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THE EXTENSION OF DARBOUX'S METHOD

By D. H. PARSONS.

INTRODUCTION.

In the long history of research into the theory of partial differential equations of the second order, many and varied procedures have been developed; almost always the objective has been the solution of boundary-value problems of various types, although methods exist which in certain cases provide general solutions. Certain procedures stand out by virtue of their ability to provide the required solution in finite terms, by means of quadratures or other processes which can, at any rate theoretically, always be carried out. Among these, the methods of Laplace, Monge, Ampere and Darboux [1], in that order of chronology, mark the development of the particular line of approach with which the present work is concerned.

In 1773, Laplace published a method, applicable to certain equations with two independent variables, which consisted essentially ot applying a series of transformations to the given equation, with the object of eventually obtaining an equation possessing an intermediate integral, which could be found by inspection. Shortly afterwards, in 1784, Monge published his work which set forth the general method of solution by seeking intermediate integrals. This procedure was extended by Ampere in 1814. A considerable interval then elapsed, in which the theory advanced very little, until in 1870 appeared the paper of Darboux [1], in which he set forth the method of integration which bears his name, and of which the methods of Monge and Ampere, and to a large extent that of Laplace, are special cases.

These methods have been only very imperfectly understood by many mathematicians; but the theory and applications have been most elegantly and simply described by Goursat [2], who uses the theory of characteristic multiplicities to illuminate the underlying principles.

Both Monge's and Darboux's methods are such that for those equations to which the methods may be successfully applied, the boundary-value problem of Cauchy may be solved explicitly, reduced to quadratures, to the integration of a completely integrable system of total differential equations, or to the integration of a system of ordinary differential equations—processes which, from the theoretical point of view, are of equal simplicity, and can always be carried out, in theory at least: and the desired result is expressed in finite terms. Herein lies the importance of these methods.

However, in their classical forms, the Monge-Ampere and Darboux procedures are applicable only to equations with two independent variables; and this limitation has led to their being relegated to positions of obscurity in the minds of most mathematicians. Attempts to produce extensions to equations with three or more independent variables have not met with much success, although Natani [3] indicates a possible extension of Monge's method to equations with any number of independent variables, belonging to a special class, while Vivanti [4] has dealt with equations having three independent variables and possessing an intermediate integral, the essential feature of the Monge-Ampere method.

The primary object of the present work is the extension of Darboux's method to equations with three independent variables. It is shown that the method can be extended to deal with a class of equations, for which a certain discriminant vanishes identically, exactly as in the case treated by Natani.

As in Goursat's account of the classical case, the idea of equations in involution, originated by Sophus Lie, is also extended to this class of equations, in order to gain insight into the reason why the method succeeds, which at first sight is far from obvious. Analogues of various classical theorems are established, and finally some examples of application of the method are given.

CHAPTER I.

NOTATIONS, FUNDAMENTAL ASSUMPTIONS, CERTAIN CONVENTIONS AND DEFINITIONS.

We begin by defining certain notations, and certain terms which will be used throughout.

Let the independent variables be x, y, z, and let the dependent variable be u. Let

$$\frac{\partial u}{\partial x} = l, \quad \frac{\partial u}{\partial y} = m, \quad \frac{\partial u}{\partial z} = n, \quad \frac{\partial^2 u}{\partial x^2} = a, \quad \frac{\partial^2 u}{\partial y^2} = b,$$

$$\frac{\partial^2 u}{\partial z^2} = c, \quad \frac{\partial^2 u}{\partial y\, \partial z} = f, \quad \frac{\partial^2 u}{\partial z\, \partial x} = g, \quad \frac{\partial^2 u}{\partial x\, \partial y} = h;$$

in general let

$$\frac{\partial^{i+j+k} u}{\partial x^i\, \partial y^j\, \partial z^k} = p_{i,j,k},$$

and in particular let

$$\frac{\partial^{2+j+k} u}{\partial x^2\, \partial y^j\, \partial z^k} = r_{2,j,k},$$

$$\frac{\partial^{1+j+k} u}{\partial x\, \partial y^j\, \partial z^k} = s_{1,j,k},$$

$$\frac{\partial^{j+k} u}{\partial y^j\, \partial z^k} = t_{0,j,k}.$$

Thus when no particular emphasis is required upon the number of derivations with respect to x, we use the letter p to denote those variables which would otherwise be denoted by r, s, or t as the case might be.

Two terms which will henceforth occur frequently, now require precise definition. An "element of contact of order n" indicates a set of values of the variables x, y, z, $p_{i,j,k}$ where i, j, k take all positive integral and zero values such that $i + j + k \leqq n$.

A "multiplicity M_q of order n", or simply a "multiplicity M_q" denotes an aggregate of elements of contact of order n, depending upon q parameters (i. e., having q dimensions) and satisfying, for any displacement in the multiplicity, the "equations of contact"

$$du - l\,dx - m\,dy - n\,dz = 0,$$
$$dp_{i,j,k} - p_{i+1,j,k}\,dx - p_{i,j+1,k}\,dy - p_{i,j,k+1}\,dz = 0,$$

for each i, j, k such that

$$i + j + k \leqq n - 1.$$

Considering equations of the second order, we shall suppose that the equation to be considered can be solved for one of the three derivatives a, b and c; and clearly there is no loss of generality in taking this derivative to be a. This assumption is justified : for we are concerned with *any* integral of the equation, and not with particular integrals, and thus if the equation be

$$\psi(x, y, z, u, l, m, n, a, b, c, f, g, h) = 0$$

we may solve for a, b or c, except on particular integrals, unless we have identically, or as a consequence of the equation itself

$$\frac{\partial\psi}{\partial a} = \frac{\partial\psi}{\partial b} = \frac{\partial\psi}{\partial c} = 0$$

At this stage we will make the assumption, which will apply throughout, that all functions with which we are concerned are analytic functions of their arguments, in the neighbourhood of certain initial values. We will, in addition, assume that the equation $\psi = 0$ is of fully reduced form; that is, that in the neighbourhood of a set of values of the arguments, satisfying the equation, at least one of

$$\frac{\partial\psi}{\partial x}, \quad \frac{\partial\psi}{\partial y}, \quad \frac{\partial\psi}{\partial z}, \quad \frac{\partial\psi}{\partial u}, \quad \frac{\partial\psi}{\partial l}, \quad \frac{\partial\psi}{\partial m}, \quad \frac{\partial\psi}{\partial n},$$
$$\frac{\partial\psi}{\partial a}, \quad \frac{\partial\psi}{\partial b}, \quad \frac{\partial\psi}{\partial c}, \quad \frac{\partial\psi}{\partial f}, \quad \frac{\partial\psi}{\partial g}, \quad \frac{\partial\psi}{\partial h}$$

is not zero. Under these circumstances, the conditions stated above would imply that the equation does not contain a, b or c. If this were so, the equation would contain f, g or h; for otherwise it would be of the first order. Suppose for example that $\frac{\partial\psi}{\partial h}$ is not zero. Then we may solve the equation for h and write

$$h + \pi(x, y, z, u, l, m, n, f, g) = 0.$$

But the change of variables

$$x + y = x'; \qquad x - y = y'; \qquad z = z'$$

renders this last equation soluble for $\frac{\partial^2 u}{\partial x'^2}$.

Consequently we will from now on restrict our attention to the equation

(1) $$a + \Psi(x, y, z, u, l, m, n, h, g, b, f, c) = 0,$$

where Ψ is analytic in the neighbourhood of a suitable set of initial values of x, y, z, u, l, m, n, h, g, b, f, c.

We will be concerned with certain functions of the elements of contact of various orders; but since we will be mainly interested in the behaviour of these functions relative to integrals of (1), it will always be assumed that these functions contain only those derivatives which involve not more than one differentiation with respect to x. For we may calculate the values of all derivatives involving more than one differentiation with respect to x, in terms of x, y, z, u and derivatives of u involving one or no x-differentiation, by means of (1) and the equations derived from (1) by successive differentiations, regarding u and each of its derivatives as a function of x, y, z. Accordingly we now introduce a quite usual convention, with a slight modification to suit this particular problem. χ being a function of the elements of contact of order n, the symbols

$$\frac{d\chi}{dx}, \quad \frac{d\chi}{dy}, \quad \frac{d\chi}{dz}$$

denote partial derivatives of χ, in calculating which u and the variables $p_{i,j,k}$ are regarded as functions of x, y, z, in accordance with the equations defining these variables. Again,

$$\left(\frac{d\chi}{dx}\right), \quad \left(\frac{d\chi}{dy}\right), \quad \left(\frac{d\chi}{dz}\right)$$

denote the same derivatives as before, but with the terms involving derivatives of u of order $n+1$ *omitted*. Thirdly

$$\frac{d\chi^*}{dx}, \quad \left(\frac{d\chi}{dx}\right)^*$$

indicate that, *after* calculating $\frac{d\chi}{dx}$ and $\left(\frac{d\chi}{dx}\right)$, we have replaced all derivatives of u involving more than one x-differentiation by their values derived from (1), as explained above. In connection with the last definition, it is important to notice that we do *not* obtain $\left(\frac{d\chi}{dx}\right)^*$ from $\frac{d\chi^*}{dx}$ in the same way as $\left(\frac{d\chi}{dx}\right)$ is obtained from $\frac{d\chi}{dx}$: for the substitution which changes $\frac{d\chi}{dx}$ into $\frac{d\chi^*}{dx}$ may introduce fresh terms which do not contain derivatives of u of order $n+1$,

We extend the same conventions to any order of differentiation. Thus if χ contains derivatives of u of order n but not of higher orders, $\left(\frac{d^{j+k}\chi}{dy^j\, dz^k}\right)$ indicates that after carrying out the indicated differentiation we omit those terms involving derivatives of u of order $n+j+k$.

To illustrate these conventions, suppose that χ is a function of the elements of contact of order n, containing no derivative of u involving more than one x-derivation, but containing at least one derivative of the n'th order. Then we may write

$$\frac{d\chi}{dx} = \left(\frac{d\chi}{dx}\right) + \sum_{i=0}^{n-1} r_{2,n-i-1,i}\frac{\partial\chi}{\partial s_{1,n-i-1,i}} + \sum_{j=0}^{n} s_{1,n-j,j}\frac{\partial\chi}{\partial t_{0,n-j,j}},$$

$$\frac{d\chi}{dy} = \left(\frac{d\chi}{dy}\right) + \sum_{i=0}^{n-1} s_{1,n-i,i}\frac{\partial\chi}{\partial s_{1,n-i-1,i}} + \sum_{j=0}^{n} t_{0,n-j+1,j}\frac{\partial\chi}{\partial t_{0,n-j,j}},$$

$$\frac{d\chi}{dz} = \left(\frac{d\chi}{dz}\right) + \sum_{i=1}^{n} s_{1,n-i,i}\frac{\partial\chi}{\partial s_{1,n-i,i-1}} + \sum_{j=1}^{n+1} t_{0,n-j+1,j}\frac{\partial\chi}{\partial t_{0,n-j+1,j-1}}.$$

while to calculate $\frac{d\chi^*}{dx}$ we substitute in the first of these equations the appropriate values of the $r_{2,n-i-1,i}$ obtained from (1) and in addition write $\left(\frac{d\chi}{dx}\right)^*$ instead of $\left(\frac{d\chi}{dx}\right)$.

At this stage we adopt a further device which will be found to give great abbreviation in writing. We introduce certain fictitious variables $s_{1,i,j}$ and $t_{0,i,j}$, in which one or both of the suffices is *negative*. These variables may be regarded as always equal to zero; furthermore the functions with which we shall be concerned do not contain these variables, so that we have always

$$\frac{\partial\chi}{\partial s_{1,i,j}} = 0 \quad\text{and}\quad \frac{\partial\chi}{\partial t_{0,i,j}} = 0$$

when i or j or both are negative.

One last remark, of great importance, must be made before we go on to obtain results. From the form of (1), and since Ψ is an analytic function of its arguments, it follows from the usual existence theorem that integrals of (1) exist such that when $x = 0$, u and l reduce to arbitrarily assigned (analytic) functions of y and z. This being so, we see that for any fixed values of x, y, z, integrals of (1) exist for which all the variables u, $s_{1,i,j}$, $t_{0,k,q}$ (i, j, k, q being, of course, positive integers or zero), up to and including any given order of derivatives, take arbitrarily chosen values at the point x, y, z.

In particular, given the values of x, y, z, u and all the derivatives of u of order up to and including n (that is, the variables $s_{1,k-i-1,i}$, $t_{0,k-j,j}$, $1 \leqq k \leqq n$), we may find integrals of (1) for which these variables take the assigned values at this point, and for which the derivatives of u of the $n+1$'th order, $s_{1,k-i,i}$ and $t_{0,k-j+1,j}$ ($i = 0, 1, .., n$, $j = 0, 1, \ldots, n, n+1$), assume any arbitrarily assigned values.

From now on, it will be understood that an "arbitrary element of contact of order n" means an element of contact in which x, y, z, u and derivatives of u involving not more than one differentiation with respect to x have been chosen arbitrarily; while the other derivatives have the values calculated from (1) and the equations derived from (1) by differentiations. Similarly the phrase "the derivatives of u of order $n+1$" will refer only to the derivatives $s_{1,n-i,i}$ and $t_{0,n-j+1,j}$. Thus we may state the remark of the last paragraph more briefly as follows.

"Given an element of contact of order n, integrals of (1) exist, admitting this element, for which the derivatives of u of order $n+1$ corresponding to the element assume arbitrary values". Again, the phrase "a function of the elements of contact of order n" will henceforth refer exclusively to functions of x, y, z, u and the variables $s_{1,k-i-1,i}$, $t_{0,k-j,j}$, $1 \leqq k \leqq n$, $0 \leqq i \leqq k-1$, $0 \leqq j \leqq k$.

We write

$$\frac{\partial \Psi}{\partial b} = B, \qquad \frac{\partial \Psi}{\partial c} = C, \qquad \frac{\partial \Psi}{\partial f} = F, \qquad \frac{\partial \Psi}{\partial g} = G, \qquad \frac{\partial \Psi}{\partial h} = H.$$

Hence, differentiating (1) $n-i-1$ times with respect to y, and i times with respect to z, and using the notation described above, we have

$$(2) \qquad r_{2,n-i-1,i} = -\left\{ \left(\frac{d^{n-1}\Psi}{dy^{n-i-1}\,dz^{i}} \right) + H s_{1,n-i,i} + G s_{1,n-i-1,i+1} \right.$$
$$\left. + B t_{0,n-i+1,i} + F t_{0,n-i,i+1} + C t_{0,n-i-1,i+2} \right\}$$

a result which we shall frequently use.

CHAPTER II.

DEFINITION OF CHARACTERISTICS : EQUATIONS TO BE SATISFIED : CONSEQUENCES OF THE DEFINITION : THE RANK OF A PARTIAL DIFFERENTIAL EQUATION : THEOREM I.

The method of Darboux, as expounded by Goursat, is inextricably linked with the theory of characteristics. Thus when contemplating an extension of the method to equations with three independent variables, the question which one naturally asks first is, what is a characteristic multiplicity? As Goursat [5] points out, there are two possibilities which are naturally suggested. If there be m independent variables we may define a characteristic of order n to be an $m-1$ dimensional multiplicity of elements of contact of order n, contained in more than one m-dimensional integral multiplicity : that is, rendering the problem of Cauchy indeterminate. This is the view adopted by Beudon [6] in his work on characteristics.

Alternatively, we may adopt a viewpoint analogous to that of Natani [7], and define a characteristic to be a multiplicity, contained in an integral multiplicity, and satisfying at least one total differential equation, distinct from the equations of contact, which may be written down in advance and is independent of the particular integral on which the characteristic is situated.

Goursat points out that this latter definition may only be adopted for a special class of equations of the second order with three independent variables; nevertheless it will appear later that for our purpose this kind of definition is the more fruitful, and that Darboux's method may only be extended to those very equations to which the definition leads.

To avoid confusion, we will refer to the $m-1$ dimensional characteristics of the first definition as Monge-characteristics, in accordance with the usual nomenclature for equations with two independent variables : to the multiplicities of the second definition we simply refer as " characteristics ", since it is with this kind that we shall be mainly concerned : while to the one-dimensional characteristics of a partial differential equation of the *first* order, which depend only on arbitrary constants and not on arbitrary functions, we refer as Cauchy-characteristics, in accordance with common practice. Meanwhile, we remark that it is well known that the two definitions lead to identical results in the case of equations with two independent variables.

But before stating the definition precisely, it is convenient to make some further remarks about " multiplicities of elements of contact ", which have been mentioned on pages 3 and 8.

Suppose that we are given an analytic integral of (1), in the form

$$u = U(x, y, z).$$

Then by differentiating this equation we may obtain, at any point x, y, z, the values of the partial derivatives of u, up to and including any desired order n. Adopting the notation for these derivatives which has been used all along, it is clear that we may thus associate with each point a set of values of the variables which compose an element of contact of order n. Then if throughout any suitable region of the space of (x, y, z) we consider the aggregate of these elements of contact of order n, it is clear from the way in which each element is obtained that the equations of contact (p. 4) are satisfied : and thus that the aggregate is in fact a multiplicity M_3 of order n (*see* p. 3). And since $U(x, y, z)$ is an integral of (1), it is natural to refer to a multiplicity M_3 of this special sort as an " integral multiplicity ".

