},$$

where now for each n there are chosen $n+1$ points $\beta^{(n)}$ on B; equation (1.4.3) persists uniformly on any closed set in D.

THEOREM 1.4.2. — *Under the conditions of theorem 1.4.1 let Γ_σ denote generically the locus $u(z) = \sigma$ $(0 < \sigma < 1)$ in* D, *and let D_σ denote the point set $0 < u(z) < \sigma$ in* D. *Let the function $f(z)$ be analytic throughout D_ρ plus the closed interiors of the curves* B. *Then there exists a sequence of rational functions $r_n(z)$ of respective degrees n, whose poles $\alpha_j^{(n)}$ lie on a locus $\Gamma_{1+\varepsilon}$, determined by interpolation to $f(z)$ in points $\beta_j^{(n)}$ on* B, *satisfying*

$$(1.4.6) \qquad \limsup_{n\to\infty}\left[\max|f(z)-r_n(z)|,\ z \text{ on B}\right]^{\frac{1}{n}} \leqq e^{\frac{-\rho+\varepsilon}{\tau}},$$

$$(1.4.7) \qquad \limsup_{n\to\infty}\left[\max|r_n(z)|,\ z \text{ on C}\right]^{\frac{1}{n}} \leqq e^{\frac{1-\rho+\varepsilon}{\tau}}$$

The function $u(z)$ can be extended harmonically across C so that $u(z)$ remains harmonic in the region bounded by B and some locus $\Gamma_{1+\varepsilon}$: $u(z) = 1+\varepsilon$ $(\varepsilon > 0)$, consisting of ν analytic Jordan curves exterior to D near to the respective C_j. We set for z interior to $\Gamma_{\rho-\varepsilon}$ $\left(\varepsilon < \frac{\rho}{2}\right)$

$$(1.4.8) \qquad f(z) - r_n(z) \equiv \frac{1}{2\pi i}\int_{\Gamma_{\rho-\varepsilon}} \frac{\omega_n(z) f(t)\, dt}{\omega_n(t)(t-z)},$$

where $\omega_n(z)$ is defined by (1.4.5) and (1.4.3) is valid with $u(z)$ replaced by $\frac{u(z)}{1+\varepsilon}$ and τ replaced by $\frac{\tau}{1+\varepsilon}\cdot$ For z on $\Gamma_\sigma\,(\sigma < \rho - \varepsilon)$ by (1.4.3) as modified, the superior limit of the n^{th} root of the second member of (1.4.8) is not greater than $\exp\left[\frac{\sigma-\rho+\varepsilon}{\tau}\right]$, whence $(\sigma \to 0)$ we have (1.4.6).

To establish (1.4.7) we use (1.4.8) together with Cauchy's integral formula for $f(z)$; for z interior to $\Gamma_{\rho-\varepsilon}$ we have

$$(1.4.9) \qquad r_n(z) = \frac{1}{2\pi i}\int_\Gamma \frac{[\omega_n(t)-\omega_n(z)]f(t)\,dt}{\omega_n(t)(t-z)},$$

and since the integrand has no singularity in z on $\Gamma_{\rho-\varepsilon}$, this formula is valid for z on C. Inequality (1.4.7) follows.

Theorem 1.4.2 applies [1938] to approximation by bounded analytic functions :

THEOREM 1.4.3. — *Assume the geometric conditions and notations of theorems* 1.4.1 *and* 1.4.2. *Let the function* $f(z)$ *be analytic throughout* D_ρ *and also throughout the closed interiors of the* B_j, *but not analytic throughout any* $D_{\rho'}$ $(\rho' > \rho)$. *Let* $\varphi_M(z)$ *denote the (or a) function analytic and in modulus not greater than* M *in* D *plus the closed interiors of the* B_j, *for which*

$$(1.4.10) \qquad m_M = [\max |f(z) - \varphi_M(z)|,\ z \text{ on } B]$$

is least. Then we have

$$(1.4.11) \qquad \limsup_{M\to\infty} m_M^{\frac{1}{\log M}} = e^{-\frac{\rho}{1-\rho}}.$$

Our hypothesis that each of the Jordan curves C_j is analytic involves no loss of generality; any given D can be mapped so that it lies interior to an analytic Jordan curve, the image of C_1; further conformal maps can be made onto regions D so that the images of C_1 and C_2 are analytic, then the images of C_1, C_2 and C_3 are analytic, etc.

If B is composed of analytic Jordan curves, the method used in theorem 1.1.1 based on (1.1.4) and (1.1.5) but now based on (1.4.6) and (1.4.7) shows that the first member of (1.4.11) is not less than the second member (of course $\varepsilon \to 0$). Equality in (1.4.11) follows from corollary 1.2.2.

If B is not composed of analytic Jordan curves [indeed, it is sufficient if B is composed of a finite or infinite number of components, provided $u(z)$ exists], the method of theorem 1.2.4 completes the proof.

THEOREM 1.4.4. — *With the topological hypothesis and notation of theorem* 1.4.3, *let* $f(z)$ *be analytic in* D_ρ *and continuous on* B *but not analytic throughout any* $D_{\rho'}$ $(0<\rho<\rho'<1)$; *let* $\varphi_M(z)$ *denote the (or a) function analytic in* D, *and in modulus not greater than* M *in* D, *for which* (1.4.10) *is least. Then* (1.4.11) *is valid.*

Theorem 1.4.4 is to be proved by use of the components of $f(z)$. For z in D, say $0<\varepsilon<u(z)<\rho-\varepsilon$, we have

$$(1.4.12) \quad f(z) \equiv \frac{1}{2\pi i}\int_{u(z)=\varepsilon} \frac{f(t)\,dt}{t-z} + \frac{1}{2\pi i}\int_{u(z)=\rho-\varepsilon} \frac{f(t)\,dt}{t-z},$$

where the integrals are taken over the loci indicated, in the positive sense with respect to the point set $\varepsilon<u(z)<\rho-\varepsilon$. The second integral in (1.4.12) represents a function analytic in D_ρ plus the interiors of the B_j satisfying the hypothesis of theorem 1.4.3, and the first integral represents a function analytic throughout D, continuous on B. Details of the proof of theorem 1.4.4 are left to the reader, and are entirely analogous to those of the proof of theorem 1.2.1; it is convenient to apply corollary 1.2.2, which indeed may be regarded as a converse of theorem 1.4.4.

CHAPTER II.

PROBLEM α : $f(z)$ NOT ANALYTIC ON E.

2.1. Approximation by bounded analytic functions. — Chapter I represents in broad outlines a relatively complete treatment of Problem A; we turn now to Problem α, i. e. degree of approximation on a point set E to a function whose properties (less than analyticity) are given on E. We are in a position to make large use of the theory of trigonometric approximation in the real domain, in the form given it by de la Vallée Poussin [1919], namely :

THEOREM 2.1.1. — *If $f(\theta)$ is a function with period 2π whose p^{th} derivative satisfies a Lipschitz condition of order $\alpha (0 < \alpha < 1)$, then there exist trigonometric polynomials*

$$T_n(\theta) \equiv \sum_{0}^{n} (a_{nk} \cos k\theta + b_{nk} \sin k\theta)$$

of orders $n = 1, 2, \ldots$, such that for all θ

$$(2.1.1) \qquad |f(\theta) - T_n(\theta)| \leq \frac{A}{n^{p+\alpha}}.$$

Conversely, if the $T_n(\theta)$ exist such that (2.1.1) holds for all θ, then $f(\theta)$ has a p^{th} derivative which satisfies a Lipschitz condition of order α.

Our fundamental theorem here concerning approximation by bounded analytic functions is [compare 1951, 1952 *c* and 1959 *e*].

THEOREM 2.1.2. — *Let the function $f(z)$ defined on the analytic Jordan curve C possess a p^{th} derivative on C which satisfies a Lipschitz condition of order $\alpha (0 < \alpha < 1)$, with respect to arc length on C. Then there exists a region D containing C and a sequence of functions $f_n(z)$ analytic in D satisfying*

$$(2.1.2) \qquad |f_n(z)| \leq AR^n, \qquad z \text{ in } D,$$

$$(2.1.3) \qquad |f(z) - f_n(z)| \leq \frac{A_1}{n^{p+\alpha}}, \qquad z \text{ on } C.$$

Conversely, if $f(z)$ is defined on C, and the functions $f_n(z)$ analytic in some region D containing C satisfy (2.1.2) and (2.1.3), then $f_n^{(p)}(z)$ exists and satisfies a Lipschitz condition of order α on C.

We establish the first part of theorem 2.1.2 when C is the unit circle by the use of theorem 2.1.1. A trigonometric polynomial $T_n(\theta)$ may be expressed on $\Gamma : |z| = 1$, by the Euler formulas

$$z^n = \cos n\theta + i \sin n\theta, \qquad z^{-n} = \cos n\theta - i \sin n\theta,$$

as a polynomial $P_n\left(z, \frac{1}{z}\right)$ in z and $\frac{1}{z}$ of degree n, and if (2.1.1) is satisfied we may write $\left|P_n\left(z, \frac{1}{z}\right)\right| \leq A_2$ on Γ. The functions $\frac{P_n\left(z, \frac{1}{z}\right)}{z^n}$ and $z^n P_n\left(z, \frac{1}{z}\right)$ are analytic respectively in the closed regions $|z| \geq 1$ and $|z| \leq 1$, and are in modulus not greater than A_2 on Γ and in those regions. Thus we have $\left|P_n\left(z, \frac{1}{z}\right)\right| \leq A_2 R^n$ on the two circles $|z| = R(>1)$ and $|z| = \frac{1}{R}$, so that same inequality persists in the annulus $D : \frac{1}{R} < |z| < R$.

The first part of theorem 2.1.2, thus established when C is Γ, follows in the general case by a conformal map of C onto Γ, so that a neighborhood of C is carried into a neighborhood of Γ. The property of a function that it has a p^{th} derivative satisfying a Lipschitz condition of order α is invariant [1942, § 5.2] under conformal transformation.

To prove the second part of theorem 2.1.2 we may assume that C is Γ and D contains a closed annulus $D_0 : \frac{1}{\rho} \leq |z| \leq \rho\,(>1)$. For z interior to D_0 we write

$$(2.1.4) \qquad \begin{cases} f_n(z) \equiv f_{n1}(z) + f_{n2}(z), \\ f_{n1}(z) \equiv \dfrac{1}{2\pi i}\displaystyle\int_{|z|=\rho} \dfrac{f_n(t)\,dt}{t-z}, \\ f_{n2}(z) \equiv \dfrac{1}{2\pi i}\displaystyle\int_{|z|=\frac{1}{\rho}} \dfrac{f_n(t)\,dt}{t-z}, \end{cases}$$

where $f_{n1}(z)$ and $f_{n2}(z)$ are analytic in $|z| \leq \rho$ and $|z| \geq \frac{1}{\rho}$ respectively with $f_{n2}(\infty) = 0$. We set further

$$(2.1.5) \qquad f_{n1}(z) = \sum_{k=0}^{\infty} a_{nk} z^k, \qquad f_{n2}(z) = \sum_{k=-1}^{-\infty} a_{nk} z^k,$$

and the Cauchy inequalities on these coefficients, computed from integrals of $\frac{f_n(z)}{z^{k+1}}$ over $|z| = \rho$ and $|z| = \frac{1}{\rho}$ respectively, are $|a_{nk}| \leq \frac{AR^n}{\rho^{|k|}}$. Then on Γ we have for the partial sums $S_{n,N}(z)$ and

$T_{n,N}(z)$ of order N of the respective series in (2.1.5)

$$(2.1.6)\qquad \begin{cases} |f_{n1}(z) - S_{n,N}(z)| \leqq \displaystyle\sum_{k=N+1}^{\infty} \frac{AR^n}{\rho^k} = \frac{A_3 R^n}{\rho^N}, \\ |f_{n2}(z) - T_{n,N}(z)| \leqq \displaystyle\sum_{k=N+1}^{\infty} \frac{AR^n}{\rho^k} = \frac{A_3 R^n}{\rho^N}. \end{cases}$$

Let us choose the positive integer λ so that $\rho^\lambda > R$, whence we deduce from (2.1.6), (2.1.4) and (2.1.3)

$$|f(z) - S_{n,\lambda n}(z) - T_{n,\lambda n}(z)| \leqq \frac{A_4}{n^{p+\alpha}}, \qquad z \text{ on } \Gamma.$$

On Γ the polynomial $S_{n,\lambda n}(z) + T_{n,\lambda n}(z)$ in z and $\frac{1}{z}$ is also a trigonometric polynomial of order λn. To be sure, these trigonometric polynomials are not defined for all orders, but we may set on Γ

$$T_k(\theta) \equiv S_{n,\lambda n}(z) + T_{n,\lambda n}(z) \qquad [\lambda\, n \leqq k < \lambda\,(n+1)].$$

Then the trigonometric polynomials $T_k(\theta)$ are defined for all $k\,(> 0)$ and satisfy on Γ

$$|f(z) - T_k(\theta)| \leqq \frac{A_5}{k^{p+\alpha}},$$

and the second part of theorem 2.1.2 is a consequence of the second part of theorem 2.1.1.

Theorem 2.1.2 is to be contrasted with corollary 1.2.3. A less general situation than that of theorem 2.1.2 deserves explicit statement :

THEOREM 2.1.3. — *Let the function $f(z)$ defined on the analytic Jordan curve C, analytic interior to C, and continuous in the closed interior of C, possess a p^{th} derivative on C which satisfies a Lipschitz condition of order α $(0 < \alpha < 1)$, with respect to arc length on C. Then there exists a region D containing C and its interior, and a sequence of functions $f_n(z)$ analytic in D satisfying (2.1.2) and (2.1.3).*

Conversely, if $f(z)$ defined on C and the functions $f_n(z)$ analytic in some region D containing C and its interior satisfy (2.1.2) and (2.1.3), then $f(z)$ is analytic interior to C, continuous in the *corresponding closed region, and* $f^{(p)}(z)$ *exists and satisfies a Lipschitz condition of order* α *on* C.

In the first part of theorem 2.1.1 the sequence $T_n(\theta)$ arises by summation of the Fourier development of $f(\theta)$ by the method of D. Jackson, so if $f(\theta)$ is the set of continuous boundary values of a function analytic in $|z| < 1$, with $z = e^{i\theta}$ on $\Gamma : |z| = 1$, the trigonometric polynomial $T_n(\theta)$ in (2.1.1) is also on Γ a polynomial in z of degree n. Thus if C in the first part of theorem 2.1.3 is Γ, the $f_n(z)$ satisfying (2.1.3) can be chosen as polynomials in z of degree n, and (2.1.2) in the closed interior of Γ_R follows by the generalyzed Bernstein lemma. Consequently, if C in theorem 2.1.3 is arbitrary, a conformal map of the interior of C onto the interior of Γ yields the existence of D and the $f_n(z)$. The second part of theorem 2.1.3 follows from the second part of theorem 2.1.2.

2.2. **Approximation by polynomials.** — Both for its intrinsic interest and for later application we proceed to consider Problem α for approximation by polynomials. Here the fundamental theorem is [1936, 1937].

THEOREM 2.2.1. — *Let* $B_1, B_2, \ldots, B_\mu$ *be mutually exterior analytic Jordan curves, and let* E *denote the sum of their closed interiors. If* $f(z)$ *is analytic in the interior points of* E *and continuous on* E, *and if* $f^{(p)}(z)$ *exists and satisfies a Lipschitz condition of order* α $(0 < \alpha < 1)$ *on* $\sum B_j$, *then there exist polynomials* $p_n(z)$ *of respective degrees* $n = 1, 2, \ldots$ *such that*

$$|f(z) - p_n(z)| \leq \frac{A}{n^{p+\alpha}}, \qquad z \text{ on } E. \tag{2.2.1}$$

Conversely, if $f(z)$ *is defined on* E *and* (2.2.1) *holds for a sequence of polynomials* $p_n(z)$ *of respective degrees* n, *then* $f(z)$ *is analytic in the interior points of* E, *continuous on* E, *and on* $\sum B_j$ *possesses a* p^{th} *derivative satisfying a Lipschitz condition of order* α $(0 < \alpha < 1)$.

The second part of theorem 2.2.1 is readily proved from the second part of theorem 2.1.3, if we identify the given $p_n(z)$ with the $f_n(z)$.

It follows from (2.2.1) that the $p_n(z)$ are uniformly bounded on each B_j, and (2.1.2) follows in an arbitrary bounded region D containing E from the generalized Bernstein lemma (§ 1.3); consequently the second part of theorem 2.1.3 is applicable.

In proving the first part of theorem 2.2.1 (the general method is due to J. H. Curtiss) we apply the first part of theorem 2.1.3 to each $B_j (1 \leqq j \leqq \mu)$, making use of the functions $f_n(z)$ and the inequalities (2.1.2) and (2.1.3); we assume, as we may do, that (2.1.2) and (2.1.3) hold for all the functions $f_n(z)$ independently of j, and we choose $\rho (> 1)$ in such a way that the regions D of theorem 2.1.2 for the various B_j in their totality contain the locus E_ρ (notation of theorem 1.3.1) which consists of μ mutually exterior analytic Jordan curves containing the respective B_j. For each n the function $f_n(z)$ shall hence forth indicate the aggregate of the previous $f_n(z)$ defined for $j = 1, 2, \ldots, \mu$, so the new $f_n(z)$ is analytic throughout the closed interior of E_ρ and satisfies

$$(2.2.2) \qquad |f_n(z)| \leqq AR^n, \qquad z \text{ in } E_\rho,$$

$$(2.2.3) \qquad |f(z) - f_n(z)| \leqq \frac{A_1}{n^{p+\alpha}}, \qquad z \text{ on } E.$$

We proceed to use the method of proof of theorem 1.3.3. and in particular we use Hermite's interpolation formula (1.3.7), now in the form $(1 < 1 + \varepsilon < \rho)$

$$(2.2.4) \quad f_n(z) - p_{n,N}(z) \equiv \frac{1}{2\pi i} \int_{E_\rho} \frac{\omega_{N+1}(z) f_n(t)\, dt}{\omega_{N+1}(t)(t-z)}, \qquad z \text{ on } E_{1+\varepsilon}.$$

From the relation (1.3.5) and from (2.2.2) and (2.2.4) follows for z on $E_{1+\varepsilon}$ and hence for z on E

$$(2.2.5) \qquad |f_n(z) - p_{n,N}(z)| \leqq \frac{A_2 R^n}{\rho_1^N},$$

where A_2 is independent of n and N and where $\rho_1 (1 < \rho_1 < \rho)$ is suitably chosen. As in the discussion of (2.1.6) we choose the integer λ so that $\rho_1^\lambda > R$, whence by (2.2.5) and (2.2.3)

$$|f(z) - p_{n,\lambda n}(z)| \leqq \frac{A_3}{n^{p+\alpha}}, \qquad z \text{ on E}.$$

Here $p_{n,\lambda n}(z)$ is a polynomial in z of degree λn, and is not defined for all degrees; however we may set

$$p_k(z) \equiv p_{n,\lambda n}(z) \qquad [\lambda n \leqq k < \lambda(n+1)],$$

so the $p_k(z)$ are defined for $k = 1, 2, \ldots,$ and (2.2.1) follows. Theorem 2.2.1 is established.

A somewhat analogous result [1952], involving a function given merely on a single Jordan curve, is

THEOREM 2.2.2. — *Let* C *be an analytic Jordan curve containing the origin in its interior. If* $f(z)$ *defined on* C *has a* p^{th} *derivative which satisfies a Lipschitz condition of order* α $(0 < \alpha < 1)$ *on* C, *then there exist polynomials* $P_n\left(z, \frac{1}{z}\right)$ *of degree n in z and* $\frac{1}{z}$ *such that*

$$|f(z) - P_n\left(z, \frac{1}{z}\right)| \leq \frac{A}{n^{p+\alpha}}, \qquad z \text{ on } C. \tag{2.2.6}$$

Conversely, if $f(z)$ *is defined on* C *and if there exist polynomials* $P_n\left(z, \frac{1}{z}\right)$ *of degree n in z and* $\frac{1}{z}$ *such that* (2.2.6) *is satisfied, then* $f^{(p)}(z)$ *exists on* C *and satisfies there a Lipschitz condition of order* α.

If $f^{(p)}(z)$ exists on C and satisfies there a Lipschitz condition of order α, we use the first part of theorem 2.1.2, and assume the functions $f_n(z)$ analytic in the closure of the region D, whose boundary consists of analytic Jordan curves C_0 (interior to C) and C_1 (containing C in its interior). As in (2.1.4) we set $f_n(z) \equiv f_{n1}(z) + f_{n2}(z)$,

$$f_{n1}(z) \equiv \frac{1}{2\pi i}\int_{C_1} \frac{f_n(t)\,dt}{t - z}, \qquad f_{n2}(z) \equiv \frac{1}{2\pi i}\int_{C_0} \frac{f_n(t)\,dt}{t - z},$$

where $f_{n1}(z)$ and $f_{n2}(z)$ are analytic respectively interior to C_1 and exterior to C_0. From (2.1.2) we deduce $|f_{n1}(z)| \leq A_2 R^n$ on any closed set interior to C_1 and $|f_{n2}(z)| \leq A_2 R^n$ on any closed set exterior to C_0. Use of equation (2.2.4), where $f_n(z)$ is replaced by $f_{n1}(z)$ and where the integral is taken over a Jordan curve interior to C_1 but containing C in its interior, proves $(\rho_1 > 1)$

$$|f_{n1}(z) - p_{n,N}(z)| \leq \frac{A_3 R^n}{\rho_1^N}, \qquad z \text{ on } C,$$

for polynomials $p_{n,N}(z)$ in z of respective degrees N, and similar reasoning proves a corresponding result involving $f_{n2}(z)$ and polynomials $P_{nN}\left(\frac{1}{z}\right)$ in $\frac{1}{z}$ of respective degrees N. As in the discussion of (2.1.6) and (2.2.5) we choose the integer λ so that $\rho_1' > R$, whence by (2.1.3)

$$\left| f(z) - \left[p_{n,\lambda n}(z) + P_{n,\lambda n}\left(\frac{1}{z}\right) \right] \right| \leq \frac{A_4}{n^{p+\alpha}}, \qquad z \text{ on C.}$$

If we now set

$$P_k\left(z, \frac{1}{z}\right) \equiv p_{n,\lambda n}(z) + P_{n,\lambda n}\left(\frac{1}{z}\right) \qquad [\lambda n \leq k < \lambda(n+1)],$$

the $P_k\left(z, \frac{1}{z}\right)$ are defined for all degrees k, and (2.2.6) follows.

To prove the converse we note by (2.2.6) that the $P_n\left(z, \frac{1}{z}\right)$ are uniformly bounded on C: $\left|P_n\left(z, \frac{1}{z}\right)\right| \leq A_1$. If $g(z)$ denotes Green's function for the exterior of C with pole at infinity and $h(z)$ the conjugate function, we have

$$\left| \frac{P_n\left(z, \frac{1}{z}\right)}{e^{n(g+ih)}} \right| \leq A_1$$

on C and exterior to C even at infinity; in particular on the locus $C_R : g(z) = \log R\,(> 0)$ exterior to C we have

$$\left| P_n\left(z, \frac{1}{z}\right) \right| \leq A_1 R^n. \tag{2.2.7}$$

If $g_0(z)$ denotes Green's function for the interior of C with pole in the origin we prove similarly (2.2.7) on the locus $C_R^0 : g_0(z) = \log R$. Then (2.2.7) holds in the annular region bounded by C_R and C_R^0, so the conclusion of the second part of theorem 2.2.2 is a consequence of theorem 2.1.2.

Such a property as (2.2.6) is [1959 *e*] intrinsically invariant under conformal transformation.

2.3. Complements. — Some complements to the preceding results are of interest.

THEOREM 2.3.1. — *Let* C *be an analytic Jordan arc, let* D *be a region containing* C, *let* $f(z)$ *be defined on* C, *and let func-*

tions $f_n(z)$ analytic in D *satisfy* (2.1.2) *and* (2.1.3). *Then on any closed subarc of* C *containing no endpoint $f^{(p)}(z)$ exists and satisfies a Lipschitz condition of order* α $(0 < \alpha < 1)$.