But, just as explained on page 5 in dealing with single elements of contact, since $U(x, y, z)$ is an integral of (1), the values of all the variables representing those derivatives of u which involve more than one differentiation with respect to x, may be calculated directly from the equation (1) and the equations derived from (1) by differentiation. Thus we see that an integral multiplicity is completely specified when we know the values, at each point, of the variables x, y, z, u, and $s_{1,k-i-1,i}$, $t_{0,k-j,j}$ $(k = 1, \dots, n; i = 0, \dots, k-1; j = 0, \dots, k)$ This being so, it is clear that the " equations of contact " written on page 4 are no longer independent total differential equations. In fact, the only equations which are independent, after we have substituted for each of the variables $p_{\alpha,\beta,\gamma}\,(\alpha \geqq 2)$, its value in terms of the other variables deduced from (1), are the following :

$$(3)\quad \begin{cases} du - s_{1,0,0}\,dx - t_{0,1,0}\,dy - t_{0,0,1}\,dz = 0, \\ ds_{1,k-i-1,i} - r_{2,k-i-1,i}\,dx - s_{1,k-i,i}\,dy - s_{1,k-i-1,i+1}\,dz = 0, \\ dt_{0,k-j,j} - s_{1,k-j,j}\,dx - t_{0,k-j+1,j}\,dy - t_{0,k-j,j+1}\,dz = 0. \\ (k = 1, \ldots, n-1,\ i = 0, \ldots, k-1,\ j = 0, \ldots, k) \end{cases}$$

in which it is understood that each $r_{2,k-i-1,i}$ is expressed in terms of the other variables by means of (1) and (2). It may readily be verified by writing the appropriate expressions derived from (1) for the remaining variables, that the other equations of contact written on page 4 are consequences of (3); though indeed this is almost obvious. We may remark that the only variables, on the above understanding regarding the $r_{2,k-i-1,i}$, which appear in (3), are the variables which make up an element of contact of order n, in the narrower sense in which we have used the phrase all along.

It is also useful to remark that every integral multiplicity of order n is certainly contained in an integral multiplicity of order $n+1$: that is, the multiplicity of order $n+1$ associated with the given integral.

Having made these remarks, for the sake of clarity in the definition, we are now in a position to define characteristics.

Definition. — A characteristic multiplicity, or more shortly a characteristic, of order $n\,(\geqq 2)$, is a multiplicity of elements of contact of order n, forming a part of at least one *integral* multiplicity M_3 [and therefore satisfying (3)], and having the property of satisfying at least one new total differential equation which we may adjoin to (3); the new equation or equations being entirely independent of the particular integral of (1) of which the characteristic forms part, and containing the *differential* of at least one partial derivative of order n.

We shall see later that in consequence of this definition, characteristics of this kind, when they exist, are of one dimension : and furthermore that the new total differential equation or equations, about whose nature we have made no assumption, must be linear.

We now proceed to determine all the total differential equations which a characteristic must satisfy.

First of all, we notice that the equations of contact (3) are solved for the differentials of u and of all the variables representing the various derivatives of u, up to and including the $n-1$'th order, and express each of these differentials as a sum of multiples of dx, dy, dz. The only variables whose differentials do not appear are those which represent derivatives of u of order n. Thus since we may substitute for all the differentials for which (3) are solved in terms of dx, dy, dz, any new total differential equation which we adjoin may be written in the form

$$X\left\{\begin{matrix} x, \ldots, t_{0,0,n}; ds_{1,n-1,0}, \ldots, ds_{1,0,n-1} \\ dt_{0,n,0}, \ldots, dt_{0,0,n}; dx, dy, dz \end{matrix}\right\} = 0;$$

and by the definition, at least one new equation must contain at least one of $ds_{1,n-i-1,i}$, $dt_{0,n-j,j} (i = 0, \ldots, n-1; j = 0, 1, \ldots, n)$.

Now since by definition the characteristic whose existence we are now assuming is contained in at least one integral multiplicity of order n, it is also (*see* p. 10) contained in at least one integral multiplicity of order $n+1$. Therefore, anywhere on the characteristic there exists at least one set of values of the variables which represent the derivatives of u of order $n+1$, such that the additional equations of contact appropriate to an integral multiplicity of order $n+1$ are satisfied : that is, the equations

$$ds_{1,n-i-1,i} = r_{2,n-i-1,i}\,dx + s_{1,n-i,i}\,dy + s_{1,n-i-1,i+1}\,dz,$$

which, putting in the values of each $r_{2,n-i-1,i}$ given by expressions (2), chapter I, we may write

$$(4) \quad \left\{\begin{aligned} & ds_{1,n-i-1,i} + \left(\frac{d^{n-1}\Psi}{dy^{n-i-1}\,dz^{i}}\right) dx \\ & \quad = \{(dy - H\,dx)\,s_{1,n-i,i} + (dz - G\,dx)\,s_{1,n-i-1,i+1} \\ & \qquad - (B\,t_{0,n-i+1,i} + F\,t_{0,n-i,i+1} + C\,t_{0,n-i-1,i+2})\,dx\} \\ & \qquad\qquad (i = 0, \ldots, n-1) \end{aligned}\right.$$

and also the equations

$$(5) \quad dt_{0,n-j,j} = s_{1,n-j,j}\,dx + t_{0,n-j+1,j}\,dy + t_{0,n-j,j+1}\,dz \quad (j = 0, 1, \ldots, n).$$

Since $n \geqq 2$, and therefore $n+1 \geqq 3$, it follows that the derivatives of u of order $n+1$ do not occur in Ψ, H, G, B, F, C; nor do they occur in $\left(\frac{d^{n-1}\Psi}{dy^{n-i-1}dz^i}\right)$, by the definition of this expression on page 6. Thus when dx, dy, dz have values appropriate to a characteristic (which we suppose known), and are regarded as constants, the expressions on the right hand sides of the $2n+1$ equations (4) and (5) may be regarded as $2n+1$ homogeneous *linear* forms in the $2n+3$ variables

$$s_{1,n-i,i} \quad (i = 0, \ldots, n) \qquad \text{and} \qquad t_{0,n-j+1,j} \quad (j = 0, \ldots, n+1),$$

which do not occur on the left sides of the equations.

Now if dx, dy, dz were such that these $2n+1$ linear forms in $2n+3$ variables were linearly independent, then clearly these forms could be made to take any arbitrary set of values, by suitable choice of the values of the variables in question. And thus we could satisfy the equations (4) and (5) for any arbitrary choice of the differentials $ds_{1,n-i-1,i}$, $dt_{0,n-j,j}$ on the left hand sides, by suitable choice of the variables $s_{1,n-i,i}$, $t_{0,n-j+1,j}$. But we have seen that integrals of (1) exist, admitting any chosen element of contact of order n, and such that the variables $s_{1,n-i,i}$, $t_{0,n-j+1,j}$ assume any arbitrary set of values. Hence it would be impossible to restrict the differentials $ds_{1,n-i-1,i}$, $dt_{0,n-j,j}$ to satisfying any total differential equation, which actually contains one at least of these differentials, without any knowledge of, and without placing any restriction upon, the nature of the integral multiplicity or multiplicities of which the characteristic in question forms a part.

Therefore in order to comply with the definition, we see that on a characteristic, dx, dy, dz must be such that the $2n+1$ linear forms on the right hand sides of the equations (4) and (5) are *not* linearly independent; in which case it is clear that the elimination of the derivatives of u of order $n+1$ from these equations, which from their nature are consistent, leads at once to a certain number of linear total differential equations, containing the differentials which appear on the left hand sides of (4) and (5), and fulfilling all the conditions laid down in the definition.

We could have taken this as the starting point, and defined a characteristic of order n to be a multiplicity such that the derivatives of order $n+1$ could be eliminated between the equations (4) and (5). This would have considerable advantage in simplicity : but although the result would be the same, such a definition would be slightly less general, and much more artificial, than the one which we have in fact adopted. We now proceed to find the conditions which must be satisfied by dx, dy, dz; and to determine the new total differential equations mentioned above.

First of all, we dismiss the possibility that $dx = 0$. For suppose that $dx = 0$. Then clearly the variables $s_{1,n-i,i}$ occur in (4) but not in (5), while the variables $t_{0,n-j+1,j}$ occur in (5) but not in (4). The matrix of the coefficients of the variables $s_{1,n-i,i}$, on the right hand sides of (4), is then the matrix of n rows and $n+1$ columns such that the element in the i'th row and the k'th column is dy if $k=i$, is dz if $k=i+1$, and is zero in all other cases. If we omit the last column, we obtain a determinant equal to dy^n, while if we omit the first column, we obtain a determinant, also of n rows and columns, equal to dz^n. Thus unless $dy = dz = 0$, the matrix is of rank n. Similarly the matrix of the coefficients of the variables $t_{0,n-j+1,j}$ on the right hand sides of the equations (5) is an exactly similar matrix, but with $n+1$ rows and $n+2$ columns; and by exactly the same reasoning, this matrix is of rank $n+1$ unless $dy = dz = 0$. Thus we see that the $2n+1$ equations are linearly independent unless $dy = dz = 0$. In this latter event, it is clear from the equations (3), (4), (5) and the hypothesis that $dx = 0$, that the multiplicity reduces to a single element of contact, all the variables remaining constant : and this is obviously inadmissible, since a characteristic is defined to be of at least one dimension. Thus in order that the equations (4) and (5) be *not* linearly independent, in the sense explained, it is necessary that $dx \neq 0$.

This being so, we consider instead of the equations (4) and (5), a system of equations entirely equivalent. For each i, we multiply (4) by dx; put $j = i$ in (5) and multiply by $-(dy - \mathrm{H}\,dx)$; put $j = i+1$ in (5) and multiply by $-(dz - \mathrm{G}\,dx)$; and add the three together, thus obtaining

$$(6)\quad \left\{\begin{aligned} & dx \left\{ ds_{1,n-i-1,i} + \left(\frac{d^{n-1}\Psi}{dy^{n-i-1}\,dz^i}\right) dx \right\} \\ & \quad - (dy - \mathrm{H}\,dx)\,dt_{0,n-i,i} - (dz - \mathrm{G}\,dx)\,dt_{0,n-i-1,i+1} \\ & = -\big[(dy^2 - \mathrm{H}\,dy\,dx + \mathrm{B}\,dx^2)\,t_{0,n-i+1,i} \\ & \quad + (2\,dy\,dz - \mathrm{G}\,dy\,dx - \mathrm{H}\,dz\,dx + \mathrm{F}\,dx^2)\,t_{0,n-i,i+1} \\ & \qquad + (dz^2 - \mathrm{G}\,dz\,dx + \mathrm{C}\,dx^2)\,t_{0,n-i-1,i+2}\big] \\ & \qquad\qquad (i = 0, 1, \ldots, n-1). \end{aligned}\right.$$

Since $dx \neq 0$, it is clear that (6) and (5) are together entirely equivalent to (4) and (5); and the variables $s_{1,n-i,i}$ do not occur in (6) Furthermore since each of the $n+1$ variables $s_{1,n-i,i}$ occurs in one, and only one, of the $n+1$ equations (5), it follows that if the n expressions on the right of the equations (6) are themselves linearly distinct, considered as linear forms in the $n+1$ variables $t_{0,n-j+1,j}$, then all the $2n+1$ equations (5) and (6) are linearly distinct. Therefore on a characteristic, we must choose dx, dy, dz in such a manner that the matrix of the coefficients of the variables $t_{0,n-j+1,j}(j=0, \ldots, n+1)$, on the right of the n equations (6), is of rank less than n. If we now write, for brevity,

$$-\alpha = (dy^2 - \mathrm{H}\, dy\, dx + \mathrm{B}\, dx^2),$$
$$-\beta = (2\, dy\, dz - \mathrm{G}\, dy\, dx - \mathrm{H}\, dz\, dx + \mathrm{F}\, dx^2),$$
$$-\gamma = (dz^2 - \mathrm{G}\, dz\, dx + \mathrm{C}\, dx^2),$$

then this matrix is the matrix of n rows and $n+2$ columns such that the element in the i'th row and the k'th column is α if $k=i$, is β if $k=i+1$, is γ if $k=i+2$, and is zero elsewhere. If we omit the last two columns, we obtain a determinant of n rows and columns equal to α^n; if we omit the first two columns, we obtain a determinant equal to γ^n; while if $\alpha=\gamma=0$, and we omit the first and last columns, we obtain a determinant equal to β^n. Therefore, in order that the matrix be of rank less than n, it is necessary that

$$\alpha = \beta = \gamma = 0;$$

that is

$$dy^2 - \mathrm{H}\, dy\, dx + \mathrm{B}\, dx^2 = 0,$$
$$2\, dy\, dz - \mathrm{G}\, dy\, dx - \mathrm{H}\, dz\, dx + \mathrm{F}\, dx^2 = 0,$$
$$dz^2 - \mathrm{G}\, dz\, dx + \mathrm{C}\, dx^2 = 0.$$

But we have shown that $dx \neq 0$; thus we may put

$$dy = \mu\, dx; \qquad dz = \nu\, dx;$$

and after dividing by dx^2 we obtain

$$\mu^2 - \mathrm{H}\mu + \mathrm{B} = 0, \tag{7}$$
$$2\mu\nu - \mathrm{G}\mu - \mathrm{H}\nu + \mathrm{F} = 0, \tag{8}$$
$$\nu^2 - \mathrm{G}\nu + \mathrm{C} = 0. \tag{9}$$

We must therefore investigate the necessary and sufficient conditions for a common solution of the equations (7)-(9). Let μ_1, μ_2 be the roots of the equation (7), and ν_1, ν_2 the roots of (9). Then

$$H = \mu_1 + \mu_2, \qquad B = \mu_1\mu_2, \qquad G = \nu_1 + \nu_2, \qquad C = \nu_1\nu_2.$$

Putting in the values of G, H, equation (8) becomes

$$2\mu\nu - (\nu_1 + \nu_2)\mu - (\mu_1 + \mu_2)\nu + F = 0.$$

If this equation be satisfied either by $\mu = \mu_1$, $\nu = \nu_1$ or $\mu = \mu_2$, $\nu = \nu_2$, we see that the necessary condition is $F = \mu_1\nu_2 + \mu_2\nu_1$. If the equation be satisfied by $\mu = \mu_1$, $\nu = \nu_2$, or $\mu = \mu_2$, $\nu = \nu_1$, we have $F = \mu_1\nu_1 + \mu_2\nu_2$. We may always suppose the former condition satisfied : for the latter is obtained from the former simply by re-labelling ν_1, ν_2 as ν_2, ν_1 respectively. Thus we may assert that the necessary condition for a common solution is $F = \mu_1\nu_2 + \mu_2\nu_1$. This condition is also sufficient, for, putting in the values of G and H, and this value for F, (8) becomes

$$(\mu - \mu_1)(\nu - \nu_2) + (\mu - \mu_2)(\nu - \nu_1) = 0,$$

and this equation is clearly satisfied by $\mu = \mu_1$, $\nu = \nu_1$ or $\mu = \mu_2$, $\nu = \nu_2$.

To interpret this condition in terms of the function Ψ, we observe that, after putting in the values of H, G. B, C, we have

$$\Delta = \begin{vmatrix} 1 & \frac{H}{2} & \frac{G}{2} \\ \frac{H}{2} & B & \frac{F}{2} \\ \frac{G}{2} & \frac{F}{2} & C \end{vmatrix}$$

$$= -\frac{1}{4}\{F - (\mu_1\nu_2 + \mu_2\nu_1)\}\{F - (\mu_1\nu_1 + \mu_2\nu_2)\}.$$

Thus, noticing once again that the second factor is obtained from the first by permuting ν_1 and ν_2, we see that the necessary and sufficient condition for the existence of a common solution of (7), (8) and (9) is that Δ, which we shall call the « discriminant » of (1), shall be zero.

We may go further than this. For the minor of Δ obtained by omitting the last row and column is $B - \frac{1}{4}H^2 \equiv -\frac{1}{4}(\mu_1 - \mu_2)^2$; and the minor obtained by omitting the second row and column is $C - \frac{1}{4}G^2 \equiv -\frac{1}{4}(\nu_1 - \nu_2)^2$. Thus if the determinant is zero, but either of $(\mu_1 - \mu_2)$, $(\nu_1 - \nu_2)$ is different from zero, Δ is of rank 2, and there are two distinct pairs of solutions of (7)-(9), namely $\mu = \mu_1$, $\nu = \nu_1$ and $\mu = \mu_2$, $\nu = \nu_2$. But if

$$\mu_1 = \mu_2, \qquad \nu_1 = \nu_2, \qquad F = \mu_1\nu_2 + \mu_2\nu_1 = 2\mu_1\nu_1,$$

Δ becomes

$$\begin{vmatrix} 1 & \mu_1 & \nu_1 \\ \mu_1 & \mu_1^2 & \mu_1\nu_1 \\ \nu_1 & \mu_1\nu_1 & \nu_1^2 \end{vmatrix}$$

in which the second and third rows are multiples of the first and Δ is clearly of rank 1. Thus we see that if Δ be zero but of rank 2, at least one of $\mu_1 - \mu_2$, $\nu_1 - \nu_2$ is non-zero; while if Δ be of rank 1, $\mu_1 = \mu_2$, $\nu_1 = \nu_2$, and (7)-(9) admit one solution only, namely

$$\mu = \mu_1 = \mu_2, \qquad \nu = \nu_1 = \nu_2.$$

Suppose now that the condition $\Delta = 0$ is satisfied, so that

$$H = \mu_1 + \mu_2, \quad B = \mu_1\mu_2, \quad G = \nu_1 + \nu_2, \quad C = \nu_1\nu_2, \quad F = \mu_1\nu_2 + \mu_2\nu_1.$$

Then choosing, for definiteness, the solution $\mu = \mu_1$, $\nu = \nu_1$, we have

$$(10) \qquad dy - \mu_1\,dx = 0, \qquad dz - \nu_1\,dx = 0.$$

Putting these values, and the values of H, G, B, F, C in the equations (6), and dividing by dx, which is not zero, we have

$$(11) \quad \begin{cases} ds_{1,n-i-1,i} + \mu_2\,dt_{0,n-i,i} + \nu_2\,dt_{0,n-i-1,i+1} + \left(\dfrac{d^{n-1}\Psi}{dy^{n-i-1}\,dz^i}\right)dx = 0 \\ (i = 0, 1, \ldots, n-1) \end{cases}$$

and clearly the equations (11) fulfil the conditions of the definition of characteristics of order n. Furthermore, from (10) we see that there is one relationship between y and x only, containing y, and one between z and x, containing z. Hence, since a characteristic is, by definition, contained in an integral multiplicity M_3, in which the independent variables are x, y and z, we see at once that characteristics of the kind which we have defined are of one dimension only. Thus we regard (10), (11) and the equations of contact (3), no longer as total differential equations, but as an incomplete system of ordinary differential equations.