Let the transformation $z = \varphi(w)$ map C onto the line segment $S: -1 \leqq w \leqq 1$, the map being conformal and one to one in suitable neighborhoods of C and S; inequalities (2.1.2) and (2.1.3) in suitably modified form persist. Green's function $g(w)$ for the w-plane slit along S with pole at infinity admits a representation analogous to (1.3.3), where the integral is taken over $S_{1+\varepsilon}$ $(\varepsilon > 0)$ and the analogue of (1.3.5) persists exterior to $S_{1+\varepsilon}$. The method of proof of the first part of theorem 2.2.1 is valid, and shows the existence of polynomials $p_n(w)$ in w of respective degrees n satisfying

$$(2.3.1) \qquad |f[\varphi(w)] - p_n(w)| \leqq \frac{A_0}{n^{p+\alpha}}, \qquad w \text{ on S}.$$

The classical transformation $w = \cos\theta$ maps S onto the axis $-\infty < \theta < +\infty$, transforms $f[\varphi(w)]$ into a periodic function of θ and $p_n(w)$ into a trigonometric polynomial in θ of order n. Theorem 2.1.1 now applies, and yields the conclusion.

We have essentially proved the first part of : *If* D *is suitably chosen, a necessary and sufficient condition for the existence of the $f_n(z)$ satisfying* (2.1.2) *and* (2.1.3) *is that $f[\varphi(\cos\theta)]$ possess a p^{th} derivative with respect to θ which satisfies a Lipschitz condition of order α with respect to θ.*

If $f[\varphi(\cos\theta)]$ satisfies this latter condition with $Z = e^{i\theta}$, there exist by theorem 2.1.1 polynomials $p_n\left(Z, \frac{1}{Z}\right)$ of degree n in Z and $\frac{1}{Z}$ satisfying on $\Gamma : |Z| = 1$ the inequality

$$\left| f[\varphi(\cos\theta)] - p_n\left(Z, \frac{1}{Z}\right) \right| \leqq \frac{A}{n^{p+\alpha}}.$$

Since $f[\varphi(\cos\theta)]$ as a function of Z is symmetric in the axis of reals, there follow on Γ

$$\left| f[\varphi(\cos\theta)] - p_n\left(\frac{1}{Z}, Z\right) \right| \leqq \frac{A}{n^{p+\alpha}},$$

$$\left| f[\varphi(\cos\theta)] - \frac{1}{2} p_n\left(Z, \frac{1}{Z}\right) - \frac{1}{2} p_n\left(\frac{1}{Z}, Z\right) \right| \leqq \frac{A}{n^{p+\alpha}}.$$

The transformation $w = \frac{1}{2}\left(Z + \frac{1}{Z}\right)$ now yields (2.3.1) and (2.1.3), also (2.1.2) by (2.3.1) and the generalized Bernstein lemma.

If the functions $f_n(z)$ of theorem 2.3.1 are given as polynomials in z of respective degrees n, inequality (2.1.2) in a suitable region D is a consequence of (2.1.3), by the generalized Bernstein lemma. Moreover, the methods that we have developed (compare the proof of the first part of theorem 2.2.1) show that if $f[\varphi(\cos\theta)]$ possesses a p^{th} derivative with respect to θ which satisfies a Lipschitz condition of order α with respect to θ, then the functions $f_n(z)$ in (2.1.3) can be chosen as polynomials of degree n in z. Thus if D is an *arbitrary* bounded region containing C, and if $f[\varphi(\cos\theta)]$ satisfies the condition just mentioned, the functions $f_n(z)$ of theorem 2.3.1 exist, as polynomials of degree n in z.

The second part of theorem 2.1.2 follows from theorem 2.3.1, for we may apply the latter to two subarcs of the Jordan curve C of theorem 2.1.2 overlapping each other at both ends.

Thus far in chapter II we have considered, concerning approximation by bounded analytic functions, sequences rather than families depending on continuous parameters. We now treat briefly extremal functions [1951], for definiteness in the situation of theorem 2.1.2; but it is clear that a similar discussion applies also in numerous other situations.

THEOREM 2.3.2. — *Let* D *be a bounded annular region and let the analytic Jordan curve* C *separate the two bounding curves of* D. *Let the function $f(z)$ be defined on* C, *and for each* $M(>0)$ *let $\varphi_M(z)$ denote the (or a) function analytic and of modulus not greater than* M *in* D *such that*

$$m_M = [\max|f(z) - \varphi_M(z)|,\ z \text{ on } C] \tag{2.3.2}$$

is least. Then a necessary and sufficient condition that $f(z)$ possess a p^{th} derivative with respect to arc length on C *which satisfies there a Lipschitz condition of order α $(0 < \alpha < 1)$, is that*

$$\log M \cdot m_M^{\frac{1}{p+\alpha}} \tag{2.3.3}$$

be bounded as $M \to \infty$.

If (2.3.3) is bounded, we choose the sequence $M = e^n$, whence

$$m_M^{\frac{1}{p+\alpha}} \leqq \frac{A_1}{n}, \qquad m_M \leqq \frac{A_1}{n^{p+\alpha}}$$

and the conclusion follows from theorem 2.1.2. In theorem 2.3.2 it is sufficient if (2.3.3) is bounded for a monotonic sequence M_n with $\frac{\log M_{n+1}}{\log M_n}$ bounded; boundedness of the original form of (2.3.3) follows if we set $\varphi_M(z) \equiv \varphi_{M_n}(z)$, $M_n \leqq M < M_{n+1}$.

To prove the converse, we note that in the first part of theorem 2.2.2, the polynomials $P_n\left(z.\ \frac{1}{z}\right)$ satisfy (2.2.7) for suitably chosen R in any bounded region D containing C but whose closure does not contain the origin; this fact appears in the proof of the second part of theorem 2.2.2. Consequently *in the first part of theorem* 2.1.2 *the region* D *may be chosen as an arbitrary annular region whose two boundary curves are separated by* C, the position of the origin being unessential. We now compare m_M defined by (2.3.2) with the measure of approximation to $f(z)$ on C of the $f_n(z)$ of theorem 2.1.2. Let n be defined as a function of M by the inequalities $AR^n \leqq M < AR^{n+1}$, in the notation of (2.1.2); we have $m_M \leqq \frac{A_1}{n^{p+\alpha}}$ in the notation of (2.1.3), whence

$$m_M^{\frac{1}{p+\alpha}} \leqq \frac{A_1^{\frac{1}{p+\alpha}}}{n}, \qquad \log M < \log A + (n+1) \log R,$$

from which the boundedness of (2.3.3) follows.

It is merely for convenience in exposition that we have supposed D in theorem 2.3.2 to be an annular region (i. e. bounded by two Jordan curves); it is sufficient, as is shown by a suitable conformal map, if D has at least one boundary component not a single point interior to C and at least one such boundary component exterior to C.

2.4. **Approximation by rational functions.** — An extension [1956*a*] of theorem 2.2.1 to a more general topological situation turns out to be useful in the sequel :

THEOREM 2.4.1. — *Let* E *be a bounded open set whose boundary* J *consists of a finite number of mutually disjoint analytic Jordan* *curves* J_j, $J = \sum J_j$. *Let* $f(z)$ *be analytic on* E, *continuous on* $E + J$, *and possess on* J *a* p^{th} *derivative which satisfies a Lipschitz condition there of order* α $(0 < \alpha < 1)$. *In the extended plane, let the set complementary to* $E + J$ *consist of the mutually disjoint regions* $E_1, E_2, \ldots, E_\nu$, *let a point* α_k *be given in each* E_k, *and for each* $n = \nu, \nu + 1, \nu + 2, \ldots$ *let positive integers* m_{nk} *be given with* $\sum_{k=1}^{\nu} m_{nk} = n$. *Suppose the numbers* $\frac{n}{m_{nk}}$ *are bounded for all k and n. Then there exist rational functions* $R_n(z)$ *of respective degrees n whose poles (of respective multiplicities not greater than* m_{nk}*) lie in the points* α_k *such that we have*

$$(2.4.1) \qquad |f(z) - R_n(z)| \leq \frac{A}{n^{p+\alpha}}, \qquad z \text{ on } E + J.$$

For z in E we have

$$(2.4.2) \qquad f(z) \equiv \sum \frac{1}{2\pi i} \int_J \frac{f(t)\,dt}{t - z},$$

where the integrals are taken over all J_j in the positive sense with respect to the regions which compose E. Although the function

$$(2.4.3) \qquad \frac{1}{2\pi i} \int_{J_j} \frac{f(t)\,dt}{t - z}$$

is defined and analytic at all points of the plane (including by continuity the point at infinity) except on J_j, it is not defined on J_j; we hereby define (2.4.3) on J_j by equation (2.4.2), of which all terms but the one (2.4.3) are previously defined. Then (2.4.3) is continuous on E, and on J_j has a p^{th} derivative which satisfies there a Lipschitz condition of order α; the function (2.4.3) is analytic throughout that one of the two regions bounded by J_j which contains a subregion of E adjacent to J_j, and is continuous in the corresponding closed region.

Each curve J_j belongs to the boundary of precisely one region E_k. For every k $(1 \leq k \leq \nu)$, we define the function

$$(2.4.4) \qquad F_k(z) \equiv \sum_j \frac{1}{2\pi i} \int_{J_j} \frac{f(t)\,dt}{t - z},$$

where the integrals are extended over the complete boundary of E_k, over each J_j of this boundary in the same sense as in (2.4.2); this function $F_k(z)$ is analytic in the interior points of the complement $C(E_k)$ of E_k (the complement contains $E+J$), continuous on $C(E_k)$, and on the boundary of E_k has a p^{th} derivative satisfying a Lipschitz condition of order α, by our definition of (2.4.3) on J_j. The equation $f(z) \equiv \sum_{k=1}^{\nu} F_k(z)$ follows from (2.4.2), for z on $E+J$.

By theorem 2.2.1 there exists a rational function $R_{m_{nk}}^{(k)}(z)$ of degree m_{nk} whose poles lie in α_k such that we have

$$(2.4.5) \qquad |F_k(z) - R_{m_{nk}}^{(k)}(z)| \leqq \frac{A_k}{m_{nk}^{p+\alpha}}, \qquad z \text{ on } C(E_k).$$

If we assume $\frac{n}{m_{nk}} \leqq A_0$ for all k and n, we have

$$\frac{A_k}{m_{nk}^{p+\alpha}} \leqq \frac{A_k A_0^{p+\alpha}}{n_0^{p+\alpha}};$$

consequently $R_n(z) \equiv \sum_{k=1}^{\nu} R_{m_{nk}}^{(k)}(z)$ is a rational function of the kind required, which satisfies (2.4.1). Theorem 2.4.1 is established.

We shall later apply:

COROLLARY 2.4.1. — *Theorem 2.4.1 remains valid if the total number of given points α_j is greater than ν, provided each E_k contains at least one α_j, the equation $\sum_j m_{nj} = n$ persists, and the quotients $\frac{n}{m_{nj}}$ are bounded for all j and n.*

We make use of but a single point α_k in each E_k in establishing (2.4.5) as before; then $R_n(z) \equiv \sum_{k=1}^{\nu} R_{nk}^{(k)}(z)$ is a rational function, of degree n but also perhaps of smaller degree, which satisfies (2.4.1).

In theorem 2.4.1 it is essential to place at least one point α_k in each E_k; if no α_k lies in a particular E_k, the function $f(z) \equiv \frac{1}{z-\beta}$, where β is a finite point in E_k (assumed bounded), cannot be uniformly approximated on J by rational functions $R_n(z)$ whose poles lie in the α_j. A sequence of such rational functions $R_n(z)$, converging uniformly on J to the function $f(z)$, would yield for the integrals over the boundary of E_k

$$0 = \lim_{n\to\infty} \int R_n(z)\, dz = \int f(z)\, dz,$$

whereas the integral of $f(z)$ is $2\pi i$.

For a precisely similar reason, in approximation on a set E bounded by a finite number of mutually disjoint analytic Jordan curves by functions analytic and bounded in a region D containing E, it is essential that every one of the subregions into which E separates the plane should contain an infinite number of boundary points of D. For instance, theorem 2.3.2 is false if D is allowed to be merely a simply connected region containing E; under such conditions if D is bounded and we choose $f(z) \equiv \frac{1}{z-\beta}$, where β lies interior to C, the relation $m_M \to 0$ is not possible as $M \to \infty$.

The functions $R_n(z)$ of theorem 2.4.1 satisfy an inequality

$$(2.4.6) \qquad |R_n(z)| \leqq A_0 R^n,$$

in a suitably chosen region containing $E + J$. Indeed, if $g_k(z)$ denotes Green's function for the region E_k with pole in α_k, and $h_k(z)$ the function conjugate to $g_k(z)$ in E_k, the function $\Phi_{nk}(z) \equiv \frac{R_n(z)}{e^{n(g_k+ih_k)}}$ is analytic in E_k even at α_k, and $|\Phi_{nk}(z)|$ is single valued and continuous in E_k. From (2.4.1) we deduce $|R_n(z)| \leqq A_0$ on J, whence $|\Phi_{nk}(z)| \leqq A_0$ on the boundary of E_k, and $|R_n(z)| \leqq A_0 R^n$ on the locus $L_k : g_k(z) = \log R\ (> 0)$ in E_k. Consequently (2.4.6) is valid in the region D_R bounded by these ν loci L_k; the L_k may be chosen as close to the points α_k as desired, merely by choosing R sufficiently large. In any subregion of D_R, as in D_R itself, the functions $R_n(z)$ are analytic and satisfy (2.4.6).

We have now at hand a converse of theorem 2.4.1: *if rational functions* $R_n(z)$ *of the prescribed kind exist satisfying* (2.4.1), *then* $f(z)$ *is analytic on* E, *continuous on* $E + J$, *and has a derivative* $f^{(p)}(z)$ *on* J *which satisfies there a Lipschitz condition of order* α. We merely apply the second part of theorem 2.1.2.

Since the points α_k of theorem 2.4.1 are entirely arbitrary in the respective E_k, and since any simply connected region whose boundary is a continuum (not a single point) can be mapped onto the interior or exterior of a circle, there follows by the method of proof of theorem 2.3.2:

THEOREM 2.4.2. — *Let* E *be a bounded open set whose boundary* J *consists of a finite number of mutually disjoint analytic Jordan curves* J_j, $J = \sum J_j$. *Let* D *be a region containing* E + J *such that each of the regions* $E_1, E_2, \ldots, E_\nu$ *composing the complement of* E + J *contains at least one component of the boundary of* D *which is a continuum not a single point. Let the function* $f(z)$ *be analytic on* E, *continuous on* E + J, *and for each* M (> 0) *let* $\varphi_M(z)$ *denote the (or a) function analytic and in modulus not greater than* M *in* D *such that*

$$(2.4.7)\qquad m_M = [\max|f(z) - \varphi_M(z)|,\ z \text{ on } E + J]$$

is least. Then a necessary and sufficient condition that $f(z)$ *possess a* p^{th} *derivative on* J *which satisfies there a Lipschitz condition of order* α $(0 < \alpha < 1)$ *is that*

$$\log M \cdot m_M^{\frac{1}{p+\alpha}}$$

be bounded as $M \to \infty$.

Thanks to a possible succession of conformal transformations, we may assume that the region D of theorem 2.4.2 is such that each E_k contains points not in the closure of D, so the points α_k in E_k exterior to D exist for application of theorem 2.4.1.

Theorems 2.4.1 and 2.4.2 both extend to the case that E + J is replaced by an arbitrary closed set E whose boundary consists of a finite number of mutually disjoint analytic Jordan curves; the new set may contain a Jordan curve which does not bound (wholly or in part) a region belonging to the new set E; compare [1956 *a*].

CHAPTER III.

PROBLEM 3 : $f(z)$ ANALYTIC ON E, MORE REFINED DEGREE OF CONVERGENCE ON E.

If the Taylor development about the origin of a function $f(z)$ has the radius of convergence ρ (> 1), the precise degree of convergence to $f(z)$ of the Taylor development on $E: |z| \leqq 1$ depends on the kinds of singularities of $f(z)$ on $|z| = \rho$, whether for instance $f(z)$ has a pole of order 17 or is relatively smooth there. Degree of convergence to $f(z)$ on E of best approximating analytic functions of prescribed norm depends likewise on the behavior of the function *on the boundary of its region of analyticity*, and we proceed to study this relationship. The Taylor development is adequate for such a study in the simplest cases, but our program involves geometric situations which seem to require more powerful tools, so we first consider a general conformal map and then a special series of rational functions. We frequently use the plane of finite points extended by the adjunction of the point at infinity.

3.1. **Conformal map of multiply connected regions.** — Here our fundamental theorem is

THEOREM 3.1.1.—*Let* D *be a region of the z-plane whose boundary consists of mutually disjoint Jordan curves* $B_1, B_2, \ldots, B_\mu$; $C_1, C_2, \ldots, C_\nu$. *Then* D *can be mapped conformally onto a region* Δ *of the* Z*-plane, one-to-one and continuously in the closures of the two regions, where* Δ *is defined by*

$$(3.1.1)\qquad \left\{\begin{array}{l} 1 < \left| \dfrac{A(Z-a_1)^{M_1}(Z-a_2)^{M_2}\ldots(Z-a_\mu)^{M_\mu}}{(Z-b_1)^{N_1}(Z-b_2)^{N_2}\ldots(Z-b_\nu)^{N_\nu}} \right| < e^{\frac{1}{\tau}}, \\ M_j > 0, \qquad N_j > 0, \qquad \sum M_j = \sum N_j = 1. \end{array}\right.$$

The images of the B_j *and* C_j *separate the* a_j *and* b_j *respectively from* Δ.

We outline the proof, whose methods are due in part to de la Vallée Poussin (who treats the case $\nu = 1$) and to Julia; details may be found in [1956].

We omit the classical case $\mu = \nu = 1$, which in fact is easily treated by the same methods, and assume $\mu \geqq 2$, which may require interchange of the roles of the B_j and C_j. By a preliminary transformation we may assume the curves B_j and C_j analytic; compare the proof of theorem 1.4.3. We assume D interior to C_1. The function $u(z)$ defined and used in theorem 1.4.1 is invariant under conformal transformation, and is central in the present proof.

The function $u(z)$ is harmonic in the closure $\overline{D}$ of D, and can be extended harmonically across each of the bounding curves of D, so as to be harmonic in a closed region D′ containing $\overline{D}$ whose boundary $B' : u(z) = -\delta_1 (< 0)$, $C' : u(z) = \delta = 1 + \delta_1$, consists of $\mu + \nu$ analytic Jordan curves B'_j and C'_j near the B_j and C_j. If $v(z)$ denotes the conjugate of $u(z)$, Green's formula corresponding to (1.4.1) for z interior to D′ becomes

$$(3.1.2)\quad \begin{cases} u(z) = \displaystyle\int_0^{\tau} \log|z-t|\,d\sigma - \int_{\tau}^{2\tau} \log|z-t|\,d\sigma + \delta, \\ d\sigma = \dfrac{|dv|}{2\pi}, \qquad \tau = \displaystyle\int_{B'} d\sigma = \int_{C'} d\sigma > 0; \end{cases}$$

the first and second integrals in (3.1.2) are taken over B′ and C′. The derivative of $u(z) + iv(z)$ does not vanish on $B = \sum B_j$ or $C = \sum C_j$, for near each point say of B the locus $u(z) = 0$ consists of a single analytic Jordan arc. We assume too that the derivative of $u(z) + iv(z)$ does not vanish on $B' + C'$ or between B′ and B, and C′ and C.

If the $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_n$ (depending on n) divide B′ and C′ respectively into n equal parts with respect to the parameter σ, we have uniformly in $\overline{D}$:

$$u_n(z) \equiv \frac{\tau}{n}\sum_1^n \log|z-\alpha_k| - \frac{\tau}{n}\sum_1^n \log|z-\beta_k| + \delta \to u(z).$$

For n sufficiently large D is approximated by $D_n : 0 < u_n(z) < 1$, a region whose boundary consists of $\mu + \nu$ Jordan curves near the B_j and C_j. Each point of D lies in D_n for n sufficiently large; each point exterior to D lies in at most a finite number of the D_n. The region D_n can be expressed

$$1 < |R_n(z)| < e^{\frac{n}{\tau}}, \qquad R_n(z) \equiv e^{\frac{n\delta}{\tau}} \prod_1^n \frac{z-\alpha_k}{z-\beta_k} = e^{\frac{n[u_n(z)+iv_n(z)]}{\tau}}$$

The transformation $w = R_n(z)$ of the z-plane onto an n-sheeted Riemann surface σ_0 over the w-plane maps D_n onto a connected set $1 < |w| < e^{\frac{n}{\tau}}$ whose boundary consists of μ circumferences of radius 1 of respective multiplicities m_j and ν circumferences of radius $e^{\frac{n}{\tau}}$ of respective multiplicities n_j; here the m_j and n_j are the numbers of α_k and β_k on the B'_j and C'_j. The $\mu+\nu$ closed regions of the z-plane complementary to D_n are mapped into $\mu+\nu$ simply connected closed regions of σ_0 covering $|w| \leqq 1$ and $|w| \geqq e^{\frac{n}{\tau}}$ respectively m_j and n_j times. We define a new Riemann surface σ_1 over the w-plane by replacing continuously each of these $\mu+\nu$ closed regions by a subregion of the Riemann surface for $z = w^{\frac{1}{m_j}}$ or $z = w^{\frac{1}{n_j}}$ likewise covering $|w| \leqq 1$ or $|w| \geqq e^{\frac{n}{\tau}}$ precisely m_j or n_j times; these new subregions of σ_1 have each a single branch point, at $w = 0$ or $w = \infty$, where all m_j or n_j sheets meet. Then σ_1 is also the topological image of the extended z-plane, and (Schwarz) can be mapped conformally onto the extended Z-plane; the transformation is of the form

$$w = S_n(Z) \equiv \frac{A_n(Z-a'_1)^{m_1}(Z-a'_2)^{m_2}\ldots(Z-a'_\mu)^{m_\mu}}{(Z-b'_1)^{n_1}(Z-b'_2)^{n_2}\ldots(Z-b'_\nu)^{n_\nu}}.$$

Here the a'_j and b'_j (depending on n) are distinct, and lie exterior to the image $\Delta_n : 1 < |S_n(Z)| < e^{\frac{n}{\tau}}$ of D_n. The transformation $R_n(z) = S_n(Z)$ of D_n can be written $Z = Z_n(z)$. We choose a'_1, a'_2, b'_1 as distinct points independent of n, as is possible by a linear transformation of the Z-plane.

As n tends to infinity, there exists a sequence of indices n such that all the $A_n^{\frac{1}{n}}$, a'_j, b'_j approach limits A, a_j, b_j; we define M_j and N_j by the equations

$$\frac{m_j}{n} \to \frac{1}{\tau}\int_{B'_j} d\sigma = M_j, \qquad \frac{n_j}{n} \to \frac{1}{\tau}\int_{C'_j} d\sigma = N_j, \qquad \sum M_j = \sum N_j = 1.$$

Thus the inequalities defining Δ_n take the limiting form (3.1.1), inequalities which define some region Δ.

The functions $Z_n(z)$ admit in D the exceptional values a_1, a_2, b_1, hence form a normal family there. Henceforth we consider only a subsequence of the subsequence of indices n already chosen, such that the $Z_n(z)$ approach a limit $Z_0(z)$ in D, uniformly on every closed subset of D. The assumption $Z_0(z) \equiv g$, a constant, leads to a contradiction, for by a suitable linear transformation of the Z-plane we may choose $g \neq \infty$, $a_2 \neq \infty$, and if necessary by a change of notation we take $g \neq a_2$. If Γ is a Jordan curve in D near B_2 and containing B_2 in its interior, the image of Γ under the transformation $Z = Z_n(z)$ contains a_2 in its interior, whence $\arg[Z_n(z) - a_2]|_\Gamma = 2\pi$, which contradicts $Z_n(z) \to g$ uniformly on Γ.

A slight modification of classical reasoning concerning the conformal mapping of variable regions now shows that the function $Z = Z_0(z)$, univalent in D, maps D onto Δ. The methods of Carathéodory or of Montel show that the map is one-to-one and continuous in the corresponding closed regions.

3.2. **A series of interpolation** [1955 *a*]. — To prepare for our further study of approximation we shall prove two lemmas.