If one or other or both of $\mu_1 - \mu_2$, $\nu_1 - \nu_2$ is different from zero, we may deduce a second system of equations which characteristics of order n may satisfy, simply by writing μ_2, ν_2 for μ_1, ν_1 respectively, and *vice versa*, in (10), (11). In this case we shall show that there are two distinct systems of characteristics of each order $n (n \geqq 2)$, while if $\mu_1 = \mu_2$, $\nu_1 = \nu_2$ there is only one system.

We have yet to establish the existence of characteristics of this kind, although we have found the equations (10) and (11) which, in addition to the equations of contact (3), they must satisfy. But this point presents no difficulty. For suppose that we have any integral of (1), so that all the variables which make up an element of contact of order $n + 1$ may be regarded as known functions of x, y, z. Then, substituting the values of the appropriate variables in μ_1 and ν_1, let us consider the system of two *ordinary* différential equations

$$(12) \qquad \frac{dy}{\mu_1(x, y, z)} = \frac{dz}{\nu_1(x, y, z)} = \frac{dx}{1}.$$

These equations define a system of (one-dimensional) *curves* in the space of (x, y, z), such that through each point x, y, z passes one curve of the system and only one (throughout, of course, a suitable region within which all the functions involved are analytic). Let us then associate with each point of one of these curves, the element of contact of order n, and the element of contact of order $n + 1$ containing the former element, belonging to the integral multiplicities of order n and $n + 1$ respectively, associated with the given integral of (1). Then the equations (3), (4), (5) and therefore (6), being satisfied throughout the integral multiplicity of order $n + 1$, are satisfied in particular along any one of the curves defined by (12). While remembering that $dx \neq 0$, the equations (12) are themselves equivalent, on this integral, to (10). And therefore, along this curve we may eliminate the derivatives of u of order $n + 1$ from (4) and (5), or (5) and (6) which are equivalent, thus showing that the equations (11) are also satisfied. Therefore the one-dimensional multiplicity M_1 consisting of any of the curves defined by (12), and the element of contact of order n associated with each point of the curve, constitutes a characteristic of order n, in accordance with the definition adopted. And since one and only one of these curves passes through any point x, y, z, we may state the result in the following concise manner, remembering that the same arguments may, of course, be applied to the second system of characteristics, by permuting μ_1 and μ_2, ν_1 and ν_2 :

« Every integral multiplicity of order n, is a *locus* of one-dimensional characteristics of order n belonging to either system, when the two systems are distinct, and to the single system when the two are confluent ».

We therefore see that a characteristic of order n is a multiplicity M_1 of elements of contact of order n, satisfying the equations of contact of order n [in which, it is understood, each derivative of the form $r_{2,j,k}$ is replaced by the appropriate expression of the form (2), chapter I], being contained in at least one integral multiplicity of order n, and satisfying either the system (10), (11), or the system obtained by permuting μ_1 and μ_2, ν_1 and ν_2, If the determinant Δ be of rank 2, there are two distinct systems of characteristics, while if Δ be of rank 1, the two systems are confluent, i. e., there is one system only.

Before stating the theorem which really summarises all the conclusions of this chapter up till now, we introduce an extremely important conception, namely the *rank* of a partial differential equation. We define the rank of the equation (1) to be the rank of the determinant Δ (p. 15). But this conception may be extended to equations of more general form than (1). Suppose in fact, with the notation of chapter I, that the given partial differential equation is

$$\psi(x, y, z, u, l, m, n, a, b, c, f, g, h) = 0; \tag{13}$$

then we adopt the following definition.

Definition. — The *rank* of the equation (13) is the rank of the determinant

$$\Delta' \begin{vmatrix} \frac{\partial\psi}{\partial a} & \frac{1}{2}\frac{\partial\psi}{\partial h} & \frac{1}{2}\frac{\partial\psi}{\partial g} \\ \frac{1}{2}\frac{\partial\psi}{\partial h} & \frac{\partial\psi}{\partial b} & \frac{1}{2}\frac{\partial\psi}{\partial f} \\ \frac{1}{2}\frac{\partial\psi}{\partial g} & \frac{1}{2}\frac{\partial\psi}{\partial f} & \frac{\partial\psi}{\partial c} \end{vmatrix},$$

which we shall call the discriminant of (13). If the equation (13) contains a, so that $\frac{\partial\psi}{\partial a} \neq 0$, we may solve for a in the form (1). Thus, applying the rule for calculating the derivative of an implicit function, we see that

$$H = \frac{\partial\Psi}{\partial h} = \frac{\frac{\partial\psi}{\partial h}}{\frac{\partial\psi}{\partial a}},$$

and similarly for g, b, f, c. Thus, Δ being the corresponding determinant for the equation solved for a, i. e., (1), we see that each element of Δ', is $\frac{\partial\psi}{\partial a}$ times the corresponding element of Δ : and clearly the rank of Δ' is the same as the rank of Δ. A similar argument applies if (13) contains b or c, by simply permuting x, y and z. If (13) does not contain a, b or c, but contains, for example, h, we have seen in chapter I that the change of variables $x_1 = x + y$, $y_1 = x - y$ renders the equation soluble for $\frac{\partial^2 u}{\partial x_1^2}$: and it is easily verified that after this substitution the rank of the corresponding determinant is the same as the rank of Δ'.

Thus we may now summarise all the results of this chapter in the following theorem :

THEOREM 1 :

(i) *If the equation* (13) *be of rank* 3, *there are no characteristics of the kind defined on page* 10.

(ii) *If* (13) *be of rank* 2, *there are two distinct systems of one-dimensional characteristics of each order* $n \geqq 2$ *of the kind defined earlier, and every integral multiplicity of order n is a locus of characteristics of order n, of either system. If the equation be solved for a, in the form* (1), *chapter I, then the equations satisfied by characteristics of order n, in addition to the equations of contact of order n, are the equations* (10), (11), *or else the system obtained by permuting* μ_1 *and* μ_2, ν_1 *and* ν_2, *where* (μ_1, ν_1), (μ_2, ν_2) *are the two distinct pairs of solutions of the three equations* (7), (8), (9).

(iii) *If* (13) *be of rank* 1, *there is one system only of one-dimensional characteristics of each order* $n \geqq 2$, *of this kind,* *and every integral multiplicity of order n is a locus of characteristics of order n of this single system. If the equation be solved for a, in the form* (1), *there is only one solution, say* $\mu = \mu_1$, $\nu = \nu_1$ *of the equations* (7)-(9), *and the equations satisfied by characteristics of order n, in addition to the equations of contact, are* (10), (11), *in which we write* $\mu_1 = \mu_2$, $\nu_1 = \nu_2$.

This is the fundamental theorem of characteristics for second order equations with three independent variables. Comparing the results with the classical case of an equation with two independent variables, and making the corresponding definition of rank, we observe that an equation with two independent variables of rank 2 has two systems of characteristics, while an equation of rank 1 has one system only, entirely in accordance with the results we have established. But there is an interesting difference to which we shall return later.

CHAPTER III.

Invariants, and the generalisation of Darboux's method.

The next problem to engage our attention is that of finding the conditions which must be satisfied by a function of the elements of contact of order n, which has the special property of remaining constant along a characteristic of one or other of the two systems [from now on, we will always assume that the equation (1) is of rank 2 or 1; and except where it is expressly stated that the characteristics are distinct, all propositions about " one or other of the two systems " will apply to the case when the two systems are coincident]. A function of this kind is called an " invariant " of the appropriate system. To be more precise, we state the following definition.

Definition. — An invariant of order n, is a function χ of the elements of contact of order n, such that the equation

$$d\chi = 0$$

is a consequence of the system of equations (10) and (11) and the equations of contact of order n, or else of the corresponding system obtained by permuting μ_1 and μ_2, ν_1 and ν_2.

Thus let χ be *any* function of the elements of contact of order $n (\geqq 2)$. Then to calculate the variation of χ along any characteristic of the system characterised by μ_1, ν_1, we substitute in $d\chi$ for each of the differentials for which the equations (10), (11) and the equations of contact are solved; and thus, taking account of these equations and of the definitions of the symbols $\left(\frac{d\chi}{dx}\right)^*$ etc.,

$$(14)\quad d\chi \equiv \frac{\partial\chi}{\partial x}dx + \frac{\partial\chi}{\partial y}dy + \frac{\partial\chi}{\partial z}dz + \frac{\partial\chi}{\partial u}du$$

$$+\sum_{k=1}^{n}\left\{\sum_{i=0}^{k-1}\frac{\partial\chi}{\partial s_{1,k-i-1,i}}ds_{1,k-i-1,i} + \sum_{j=0}^{k}\frac{\partial\chi}{\partial t_{0,k-j,j}}dt_{0,k-j,j}\right\}$$

$$= \left\{\left(\frac{d\chi}{dx}\right)^* + \mu_1\left(\frac{d\chi}{dy}\right) + \nu_1\left(\frac{d\chi}{dz}\right)\right\}dx$$

$$+\sum_{i=0}^{n-1}\frac{\partial\chi}{\partial s_{1,n-i-1,i}}\left\{-\mu_2\,dt_{0,n-i,i} - \nu_2\,dt_{0,n-i-1,i+1} - \left(\frac{d^{n-1}\Psi}{dy^{n-i-1}\,dz^{i}}\right)dx\right\}$$

$$+\sum_{j=0}^{n}\frac{\partial\chi}{\partial t_{0,n-j,j}}dt_{0,n-j,j} \qquad [\text{using}(10)\text{-}(11)]$$

$$= \left\{\left(\frac{d\chi}{dx}\right)^* + \mu_1\left(\frac{d\chi}{dy}\right) + \nu_1\left(\frac{d\chi}{dz}\right) - \sum_{i=0}^{n-1}\left(\frac{d^{n-1}\Psi}{dy^{n-i-1}\,dz^{i}}\right)\frac{\partial\chi}{\partial s_{1,n-i-1,i}}\right\}dx$$

$$+\sum_{j=0}^{n}\left\{\frac{\partial\chi}{\partial t_{0,n-j,j}} - \mu_2\frac{\partial\chi}{\partial s_{1,n-j-1,j}} - \nu_2\frac{\partial\chi}{\partial s_{1,n-j,j-1}}\right\}dt_{0,n-j,j}$$

(after rearranging and using the negative suffix convention).

Now given an element of contact of order n, we have seen that integrals of (1) exist, admitting this element, for which the derivatives of u of order $n+1$ assume any arbitrary values. Furthermore, it is clear from the equations (5) that, having chosen the ratios $dx{:}dy{:}dz\ (dx \neq 0)$, we may certainly choose the derivatives of order $n+1$ in such a way that the $n+1$ ratios $\frac{dt_{0,n-j,j}}{dx}$ assume any arbitrarily chosen set of values. But the ratios $dx{:}dy{:}dz$ are determined by (10) in terms of elements of contact of order 2; and are therefore fixed when an element of contact of order n is assigned.

Thus we may assert that given an element of contact of order n, we may always find an integral, admitting this element, for which the ratios $\frac{dt_{0,n-j,j}}{dx}$, corresponding to the characteristic of order n of this system lying on the integral and containing the given element, assume *any* set of values given in advance.

Thus in order that $d\chi = 0$ on *any* characteristic, it is neccessary that the coefficient of each $dt_{0,n-j,j}$ $(j = 0, \ldots, n)$ on the right hand side of (14) be zero : and then since $dx \neq 0$ on a characteristic, we may also equate to zero the coefficient of dx in (14). Therefore in order that χ may be an invariant of the first system, it is necessary, and from the equation (14) it is also obviously sufficient, that χ should be an integral of the partial differential equations of the first order

$$(15) \quad \begin{cases} \left(\frac{d\chi}{dx}\right)^* + \mu_1\left(\frac{d\chi}{dy}\right) + \nu_1\left(\frac{d\chi}{dz}\right) - \sum_{i=0}^{n-1}\left(\frac{d^{n-1}\Psi}{dy^{n-i-1}\,dz^i}\right)\frac{\partial\chi}{\partial s_{1,n-i-1,i}} = 0, \\ \frac{\partial\chi}{\partial t_{0,n-i,i}} - \mu_2\frac{\partial\chi}{\partial s_{1,n-i-1,i}} - \nu_2\frac{\partial\chi}{\partial s_{1,n-i,i-1}} = 0 \\ \qquad (i = 0, \ldots, n) \end{cases}$$

and clearly an invariant of order n, of the other system, satisfies the same system with μ_1 and μ_2, ν_1 and ν_2 permuted.

We may remark here that from the definitions in chapter I of the symbols $\frac{d\chi}{dx}^*$ etc., and the equations such as (2), chapter I, the equation

$$\frac{d\chi}{dx}^* + \mu_1\frac{d\chi}{dy} + \nu_1\frac{d\chi}{dz} \equiv 0$$

is a direct consequence of the equations (15).

We will frequently be dealing with characteristics, and with invariants, of different orders. Thus we will usually refer to the system of equations (15) as written, as the "system (15) of order n". But when it is a question of distinguishing between one system and the other, we simply refer to the system characterised by (10) as the " first system " and the system obtained by permuting μ_1, and μ_2, ν_1 and ν_2 as the " second system ".

We now prove two further results.

THEOREM 2. — *Every characteristic of order* $n+1$ $(n \geqq 2)$ *(of either system when the two are distinct) contains a characteristic of order* n.

For suppose that we know a characteristic of order $n+1$ of, say, the first system. Then along this characteristic, the equations (10) are satisfied, and also the equations of contact appropriate to any multiplicity of order $n+1$: that is to say, (3), (4) and (5). And therefore the equations (11) are also satisfied : for we have seen that when (10) are satisfied, (11) are consequences of (4) and (5). Thus the multiplicity of order n contained in the characteristic multiplicity of order $n+1$, is itself a characteristic multiplicity of order n : and the theorem is established.

From this last result it follows at once that any " invariant of order n ", as defined above, is also an invariant of the characteristics of any higher order $n+m$. For a characteristic of order $n+m$ contains one of order $n+m-1$: one of order $n+m-2$: . . . : and finally one of order n. And χ being constant along the last, is also constant along the first.

Suppose now (which is by no means always the case) that one system of characteristics, say the first, possesses three invariants ξ, η, ζ, of order not exceeding $n\,(n \geqq 2)$, one at least of which is of order n. Then by the corollary to theorem 2, we see that ξ, η, ζ are all invariants of the first system of order n. Consider now any integral of (1), which we have proved to be a locus of characteristics of the first system. Let us then substitute for u and the partial derivatives of order up to and including n, in ξ, η and ζ, the values, in terms of x, y and z, appropriate to the integral, and make the same substitution in μ_1, ν_1. We thus obtain three functions $\xi(x, y, z)$, $\eta(x, y, z)$, $\zeta(x, y, z)$. But since ξ, η and ζ are, by hypothesis, constant along any characteristic of order n of the first system, we have, along the characteristic $\left(\text{the symbols } \frac{\partial \xi}{\partial x} \text{ etc.,}\right.$ now referring to the functions ξ etc., after substituting for u and the partial derivatives$\Big)$

$$d\xi \equiv \frac{\partial \xi}{\partial x}dx + \frac{\partial \xi}{\partial y}dy + \frac{\partial \xi}{\partial z}dz$$
$$\equiv \left\{\frac{\partial \xi}{\partial x} + \mu_1(x, y, z)\frac{\partial \xi}{\partial y} + \nu_1(x, y, z)\frac{\partial \xi}{\partial z}\right\}dx = 0,$$

i. e.

$$\frac{\partial \xi}{\partial x} + \mu_1 \frac{\partial \xi}{\partial y} + \nu_1 \frac{\partial \xi}{\partial z} = 0,$$

and similarly

$$\frac{\partial \eta}{\partial x} + \mu_1 \frac{\partial \eta}{\partial y} + \nu_1 \frac{\partial \eta}{\partial z} = 0,$$

$$\frac{\partial \zeta}{\partial x} + \mu_1 \frac{\partial \zeta}{\partial y} + \nu_1 \frac{\partial \zeta}{\partial z} = 0.$$

Hence, the values of u and all partial derivatives having been replaced by the functions of x, y, z appropriate to the integral, we have

$$\frac{\partial(\xi, \eta, \zeta)}{\partial(x, y, z)} = 0;$$

and therefore the given integral satisfies an equation of the form

$$(16) \qquad \chi(\xi, \eta, \zeta) = 0,$$

for some form of the function χ. Thus, once again expressing ξ, η, ζ in terms of the elements of contact of order n, we see that (16) is a partial differential equation of order n, which must be satisfied, for some form of the function χ, by each integral of (1). This important result is entirely in accordance with the result for equations with two independent variables.

Now the essence of Darboux's method, in the classical case, is that, given the data of Cauchy along a non-characteristic curve, and knowing two invariants of one system of characteristics, we may deduce a new ordinary differential equation which enables the characteristics of the opposite system (or the single system, if the two be confluent) to be determined. We shall therefore investigate the corresponding possibility.

It is a well known result in Cauchy's problem, in three independent variables, that if u and one of the derivatives of u of the first order be given at every point of an analytic surface, then in general this data specifies an unique integral of (1); and the values of all the derivatives of u of any order may be calculated at any point of the surface, by using the equations of contact, the equation (1) itself, and equations derived from (1) by partial differentiation (for example, this latter fact is obvious if u and $s_{1,0,0}$ be given as funcs tions of y and z when $x = 0$). One of the conditions sufficient in order that this may be true is that the surface shall *not* correspond to a *Monge* characteristic of the integral in question : and it is a standard result that the surface $\Xi(x, y, z) = 0$ is a Monge characteristic corresponding to a particular integral of (1) if the function Ξ satisfies the equation

$$(17) \quad \left\{\frac{\partial\Xi}{\partial x}\right\}^2 + H\frac{\partial\Xi}{\partial x}\frac{\partial\Xi}{\partial y} + G\frac{\partial\Xi}{\partial x}\frac{\partial\Xi}{\partial z} + B\left\{\frac{\partial\Xi}{\partial y}\right\}^2 + F\frac{\partial\Xi}{\partial y}\frac{\partial\Xi}{\partial z} + C\left\{\frac{\partial\Xi}{\partial z}\right\}^2 = 0,$$

in which we substitute for u and each of the derivatives of u the values appropriate to the integral in question.

Thus we now assume that we are given the Cauchy data in such a manner that the problem is determinate. To be definite, we will suppose that the coordinates x, y, z of a point in a suitable closed region of the surface $\Xi(x, y, z) = 0$ are expressed in terms of two parameters v and w, in such a way that not all three of the Jacobians $\frac{\partial(y, z)}{\partial(v, w)}, \frac{\partial(z, x)}{\partial(v, w)}, \frac{\partial(x, y)}{\partial(v, w)}$, are zero, throughout the corresponding region of the plane of v, w : and that from the given data, the equation (1) and the equations derived from (1), the values of u and all the derivatives of u of order up to and including $n\,(\geqq 2)$ have been expressed as analytic functions of v and w throughout this region : and further that, putting in the value in terms of v and w of each of the appropriate variables, we have

$$\Delta_1(v, w) \equiv \left\{\frac{\partial\Xi}{\partial x}\right\}^2 + H\frac{\partial\Xi}{\partial x}\frac{\partial\Xi}{\partial y} + G\frac{\partial\Xi}{\partial x}\frac{\partial\Xi}{\partial z} + B\left\{\frac{\partial\Xi}{\partial y}\right\}^2 + F\frac{\partial\Xi}{\partial y}\frac{\partial\Xi}{\partial z} + C\left\{\frac{\partial\Xi}{\partial z}\right\}^2 \neq 0.$$

We have thus a two-dimensional multiplicity of order n : and for brevity we henceforth refer to this as the « initial multiplicity ».

We suppose all along that the equation (1) with which we are concerned is of rank 2 or 1. Then we have seen, in the last chapter, that there are two systems of one-dimensional characteristics of each order n, of the kind which we have discussed hitherto : and furthermore, that the integral multiplicity of order n, corresponding to the required integral, is a locus of characteristics of either system. Thus we may assert that this integral multiplicity is in fact the locus of those characteristics of either system, contained in the integral multiplicity, which emanate from each element of the initial multiplicity, *provided* that we can establish that these latter characteristics do in fact form a multiplicity M_3, and are not entirely contained in the initial multiplicity. To prove this point, we will prove that there is a one-one correspondence between the points of the space of x, y, z, and the points of the curves (12) which pass through each point of the surface $\Xi = 0$. In this connection, we may remark that since the given boundary data makes the problem determinate, we may regard u and each derivative of u as known analytic functions of x, y, z; and then, substituting these values in μ_1, μ_2, ν_1, ν_2, we may regard the equations (12) as defining a known system of curves, one of which passes through each point of the space of x, y, z : each curve, of course, corresponding to a characteristic contained in the required integral multiplicity.

Thus let us consider the curve of the system (12) which passes through the point of the surface $\Xi = 0$ which is specified by the parameters (v, w). Let the position of a point on this curve be specified by a new parameter θ, chosen in any convenient way, which takes the value zero at the point (v, w) on the given surface. For example, we could take θ to be the arc-lengh along the curve, measured from the point (v, w) on the surface. Thus we may find a function $\lambda(x, y, z)$ (different from zero, since $dx \neq 0$) such that along the curve we have

$$\frac{dy}{\mu_1(x, y, z)} = \frac{dz}{\nu_1(x, y, z)} = \frac{dx}{1} = \lambda(x, y, z)\, d\theta.$$

Then clearly the point (x, y, z) on a curve passing through any point on the surface is entirely specified by the three parameters (θ, v, w) : and we now proceed to calculate the Jacobian $\frac{\partial(x, y, z)}{\partial(\theta, v, w)}$, at any point *on* the surface $\Xi = 0$, that is; at any point where $\theta = 0$. To this end we remark that from the equations written above it is clear that for any values of θ, v, w, we have

$$\frac{\partial x}{\partial \theta} = \lambda; \quad \frac{\partial y}{\partial \theta} = \lambda\mu_1; \quad \frac{\partial z}{\partial \theta} = \lambda\nu_1;$$

while, on the surface, we obtain by differentiating $\Xi = 0$ with respect to v and w respectively

$$\frac{\partial \Xi}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial \Xi}{\partial y}\frac{\partial y}{\partial v}+\frac{\partial \Xi}{\partial z}\frac{\partial z}{\partial v}=0,$$
$$\frac{\partial \Xi}{\partial x}\frac{\partial x}{\partial w}+\frac{\partial \Xi}{\partial y}\frac{\partial y}{\partial w}+\frac{\partial \Xi}{\partial z}\frac{\partial z}{\partial w}=0;$$

and thus, solving these equations, we see that a function $\rho(x, y, z)$ must exist, such that

$$\frac{\partial(y, z)}{\partial(v, w)}=\rho\frac{\partial \Xi}{\partial x}, \qquad \frac{\partial(z, x)}{\partial(v, w)}=\rho\frac{\partial \Xi}{\partial y}, \qquad \frac{\partial(x, y)}{\partial(v, w)}=\rho\frac{\partial \Xi}{\partial z},$$

and clearly $\rho \neq 0$, since by hypothesis one of the three expressions on the left of these last equations is not zero. Hence when $\theta = 0$, we have

$$\frac{\partial(x, y, z)}{\partial(\theta, v, w)}=\begin{vmatrix} \lambda & \lambda\mu_1 & \lambda\nu_1 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}=\lambda\rho\left\{\frac{\partial \Xi}{\partial x}+\mu_1\frac{\partial \Xi}{\partial y}+\nu_1\frac{\partial \Xi}{\partial z}\right\}.$$

But, putting in the values of H, G, B, F, C in terms of μ_1, μ_2, ν_1, ν_2, in the expressions $\Delta_1(v, w)$, we see that

$$\Delta_1(v, w) \equiv \left(\frac{\partial \Xi}{\partial x}\right)^2+\mu_1\mu_2\left(\frac{\partial \Xi}{\partial y}\right)^2+\nu_1\nu_2\left(\frac{\partial \Xi}{\partial z}\right)^2+(\mu_1\nu_2+\mu_2\nu_1)\left(\frac{\partial \Xi}{\partial y}\right)\left(\frac{\partial \Xi}{\partial z}\right)$$
$$+(\nu_1+\nu_2)\left(\frac{\partial \Xi}{\partial z}\right)\left(\frac{\partial \Xi}{\partial x}\right)+(\mu_1+\mu_2)\left(\frac{\partial \Xi}{\partial x}\right)\left(\frac{\partial \Xi}{\partial y}\right)$$
$$\equiv\left(\frac{\partial \Xi}{\partial x}+\mu_1\frac{\partial \Xi}{\partial y}+\nu_1\frac{\partial \Xi}{\partial z}\right)\left(\frac{\partial \Xi}{\partial x}+\mu_2\frac{\partial \Xi}{\partial y}+\nu_2\frac{\partial \Xi}{\partial z}\right)\neq 0,$$

and therefore in particular

$$\frac{\partial \Xi}{\partial x}+\mu_1\frac{\partial \Xi}{\partial y}+\nu_1\frac{\partial \Xi}{\partial z}\neq 0.$$

Therefore, since $\lambda \neq 0$, $\rho \neq 0$, we have

$$\frac{\partial(x, y, z)}{\partial(\theta, v, w)}\neq 0,$$

whenever $\theta = 0$ and v, w lie in the prescribed region of the v, w plane. But since we assume that all the functions with which we are dealing are analytic, it follows that there is a region of the space of (θ, v, w) which includes the region of the plane $\theta = 0$ in its interior, and throughout which the Jacobian is different from zero. Consequently, we may express θ, v, w in terms of x, y, z throughout a region of the space of (x, y, z) which includes the given region of the surface $\Xi = 0$: and there is thus an unique correspondence between the points of the space of (x, y, z) and the points of the curves of the system (12) which emanate from each point of the initial multiplicity.

By exactly the same reasoning we may establish the same result for the system of curves of the second system, obtained by writing μ_2, ν_2 for μ_1, ν_1 in (12), emanating from each element of the initial multiplicity; and the desired result is thus obtained. In other words, we may now assert that the required integral multiplicity of order n is the locus of the characteristics of order n, of either system, which are contained in the integral multiplicity and emanate from each element of the initial multiplicity.

Now it is clear that if by any means we could determine these characteristics explicitly, the problem of finding the integral would be solved. For if, with the notation which we have adopted in this chapter, we could solve for the various elements of a characteristic of order n, emanating from the initial multiplicity, in the form

$$x = x(\theta, v, w), \qquad y = y(\theta, v, w), \qquad z = z(\theta, v, w), \qquad u = u(\theta, v, w),$$
$$s_{1\,k-i-1,i} = s_{1\,k-i,-1,i}(\theta, v, w), \qquad t_{0,k-j,j} = t_{0,k-j,j}(\theta, v, w),$$
$$(k = 1, \ldots, n;\ i = 0, \ldots, k-1;\ j = 0, \ldots, k)$$

then since $\dfrac{\partial(x, y, z)}{\partial(\theta, v, w)} \neq 0$, we could eliminate θ, v, w from the first four equations and obtain the required integral in the form

$$u = u(x, y, z).$$

We shall now see how, in certain cases, we may carry out this determination : the method of procedure to be adopted being clearly suggested by Darboux's procedure for equations with two independent variables, and the results which we have obtained so far.

Thus suppose for some integer $n(\geq 2)$ one or other of the systems of characteristics possesses three invariants ξ, η, ζ. Then we have seen that when the variables involved take the values corresponding to an integral multiplicity of (1), ξ, η, ζ satisfy an equation of the form (16). Now we may determine the precise form of the function χ in (16) from the boundary conditions. For suppose that we substitute for each of the variables which make up an element of contact of order n, the values in terms of the parameters v and w which these variables assume on the initial multiplicity, thus obtaining three functions $\xi(v, w)$, $\eta(v, w)$, $\zeta(v, w)$. In general (in other words, except possibly for special boundary conditions), we may then express one of these functions uniquely in terms of the other two, say

$$\xi(v, w) + \varphi\{\eta(v, w), \zeta(v, w)\} \equiv 0,$$

the form of the function φ now being known; and since we have proved that there is a functional relationship between ξ, η, ζ valid for all x, y, z, it follows that the required integral satisfies, for all x, y, z, the equation

$$(18) \qquad \chi \equiv \xi + \varphi(\eta, \zeta) = 0;$$

and we may remark that χ is itself an invariant. The foregoing reasoning would only be defective if every integral of (1) were such that, on substituting for u and its derivatives their values appropriate to the integral, there were always two or three functional relationships between ξ. η and ζ, so that either, say, $\xi = X(\zeta)$, $\eta = Y(\zeta)$, or else ξ, η and ζ are constants. Both these circumstances we shall shortly show to be impossible.

Since, by hypothesis, at least one of ξ, η, ζ contains a derivative of order n, it is clear that, in general, (18) is an equation of order n. But we must now prove, [denoting the left side of (18) by χ], that in general, one at least of $\frac{\partial \chi}{\partial s_{1,n-i-1,i}}$ $(i = 0, 1, \ldots, n-1)$ is not zero. For it is conceivable that, whatever the integral of (1), the corresponding form of the function φ in (18) might be such that each $\frac{\partial \chi}{\partial s_{1,n-i-1,i}} = 0$, when we substitute for u and each derivative of u the values appropriate to the integral, in terms of x, y z.

Now integrals of (1) exist admitting any arbitrary element of contact of order n. Thus for general values of the variables, we may suppose one at least of $\frac{\partial\xi}{\partial s_{1,n-i-1,i}}$, $\frac{\partial\eta}{\partial s_{1,n-i-1,i}}$, $\frac{\partial\zeta}{\partial s_{1,n-i-1,i}}$ $(i=0, 1, \ldots, n-1)$, to be non-zero. For if this were not the case, it follows from (15) that none of ξ, η, ζ would contain a derivative of order n, contrary to hypothesis. Thus the matrix of n rows and three columns, which has $\frac{\partial\xi}{\partial s_{1,n-i-1,i}}$, $\frac{\partial\eta}{\partial s_{1,n-i-1,i}}$, $\frac{\partial\zeta}{\partial s_{1,n-i-1,i}}$ in the $i+1$'th row and respectively the first, second, and third column, is of rank 1, 2 or 3.

Let us therefore suppose, if possible, that for every integral of (1), the function φ in (18) is such that each $\frac{\partial f}{\partial s_{1\,n-i-1\,i}}$ is zero on the integral. This requires

$$\frac{\partial\xi}{\partial s_{1,n-i-1,i}}+\frac{\partial\varphi}{\partial\eta}\frac{\partial\eta}{\partial s_{1,n-i-1,i}}+\frac{\partial\varphi}{\partial\zeta}\frac{\partial\zeta}{\partial s_{1,n-i-1\,i}}=0 \qquad (i=0, 1, \ldots, n-1).$$

If the matrix described above be of rank 3, then these n equations are inconsistent. If it be of rank 2, and if the equations be consistent, then we may solve them uniquely for $\frac{\partial\varphi}{\partial\eta}$ and $\frac{\partial\varphi}{\partial\zeta}$ in terms of elements of contact of order n. But since, by hypothesis, η and ζ are independent functions of these elements, it follows that we may express two of the variables composing the element of contact in terms of η, ζ and the remaining variables. Doing this, we solve the n equations above, in the form

$$\frac{\partial\varphi}{\partial\eta}=A, \qquad \frac{\partial\varphi}{\partial\zeta}=B,$$

where A and B are functions of η, ζ, and all except two of the variables composing an element of contact of order n. But since φ is a function of η and ζ only, this is impossible unless A and B are, in fact, functions of η and ζ only. Hence we suppose that we have

$$\frac{\partial\varphi}{\partial\eta}=A(\eta, \zeta); \qquad \frac{\partial\varphi}{\partial\zeta}=B(\eta, \zeta).$$

But even if the integrability condition were satisfied, we would then have φ in the form $\varphi=U(\eta, \zeta)-\lambda$ where the form of U is fixed, and λ is an arbitrary constant. Thus we would have the result that every integral of (1) satisfies an equation of fixed form

$$R\equiv\xi+U(\eta, \zeta)=\lambda,$$

for some value of λ; and we observe that R is also an invariant.

Now this is impossible. For suppose that R contains a derivative of order $s \leqq n$, which we may denote by p. Then, for general values of the variables, $\frac{\partial R}{\partial p}$ must be non-zero. Hence, differentiating the equation $R = \lambda$ with respect to y or z, we obtain a linear relationship between the derivatives of order $s+1$, containing at least one of these latter, and satisfied by every integral of (1), contradicting the fact that integrals of (1) exist, admitting any element of contact of order $s+1$.

It remains only to examine the case when the matrix described earlier is of rank 1. If this be so, we see that, with our hypothesis, and again expressing two of the variables in terms of η, ζ and the remaining ones, φ must satisfiy, for any integral of (1), an equation of the form

$$A\frac{\partial \varphi}{\partial \eta} + B\frac{\partial \varphi}{\partial \zeta} = C,$$

where A, B, C are functions of η, ζ, and all but two of the variables composing the element of contact of order n, and at least one of A, B, C is non-zero for general values of the variables. Clearly the equation can only be satisfied if one of A, B is non-zero. Since φ depends only on η and ζ, it is easy to show that either we deduce two equations of the form $\frac{\partial \varphi}{\partial \eta} = A_1(\eta, \zeta)$, $\frac{\partial \varphi}{\partial \zeta} = B_1(\eta, \zeta)$, which we have just shown to be impossible, or else the ratios A:B:C depend only on η, ζ, so that we may rewrite the single equation satisfied by φ in the form

$$K(\eta, \zeta)\frac{\partial \varphi}{\partial \eta} + L(\eta, \zeta)\frac{\partial \varphi}{\partial \zeta} = M(\eta, \zeta),$$

not both of K and L being identically zero. But such an equation may be integrated, in the form

$$\varphi = U(\eta, \zeta) + \Phi\{V(\eta, \zeta)\},$$

where $U(\eta, \zeta)$ and $V(\eta, \zeta)$ are functions of known form, and Φ is arbitrary. We would thus have the result that any integral of (1) satisfies an equation of the form

$$\xi + U(\eta, \zeta) + \Phi\{V(\eta, \zeta)\} = 0,$$

or, writing $P = \xi + U(\eta, \zeta)$, $Q = V(\eta, \zeta)$, and noticing that P and Q are again invariants,

$$P + \Phi(Q) = 0.$$

Now if P and Q are not independent invariants, we have an equation $P = \text{Cte}$, satisfied by every integral of (1), which we have shown to be impossible. If P and Q are independent, at least one must contain a derivative of u, otherwise we would have an expression for the general integral of (1), containing only one arbitrary function, which is impossible. Thus without loss of generality we may suppose that the derivative of highest order in P or Q is of order n (n not necessarily being the same as before). Let this derivative be p. Then either $\frac{\partial P}{\partial p}$ or $\frac{\partial Q}{\partial p}$ must be non-zero for general values of the variables, and therefore not both of $\frac{dP}{dy}, \frac{dQ}{dy}$, and not both of $\frac{dP}{dz}, \frac{dQ}{dz}$ are identically zero. But, differentiating the equation $P + \Phi(Q) = 0$ with respect to x, y and to z, and using (1) as all along, we obtain

$$\frac{dP}{dx}^{*} + \Phi'(Q)\frac{dQ}{dx}^{*} = 0,$$

$$\frac{dP}{dy} + \Phi'(Q)\frac{dQ}{dy} = 0,$$

$$\frac{dP}{dz} + \Phi'(Q)\frac{dQ}{dz} = 0.$$

It follows that, for every integral of (1), and therefore for arbitrary elements of contact of order $n + 1$, the matrix

$$\begin{bmatrix} \frac{dP}{dx}^{*} & \frac{dQ}{dx}^{*} \\ \frac{dP}{dy} & \frac{dQ}{dy} \\ \frac{dP}{dz} & \frac{dQ}{dz} \end{bmatrix}$$

must be of rank 1 : and this may be shown to contradict the hypothesis that P and Q are independant invariants.