LEMMA 3.2.1. — *If the positive numbers $m_1, m_2, \ldots, m_\mu$ are given with $\sum m_j = 1$, there exist positive integers N_{nj} for $j = 1, 2, \ldots, \mu$; $n = 1, 2, \ldots$ which satisfy the relations*

$$\sum_{j=1}^{\mu} N_{nj} = n, \tag{3.2.1}$$

$$N_{nj} \leq N_{n+1,j} \leq N_{nj} + 1, \tag{3.2.2}$$

$$|N_{nj} - nm_j| \leq A \qquad (j = 1, 2, \ldots, \mu;\ n = 1, 2, \ldots). \tag{3.2.3}$$

In the case $\mu = 2$ it is sufficient to set $N_{n1} = [nm_1]$, the largest integer not greater than nm_1, and $N_{n2} = n - [nm_1]$. For $\mu > 2$ it is sufficient to iterate this process.

LEMMA 3.2.2. — *Let the finite points $a_1, a_2, \ldots, a_\mu$ be given, and relative positive weights $m_1, m_2, \ldots, m_\mu$, $\sum m_j = 1$. Set*

$$U(z) \equiv (z - a_1)^{m_1}(z - a_2)^{m_2}\ldots(z - a_\mu)^{m_\mu}.$$

Then there exists a sequence of points $\alpha_1, \alpha_2, \ldots$ each of which is some a_j, such that on every compact set E containing no a_j we have

$$0 < A_1 < \left|\frac{(z - \alpha_1)(z - \alpha_2)\ldots(z - \alpha_n)}{[U(z)]^n}\right| < A_2. \tag{3.2.4}$$

We define the α_j according to the properties established in lemma 3.2.1, namely so that among the points $\alpha_1, \alpha_2, \ldots, \alpha_n$ there are precisely N_{nj} which coincide with a_j. Then for each n we have

$$\prod_1^n (z-\alpha_j) \equiv \prod_{j=1}^{\mu} (z-a_j)^{N_{nj}}.$$

From (3.2.3) follows for z on E and for $j = 1, 2, \ldots, \mu$

$$|N_{nj} \log|z-a_j| - nm_j \log|z-a_j|| \leqq A_3,$$

$$\left|\sum_{j=1}^{\mu} \log|z-a_j|^{N_{nj}} - n\sum_{j=1}^{\mu} \log|z-a_j|^{m_j}\right| \leqq \mu A_3,$$

and (3.2.4) follows. The set E may be taken as a compact set of the extended z-plane.

We are now in a position to consider our series development [1955 a], a generalization of the Taylor development :

THEOREM 3.2.1. — *Suppose given the finite points $a_1, a_2, \ldots, a_\mu$; $b_1, b_2, \ldots, b_\nu$, and the positive relative weights $m_1, m_2, \ldots, m_\mu$; $n_1, n_2, \ldots, n_\nu$, $\sum m_j = \sum n_j = 1$. We set*

$$(3.2.5) \qquad u(z) \equiv \frac{(z-a_1)^{m_1}(z-a_2)^{m_2}\ldots(z-a_\mu)^{m_\mu}}{(z-b_1)^{n_1}(z-b_2)^{n_2}\ldots(z-b_\nu)^{n_\nu}},$$

and for every σ (>0) denote by E_σ the set $|u(z)| < \sigma$. A function $f(z)$ analytic on E_ρ but not analytic throughout any $E_{\rho'}$ ($\rho' > \rho$), can be expanded on E_ρ in a series

$$(3.2.6) \quad f(z) \equiv \sum_{n=0}^{\infty} c_n u_n(z), \qquad u_0(z) \equiv 1, \qquad u_n(z) \equiv \frac{(z-\alpha_1)\ldots(z-\alpha_n)}{(z-\beta_1)\ldots(z-\beta_n)},$$

which converges uniformly on every E_σ ($\sigma < \rho$). The α_n, each of which is an a_j, are to be chosen to satisfy (3.2.4), and the β_n, each of which is some b_j, are to be chosen to satisfy the analogue of (3.2.4); consequently on every compact set containing no a_j or b_j we have

$$(3.2.7) \qquad 0 < A_3 < \left|\frac{u_n(z)}{[u(z)]^n}\right| < A_7.$$

The coefficients c_n in (3.2.6) satisfy

$$\limsup |c_n|^{\frac{1}{n}} = \frac{1}{\rho}, \tag{3.2.8}$$

and for every $\sigma\,(<\rho)$ we have

$$\left\{\begin{aligned} &\limsup [\max |f(z) - S_n(z)|,\ z \text{ on } E_\sigma]^{\frac{1}{n}} = \frac{\sigma}{\rho}, \\ &S_n(z) \equiv \sum_0^n c_k u_k(z). \end{aligned}\right. \tag{3.2.9}$$

Inequality (3.2.7) is quite powerful, and a weaker inequality although sufficient for Problem A would not suffice for our later uses of theorem 3.2.1 in Problem β. With (3.2.7) the series (3.2.6) possesses most of the important properties (i. e. for present purposes) of Taylor's series. Series (3.2.6) has the well-known form of a series of interpolation (Newton's series); the c_n can be found formally from (3.2.6) by setting successively $z = \alpha_1$, $z = \alpha_2$, ..., with differentiation a suitable number of times when the $\alpha_1, \alpha_2, \ldots, \alpha_n$, are not all distinct. The series (3.2.6) has a meaning even if one of the points a_j or b_j is infinite; in (3.2.5) and (3.2.6) a linear factor of $u_n(z)$ corresponding to an infinite value of α_j or β_j is simply to be omitted.

As we have said, choose all the α_j and β_j finite, and E_ρ bounded, with $\sigma < \sigma' < \rho$. If Γ_σ denotes generically the locus $|u(z)| = \sigma$, we have for z in $E_{\sigma'}$

$$f(z) - S_n(z) \equiv \frac{1}{2\pi i}\int_{\Gamma_{\sigma'}} \frac{\omega_n(z) f(t)\,dt}{\omega_n(t)(t-z)}, \tag{3.2.10}$$

$$\omega_n(z) \equiv u_n(z)(z - \alpha_{n+1}). \tag{3.2.11}$$

As a consequence of (3.2.7) there follows

$$0 < A_5 < \left| \frac{\omega_n(z)}{[u(z)]^n} \right| < A_6 \tag{3.2.12}$$

on any compact set containing no a_j nor b_j. It is now clear from (3.2.10), by allowing σ' to approach ρ, that the first member of (3.2.9) is not greater than the second member.

The coefficients c_n are readily estimated from (3.2.9) and (3.2.7), since $S_n(z) - S_{n-1}(z) \equiv c_n u_n(z)$. It follows that the first member of (3.2.8) is not greater than the second member. If the first member of (3.2.8) or (3.2.9) is less than the second member, that is true of both (3.2.8) and (3.2.9), whence by (3.2.7) the sequence $S_n(z)$ converges uniformly throughout the interior of some $E_{\rho'}$ $(\rho' > \rho)$ contrary to our hypothesis on $f(z)$. Theorem 3.2.1 is established. The case $\rho = \infty$ is not excluded here.

We add the remark that (3.2.8) and (3.2.7) yield

$$\text{(3.2.13)} \qquad \limsup_{n \to \infty} [\max |S_n(z)|,\ z \text{ on } E_\sigma]^{\frac{1}{n}} = \frac{\sigma}{\rho} \qquad (\sigma > \rho).$$

We have at hand in theorems 3.2.1 and 3.1.1 a new method for the proof of theorems 1.4.3 and 1.4.4, namely precisely the method of use of theorem 1.1.1 to establish theorems 1.2.2 and 1.2.3, as the reader may verify. We have preferred to base theorems 1.4.3 and 1.4.4 on the simpler expansion properties expressed in theorem 1.4.2, which is much more elementary in the sense that it does not involve theorem 3.1.1.

The method of proof of the generalized Bernstein lemma (§ 1.3) gives here too a useful result :

LEMMA 3.2.3. — *Let $R_n(z)$ be a rational function of degree n whose poles lie in the points $\beta_1, \beta_2, \ldots, \beta_n$, and suppose $|R_n(z)| \leq M$ on Γ_σ. Then for $\tau > \sigma$ we have*

$$\text{(3.2.14)} \qquad |R_n(z)| \leq \frac{A_0 M \tau^n}{\sigma^n}, \qquad z \text{ on } \Gamma_\tau,$$

where the constant A_0 does not depend on n, z, σ, τ, or $R_n(z)$, except that σ and τ are to have specific finite upper and lower (positive) bounds.

The function $\dfrac{R_n(z)}{u_n(z)}$ is analytic on the set $|u(z)| \geq \sigma$, even in the points β_j; on Γ_σ we have in the notation of (3.2.7)

$$\left|\frac{R_n(z)}{u_n(z)}\right| = \left|\frac{R_n(z)}{[u(z)]^n}\,\frac{[u(z)]^n}{u_n(z)}\right| \leq \frac{M}{A_3 \sigma^n}.$$

This inequality, valid on Γ_σ, is also valid on the set $|u(z)| \geq \sigma$;

and in particular on Γ_τ we have again by (3.2.7)

$$|R_n(z)| = \left| \frac{R_n(z)}{u_n(z)} \frac{u_n(z)}{[u(z)]^n} [u(z)]^n \right| \leqq \frac{A_1 M \tau^n}{A_3 \sigma^n},$$

which is (3.2.14).

3.3. **Problem** β. — We now introduce the notation that a function $f(z)$ analytic in a one-sided neighborhood of an analytic Jordan curve C, continuous on C, is *of class* $L(p, \alpha)$ *on* C, where $p\ (\geqq 0)$ is integral and $0 < \alpha < 1$, *if* $f(z)$ *has a one-dimensional* p^{th} *derivative on* C *which satisfies there a Lipschitz condition of order* α. It is immaterial here [1949 *a*, theorem 2.4] whether $f^{(p)}(z)$ is taken on C as a one-dimensional derivative with respect to z or to arc-length, or indeed a two-dimensional derivative with respect to z.

For negative integral values of p and $0 < \alpha < 1$, we say that $f(z)$ is of class $L(p, \alpha)$ on the analytic Jordan curve C provided $f(z)$ is analytic in a one-sided neighborhood of C, and C can be expressed as the level locus $u(z) = 1$ of a non-constant function $u(z)$ harmonic and without critical points in an annulus containing C, and where in the one-sided neighborhood of C we have $|f(z)| \leqq A(1-\rho)^{p+\alpha}$ on the locus $u(z) = \rho$ $(\rho_0 < \rho < 1)$; this condition is [1950, theorem 5.3] independent of any particular $u(z)$. To be sure, this requirement is a restriction on the behavior of $f(z)$ not on C but in a one-sided neighborhood of C. Nevertheless, as Hardy and Littlewood have shown if C is the unit circle, and as also is true if C is an arbitrary analytic Jordan curve [1942, § 5.2], whenever $f(z)$ is of class $L(p, \alpha)$ on C $(0 < \alpha < 1)$, the derivative and integral (if single valued) of $f(z)$ are of respective classes $L(p-1, \alpha)$ and $L(p+1, \alpha)$ on C when suitably defined on C if necessary; so even in the case $p < 0$ the class $L(p, \alpha)$ is closely related to behavior on C. The class $L(p, \alpha)$ is invariant under a one-to-one conformal map of a region containing C [compare 1950, § 5].

Our main theorem on Problem β is [1958] :

THEOREM 3.3.1. — *Let* D *be a finite region whose boundary consists of mutually disjoint Jordan curves* $B_1, B_2, \ldots, B_\mu$; $C_1, C_2, \ldots, C_\nu$, *and let* $U(z)$ *be the function harmonic in* D, *continuous in the closure of* D, *and equal to zero and unity* *on* $B = \sum B_j$ *and* $C = \sum C_j$ *respectively. For every* σ $(0 < \sigma < 1)$, *let* Γ_σ *denote the locus* $U(z) = \sigma$ *in* D *and let* D_σ *denote the subregion* $0 < U(z) < \sigma$ *of* D, *whose boundary is* $B + \Gamma_\sigma$.

If Γ_ρ *has no multiple point, and if the function* $f(z)$ *is analytic in* D_ρ, *continuous on* B, *and of class* $L(p, \alpha)$ *on* Γ_ρ $(0 < \alpha < 1)$, *then there exist functions* $f_n(z)$ *analytic in* D *and continuous on* B *such that* $(n = 1, 2, 3, \dots)$

$$(3.3.1) \qquad |f(z) - f_n(z)| \leq \frac{A_1 e^{-\frac{n\rho}{\tau}}}{n^{p+\alpha}}, \qquad z \text{ on } B,$$

$$(3.3.2) \qquad |f_n(z)| \leq \frac{A_2 e^{\frac{n(1-\rho)}{\tau}}}{n^{p+\alpha}}, \qquad z \text{ in } D,$$

where $2\pi\tau$ *is the total variation along* Γ_ρ *of the function conjugate to* $U(z)$.