We have thus proved that, the function φ in (18) being chosen appropriately for any particular integral of (1), it is impossible that each $\frac{\partial \chi}{\partial s_{1, n-i-1, i}}$ is always zero. This may occur for special integrals of (1) : but for general integrals of (1), we may suppose that at least one of the $\frac{\partial\chi}{\partial s_{1,n-i-1,i}}$ is non-zero.

It may happen that we know a number of invariants, say of the first system, of different orders, and are thus able to form a number of equations such as (18), of different orders, which the required integral satisfies. We shall therefore now show how, from an equation such as (18), of order n, we may deduce $q+2$ new ordinary differential equations for the determination of the characteristics of the opposite system of order $n+q (q \geqq 0)$. Now, since the integral which we are seeking satisfies (18), and since, on this integral, all variables are functions of x, y, z, it follows that the integral must also satisfy the equation obtained by differentiating (18) any number of times with respect to x, y and z. No advantage is gained by differentiating with respect to x : for we have seen that the identity

$$\frac{d\chi}{dx}^* + \mu_1 \frac{d\chi}{dy} + \nu_1 \frac{d\chi}{dz} \equiv 0$$

is an algebraic consequence of (15), and thus, when we use (1) and the equations derived from (1), all equations derived from (18) by differentiation are consequences of those among them involving no differentiation with respect to x.

Differentiating (18) $q-k+1$ times with respect to y, and k times with respect to $z (0 \leqq k \leqq q+1, \; q \geqq 0)$ we obtain

$$(19) \qquad \left(\frac{d^{q+1}\chi}{dy^{q-k+1}\, dz^k}\right) + \sum_{i=0}^{n-1} \frac{\partial\chi}{\partial s_{1,n-i-1,i}} s_{1,n+q-i-k,i+k} + \sum_{j=0}^{n} \frac{\partial\chi}{\partial t_{0,n-j,j}} t_{0,n+q-j-k+1,j+k} = 0.$$

But along any characteristic of order $n+q$, of the second system, we have

$$dy = \mu_2\, dx, \qquad dz = \nu_2\, dx,$$

and therefore

$$dt_{0,n+q-i-k,i+k} = \{ s_{1,n+q-i-k,i+k} + \mu_2 t_{0,n+q-i-k+1,i+k} + \nu_2 t_{0,n+q-i-k,i+k+1} \}\, dx$$
$$(i = 0, 1, \ldots, n-1;\; k = 0, 1, \ldots, q+1).$$

Since $dx \neq 0$, we may substitute for each $s_{1,n+q-i-k,i+k}$ in the equations (19) from these last equations. Doing so, and rearranging with the help of the negative suffix convention, we obtain

$$\left(\frac{d^{q+1}\chi}{dy^{q-k+1}\,dz^k}\right)dx + \sum_{i=0}^{n-1} \frac{\partial\chi}{\partial s_{1,n-i-1,i}}\,dt_{0,n+q-i-k,i+k}$$
$$+\sum_{j=0}^{n}\left\{\frac{\partial\chi}{\partial t_{0,n-j,j}} - \mu_2\frac{\partial\chi}{\partial s_{1,n-j-1,j}} - \nu_2\frac{\partial\chi}{\partial s_{1,n-j,j-1}}\right\}$$
$$\times\, dx\, t_{0,n+q-j-k+1,j+k} = 0.$$
$$(k = 0,\ 1,\ \ldots,\ q+1).$$

Then, taking account of (15), which are satisfied since χ is an invariant of the first system, we have

$$(20)\qquad \begin{cases} \displaystyle\sum_{i=0}^{n-1} \frac{\partial\chi}{\partial s_{1,n-i-1,i}}\,dt_{0,n+q-i-k,i+k} + \left(\frac{d^{q+1}\chi}{dy^{q-k+1}\,dz^k}\right)dx = 0 \\ (k = 0,\ 1,\ \ldots,\ q+1). \end{cases}$$

Since at least one of the $\dfrac{\partial\chi}{\partial s_{1,n-i-1,i}}$ is non-zero, it is easy to see that the matrix of the coefficients of the $n+q+1$ differentials $dt_{0,n-j+q,j}$ $(j = 0,\ 1,\ \ldots,\ n+q)$, in the $q+2$ equations (20) is always of rank $q+2$. These equations are thus distinct, and we have $q+2$ new ordinary differential equations which are satisfied by the characteristics of the second system, of order $n+q$, contained in the required common integral of (1) and (18).

Suppose now that we have determined a number of invariants of the first system, of order not exceeding $n\,(n \geqq 2)$, say $\xi_1, \ldots, \xi_s, \eta, \zeta$, one at least being of order n. Then, using the boundary conditions, we may form s equations of the same form as (18), say

$$\chi_i \equiv \xi_i + \varphi_i(\eta, \zeta) = 0,$$

the equations being of various orders, but at least one being of order n. Then, applying the procedure indicated above, we may construct a number of ordinary differential equations satisfied by the characteristics of order n, of the second system. These equations will not, in general, all be distinct; but if there be $n+1$ distinct equations, we may solve these latter for the $n+1$ differentials $dt_{0,n-j,j}(j = 0,\ 1,\ \ldots,\ n)$; and, adjoining these new equations to the equations of characteristics of the second system of order n, we have a complete set of ordinary differential equations, which, integrating and using the boundary conditions, determines the required integral.

This is, in essence, the extension of Darboux's method, though there are a number of points still to be examined before we give examples of the method.

We notice that we may write down at once a number of first integrals of the system of ordinary differential equations for the second system. Firstly, any invariant of the second system, of order not exceeding n is, by définition, a first integral of this system. So that even though we may, in theory, solve the problem by using the invariants of the first system, in practice it is well worth determining all the invariants of the second system as well, in order to simplify the final integration. Secondly, suppose that we are using, among others, an equation of the form (18) of order $p \leqq n$, and that we are attempting to determine the characteristics of order n of the second system. Then, in addition to the various ordinary differential equations satisfied by this system, we have the finite equation (18) itself, and $\frac{1}{2}(n-p)(n-p+3)$ other finite equations

$$\frac{d^s \chi}{dy^{s-t}\,dz^t} = 0 \qquad (s = 1, 2, \ldots, n-p;\ t = 0, 1, \ldots, s)$$

which the common integral of (1) and (18) must satisfy, and which are therefore satisfied, in particular, along the characteristics of the second system contained in the common integral [if $p = n$ we, of course, simply have the equation (18) itself]. These equations and (18) itself, all of which may easily be seen to be distinct, by virtue of the hypothesis that one of the $\frac{\partial \chi}{\partial s_{1,\,n-t-1,\,t}}$ is non-zero, may be solved for $\frac{1}{2}(n-p+1)(n-p+2)$ of the variables composing the element of contact of order n; and we may therefore reduce the number of ordinary differential equations by this number.

In the particular case when $p = n$, so that we are seeking to determine the characteristics of order n, using an equation such as (18) of order n, we shall show directly that χ is a first integral of the equations (20), which in this case reduce to two, namely

$$(21)\qquad \sum_{i=0}^{n-1} \frac{\partial \chi}{\partial s_{1,n-i-1,i}} dt_{0,n-i,i} + \left(\frac{d\chi}{dy}\right) dx = 0,$$

$$(22)\qquad \sum_{i=1}^{n} \frac{\partial \chi}{\partial s_{1,n-i,i-1}} dt_{0,n-i,i} + \left(\frac{d\chi}{dz}\right) dx = 0.$$

together with the equations of characteristics of the second system. Suppose that we calculate the variation of the function χ along a characteristic of the second system, without assuming that the integral satisfies (18). Then, by the reasoning by which we obtained (14), but permuting μ_1 and μ_2, ν_1 and ν_2

$$d\chi = \left\{ \left(\frac{d\chi}{dx}\right)^* + \mu_2 \left(\frac{d\chi}{dy}\right) + \nu_2 \left(\frac{d\chi}{dz}\right) - \sum_{i=0}^{n-1} \left(\frac{d^{n-1}\Psi}{dy^{n-i-1}\, dz^i}\right) \frac{\partial \chi}{\partial s_{1,n-i-1,i}} \right\} dx$$
$$+ \sum_{j=0}^{n} \left\{ \frac{\partial \chi}{\partial t_{0,n-j,j}} - \mu_1 \frac{\partial \chi}{\partial s_{1,n-j-1,j}} - \nu_1 \frac{\partial \chi}{\partial s_{1,n-j,j-1}} \right\} dt_{0,n-j,j}.$$

But again, χ being an invariant of the first system and satisfying (15), we substitute for each $\dfrac{\partial \chi}{\partial t_{0,n-j,j}}$ and for

$$\left(\frac{d\chi}{dx}\right)^* - \sum_{i=0}^{n-1} \left(\frac{d^{n-1}\Psi}{dy^{n-i-1}\, dz^i}\right) \frac{\partial \chi}{\partial s_{1,n-i-1,i}} \quad \text{from (15)},$$

and obtain

$$(23)\qquad d\chi = (\mu_2 - \mu_1) \left\{ \sum_{i=0}^{n-1} \frac{\partial \chi}{\partial s_{1,n-i-1,i}} dt_{0,n-i,i} + \left(\frac{d\chi}{dy}\right) dx \right\}$$
$$+ (\nu_2 - \nu_1) \left\{ \sum_{i=1}^{n} \frac{\partial \chi}{\partial s_{1,n-i,i-1}} dt_{0,n-i,i} + \left(\frac{d\chi}{dz}\right) dx \right\}.$$

Therefore it is apparent from (23) that

$$d\chi = 0$$

is a consequence of the equations of characteristics of the second system, and the two equations (21) and (22).

We may remark from (23) that when the characteristics are distinct, so that one of $(\mu_2 - \mu_1)$, $(\nu_2 - \nu_1)$ is non-zero, then instead of the equations (21) and (22), we may write one of these and the equation $d\chi = 0$. But it is better to write (21) and (22), and to solve the finite equation $\chi = 0$ for one of the variables $s_{1, n-i-1, i}$ $(i = 0, \ldots, n-1)$, $\left(\text{which is possible since one of the } \frac{\partial \chi}{\partial s_{1, n-i-1, i}} \text{ is non-zero}\right)$, and thence eliminate this variable from the system of ordinary differential equations.

It may happen that an equation such as (18), in general of order p, may be of lower order for particular boundary conditions. For example, if ξ, η, ζ are three invariants of the first system, ξ being of order p but η and ζ being of order $q < p$, then in general, an integral satisfies an equation of the form

$$\chi(\xi, \eta, \zeta) = 0,$$

of order p : but a particular integral may satisfy an equation of the form

$$\chi(\eta, \zeta) = 0, \qquad \text{or} \qquad \eta + \varphi(\zeta) = 0,$$

of order $q < p$. This circumstance, when it arises, is highly advantageous. For from the results established we see that the number of new ordinary differential equations which we can deduce for the characteristics of order n from an equation (18) of order q is $(p - q)$ greater than the number deduced from an equation of order p. If it should happen that the equation $\chi = 0$ contains only partial derivatives of the first order, then this equation may always be solved, say by Cauchy's method, and we could thus obtain the required integral of (1). Finally, if the equation contains no derivative, but contain u, we would have the required integral of (1) explicitly.

Just as in the case of equations with two independent variables [8], there is an important simplification in the procedure when the two systems of characteristics are confluent. For all the reasoning hitherto applies whether the characteristics are distinct or confluent: but when they are confluent, *every* invariant is constant along the system of characteristics which we seek to determine, by the method described above. From this remark it follows that, provided that we can determine a sufficient number of invariants, we do *not* need to determine the exact form of the functions φ in the equations of the form (18), in order to be able to integrate the system of ordinary differential equations which determine the characteristics. For if ξ, η, ζ are three *known* invariants of the single system of characteristics of order n, then for *any* function $\chi(\xi, \eta, \zeta)$ we have

$$\frac{\partial\chi}{\partial s_{1,n-i-1,i}} = \frac{\partial\chi}{\partial\xi}\frac{\partial\xi}{\partial s_{1,n-i-1,i}} + \frac{\partial\chi}{\partial\eta}\frac{\partial\eta}{\partial s_{1,n-i-1,i}} + \frac{\partial\chi}{\partial\zeta}\frac{\partial\zeta}{\partial s_{1,n-i-1,i}}$$
$$(i = 0, \ldots, n-1),$$
$$\left(\frac{d^{q+1}\chi}{dy^{q-k+1}\,dz^k}\right) = \frac{\partial\chi}{\partial\xi}\left(\frac{d^{q+1}\xi}{dy^{q-k+1}\,dz^k}\right) + \frac{\partial\chi}{\partial\eta}\left(\frac{d^{q+1}\eta}{dy^{q-k+1}\,dz^k}\right) + \frac{\partial\chi}{\partial\zeta}\left(\frac{d^{q+1}\zeta}{dy^{q-k+1}\,dz^k}\right).$$

Thus the left-hand sides of the $q+2$ equations (20), or the two equations (21) and (22) if $q = 0$, for the function χ are the sums of three corresponding expressions for ξ, η, ζ (which do not depend on the boundary conditions), the coefficients of the three expressions being $\frac{\partial\chi}{\partial\xi}$, $\frac{\partial\chi}{\partial\eta}$, $\frac{\partial\chi}{\partial\zeta}$. But along a characteristic of the single system, ξ, η, ζ are constant; and thus $\frac{\partial\chi}{\partial\xi}$, $\frac{\partial\chi}{\partial\eta}$, $\frac{\partial\chi}{\partial\zeta}$ are also constant. Hence if there are enough invariants for us to construct $n+1$ new ordinary differential equations of order n, then we may integrate the system of ordinary differential equations thereby obtained, treating $\frac{\partial\chi}{\partial\xi}$, $\frac{\partial\chi}{\partial\eta}$, $\frac{\partial\chi}{\partial\zeta}$ as parameters in the integration, which may thus be carried out *before* the boundary conditions are specified.

This is the basis of the extension of Darboux's method. We see that the success or failure of the method depends upon the existence of a sufficient number of invariants of one or other of the systems of characteristics, of some order n.

We must observe that the procedure of deducing equations of the form (20), or as a particular case (21) and (22), is fundamental. There in no question of finding enough invariants, and thence enough equations of the form (18), of order not exceeding n, to solve these equations for the variables $t_{0,n-j,j}$ ($j = 0, 1, \ldots, n$), in terms of the remaining variables, and integrating the equations of characteristics of order n, which would then be of sufficient number. For suppose that we have any number s, $s \geqq n+1$, of invariants of the first system of orders not exceeding n, say $\chi_1, \chi_2, \ldots, \chi_s$. Consider then the matrix which has $\frac{\partial \chi_k}{\partial s_{1,n-i-1,i}}$ in the $i+1$'th row and k' th column ($0 \leqq i \leqq n-1$), and has $\frac{\partial \chi_k}{\partial t_{0,n-j,j}}$ in the $n+j+1$'th row and k' th column ($0 \leqq j \leqq n$). Then by virtue of (15), we see that every row of this matrix is a sum of multiples of the first n rows, and the matrix is of rank not exceeding n. Consequently, it is impossible to solve the equations $\chi = 0$ for the $n+1$ variables $t_{0,n-j,j} (j = 0, 1, \ldots, n)$. Similarly, we may see that if we find all the invariants of both systems, when they are distinct, of order not exceeding n, and then form the equations such as (18) for both systems, we can never solve the problem by solving these equations for all the derivatives of order n, and then treating the equations of contact as an integrable system of total differential equations. For if we have any number of invariants $\chi_1, \ldots, \chi_s$ of one system, and $\Xi_1, \ldots, \Xi_r$ of the other system, of order not exceeding n, the matrix whose elements are the partial derivatives of the invariants with respect to the $2n+1$ variables $s_{1,n-i-1,i}$, $t_{0,n-j,j}$ is easily seen to be of rank not exceeding $2n$, by virtue of (15) and the analogous system with μ_1 and μ_2, ν_1 and ν_2 permuted. There is thus no analogue of the simplification which sometimes occurs in the case of equations with two independent variables. Another contrast with the classical case is that, whereas in the latter the knowledge of any two invariants of one system is sufficient to solve the problem, in this case the knowledge of three invariants is not. For we have seen that from three invariants of order not exceeding n, one at least being of order n, we deduce $q+2$ new equations for the characteristics of order $n+q$, while $n+q+1$ new equations are required.

Thus we see that, given an equation of the form (1), the procedure is as follows:

(i) We form the determinant Δ (chap. II, p. 15). If this determinant is of rank 3 (i. e., not zero), the method cannot be applied. If however the determinant be of rank 2 or 1, we proceed to the net step.

(ii) We set up the equations (15) of order 2 for one system of characteristics and, applying the standard procedures for solving systems of linear, homogeneous, partial differential equations of the first order, we find all the integrals of the system. If these be sufficient in number and of such a nature that we can deduce three distinct new equations, of the form (21) and (22) with $n = 2$ (having, of course, used the boundary conditions to determine the form of the functions χ_i except, as explained above, when the characteristics are confluent) we may determine the characteristics of the other system of order 2 by the aid of equations of the form (21) and (22). Failing the existence of enough suitable invariants to solve the problem in this way, we

(iii) repeat the procedure with the equations (15) appropriate to the other system, in an attempt to determine the characteristics of the first system of order 2. Failing this, we

(iv) repeat the procedure with the equations (15) for the first system, and then the equations (15) for the second system, of order 3 : and so on until the problem can be solved, *if* indeed such a stage can ever be reached.

Just as in Darboux's original method, there is no way of telling in advance whether or not the method can be successful, nor of determining what order of characteristics we may have to consider, in order to solve the problem in those cases when the method succeeds.

From the results obtained, it appears thats as n increases, so does the number of invariants which are required. But though this might seem to make the work involved quite prohibitively long, we shall later see how the knowledge of invariants of a certain order enables us to determine new invariants of higher order. And indeed there are a number of results, most of them similar to corresponding results for equations with two independent variables, which may be used to simplify the search for invariants, and in certain cases to place an upper limit on the number of invariants of each order which could exist [9]. Some of these we shall obtain in chapter V.

Meanwhile, we notice that there are certain aspects of the theory which are not entirely satisfying. The crucial fact, on which the success of this method depends, is the vanishing of the variables representing derivatives of u of order $n + q + 1$, from the ordinary differential equations which lead to the equations (20). At first sight there seems to be an element of luck in the fact that the coefficients of these variables conveniently vanished. And in order to obtain a full understanding of the theory of these equations, we follow the line of investigation adopted by Goursat [10], in dealing with equations with two independent variables : in other words, we extend the conception, originated by Sophus Lie [11], of « equations in involution ». In doing so, we shall also see how the knowledge of a number of invariants, in itself inadequate to determine the characteristics of any order, may yet be used to determine an infinity of integrals of (1), depending upon an arbitrary function.

CHAPTER IV.

EQUATIONS IN INVOLUTION : COMMON INTEGRALS OF SUCH EQUATIONS : COMMON SYSTEM OF CHARACTERISTICS FOR A SYSTEM IN INVOLUTION.

Suppose now that we are seeking a common solution of the equation (1) and another partial differential equation of order $n\,(\geqq 2)$. We may always assume that the second equation contains no partial derivatives of u which involve more than one differentiation with respect to x; for we may always substitute for these derivatives their values in terms of the other variables, obtained from (1) and equations obtained by differentiating (1). Thus we write the second equation in the form

$$\chi(x, \ldots, t_{0,0,n}) = 0, \tag{24}$$

and we suppose this equation to be of fully reduced form. Then since on a common integral, whose existence we assume, all the variables involved are analytic functions of x, y, z, the derivatives of u of order $n+1$, appropriate to such an integral, satisfy firstly the various equations obtained by differentiating (1) $n-1$ times, and secondly the three equations

$$\frac{d\chi}{dx} = 0; \qquad \frac{d\chi}{dy} = 0; \qquad \frac{d\chi}{dz} = 0.$$

Then, as mentioned above, all variables $r_{2,j,k}$ may be expressed by means of (1) and equations derived from (1) and substituting for these variables in the first of the three equations written above, we have finally three equations relating the $2n+3$ variables $s_{1,n-i,i}$, $t_{0,n-j+1,j}$ $(i = 0, \ldots, n;\ j = 0, \ldots, n+1)$, namely

$$\frac{d\chi}{dx}^{*} = \frac{d\chi}{dy} = \frac{d\chi}{dz} = 0.$$

Using the expressions of the form (2), Chapter I, we may write these equations more fully in the form

$$(25)\quad \frac{d\chi}{dx}^{*} \equiv \left(\frac{d\chi}{dx}\right)^{*} - \sum_{i=0}^{n-1}\left(\frac{d^{n-1}\Psi'}{dy^{n-i-1}dz^{i}}\right)\frac{\partial\chi}{\partial s_{1,n-i-1,i}}$$
$$+\sum_{i=0}^{n}\left\{\frac{\partial\chi}{\partial t_{0,n-i,i}} - H\frac{\partial\chi}{\partial s_{1,n-i-1,i}} - G\frac{\partial\chi}{\partial s_{1,n-i,i-1}}\right\}s_{1,n-i,i}$$
$$-\sum_{j=0}^{n+1}\left\{B\frac{\partial\chi}{\partial s_{1,n-j-1,j}} + F\frac{\partial\chi}{\partial s_{1,n-j,j-1}} + C\frac{\partial\chi}{\partial s_{1,n-j+1,j-2}}\right\}t_{0,n-j+1,j} = 0,$$

$$(26)\quad \frac{d\chi}{dy} \equiv \left(\frac{d\chi}{dy}\right) + \sum_{i=0}^{n}\frac{\partial\chi}{\partial s_{1,n-i-1,i}}s_{1,n-i,i} + \sum_{j=0}^{n+1}\frac{\partial\chi}{\partial t_{0,n-j,j}}t_{0,n-j+1,j} = 0,$$

$$(27)\quad \frac{d\chi}{dz} \equiv \left(\frac{d\chi}{dz}\right) + \sum_{i=0}^{n}\frac{\partial\chi}{\partial s_{1,n-i,i-1}}s_{1,n-i,i} + \sum_{j=0}^{n+1}\frac{\partial\chi}{\partial t_{0,n-j+1,j-1}}t_{0,n-j+1,j} = 0.$$

Définition. — Now extending the well known definition, given by Goursat [12] for equations with two independent variables, we say that the equations (1) and (24) are « in involution » if the equations (25), (26), (27), regarded as linear equations in the $2n+3$ variables which represent the derivatives of u of order $n+1$, reduce to *two* distinct equations, in the neighbourhood of an arbitrary element of contact of order n satisfying (24).

In order that this may be so, it is clearly necessary and sufficient that three functions of the elements of contact of order n, λ, μ, ν should exist, not all zéro, and such that

$$(28)\qquad \lambda\frac{d\chi}{dx}^{*} + \mu\frac{d\chi}{dy} + \nu\frac{d\chi}{dz} \equiv 0.$$

First of all we dismiss the possibility that $\lambda = 0$. For suppose that $\lambda = 0$, but that one of μ, ν is not zero; say the former. Then equating to zero the coefficients of $s_{1,n-i,i}$ and $t_{0,n-j+1,j}$, we obtain

$$\mu\frac{\partial\chi}{\partial s_{1,n-i-1,i}} + \nu\frac{\partial\chi}{\partial s_{1,n-i,i-1}} = 0,$$
$$\mu\frac{\partial\chi}{\partial t_{0,n-j,j}} + \nu\frac{\partial\chi}{\partial t_{0,n-j+1,j-1}} = 0;$$

and putting $i = 0, \ldots, n-1$ successively, and $j = 0, \ldots, n$

successively, and using the negative suffix convention, we obtain

$$\frac{\partial\chi}{\partial s_{1,n-i-1,i}} = \frac{\partial\chi}{\partial t_{0,n-j,j}} = 0 \qquad (i = 0, \ldots, n-1;\ j = 0, \ldots, n).$$

But the equation (24) is, by hypothesis, of fully reduced form, so that the partial derivative of χ with respect to at least one of the variables composing the element of contact of order n, is not zero. We may therefore suppose (24) solved for one of these variables, which clearly does not affect the property of being in involution with (1), as defined, and consequently if each $\frac{\partial\chi}{\partial s_{1,n-i-1,i}}$ and $\frac{\partial\chi}{\partial t_{0,n-j,j}}$ be zero for values of the variables, which are arbitrary subject to (24) being satisfied, they must be identically zero, and not merely zero as a consequence of (24). It follows that (24) contains no derivatives of u of order n, contrary to hypothesis. Hence we must have $\lambda \neq 0$; and we may divide (28) by λ, or else take $\lambda = 1$, which is the same thing. Thus taking $\lambda = 1$, and equating to zero the coefficient of each $s_{1,n-i,i}$, $t_{0,n-j+1,j}$ in (28), and also equating to zero the term independent of these variables, we obtain

$$(29)\quad \left(\frac{d\chi}{dx}\right)^{*} + \mu\left(\frac{d\chi}{dy}\right) + \nu\left(\frac{d\chi}{dz}\right) - \sum_{i=0}^{n-1}\left(\frac{d^{n-1}\Psi}{dy^{n-i-1}dz^{i}}\right)\frac{\partial\chi}{\partial s_{1,n-i-1,i}} = 0,$$

$$(30)\quad \frac{\partial\chi}{\partial t_{0,n-i,i}} + (\mu - \mathrm{H})\frac{\partial\chi}{\partial s_{1,n-i-1,i}} + (\nu - \mathrm{G})\frac{\partial\chi}{\partial s_{1,n-i,i-1}} = 0 \qquad (i = 0, \ldots, n),$$

$$(31)\quad \mu\frac{\partial\chi}{\partial t_{0,n-j,j}} + \nu\frac{\partial\chi}{\partial t_{0,n-j+1,j-1}} - \mathrm{B}\frac{\partial\chi}{\partial s_{1,n-j-1,j}} - \mathrm{F}\frac{\partial\chi}{\partial s_{1,n-j,j-1}} - \mathrm{C}\frac{\partial\chi}{\partial s_{1,n-j+1,j-2}} = 0 \qquad (j = 0, \ldots, n+1).$$

Then, substituting from the equations (30) in each equation (31), we have

$$(32)\quad (\mu^2 - \mathrm{H}\mu + \mathrm{B})\frac{\partial\chi}{\partial s_{1,n-j-1,j}} + (2\mu\nu - \mathrm{G}\mu - \mathrm{H}\nu + \mathrm{F})\frac{\partial\chi}{\partial s_{1,n-j,j-1}} + (\nu^2 - \mathrm{G}\nu + \mathrm{C})\frac{\partial\chi}{\partial s_{1,n-j+1,j-2}} = 0 \qquad (j = 0, \ldots, n+1)$$

If $(\mu^2 - \mathrm{H}\mu + \mathrm{B})$ is not zero, we put $j = 0, 1, \ldots, n-1$ successively; if $(\nu^2 - \mathrm{G}\nu + \mathrm{C})$ is not zero, we take $j = n+1, n, \ldots, 2$ successively; while if both the preceding expressions are zero but $(2\mu\nu - \mathrm{G}\mu - \mathrm{H}\nu + \mathrm{B})$ is different from zero, we take $j = 1, \ldots n$.

In any of these three events, taking account of the negative suffix convention, we see that

$$\frac{\partial \chi}{\partial s_{1,n-i-1,i}} = 0 \qquad (i = 0, \ldots, n-1)$$

and thence from (30)

$$\frac{\partial \chi}{\partial t_{0,n-j,j}} = 0 \qquad (j = 0, \ldots, n)$$

Thus, by the same reasoning as before, the hypothesis that the equation (24) is of order n is contradicted, unless we have

$$\begin{aligned} (\mu^2 - \mathrm{H}\mu + \mathrm{B}) &= 0, \\ (2\mu\nu - \mathrm{G}\mu - \mathrm{H}\nu + \mathrm{F}) &= 0, \\ (\nu^2 - \mathrm{G}\nu + \mathrm{C}) &= 0, \end{aligned}$$

equations which we recognise at once as the equations (7)-(9), chapter II. Thus the reasoning of chapter II follows at once; and in particular we see that it is impossible for the equations (1) and (24) to be in involution unless the condition $\Delta = 0$ is satisfied : in other words, unless the equation (1) is of rank 2 or 1. If this condition be satisfied, the quantities H, G, B, F, C have the values found in chapter II. Then if we choose $\mu = \mu_1$, $\nu = \nu_1$, and substitute these values in (29) and (30), we obtain precisely the equations (15), chapter III; while from the way in which we obtained the equations (32), it is clear that equations (31) are consequences of (30) and (7)-(9). Similarly, the choice $\mu = \mu_2$, $\nu = \nu_2$ leads to the equations (15), with μ_1 and μ_2, ν_1 and ν_2 permuted.

There is, however, a significant difference in the way in which we interpret the equations (15). In order that χ might be an *invariant* of order n of the first system, it was necessary for χ to satisfy the equations (15) *identically*, since integrals of (1) exist for which the variables which figure in (15) of order n assume arbitrary values, and characteristics of each system emanate from any element of contact contained in an integral of (1). But in order that (1) and (24) may be in *involution*, it is clearly sufficient that the equations (15), or the system obtained by permuting μ_1 and μ_2, ν_1 and ν_2, should be satisfied as a *consequence* of the equation (24) itself. Thus we may summarise the result in the statement of the following theorem :

THEOREM 3. — *In order that the two equations* (1) *and* (24) *be in involution, it is necessary and sufficient :*

(i) *that the equation* (1) *be of rank* 2 *or* 1 *;*

(ii) *that the function* χ *should satisfy either* (15), *or* (15) *with* μ_1 *and* μ_2, ν_1 *and* ν_2 *permuted, either identically or as a consequence of* (24) *itself.*

It is an immediate consequence of this result that any equation of the form (18), that is to say an equation $\chi = 0$ in which the function χ is an invariant of either system, is in involution with (1).

We now proceed to show that two equations in involution have a system of common integrals, which depend on an arbitrary function [13].

Suppose that we are given, instead of the full Cauchy data, the value of u in terms of the two parameters v and w, at each point of the surface $\Xi = 0$ (*see* chap. III). Then we will now show that if the equations (1) and (24) are in involution, we can determine a common integral of the two equations, such that u takes the given values on the surface $\Xi = 0$.

We assume for one of the derivatives of u of the first order, an undetermined form, say

$$s_{1,0,0} = S(v, w).$$

Then, as explained in chapter III, we may calculate by means of the equations of contact, the equation (1) itself and the equations derived from (1) by differentiations, the values of all the derivatives of u of order up to and including n, in terms of v, w, $S(v, w)$ and the partial derivatives of S with respect to v and w, of orders in general up to and including $n-1$. Substituting these values, together with the given values of u, and the values of x, y, z in terms of v and w, in the equation (24), we obtain in general a partial differential equation of order $n-1$, with one dependent variable S, and two independent variables v and w : though in particular cases it may happen that the order of the equation is lower than $n-1$, or even that the equation is a finite equation for S. The integrals of this equation depend in general upon $n-1$ arbitrary functions of *one* variable : that is, we can in general specify the values of S and $n-2$ partial derivatives of S along a *curve* in the plane of (v, w).

But selecting any integral whatever of the latter partial differential equation, we have in conjunction with the given value of u in terms of v and w, the data of Cauchy in the usual form : and this data specifies uniquely an integral of (1), provided that it does not correspond to a Monge characteristic. This latter event might arise for a particular surface $\Xi = 0$ and for a particular specification of u on the surface : but in general it cannot occur : for from the equation (17), chapter III, we observe, for example, that a plane $x =$ Cte can never be a Monge characteristic. And from the way in which we have calculated the boundary conditions which the integral satisfies, it is clear that if we substitute for all the variables in the function χ on the left hand side of (24), the value appropriate to this integral of (1), then at any point on the surface $\Xi = 0$ we have

$$\chi = 0.$$

But we have seen that any integral multiplicity of (1) is the locus of characteristics of either system, emanating from each element of the initial multiplicity associated with the surface $\Xi = 0$.

Hence if χ be an *invariant*, say of the first system, and therefore in involution with (1), then if we substitute the values of all variables in χ, appropriate to this particular integral, we have along each characteristic of order n of the first system, emanating from the initial multiplicity,

$$d\chi = 0;$$

and thus since $\chi = 0$ on the surface $\Xi = 0$, it follows that the equation (24) is satisfied at *every* point by the integral which we have determined, which is thus a common integral, taking the specified values on the surface $\Xi = 0$.

If, however, the function χ is *not* an invariant of either system, although the equations (1) and (24) *are* in involution — that is, if the function χ satisfy the equations (15) (or those obtained by permuting μ_1 and μ_2, ν_1 and ν_2), *not* identically, but as a consequence of (24), then we have to proceed rather differently to establish this result. Let us suppose, for definiteness, that χ satisfies (15) of order n as a consequence of (24). Having determined the integral of (1) by the procedure of the last page, and having seen that the equation (24) is satisfied at any point of the surface $\Xi = 0$, suppose that we substitute in the function χ the values of each variable, appropriate to a point on a characteristic of the first system, contained in the integral multiplicity which we have determined, and emanating from the point on the initial multiplicity specified by the parameters ϵ, ϖ. And let us suppose that the point of this characteristic is specified by the parameter θ, as in chapter III. χ is then a function of θ, whose derivative we require to calculate at any point of the characteristic. But before doing so, we may place certain restrictions on the form of the function χ.

We may always regard the equation (24) as solved for one of the variables $s_{1,n-i-1,i}$, $i = 0, \ldots, n-1$. For if, for arbitrary values of the variables, satisfying (24), each $\frac{\partial \chi}{\partial s_{1,n-i-1\,i}}$ were zero, then since χ by hypothesis satisfies the equations (15), each $\frac{\partial \chi}{\partial t_{0,n-j\,j}}$ $(j = 0, \ldots, n)$ would also be zero, and by the same reasoning as earlier, the hypothesis that (24) is of order n would be contradicted. Suppose then that for some particular integer p, $\frac{\partial \chi}{\partial s_{1,n-p-1\,p}} \neq 0$. We may thus solve the equation (24) for $s_{1,n-p-1,p}$; and from the definition which we have adopted, it is clear that solving for one of the variables cannot affect the property of being in involution with (1).

Thus we may write the equation (24) in the form

$$(33) \qquad \chi = s_{1\,n-p-1,p} + \varphi(x, \ldots, t_{0,0,n}) = 0,$$

in which the function φ does not contain $s_{1,n-p-1,p}$. Then substituting this value of χ in the equations (15), chapter III, we obtain

$$(34) \quad \left(\frac{d\varphi}{dx}\right)^* + \mu_1\left(\frac{d\varphi}{dy}\right) + \nu_1\left(\frac{d\varphi}{dz}\right) - \sum_{i=0}^{n-1}\left(\frac{d^{n-1}\psi}{dy^{n-i-1}\,dz^i}\right)\frac{\partial\varphi}{\partial s_{1,n-i-1,i}} - \left(\frac{d^{n-1}\psi}{dy^{n-p-1}\,dz^p}\right) = 0$$

in which the term $i = p$ is omitted from the summation; and also

$$(35) \quad \begin{cases} \dfrac{\partial\varphi}{\partial t_{0\,n-i\,i}} - \mu_2\dfrac{\partial\varphi}{\partial s_{1\,n-i-1,i}} - \nu_2\dfrac{\partial\varphi}{\partial s_{1,n-i\,i-1}} = 0 \\ (i = 0, \ldots n;\ i \neq p;\ i \neq p+1) \end{cases}$$

and finally

$$(36)\qquad \frac{\partial\varphi}{\partial t_{0,n-p,p}} - \mu_2 - \nu_2 \frac{\partial\varphi}{\partial s_{1,n-p,p-1}} = 0,$$

$$(37)\qquad \frac{\partial\varphi}{\partial t_{0,n-p-1,p+1}} - \mu_2 \frac{\partial\varphi}{\partial s_{1,n-p-2,p+1}} - \nu_2 = 0.$$

Now if $n > 2$, it is clear that $s_{1,n-p-1,p}$ does not occur in any of the equations (35), (36), (37) : and therefore these equations cannot be satisfied as a consequence of (33). Thus they must be satisfied identically by φ. But, by hypothesis, χ satisfies (34)-(37) only as a consequence of (33) : and thus (34) must be a consequence of (33). That is, φ must be such that (34) is satisfied identically when we write $-\varphi$ for $s_{1,n-p-1,p}$.