Reciprocally, if $f(z)$ *is defined on* B, *if the* $f_n(z)$ *are analytic in* D, *and if* (3.3.1) *and* (3.3.2) *are valid for the boundary values of the* $f_n(z)$ *on* B *and* C, *where* p *is an integer and* $0 < \alpha < 1$, *then* $f(z)$ *on* B *represents the boundary values of a function analytic in* D_ρ *and of class* $L(p-1, \alpha)$ *on* Γ_ρ.

Thanks to a conformal map (theorem 3.1.1) it is no loss of generality to suppose D interior to C_1 and defined by $d_0 < \log |u(z)| < d_1$, where $u(z)$ is defined by (3.2.5) and the a_j and b_j are finite. We use the series (3.2.6) of theorem 3.2.1.

For z in D_ρ, the function $f(z)$ can be expressed

$$f(z) \equiv \varphi_1(z) + \varphi_2(z), \qquad \varphi_1(z) \equiv \sum \varphi_{k1}(z), \qquad \varphi_2(z) \equiv \sum \varphi_{k2}(z),$$
$$\varphi_{k1}(z) \equiv \frac{1}{2\pi i}\int_{\gamma_k} \frac{f(t)\,dt}{t-z}, \qquad \varphi_{k2}(z) \equiv \frac{1}{2\pi i}\int_{B_k} \frac{f(t)\,dt}{t-z},$$

where γ_k (depending on z) is a suitably chosen rectifiable Jordan curve in D_ρ near Γ_ρ, precisely one such curve near each component of Γ_ρ, z interior to the regions bounded by B and the γ_k, and where the integrals are taken in the positive sense with respect to those regions. The function $\varphi_1(z)$ is analytic throughout the set D'_0 : $|u(z)| < e^{d_1}$, and can be represented there by a series (3.2.6). Moreover $\varphi_2(z)$ is analytic on Γ_ρ, so $\varphi_1(z)$ is of class $L(p, \alpha)$ there.

The poles β_j of $u_n(z)$ are to be found among the b_k, so lie exterior to D'_0. Suppose now $p \geqq 0$. If $S_n(z)$ is the sum of the first $n+1$ terms of the development (3.2.6), we have for z in D'_0 the interpolation formula

$$(3.3.3)\quad \begin{cases} \varphi_1(z) - S_n(z) = \dfrac{1}{2\pi i}\displaystyle\int_\gamma \dfrac{\omega_n(z)\,\varphi_1(t)\,dt}{\omega_n(t)\,(t-z)}, \qquad \gamma = \sum \gamma_k, \\ \omega_n(z) \equiv (z - \alpha_{n+1})\,u_n(z); \end{cases}$$

here $S_n(z)$ is the unique rational function of degree n whose poles lie in the set $\beta_1, \beta_2, \ldots, \beta_n$ and which coincides with $\varphi_1(z)$ in the points $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}$. A particular case of (3.3.3) occurs if $\varphi_1(z)$ is replaced by an arbitrary rational function $R_n(z)$ of degree n whose poles lie in the set $\beta_1, \beta_2, \ldots, \beta_n$

$$(3.3.4)\qquad 0 = \frac{1}{2\pi i}\int_\gamma \frac{\omega_n(z)\,R_n(t)\,dt}{\omega_n(t)\,(t-z)}, \qquad z \text{ in } D'_0.$$

We shall use a combination of (3.3.3) and (3.3.4):

$$(3.3.5)\quad \varphi_1(z) - S_n(z) \equiv \frac{1}{2\pi i}\int_\gamma \frac{\omega_n(z)\,[\varphi_1(t) - R_n(t)]\,dt}{\omega_n(t)\,(t-z)}, \qquad z \text{ in } D'_0.$$

Under the present circumstances ($p \geqq 0$) the integral in (3.3.5) may be taken over D_ρ, and it follows from corollary 2.4.1 that functions $R_n(z)$ exist such that ($n > 0$)

$$(3.3.6)\qquad |\varphi_1(z) - R_n(z)| \leqq \frac{A}{n^{p+\alpha}}, \qquad z \text{ on } \Gamma_\rho.$$

When we note the equations

$$U(z) = \frac{\log|u(z)| - d_0}{d_1 - d_0}, \qquad \tau = \frac{1}{d_1 - d_0};$$

the inequality

$$(3.3.7)\qquad |\varphi_1(z) - S_n(z)| \leqq \frac{A_1 e^{-\frac{n\rho}{\tau}}}{n^{p+\alpha}}, \qquad z \text{ on } B,$$

follows from (3.3.6), (3.3.5) and (3.2.12). We now set

$$S_{n+1}(z) - S_n(z) \equiv c_{n+1}\,u_{n+1}(z),$$

whence from (3.3.7)

$$|c_{n+1}\,u_{n+1}(z)| \leqq \frac{2A_1 e^{-\frac{n\rho}{\tau}}}{n^{p+\alpha}}, \qquad z \text{ on } B,$$

and by (3.2.7)

$$|c_{n+1}u_{n+1}(z)| \leqq \frac{A_2 e^{\frac{n(1-\rho)}{\tau}}}{n^{p+\alpha}}, \qquad z \text{ on } C.$$

Purely algebraic inequalities now show

$$(3.3.8) \qquad |S_{n+1}(z)| \leqq \frac{A_3 e^{\frac{n(1-\rho)}{\tau}}}{n^{p+\alpha}}, \qquad z \text{ on } C.$$

Since $f(z)$ is continuous on B, so also is $\varphi_2(z) \equiv f(z) - \varphi_1(z)$, so if we set $f_n(z) \equiv S_n(z) + \varphi_2(z)$, we deduce (3.3.1) and (3.3.2) from (3.3.7) and (3.3.8).

In the case $p < 0$ we use (3.3.3) where γ is chosen as Γ_r, $r = r_n = \rho\left(1 - \frac{1}{n}\right)$. For z on B and $n > 0$ we have by (3.2.7)

$$|\varphi_1(z) - S_n(z)| \leqq \frac{A\, e^{-\frac{n\rho}{\tau}}}{n^{p+\alpha}},$$

which in form is identical with (3.3.7). As in the previous treatment of (3.3.7) we deduce (3.3.1) and (3.3.2).

The proof of the first part of theorem 3.3.1 is complete; we proceed to discuss the second part. With $p \geqq 1$ $(0 < \alpha < 1)$ and with (3.3.1) and (3.3.2) as hypothesis, we write for n sufficiently large

$$(3.3.9) \qquad |f_{n+1}(z) - f_n(z)| \leqq \frac{A_3 e^{-\frac{n\rho}{\tau}}}{n^{p+\alpha}}, \qquad z \text{ on } B,$$

$$(3.3.10) \qquad |f_{n+1}(z) - f_n(z)| \leqq \frac{A_4 e^{\frac{n(1-\rho)}{\tau}}}{n^{p+\alpha}}, \qquad z \text{ on } C,$$

where the boundary values are used on C; these exist for almost all values of the conjugate of $U(z)$. If we set

$$M_n = [\max |f_{n+1}(z) - f_n(z)|, \ z \text{ on } \Gamma_\rho],$$

the two-constant theorem applied to the function $f_{n+1}(z) - f_n(z)$ for the respective loci $U(z) = 0, \rho, 1$, is

$$\begin{vmatrix} \log A_3 - \frac{n\rho}{\tau} - (p+\alpha)\log n & 0 & 1 \\ \log M_n & \rho & 1 \\ \log A_4 - \frac{n(1-\rho)}{\tau} - (p+\alpha)\log n & 1 & 1 \end{vmatrix} \leqq 0.$$

Subtraction of the first row from the third row yields

$$M_n \leqq \frac{A_5}{n^{p+\alpha}}.$$

The sequence $f_n(z)$ converges uniformly throughout the closure of D_ρ, by (3.3.1) to a function coinciding with $f(z)$ on B, because we have for z on Γ_ρ

$$(3.3.11)\quad |f(z)-f_n(z)| \leqq |f_{n+1}(z)-f_n(z)| + |f_{n+2}(z)-f_{n+1}(z)| + \ldots$$
$$\leqq M_n + M_{n+1} + \ldots \leqq \frac{A_6}{n^{p+\alpha-1}}.$$

It follows from (3.3.11) and (3.3.2), by virtue of theorem 2.1.2 and of theorem 2.3.1 if Γ_ρ has multiple points, that $f(z)$ is of class $L(p-1, \alpha)$ on Γ_ρ.

In the case $p < 1$ we again use the two-constant theorem, now to determine a bound for

$$M_n(r) = [\max |f_{n+1}(z) - f_n(z)|, \ z \text{ on } \Gamma_r] \qquad (0 < r < \rho),$$

by means of (3.3.9) and (3.3.10). We obtain

$$M_n(r) \leqq \frac{A_6\, e^{-\frac{n(\rho-r)}{\tau}}}{n^{p+\alpha}},$$

where A_6 is independent of r near ρ. For z on Γ_r we find (for instance by comparing the series with an improper definite integral)

$$|f(z)| \leqq A_7 \sum_{m=2}^{\infty} \frac{e^{-\frac{m(\rho-r)}{\tau}}}{m^{p+\alpha}} \leqq A_8(\rho - r)^{p+\alpha-1},$$

where A_8 is independent of r, so $f(z)$ is of class $L(p-1, \alpha)$ on Γ_ρ, and theorem 3.3.1 is established. There is a discrepancy of unity in the classes of functions in the first and second parts of theorem 3.3.1, but that is inherent in the problem itself, as examples show [1949 *a*].

In the second part of theorem 3.3.1 we may replace (3.3.1) and (3.3.2) as hypothesis by (3.3.9) and (3.3.10), and define $f(z)$ on B as the limit of the convergent sequence $f_n(z)$.

Theorem 1.4.4 is not included in theorem 3.3.1, but may be proved by the same method.

3.4. **Problem β, continued.** — In the second part of theorem 3.3.1 we have assumed (3.3.1) and (3.3.2) without restriction on τ, but in the first part of theorem 3.3.1 we have written those inequalities where $2\pi\tau$ denotes the total variation of the conjugate of $U(z)$ along Γ_ρ, and for the discrete values $n = 1, 2, \ldots$. The form may be readily changed; if we set $f_x(z) \equiv f_n(z)$ for $n \leqq x < n+1$, (3.3.1) and (3.3.2) yield ($x \geqq 1$)

$$|f(z) - f_x(z)| \leqq \frac{A'_1 e^{-\frac{x\rho}{\tau}}}{x^{p+\alpha}}, \qquad z \text{ on B},$$

$$|f_x(z)| \leqq \frac{A'_2 e^{\frac{x(1-\rho)}{\tau}}}{x^{p+\alpha}}, \qquad z \text{ in D},$$

and a change of variable $\lambda = \frac{x}{\tau}$, with a change in notation of $f_x(z)$ and a possible auxiliary definition of the new $f_\lambda(z)$ for small λ, yields ($\infty > \lambda \geqq 1$)

$$(3.4.1) \qquad |f(z) - f_\lambda(z)| \leqq \frac{A''_1 e^{-\lambda\rho}}{\lambda^{p+\alpha}}, \qquad z \text{ on B},$$

$$(3.4.2) \qquad |f_\lambda(z)| \leqq \frac{A''_2 e^{\lambda(1-\rho)}}{\lambda^{p+\alpha}}, \qquad z \text{ in D}.$$

Of course (3.3.1) and (3.3.2) follow for arbitrary τ from (3.4.1) and (3.4.2).

It is sufficient in the second part of theorem 3.3.1 so far as concerns (3.3.1) and (3.3.2) if (3.4.1) and (3.4.2) are satisfied for a monotonic sequence of values λ_n with $\lambda_{n+1} - \lambda_n$ bounded; the original forms of (3.4.1) and (3.4.2) follow if we set $f_\lambda(z) \equiv f_{\lambda_n}(z)$, $\lambda_n \leqq \lambda < \lambda_{n+1}$.

In theorem 3.3.1 the functions $f(z)$ and $f_n(z)$ may be analytic on larger point sets :

THEOREM 3.4.1. — *If the hypothesis of theorem 3.3.1 is modified so that the Jordan curves B_j are analytic, and that $f(z)$ is analytic on and within each B_j, then the functions $f_n(z)$ can also be chosen analytic on and within each B_j.*

The map of D onto a canonical region used in the proof of theorem 3.3.1 is one-to-one and conformal not merely in D + B but also in the closure of a suitably chosen set $D_{-\varepsilon}: -\varepsilon < U(z) < 0$ consisting of μ regions. The functions $f(z)$ and $f_n(z)$ in (3.3.1) and (3.3.2) are analytic throughout the closure of $D_{-\varepsilon}$, and (as follows from the proof of theorem 3.3.1) for arbitrary $\delta (> 0)$ we have in the z-plane of theorem 3.4.1

$$(3.4.3) \qquad [\max |f(z) - f_n(z)|,\ z \text{ on } \Gamma_{-\varepsilon}] \leqq A_1 e^{-\frac{n(\varepsilon+\rho-\delta)}{\tau}}.$$

We split $f(z)$ and the $f_n(z)$ into their components by integrating over a locus Γ_σ, where Γ_σ is in D near Γ_ρ, and integrating also over B or $\Gamma_{-\varepsilon}$ indifferently. For z in D_σ we have $f_n(z) \equiv f_{n1}(z) + f_{n2}(z)$, with

$$f(z) \equiv \frac{1}{2\pi i}\int_{\Gamma_\sigma} \frac{f(t)\,dt}{t-z}, \qquad f_{n1}(z) \equiv \frac{1}{2\pi i}\int_{\Gamma_\sigma} \frac{f_n(t)\,dt}{t-z},$$

$$0 \equiv \frac{1}{2\pi i}\int_{\Gamma_{-\varepsilon}} \frac{f(t)\,dt}{t-z}, \qquad f_{n2}(z) \equiv \frac{1}{2\pi i}\int_{\Gamma_{-\varepsilon}} \frac{f_n(t)\,dt}{t-z},$$

$$(3.4.4) \qquad f_{n2}(z) \equiv \frac{-1}{2\pi i}\int_{\Gamma_{-\varepsilon}} \frac{[f(t) - f_n(t)]\,dt}{t-z}.$$

Equation (3.4.4) is valid throughout the closure of D, so by virtue of (3.4.4) and (3.4.3) we may now replace $f_n(z)$ in (3.3.1) and (3.3.2) by $f_{n1}(z)$, with suitable modifications of A_1 and A_2 if necessary, which completes the proof of theorem 3.4.1.

We have phrased both theorems 3.3.1 and 3.4.1 to deal merely with suitably chosen sequences $f_n(z)$, not necessarily extremal, but it is clear that for functions of best approximation, analytic and of modulus not greater than sufficiently large M in D, the analogues of (3.4.1) and (3.4.2) hold.

In various cases included under the second part of theorem 3.3.1, inequality (3.3.2) is a consequence of (3.3.1). For example let B consist of a finite number of mutually exterior Jordan curves B_k and suppose polynomials $f_n(z)$ of respective degrees $n = 1, 2, \ldots$ are given such that

$$(3.4.5) \qquad |f(z) - f_n(z)| \leqq \frac{A_1 e^{-n\delta}}{n^{p+\alpha}}, \qquad z \text{ on B},$$

is valid, $0 < \alpha < 1$, $\delta > 0$. Then we have for n sufficiently large

$$(3.4.6) \qquad |f_{n+1}(z) - f_n(z)| \leqq \frac{A_2 e^{-n\delta}}{n^{p+\alpha}}, \qquad z \text{ on B},$$

and $f_{n+1}(z) - f_n(z)$ is a polynomial of degree $n+1$. Let $g(z)$ denote Green's function with pole at infinity for the infinite region D′

bounded by B, and let g_σ denote generically the locus $g(z)=\sigma(>0)$ in D'. From the generalized Bernstein lemma (§ 1.3) follows by (3.4.6) for n sufficiently large

$$(3.4.7) \qquad |f_{n+1}(z)-f_n(z)| \leq \frac{A_2 e^{n(\gamma-\delta)}}{n^{p+\alpha}}, \qquad z \text{ on } g_\gamma.$$

Inequalities (3.4.6) and (3.4.7), provided $\gamma > \delta$, can be identified with (3.3.9) and (3.3.10), which are sufficient for the application of the second part of theorem 3.3.1. Here D is bounded by B and g_γ $(\gamma > \delta)$, with

$$U(z) \equiv \frac{g(z)}{\gamma}, \qquad \tau = \frac{1}{\gamma}, \qquad \delta = \frac{\rho}{\tau}, \qquad \gamma - \delta = \frac{1-\rho}{\tau},$$

whence Γ_ρ of theorem 3.3.1 is g_δ, namely the locus $g(z)=\gamma\rho=\delta$. The conclusion [1937], a consequence merely of (3.4.5), is that $f(z)$ *is of class* $L(p-1, \alpha)$ *on* g_δ *provided* (*if* $p>0$) g_δ *has no multiple points;* this conclusion is independent of the auxiliary number $\gamma\,(>\delta)$.

Theorem 3.3.1 thus applies under suitable conditions to approximation by polynomials; it may apply also to approximation by more general rational functions. As an illustration, suppose with the conditions and notation of theorem 3.2.1 [other than the hypothesis on $f(z)$] that for a function $f(z)$ defined merely on B : $|u(z)|=e^{d_0}$ we have $d>d_0$ $(0<\alpha<1)$ for $n=1, 2, 3, \ldots$

$$(3.4.8) \qquad |f(z)-f_n(z)| \leq \frac{A_1 e^{-n(d-d_0)}}{n^{p+\alpha}}, \qquad z \text{ on B},$$

where $f_n(z)$ is a rational function of degree n whose poles lie in the set $\beta_1, \beta_2, \ldots, \beta_n$; of course $f_n(z)$ need not be determined by interpolation, but may be for instance the rational function of the prescribed type of best approximation to $f(z)$ on B in the sense of Tchebichef with continuous norm function. From (3.4.8) we have for n sufficiently large

$$(3.4.9) \qquad |f_{n+1}(z)-f_n(z)| \leq \frac{2A_1 e^{-n(d-d_0)}}{n^{p+\alpha}}, \qquad z \text{ on B}.$$

From lemma 3.2.3 there follows $(d_1>d,\ C : |u(z)|=e^{d_1})$ for n sufficiently large

$$(3.4.10) \qquad |f_{n+1}(z)-f_n(z)| \leq \frac{A_2 e^{n(d_1-d)}}{n^{p+\alpha}}, \qquad z \text{ on C}.$$

In the notation of theorem 3.3.1 for $U(z)$ we have

$$U(z) \equiv \frac{\log|u(z)| - d_0}{d_1 - d_0}, \qquad \tau = \frac{1}{d_1 - d_0},$$

so (3.4.9) and (3.4.10) can be identified with (3.3.9) and (3.3.10), whence *it follows merely as a consequence of* (3.4.8) *that* $f(z)$ *can be extended from* B *so as to be analytic in* $|u(z)| < e^d$, *and of class* $L(p-1, \alpha)$ *on the locus* $|u(z)| = e^d$.

A remark is appropriate here regarding choice of approximating functions in general. In the geometric situation of theorem 3.3.1, it is apparent that so far as concerns degree of approximation and norm of approximating functions as we have measured them, the rational functions of theorem 3.2.1 are as effective for approximation as are any possible family of analytic functions. Likewise in the geometric situation of theorem 1.3.3 polynomials in z are as effective as any set of analytic functions can be.

CHAPTER IV.

GENERALIZATIONS AND EXTENSIONS. OPEN PROBLEMS.

We have given in the foregoing chapters a presentation in detail of some of the most striking results to date of the theory of approximation by bounded analytic functions. We shall now indicate without proof some of the wider ramifications, for the known results are by no means limited to those set forth above.

4.1. Geometric situations. — Although theorems 1.4.1-1.4.4 admit rather general geometric configurations, other interesting configurations are not included. For instance if D is an annular region and if E is either a Jordan curve or an annular region in D which separates the two boundary components of D, it is appropriate to consider approximation on E by functions analytic and bounded in D to a function analytic on E but not analytic throughout D. This configuration, and others much more general, have been studied [1944] in connection with Problem A (relation of regions of analyticity to geometric degree of convergence), have been studied in connection with Problem α in theorem 2.1.2 and elsewhere [1956 *a*], and can be studied by similar methods in connection with Problem β, though that has not as yet been done except [1942, § 8.1] when the boundaries of the sets involved are concentric circles and [1959 *b*] in a few other cases.

In chapter II we have limited ourselves to approximation on *analytic* Jordan curves and sets bounded by such curves. Although some slight progress has been made, Problem α is on the whole still open for more general curves. In particular, if C is a Bernoullian lemniscate or other analytic curve with but a single double point, no precise analogue of theorem 2.1.2 is known. Naturally, if C together with the finite regions which it bounds lies in a region D, and if functions $f_n(z)$ analytic in D satisfy (2.1.2) and (2.1.3), theorem 2.3.1 can be applied to each subarc of C, and shows that $f(z)$ is of class $L(p, \alpha)$ on C; but this remark is far from supplying a necessary and sufficient condition for (2.1.2) and (2.1.3). As a consequence of this lack, Problem β as discussed in chapter III (e. g. theorem 3.3.1) is satisfactorily treated provided Γ_ρ has no multiple points, but not if Γ_ρ has multiple points; although we have not emphasized the fact, use of class $L(p, \alpha)$ for $p < 0$ presents no difficulty if Γ_ρ has multiple points, but this is not true for $p \geqq 0$.

Likewise Problem β deserves deeper study concerning more general sets; can theorems 3.3.1 and 3.4.1 be extended to include measure of approximation (3.3.1) on an arbitrary continuum or on several continua? Results are available [1942, § 8.2] for approximation by polynomials and [1959] by bounded analytic functions on a line segment.

Hitherto we have interpreted Problem A as approximation on a set E by functions $f_n(z)$ analytic and bounded in a region D containing E. A possible extension is to admit common boundary points of D and E, with suitable behavior of the $f_n(z)$ in such points. Under suitable conditions the results of chapter I can be extended to include this situation, for both direct and indirect theorems [1954].

4.2. Continuity classes. — In chapters II and III we have for simplicity restricted ourselves to the use of classes $L(p, \alpha)(0<\alpha<1)$; this is to some extent a reflection of the fact that theorem 2.1.1 is elegant and satisfying for such classes with $p \geqq 0$, but does not extend to the class $L(p, \alpha)$ with $\alpha = 1$. Indeed, for $\alpha = 1$, the first part of theorem 2.1.1 is valid but the second part is false. To fill this gap, Zygmund [1945] introduced into the theory of degree of trigonometric approximation the condition

$$(4.2.1) \qquad |f(\theta + h) + f(\theta - h) - 2f(\theta)| \leq A\,|h|,$$

which for a continuous function $f(\theta)$ of period 2π is necessary and sufficient for approximation to $f(\theta)$ by trigonometric polynomials of order n with error not greater than $\frac{A}{n}$; a necessary and sufficient condition for approximation to $f(\theta)$ by trigonometric polynomials of order n with error not greater than $\frac{A}{n^{p+1}}$ is that $f^{(p)}(\theta)$ exist and be continuous, and satisfy the analogue of (4.2.1). This same condition, interpreted on an analytic Jordan curve C in terms of arc length, say that $f(z)$ is of class Z_p, where 0 lies interior to C, is necessary and sufficient [1950, § 4] that a function $f(z)$ can be approximated on C by polynomials of degree n in z and $\frac{1}{z}$ with error not greater than $\frac{A}{n^{p+1}}$; if polynomials of degree n in z are used, $f(z)$ must also be the continuous set of boundary values on C of a function analytic interior to C. Throughout the discussion of chapters II and III, all results remain valid if the class $L(p, \alpha)$ is replaced by class Z_p, and if the exponent $p + \alpha$ of n is replaced by $p + 1$.

Also for negative $p (\neq -1)$ the class $L(p, \alpha)$ already introduced (§ 3.3) can be replaced by a similarly defined class Z_p, and the results of chapter III persist for the new class if $p + \alpha$ is replaced by $p + 1$. But $p = -1$ is genuinely exceptional, in that a suitable class $L(-1, 1)$ is otherwise defined and studied [1950, 1957] yet with analogous conclusions.