Again if $n = 2$, we may still arrange that $s_{1,n-p-1,p}$ does not occur in Ψ, μ_1, μ_2, ν_1, ν_2. For we may then substitute for $s_{1,n-p-1,p}$ (which is in this case one of the variables h, g) in the equation (1) *from* the equation (33) itself, in terms of the remaining variables. And from the definition which we have adopted, it is clear that this procedure does not affect the fact that the two equations are in involution. Hence once again we see that the equations (35), (36), (37) are satisfied identically, but (34) only as a consequence of (33).

Furthermore, whether $n > 2$ or $n = 2$, we may observe that since $s_{1,n-p-1,p}$ does not occur in Ψ, only the terms $\left(\frac{d^{n-1}\Psi}{dy^{n-i-1}dz^{i}}\right)$, $\left(\frac{d^{n-1}\Psi}{dy^{n-p-1}dz^{p}}\right)$ in (34) can contain $s_{1,n-p-1,p}$. And this being so, we may see at once by writing the expressions for these quantities in full, that each is at the most a *quadratic* expression in $s_{1,n-p-1,p}$, so that only the first and second powers of $s_{1,n-p-1,p}$ can occur in (34).

We now turn our attention to calculating the derivative of χ with respect to the variable θ, at any point of a characteristic of the first system contained in the integral multiplicity of order n, associated with the integral of (1) which we have determined by the procedure described above : the value of each variable in terms of the parameter θ, which specifies the position of an element of the characteristic, having been, as explained earlier, substituted in the function χ.

Then using the expression (14), chapter III (which gives the variation of *any* function χ containing derivatives of u of order n

but no higher, along a characteristic of order n of the first system), but putting in (14) the particular form of the function χ written on the left of (33), using the equations (35), (36), (37), which, as we have seen, are satisfied identically by φ, and dividing by $d\theta$, we have

$$(38)\quad \begin{cases} \dfrac{d\chi}{d\theta} = \left\{ \left(\dfrac{d\varphi}{dx}\right)^{\star} + \mu_1\left(\dfrac{d\varphi}{dy}\right) + \nu_1\left(\dfrac{d\varphi}{dz}\right) \right. \\ \qquad \left. - \displaystyle\sum_{\iota=0}^{n-1}\left(\dfrac{d^{n-1}\Psi}{dy^{n-\iota-1}dz^{\iota}}\right)\dfrac{\partial\varphi}{\partial s_{1,n-\iota-1,\iota}} - \left(\dfrac{d^{n-1}\Psi}{dy^{n-p-1}dz^{p}}\right)\right\}\dfrac{dx}{d\theta} \\ (i = p \text{ being omitted from the summation}). \end{cases}$$

But the first factor on the right hand side of (38) is the same as the left hand side of (34); and is therefore zero whenever (33) is satisfied : that is, whenever $\chi = 0$. Furthermore, we have seen that the left side of (34) is of degree not exceeding 2 in $s_{1,n-p-1,p}$. Thus if we write

$$s_{1,n-p-1,p} = \chi - \varphi$$

in (38) we see, remenbering that all the variables involved are analytic functions of θ along the characteristic, that we may write (38) in the form

$$(39)\qquad \frac{d\chi}{d\theta} = \mathrm{P}(\theta)\chi + \mathrm{Q}(\theta)\chi^2,$$

P and Q being analytic functions of θ.

But we know that $\chi = 0$ when $\theta = 0$; that is, on the initial multiplicity. And it follows from the usual existence theorem for ordinary differential equations of the form (39), that the only solution of (39) which is zero when $\theta = 0$ is

$$\chi \equiv 0.$$

Therefore $\chi = 0$ at every point of the characteristic, and therefore throughout the integral multiplicity. In other words, the integral of (1) which we have determined, and which takes the prescribed values on the surface $\Xi = 0$, satisfies the equation (33) at every point: and the required result is thus established.

We may observe that instead of assigning the value of u at each point of the surface $\Xi = 0$, and then calculating the value of one of the derivatives of u of the first order in such a way as to make χ zero at every point of the surface, we could reverse the procedure. That is to say, given the value at each point of $\Xi = 0$ of one of the derivatives of u of the first order (corresponding to a direction not tangential to the surface), we could determine u by a procedure exactly analogous to that which we employed to determine l, leading to a partial differential equation with two independent variables, of order n, instead of the equation of order $n - 1$ obtained earlier. All the rest of the argument would then apply unchanged.

We may summarise these results in the following brief statement.

THEOREM 4. — *If the equations* (1) *and* (24) *are in involution, there are an infinity of common integrals of the two equations, depending on an arbitrary function.*

Next we consider the characteristics of equations in involution. Suppose that the function χ in the equation (24) satisfies the equations (15) of order n, chapter III, either identically or as a consequence of the equation (24) itself. Then the reasoning of chapter III applies unchanged, to show that along any characteristic multiplicity of order $n + q$ of the second system, contained in an integral multiplicity associated with a *common* integral of the two equations, the $q + 2$ equations (20), or if $q = 0$, the equations (21) and (22), chapter III, are satisfied.

Now in deriving these latter equations, we do *not* substitute for any of the variables, values derived from the equation (1). Furthermore, although we have shown that there are an infinity of common integrals of (1) and (24), the equations (20), or (21) and (22) are entirely independent of the particular common integral with which the characteristic of the equation (1) in question is associated. Thus we see that all the conditions of the definition of a characteristic of order $n + q$ of chapter II, are satisfied in relation to the equation (24) : and therefore every characteristic of the second system of order $n + q$, $q \geqq 0$, associated with a common integral of the two equations, is also a characteristic of the equation (24) [**14**]. And hence we may state the following very important result; bearing in mind that by exactly the same reasoning we may establish that if χ satisfies (15), with μ_1 and μ_2, ν_1 and ν_2 permuted, then the characteristics of the first system associated with a common integral are characteristics of (24) :

THEOREM 5. — *If the two equations* (1) *and* (24) *are in involution, there is a common system of characteristics of each order* $n+q$, $q \geqq 0$, *associated with the common integrals : if the function* χ *on the left of* (24) *satisfies* (15), *then the common characteristics belong to the second system; while if* χ *satisfies* (15) *with* μ_1, *and* μ_2, ν_1 *and* ν_2 *permuted, the common characteristics belong to the first system.*

This result now fully explains why it is that the knowledge of an equation in involution with (1) enables us to write down further ordinary differential equations for the determination of characteristics on the required integral of (1).

It should be noticed that we have *not* proved that when χ satisfies (15), *every* characteristic of the second system is a common characteristic : and indeed this is obviously not true. For we may take any element of contact of order n, for which $\chi \neq 0$. Then there are integrals of (1) admitting this element, and therefore characteristics of both systems which are not characteristics of (24). The result of theorem 5 applies only to characteristics contained in a common integral multiplicity.

It may happen that we know a number of equations, each in involution with (1), the function on the left of each satisfying the same system of equations, (15), chapter III, though not necessarily of the same order. Suppose that the one of highest order is of order n. Then by the methods of chapter III, we may construct, from each, a number of ordinary differential equations for the determination of characteristics of order n of the second system [of the first system, if μ_1 and μ_2, ν_1 and ν_2 are permuted in (15)], together with a number of finite equations. These equations may be inconsistent, in which case the system has no common integral. On the other hand, it may happen that they are consistent, and of sufficient number to determine the characteristics of order n, in terms of a number of arbitrary constants, so that each of the variables is expressed as a function of a parameter, representing displacements along the characteristic, and a certain number of arbitrary constants.

If these constants be two or more in number, say $p+2$, where

$p \geqq 0$, and if two of them, and the parameter of the characteristics, may be eliminated in such a way as to express u and the various partial derivatives in terms of x, y, z, and p arbitrary constants, then the characteristics so determined form a multiplicity M_3, depending on p arbitrary constants. From the way in which this multiplicity is determined, and the hypothesis that the various equations are consistent, it follows that the equations in involution with (1) are satisfied throughout the mutiplicity M_1: and we thus have a common integral of the system in involution, depending on p arbitrary constants ($p \geqq 0$). But if there be only one arbitrary constant, or none, the system has no common integral : for every integral multiplicity M_3 is a locus of one-dimensional characteristics, which must therefore depend on at least two arbitrary constants.

It must be observed that the determination of characteristics in this way can never be effected when we only have one equation in involution with (1). For suppose that we have a single equation of order n, say (24), in involution with (1), χ satisfying (15), chapter III, and suppose that we are attempting to determine the characteristics of the second system of order $n+q (q \geqq 0)$. Then we have seen that we may deduce the $q+2$ equations (20), chapter III, for the characteristics in question, whereas we require $n+q+1$ new ordinary differential equations, which could be solved for the differentials $dt_{0, n+q-j, j}$, $j=0, 1, \ldots, n+q$. We also have $q+1$ finite equations,

$$\chi(q, k) \equiv \frac{d^q \chi}{dy^{q-k} dz^k} = 0 \qquad (k=0, 1, \ldots, q)$$

which contain the variables representing derivatives of order $n+q$; and we might conjecture that the equations $d\chi(q, k) = 0$ provide new ordinary differential equations of the kind required. But this is not so. For, forming the equations $d\chi(q, k) = 0$, the terms containing the differentials of the variables representing derivatives of order $n+q$ constitute a linear form in these differentials, which, using (15), chapter III, may easily be shown to be a sum of multiples of the linear forms in these differentials on the left of two of the equations (20), chapter III, and on the left of the equations of characteristics of the second system of order $n+q$, i. e., (11) chapter II, writing $n+q$ for n and μ_1, ν_1 for μ_2, ν_2. Thus the equations

$d\chi(q, k) = 0$ are not distinct from the ordinary differential equations already known for the characteristics of the second system of order $n + q$, in accordance with the result which we proved earlier for the special case $q = 0$.

CHAPTER V.

Some results regarding the invariants : some special devices for solving the problem in particular cases.

Reverting to the actual extension of Darboux's method, the theory of which we have established in chapter III, it is obvious that the difficulty of determining whether a sufficient number of invariants of a certain order exists, and of finding these invariants when they do exist, is much greater than is the case when dealing with equations with two independent variables. And just as in the latter case, there is no test to establish whether or not the method will be successful in dealing with any given equation of rank 2 or 1. There are, however, a number of results which are useful, mostly analogous to corresponding classical theorems for equations with two independent variables [15].

Suppose firstly that we have three invariants ξ, η, ζ of the first system, of order n, at least one actually containing a derivative of order n. Then we have seen that every integral of (1) satisfies an equation

$$\chi(\xi, \eta, \zeta) = 0,$$

for some form of χ. And conversely, for any form of χ, it is clear that χ is itself an invariant, and from the results of the last chapter it follows that the equation (1) and the equation $\chi = 0$ are in involution, and therefore possess an infinity of common integrals. But any common integral must also satisfy the equations obtained by differentiating with respect to y and to z (we have seen, chapter III, that the equation $\frac{d\chi}{dx}^{\star} = 0$ is an algebraic consequence of $\frac{d\chi}{dy} = 0$, $\frac{d\chi}{dz} = 0$, so that no further information is obtained by differentiating with respect to x), that is, the equations

$$\frac{\partial\chi}{\partial\xi}\frac{d\xi}{dy}+\frac{\partial\chi}{\partial\eta}\frac{d\eta}{dy}+\frac{\partial\chi}{\partial\zeta}\frac{d\zeta}{dy}=0,$$

$$\frac{\partial\chi}{\partial\xi}\frac{d\xi}{dz}+\frac{\partial\chi}{\partial\eta}\frac{d\eta}{dz}+\frac{\partial\chi}{\partial\zeta}\frac{d\zeta}{dz}=0.$$

Since χ is an arbitrary function of ξ, η, ζ, we see that $\frac{\partial\chi}{\partial\xi}$, $\frac{\partial\chi}{\partial\eta}$, $\frac{\partial\chi}{\partial\zeta}$ can take arbitrary values, and in particular none is identically zero. Thus, solving the above equations, we have

$$\frac{\frac{\partial\chi}{\partial\eta}}{\frac{\partial\chi}{\partial\xi}}=\frac{\frac{d(\zeta,\,\xi)}{d(y,\,z)}}{\frac{d(\eta,\,\zeta)}{d(y,\,z)}}.$$

$$\frac{\frac{\partial\chi}{\partial\zeta}}{\frac{\partial\chi}{\partial\xi}}=\frac{\frac{d(\xi,\,\eta)}{d(y,\,z)}}{\frac{d(\eta,\,\zeta)}{d(y,\,z)}},$$

Now any characteristic of the first system of order $n+1$ is, by definition, contained in at least one integral multiplicity, and therefore associated with at least one choice of χ above, and also contains a characteristic of order n (theorem 2), along which ξ, η, ζ, and therefore $\frac{\partial\chi}{\partial\xi}$, $\frac{\partial\chi}{\partial\eta}$, $\frac{\partial\chi}{\partial\zeta}$ are constant. It therefore follows at once that the functions $\frac{\frac{d(\zeta,\,\xi)}{d(y,\,z)}}{\frac{d(\eta,\,\zeta)}{d(y,\,z)}}$, $\frac{\frac{d(\xi,\,\eta)}{d(y,\,z)}}{\frac{d(\eta,\,\zeta)}{d(y,\,z)}}$ are new invariants of the first system of order $n+1$.

It can be shown [16] that, starting with three invariants of order n, we may deduce $q+1$ distinct invariants of order $n+q$: and from this, and the results of chapter III, we might conjecture that we could find $2(q+1)$ new ordinary differential equation for the determination of the characteristics of the opposite system, of order $n+q$, contained in the required integral. But this is not so, for the $2(q+1)$ equations will not be distinct.

In fact, knowing the three invariants ξ, η, ζ, we have seen that we may determine the form of χ in the equation $\chi(\xi,\eta,\zeta)=0$ from the boundary conditions. Then it is easy to show [16] that the only new ordinary differential equations, for the characteristics of order $n+q$ of the opposite system, which may be deduced from the knowledge of the three invariants ξ, η, ζ of order n, of the first system, are precisely the $q+2$ equations (20), which are deduced, after determining the form of χ, by the procedure of chapter III.

Nevertheless, although deducing further invariants of higher order from three known ones gives no theoretical advantage, in practice it is well worth while. For if we are trying to determine a system of characteristics of order n, we require a number of invariants of the opposite system, of order not exceeding n, which necessitates the integration of a simultaneous system of linear partial differential equations, i. e., (15). The more integrals of this system we know, the easier is the task of finding the remaining integrals. Hence if we know three, of order less than n, it is well worth deducing, by a process which, as we have seen, requires only differentiations, further integrals of each order up to and including n.

Next we discuss an artifice which is often very valuable when one or other of the systems of characteristics of order 2 possesses at least three invariants, whether or not there are enough invariants of order 2 to solve the problem directly. Suppose that ξ, η, ζ are three distinct invariants of, say, the first system of order 2, one at least of which contains a derivative of u of order 2. Then we have seen that by using the boundary conditions we may determine an equation

$$\chi(\xi, \eta, \zeta) = 0 \tag{40}$$

which is satisfied by the required integral : and in general, that is, except possibly on exceptional integrals of (1), the reasoning of chapter III shows that one or other or both of $\frac{\partial \chi}{\partial h}$, $\frac{\partial \chi}{\partial g}$ is not zero.

Now it may well happen that it is easier to solve the boundary problem for the equation (40) than for the equation (1). In such an event, it may be desired to apply the extension of the method of Darboux, which we have developed, to (40) : and accordingly we will now show that this equation is of rank 2.

ξ, η, ζ, and therefore χ, being invariants of the first system of order 2, and therefore satisfying equations (15) of order 2, we have [writing $n=2$ and $t_{0,2,0}=b$ etc., in the equations (15) as written in chapter III, and taking account of the negative suffix convention]

$$(41)\quad \begin{cases} \dfrac{\partial\chi}{\partial b} - \mu_2\dfrac{\partial\chi}{\partial h} = 0, \\ \dfrac{\partial\chi}{\partial f} - \mu_2\dfrac{\partial\chi}{\partial g} - \nu_2\dfrac{\partial\chi}{\partial h} = 0, \\ \dfrac{\partial\chi}{\partial c} - \nu_2\dfrac{\partial\chi}{\partial g} = 0; \end{cases}$$

and therefore

$$\begin{vmatrix} 0 & \dfrac{1}{2}\dfrac{\partial\chi}{\partial h} & \dfrac{1}{2}\dfrac{\partial\chi}{\partial g} \\ \dfrac{1}{2}\dfrac{\partial\chi}{\partial h} & \dfrac{\partial\chi}{\partial b} & \dfrac{1}{2}\dfrac{\partial\chi}{\partial f} \\ \dfrac{1}{2}\dfrac{\partial\chi}{\partial g} & \dfrac{1}{2}\dfrac{\partial\chi}{\partial f} & \dfrac{\partial\chi}{\partial c} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & \dfrac{1}{2}\dfrac{\partial\chi}{\partial h} & \dfrac{1}{2}\dfrac{\partial\chi}{\partial g} \\ \dfrac{1}{2}\dfrac{\partial\chi}{\partial h} & \mu_2\dfrac{\partial\chi}{\partial h} & \dfrac{1}{2}\left(\mu_2\dfrac{\partial\chi}{\partial g} + \nu_2\dfrac{\partial\chi}{\partial h}\right) \\ \dfrac{1}{2}\dfrac{\partial\chi}{\partial g} & \dfrac{1}{2}\left(\mu_2\dfrac{\partial\chi}{\partial g} + \nu_2\dfrac{\partial\chi}{\partial h}\right) & \nu_2\dfrac{\partial\chi}{\partial g} \end{vmatrix} = 0;$$

for subtracting μ_2 times the first row from the second, and ν_2 times the first row from the last, we obtain a determinant whose second and third rows are proportional. But since one or both of $\dfrac{\partial\chi}{\partial h}$, $\dfrac{\partial\chi}{\partial g}$ is non-zero, we see that one or both of the minors obtained by omitting the second row and column, or the third row and column of the discriminant, is not zero, and the equation is therefore of rank 2. Thus we may always attempt to find the required integral by applying the method which we have developed to (40) : and we also have the curious result that the characteristics of (40) are always distinct.