Condition (4.2.1) is not the expression of a modulus of continuity, although any function which satisfies a Lipschitz condition of order unity also satisfies (4.2.1). For a function $f(\theta)$ with given modulus of continuity $\omega(\delta)$ and period 2π, de la Vallée Poussin [1919] following D. Jackson shows that there exist trigonometric polynomials of respective orders $n = 1, 2, \ldots$ approximating $f(\theta)$ with error not greater than $A\omega\left(\frac{1}{n}\right)$. Conversely, if there exist such polynomials approximating $f(\theta)$ with error not greater than $\Omega(n)$, he derives an expression for a modulus of continuity of $f(\theta)$, under suitable conditions on $\Omega(n)$. Both these results have precise analogues in approximation by polynomials in z and by bounded analytic functions; every result proved in chapters II and III concerning Lipschitz conditions admits a corresponding generalization [1949 *a*, 1951 *a*].

The behavior of continuity classes as such under conformal mapping is studied in [1949 *a*, 1959 *d*, 1959 *e*].

4.3. **Other norms.** — In chapter I we have considered the Tchebichef norm and Tchebichef measure of approximation, as being the most fundamental. It is obviously appropriate to consider other norms and measures of approximation, say p^{th} power integrals with or without weight function. Such a theory can be developed [1949], and it is not necessary to use simultaneously the same kinds of norms and measures of approximation. The results are wholly analogous to those already set forth (chapter I) on Problem A.

The theory just mentioned is of particular elegance and interest if the integrals of squares are used. For instance, if D is the region $|z|<r_1(>1)$ and E is the set $|z|\leqq r_0(<1)$, suppose $f(z)\equiv\sum_0^\infty a_k z^k$ given, where the series has unit radius of convergence. The extremal function $f_M(z)\equiv\sum_0^\infty b_k z^k$ of norm $\|f_M\|$ in D,

$$(4.3.1)\qquad \|f_M\|^2=\frac{1}{2\pi}\int_{|z|=r_1}|f_M(z)|^2\,|dz|=\sum_0^\infty |b_k|^2 r_1^{2k},$$

which shall be not greater than a prescribed M, while the measure of approximation m_M of $f_M(z)$ to $f(z)$ on E,

$$(4.3.2)\qquad m_M=\frac{1}{2\pi}\int_{|z|=r_0}|f(z)-f_M(z)|^2\,|dz|=\sum_0^\infty |a_k-b_k|^2 r_0^{2k},$$

is least, is given explicitly by

$$b_k=\frac{a_k r_0^{2k}}{r_0^{2k}+\lambda r_1^{2k}},$$

where λ is determined by the equation

$$\sum_{k=0}^{\infty}\left(\frac{|a_k|\, r_0^{2k}}{r_0^{2k}+\lambda r_1^{2k}}\right)^2 r_1^{2k} = \mathrm{M},$$

and the measure of approximation satisfies

$$m_{\mathrm{M}}^2 = \lambda^2 \sum_{k=0}^{\infty}\left(\frac{|a_k|\, r_1^{2k}}{r_0^{2k}+\lambda_1^{2k}}\right)^2 r_0^{2k}.$$

It follows that $f_{\mathrm{M}}(z)$ is analytic not merely throughout D but even throughout the larger region $|z| < \frac{r_1^2}{r_0^2}$. In fact the singularities of $f_{\mathrm{M}}(z)$ are closely related in position and character to those of $f(z)$ — this relation deserves further investigation for the norms just used as well as others.

Whether or not D and E are more general, there is a close relation between norms and measures of approximation defined by integrals of squares such as (4.3.1) and (4.3.2), and on the other hand a sequence of functions introduced by S. Bergman, functions that are mutually orthogonal not merely in D but also in E. This relation, and its connection with certain Fredholm integral equations, have been investigated by P. Davis [1952 *d*].

Although the study of approximation by functions of minimum (non-Tchebichef) norm has been carried to a certain point, investigation of boundary behavior (Problems α and β) has not yet been undertaken on a broad scale. Other questions, such as behavior of zeros of approximating functions near the boundary of a region of convergence, and overconvergence in the sense of Ostrowski, and even lacunary series, have been broadly treated [1946 *a*] but thus far without application as specific as is possible. Compare [1959 *f*].

Specific determination of the numerical constants involved in the conclusions throughout the theory would desirable.

4.4. Interpolation by functions of least norm. — A problem complementary to the one we have been studying throughout the present essay is the following [1938] :

PROBLEM I. — *Given a region* D, *a point set* E *in* D, *and a function* $f(z)$ *analytic on* E *but not throughout* D; *for each* $m (> 0)$

let $F_m(z)$ *denote the (or a) function analytic in* D *such that* $|f(z)-F_m(z)| \leqq m$ *on* E *and for which* [l. u. b. $|F_m(z)|$, z *in* D] $= M_m$ *is least. To study the convergence of* $F_m(z)$. Our previous results are clearly of significance here, and yield also optimum results.

Related to approximation by functions of least norm is a problem of interpolation [1938], which we formulate as

Problem II. — *Given a region* D, *points*

$$(4.4.1)\qquad \left\{\begin{array}{l} \beta_{11}, \\ \beta_{21},\quad \beta_{22}, \\ \dots\dots\dots, \\ \beta_{n1},\quad \beta_{n2},\quad \dots,\quad \beta_{nn}, \\ \dots\dots\dots\dots\dots\dots\dots, \end{array}\right.$$

in D, *and a function* $f(z)$ *not analytic throughout* D *but analytic in each point* β_{nk}; *let* $f_n(z)$ *be the function of least norm in* D *which coincides with* $f(z)$ *in the points* $\beta_{n1}, \beta_{n2}, \dots, \beta_{nn}$; *to study the convergence of the sequence* $f_n(z)$.

Perhaps the simplest non-trivial illustration of Problem II is that in which the points (4.4.1) are all identical; we prove

Theorem 4.4.1. — *Let* $f(z)$ *be analytic in the region* $|z| < \rho$, *but not analytic throughout any region* $|z| < \rho'$ $(\rho' > \rho)$. *Let* $f_n(z)$ *be the function analytic in* D : $|z| < R (> \rho)$ *which coincides with* $f(z)$ *in the origin counted of multiplicity* n, *the least upper bound of whose modulus in* D *is a minimum. Then we have*

$$(4.4.2)\qquad \limsup_{n \to \infty} [\max |f(z) - f_n(z)|, \ \textit{for} \ |z| \leqq r]^{\frac{1}{n}} = \frac{r}{\rho} \qquad (r < \rho),$$

$$(4.4.3)\qquad \limsup_{n \to \infty} [\text{l. u. b.} \, |f_n(z)|, \qquad \textit{for} \ |z| \leqq r]^{\frac{1}{n}} = \frac{r}{\rho} \qquad (\rho \leqq r \leqq R).$$

The existence and uniqueness of the $f_n(z)$ are known [1935, § 10.3, theorem 8].

With the notation (1.1.1) and (1.1.3) we have (1.1.4) and (1.1.5), the extremal property of the $f_n(z)$ yields

$$(4.4.4)\qquad [\text{l. u. b.} \, |f_n(z)|, \ \text{for} \ |z| \leqq R] \leqq [\max |S_{n-1}(z)|, \ \text{for} \ |z| = R],$$

whence by (1.1.5)

$$(4.4.5)\quad \lim_{n\to\infty}\sup[\text{l. u. b. } |f_n(z)-S_{n-1}(z)|,\ \text{for } |z|\leqq R]^{\frac{1}{n}}\leqq\frac{R}{\rho}.$$

The function $f_n(z)-S_{n-1}(z)$ has a zero of multiplicity at least n in the origin, so by an obvious extension of Schwarz's lemma we have

$$(4.4.6)\quad \lim_{n\to\infty}\sup[\max|f_n(z)-S_{n-1}(z)|,\ \text{for } |z|\leqq r]^{\frac{1}{n}}\leqq\frac{r}{\rho}\qquad (r\leqq R).$$

Equation (1.1.4) and inequality (4.4.6) show that the first member of (4.4.2) is not greater than the second member; equation (1.1.5) and inequality (4.4.6) show that the first member of (4.4.3) is not greater than the second member. Equality in (4.4.2) and (4.4.3) follows from corollary 1.1.1.

Let theorem 4.4.1 be modified by requiring that $f_n(z)$ shall be the function analytic in D which coincides with $f(z)$ in the origin counted of multiplicity n, whose norm is least, where now for an arbitrary $\varphi(z)\equiv\sum_0^\infty c_n z^n$ we define the norm of $\varphi(z)$ by

$$\|\varphi(z)\|^2=\sum_0^\infty |a_n|^2R^{2n}.$$

Then for every n we have $f_n(z)\equiv S_{n-1}(z)$, and the convergence properties of $S_n(z)$ are known (compare § 1.1). Since $f_n(z)$ coincides with $f(z)\equiv a_0+a_1z+\dots$ in the origin counted of multiplicity n, we have

$$f_n(z)\equiv a_0+a_1z+\dots+a_{n-1}z^{n-1}+c_nz^n+c_{n+1}z^{n+1}+\dots,$$

where the coefficients c_n, c_{n+1}, ..., to be determined so as to minimize $\|f_n(z)\|$, must all vanish; consequently

$$f_n(z)\equiv S_{n-1}(z).$$

Theorem 4.4.1 admits of large extensions [1939, 1955] to other regions D, to other norms, and to sets of points (4.4.1) which are given with certain asymptotic conditions.

Investigations of Problem II have thus far been primarily concerned with regions of analyticity and of convergence (Problem A); the more delicate questions of boundary behavior (Problems α and β) remain open.

One further phase of Problem II deserves mention if the points (4.4.1) are independent of n. To determine or to study the function $f(z)$ analytic and of minimum norm (according to any classical definition) in D, taking on prescribed values B_k in the points β_k :

$$f(\beta_k) = B_k \qquad (k = 1, 2, \ldots), \tag{4.4.7}$$

one may consider first the finite problem, that of studying the function $f_n(z)$ analytic and of minimum norm, satisfying

$$f_n(\beta_k) = B_k \qquad (k = 1, 2, \ldots, n) \tag{4.4.8}$$

and then allowing n to become infinite. This method is entirely effective if $\|f(z)\|$ satisfies simple requirements, for instance if $\|f(z)\| = [\text{l. u. b.} \ |f(z)| \text{ in D}]$ or if $\|f(z)\|^2 = \iint_{\mathrm{D}} |f(z)|^2 \, d\mathrm{S}$; compare [1935, 1954 a].

4.5. **Extremal problems.** — If a closed point set E lies interior to a region D, and if $f(z)$ is given on E whether analytic there or not, it is appropriate to study best approximation to $f(z)$ on E as measured in any one of a variety of ways by functions $\varphi_{\mathrm{M}}(z)$ analytic and of norm (in some sense) on D not greater than M. As M becomes infinite, the sequence $\varphi_{\mathrm{M}}(z)$ may well converge on E to some extremal function. In addition the $\varphi_{\mathrm{M}}(z)$ may be required to satisfy conditions of interpolation in D, conditions which may or may not vary with M and which may or may not require equality of $\varphi_{\mathrm{M}}(z)$ to $f(z)$ in certain points [compare 1935, § 11.3-11.5]. It is not to be expected that such a general problem can be fully treated. But progress has been made especially in the cases where norm and measure of approximation are taken in the sense of Tchebichef [1935, § 11.7, 11.8], and where norm and measure of approximation are expressed by integrals of squares of moduli [1950 a]. In the latter case, orthogonality conditions enter naturally and are highly convenient as a tool. Interesting open questions remain as to other norms and measures of approximation (Problem A), and even in the cases hitherto treated as to behavior on the boundary of regions of convergence (Problems α and β).

By way of historical perspective, it is of interest to note that C. Runge's theorem on polynomial approximation was published seventy-five years ago (1885), P. Montel's book on series of polynomials appeared fifty years ago (1910), and the present writers's book on interpolation and approximation dates from twenty-five years ago [1935].

This essay should not be concluded without some mention of problems for harmonic functions analogous to those here treated for analytic functions. This topic has received some treatment [1944, 1949 *a*, 1950, 1954 *b*, 1960], but for it numerous open problems also remain.

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Chapter IV.

Generalizations and extensions. Open problems.