We now establish a most important result, which in certain cases greatly facilitates the finding of the required solution of (1) : once again, this result generalises a classical theorem, due to Goursat.

THEOREM 6. — *If one system of characteristics of order* 2 *possesses* five *distinct invariants, then* (*whether the characteristics be distinct or confluent*) *the required integral of the equation* (1) *can always be found by the integration of a partial differential equation of the* first *order.*

To prove this, let us for definiteness suppose that ξ_1, ξ_2, ξ_3, η, ζ are five distinct invariants of the first system of order 2. Then by using the boundary conditions in the usual way, we form three distinct equations, say

$$(42) \qquad \chi_1(\xi_1, \eta, \zeta) = \chi_2(\xi_2, \eta, \zeta) = \chi_3(\xi_3, \eta, \zeta) = 0$$

each of which the required integral of (1) satisfies.

Now the functions χ_1, χ_2, χ_3 are themselves invariants of the first system of order 2, and therefore each satisfies the three equations (41). From this it follows that each of the last three rows of the matrix

$$\begin{bmatrix} \frac{\partial \chi_1}{\partial h} & \frac{\partial \chi_2}{\partial h} & \frac{\partial \chi_3}{\partial h} \\ \frac{\partial \chi_1}{\partial g} & \frac{\partial \chi_2}{\partial g} & \frac{\partial \chi_3}{\partial g} \\ \frac{\partial \chi_1}{\partial b} & \frac{\partial \chi_2}{\partial b} & \frac{\partial \chi_3}{\partial b} \\ \frac{\partial \chi_1}{\partial f} & \frac{\partial \chi_2}{\partial f} & \frac{\partial \chi_3}{\partial f} \\ \frac{\partial \chi_1}{\partial c} & \frac{\partial \chi_2}{\partial c} & \frac{\partial \chi_3}{\partial c} \end{bmatrix}$$

is a sum of multiples of the first two rows; and the matrix is therefore of rank not exceeding 2.

Therefore it follows that we may eliminate the five derivatives h, g, b, f, c between the three equations (42), obtaining either one equation or two equations of the *first order*, which must be satisfied by the required integral of (1). Hence this integral may always be found by the integration of this equation (or of either one, if there be two) of the first order. We shall consider an example of this type in chapter VII.

Lastly, we mention another circumstance which sometimes arises for particular boundary conditions, which may enable us to solve the problem, even when the number of invariants is inadequate to do so in general. If we have three invariants, ξ, η, ζ say, of the first system, of orders not exceeding n, then it may happen, for particular boundary conditions, that on the initial multiplicity there are two functional relationships between ξ, η, ζ, instead of one, say

$$\xi + \varphi(\zeta) = 0, \qquad \eta + \psi(\zeta) = 0.$$

Then since $\xi + \varphi(\zeta)$ and $\eta + \psi(\zeta)$ are also invariants, and are therefore constant along the characteristics of the first system of order n emanating from the initial multiplicity, we may assert that both these equations are satisfied throughout the required integral. Proceeding as before, we are then able to deduce new ordinary differential equations for the characteristics of the second system from both these equations; and clearly we may obtain a larger number for this particular integral than would be the case for general integrals of (1), which satisfy only one equation of the form $\chi(\xi, \eta, \zeta) = 0$. This larger number of ordinary differential equations may enable us to determine the characteristics of the second system, for this particular integral, even when we have not enough invariants of the first system to do so in general.

Similarly, it may even happen, for special boundary conditions, that ξ, η and ζ are constant on the initial multiplicity, say

$$\xi = \alpha, \qquad \eta = \beta, \qquad \zeta = \gamma,$$

where α, β, γ are constants. Then, since $\xi - \alpha$, $\eta - \beta$, $\zeta - \gamma$ are also invariants, by the reasoning of the last paragraph we may assert that these three equations are satisfied throughout the required integral, and deduce three new sets of ordinary differential equations for the characteristics of the second system.

CHAPTER VI.

Characteristics of the first order.

The theory of characteristics of order 2 and above, which we have built up in preceding chapters, has been sufficient for us to extend the method of Darboux to equations with three independent variables, provided that they be of rank 2 or 1, in accordance with the definition of chapter II. But the theory would be incomplete without a mention of the corresponding generalisation of the idea of characteristics of the first order. This latter topic is closely linked with the extension of the method of Monge-Ampere to equations with three independent variables, which has been fully discussed elsewhere, particularly by Vivanti. Thus we will deal very briefly with characteristics of the first order : but it is of interest to see how these are related to the characteristics which we defined in chapter II; and we shall also see how the equations of characteristics of the first order may sometimes be useful in determining certain invariants of the characteristics of higher orders.

The definition of chapter II may be extended to the case when $n = 1$, without any change whatever : in this case the equations of contact (3), reduce to the single equation

$$du - s_{1,0,0}\, dx - t_{0,1,0}\, dy - t_{0,0,1}\, dz = 0 \tag{43}$$

(we avoid denoting the derivatives of u of the first order by l, m, n, owing to the possible ambiguity with the integer n), while corresponding to (4) and (5) are

$$ds_{1,0,0} = -\Psi\, dx + h\, dy + g\, dz, \tag{44}$$

$$\begin{cases} dt_{0,1,0} = h\, dx + b\, dy + f\, dz, \\ dt_{0,0,1} = g\, dx + f\, dy + c\, dz. \end{cases} \tag{45}$$

Now the equation (44) is in general *not* linear in the five derivatives of u of the second order, h, g, b, f, c; and therefore we cannot apply all the arguments of chapter II unchanged. Nevertheless, if dx, dy, dz have the values appropriate to a characteristic of the first order, which we suppose known, and if dx, dy, dz and the variables making up an element of contact of the first order be regarded as constants, then the argument of chapter II, page 12, in which we substitute the phrase " independent functions of the derivatives of u of order 2 " for " linearly independent forms ", shows that the three expressions on the right hand sides of (44) and (45) cannot be independent functions of the five variables h, g, b, f, c. Furthermore, this being so, the reasoning of chapter II applies unchanged, writing $n = 1$, to show that the condition that the three expressions are not independent requires $dx \neq 0$.

From the usual theory of Jacobians, the condition that the three equations (44) and (45) cannot be solved for three of h, g, b, f, c is that the matrix

$$\begin{bmatrix} dy - \mathrm{H}\, dx & dz - \mathrm{G}\, dx & -\mathrm{B}\, dx & -\mathrm{F}\, dx & -\mathrm{C}\, dx \\ dx & 0 & dy & dz & 0 \\ 0 & dx & 0 & dy & dz \end{bmatrix}$$

should be of rank 2 (clearly it is not of rank 1, since $dx \neq 0$). Thus equating to zero the determinants formed respectively from the first three columns, from the first, second and fourth, and from the first, second and last, we obtain (after dividing by $dx \neq 0$)

$$dy^2 - H\,dy\,dx + B\,dx^2 = 0,$$
$$2\,dy\,dz - G\,dy\,dx - H\,dz\,dx + F\,dx^2 = 0,$$
$$dz^2 - G\,dz\,dx + C\,dx^2 = 0.$$

But writing $dy = \mu dx$, $dz = \nu dx$, we have precisely the equations (7)-(9), chapter II; and reasoning exactly as in chapter II we see that there can be no characteristics of the first order unless the equation (1) is of rank 2 or 1. Furthermore, we again see that dx, dy, dz must satisfy the equations

$$(46) \qquad dy - \mu_1\,dx = 0, \qquad dz - \nu_1\,dx = 0,$$

or else

$$(47) \qquad dy - \mu_2\,dx = 0, \qquad dz - \nu_2\,dx = 0,$$

(μ_1, ν_1) and (μ_2, ν_2) being the two pairs of solutions of (7)-(9).

Now in this case, since in general the equations (46) and (47) contain derivatives of u of order 2, we cannot in general regard these as being total differential equations appropriate to characteristics of the first order. Nevertheless, it may, under special conditions, occur that we can eliminate h, g, b, f, c between the five equations of one or other, or both, of the sets (44), (45), (46) and (44), (45), (47), obtaining thus one or more total differential equations (not necessarily linear), satisfying the conditions of the definition, where $n = 1$.

In general this elimination cannot be carried out : and thus in general the equation (1) has no characteristics of the first order. But if it should occur that the elimination can be carried out, for all values of the variables which compose an element of contact of order 1, for one or other of the sets of equations mentioned in the last paragraph, then we say that there is a system of characteristics of the first order associated with the first system, or with the second system as the case may be : if the elimination be possible for both sets, there are two systems of characteristics of the first order.

We are not primarily concerned with the conditions for the existence of characteristics of the first order. But if one or two systems of this kind exist, that is to say, if certain total differential equations which satisfy the conditions of the definition are identical consequences of one or both of the systems of equations (44), (45), (46) and (44), (45), (47), then the reasoning of chapter II applies unchanged, to establish that these characteristics are of one dimension only, and that the integral multiplicity of the first order associated with any integral of (1) is a *locus* of characteristics of the first order (of either system, if two distinct systems exist).

Furthermore, we know that along any characteristic of order 2, of, say, the first system, the equations (44), and (45) together with (46) are satisfied. Hence if a system of characteristics of the first order exists, associated with the first system, then we may at once extend the result of theorem 2, showing that every characteristic of the first system of order 2 contains exactly one characteristic of the first order of the same system. Again, since a characteristic of the first order is by definition contained in at least one integral multiplicity of the first order, which in turn is contained in a multiplicity of the second order, and since the equations (46) or (47) are satisfied, for the values of the derivatives of the second order appropriate to the integral in question, it follows that every characteristic of the first order is contained in at least one characteristic of the second order.

Now suppose that either one or two systems of characteristics of the first order exists; and suppose that one system possesses an *invariant*. That is, suppose that there is a function χ of the elements of contact of the first order, such that

$$d\chi = 0$$

is a consequence of the ordinary differential equations which the system of characteristics of the first order satisfies. Then since every characteristic of the second order of the corresponding system contains a characteristic of the first order, it follows at once that χ is also an invariant of this system of characteristics of the *second* order; and therefore χ satisfies (15) of order 2, or else the corresponding system with μ_1 and μ_2, ν_1 and ν_2 permuted.

Conversely, suppose that χ is an invariant of the system of characteristics of order 2, of the same system as the characteristics of order 1 whose existence we assume, but that χ depends only on elements of contact of the first order. Then since every characteristic of the first order is contained in at least one of the second order, along which latter χ is by hypothesis constant, it follows that χ is also an invariant of the system of characteristics of the first order.

We assume always that there are not three invariants of a system of characteristics of the first order; for if there were, the solution of (1) could always be effected by the method of Monge-Ampere extended, with which we are not concerned. Nevertheless, we have shown by the reasoning of the last two paragraphs that when a system of the first order exists, we may find all those integrals of the corresponding system (15), which depend only on elements of contact of the first order, by setting up the differential equations of the characteristics of the first order, and seeking invariants directly.

And when one or two of these exist, it may in certain cases be simpler to find them in this manner rather than by starting with (15) of order 2. Furthermore, knowing these one or two integrals in advance, we may use them to simplify the integration of the system (15). In this lies the only utility of characteristics of the first order in relation to the extension of the method of Darboux : and it will be noticed that all the results which we have obtained in this chapter are analogous to classical results, for equations with two independent variables.

CHAPTER VII.

Examples.

To illustrate the foregoing theory, we now consider two examples, the solutions of which exhibit all the essential features of the method. For the sake of brevity, we now revert to the convenient notation l, m, n for the derivatives of u of the first order $s_{1,0,0}$, $t_{0,1,0}$, $t_{0,0,1}$, respectively, since there will now be no risk of ambiguity between the derivative n and the integer n.

Example 1. — It is required to find a solution of the equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\dfrac{\partial^2 u}{\partial x\partial y}\dfrac{\partial^2 u}{\partial x\partial z}}{1-\dfrac{\partial^2 u}{\partial y\partial z}}+\left\{\frac{\dfrac{\partial^2 u}{\partial x\partial y}}{1-\dfrac{\partial^2 u}{\partial y\partial z}}\right\}^2=0,$$

such that when $x=0$, $u=\frac{1}{2}z^2$ and $\frac{\partial u}{\partial x}=y^2+yz$.

Writing the given equation in the notation which we have adopted all along, we have

$$(48)\qquad a+\frac{hg}{1-f}+\left\{\frac{h}{1-f}\right\}^2=0;$$

so that

$$H=\frac{g}{1-f}+\frac{2h}{(1-f)^2},\qquad G=\frac{h}{1-f},$$
$$F=\frac{hg}{(1-f)^2}+\frac{2h^2}{(1-f)^3},\qquad B=C=0;$$

and since each term obtained in differentiating (48) once with respect to y or z must contain a derivative of u of the third order, we have in accordance with the definition of these symbols in chapter I,

$$\left(\frac{d\Psi}{dy}\right)=\left(\frac{d\Psi}{dz}\right)=0.$$

Thus the equations (7)-(9), chapter II are in this case

$$\mu^2-\left\{\frac{g}{1-f}+\frac{2h}{(1-f)^2}\right\}\mu=0,$$
$$2\mu\nu-\frac{h}{1-f}\mu-\left\{\frac{g}{1-f}+\frac{2h}{(1-f)^2}\right\}\nu+\left\{\frac{hg}{(1-f)^2}+\frac{2h^2}{(1-f)^3}\right\}=0,$$
$$\nu^2-\frac{h}{1-f}\nu=0,$$

which we may rearrange as

$$\mu\left[\mu-\left\{\frac{g}{1-f}+\frac{2h}{(1-f)^2}\right\}\right]=0,$$
$$\left[\mu-\left\{\frac{g}{1-f}+\frac{2h}{(1-f)^2}\right\}\right]\left\{\nu-\frac{h}{1-f}\right\}+\mu\nu=0,$$
$$\left\{\nu-\frac{h}{1-f}\right\}\nu=0.$$

This latter form of the equations shows clearly that they are consistent, so that there is no need to verify that the equation (48) is of rank 2. Thus we have two pairs of values

$$\mu_1 = \frac{g}{1-f} + \frac{2h}{(1-f)^2}, \qquad \nu_1 = 0;$$

and

$$\mu_2 = 0, \qquad \nu_2 = \frac{h}{1-f}.$$

The equations (15) of order 2 are therefore in this case [writing $n=2$ and $t_{0,2,0}=b$ etc., in the equations (15) as written in chapter III, and putting in the values of $\mu_1, \nu_1, \mu_2 \nu_2, \left(\frac{d\Psi}{dy}\right), \left(\frac{d\Psi}{dz}\right)$] the following four équations :

$$\begin{aligned} V_1(\chi) \equiv \frac{\partial \chi}{\partial x} &+ \left\{ \frac{g}{1-f} + \frac{2h}{(1-f)^2} \right\} \frac{\partial \chi}{\partial y} \\ &+ \left[l + \left\{ \frac{g}{1-f} + \frac{2h}{(1-f)^2} \right\} m \right] \frac{\partial \chi}{\partial u} \\ &+ \frac{h^2}{(1-f)^2} \frac{\partial \chi}{\partial l} + \left[h + \left\{ \frac{g}{1-f} + \frac{2h}{(1-f)^2} \right\} b \right] \frac{\partial \chi}{\partial m} \\ &+ \left[\frac{g}{1-f} + \frac{2fh}{(1-f)^2} \right] \frac{\partial \chi}{\partial n} = 0, \end{aligned}$$

$$V_2(\chi) \equiv \frac{\partial \chi}{\partial b} = 0,$$

$$V_3(\chi) \equiv \frac{\partial \chi}{\partial f} - \frac{h}{1-f} \frac{\partial \chi}{\partial h} = 0,$$

$$V_4(\chi) \equiv \frac{\partial \chi}{\partial c} - \frac{h}{1-f} \frac{\partial \chi}{\partial g} = 0,$$

Forming the equation

$$V_2 \{ V_1(\chi) \} - V_1 \{ V_2(\chi) \} = 0,$$

we obtain at once

$$V_5(\chi) \equiv \frac{\partial \chi}{\partial m} = 0;$$

and thence the combination

$$V_5 \{ V_1(\chi) \} - V_1 \{ V_5(\chi) \} = 0$$

gives

$$V_6(\chi) \equiv \frac{\partial \chi}{\partial u} = 0.$$

Then, using the equations $V_5(\chi) = V_6(\chi) = 0$, we see that the equation $V_4(\chi) = 0$ becomes

$$V_7(\chi) \equiv \frac{\partial\chi}{\partial x} + \left\{ \frac{g}{1-f} + \frac{2h}{(1-f)^2} \right\} \frac{\partial\chi}{\partial y} + \frac{h^2}{(1-f)^2} \frac{\partial\chi}{\partial l} + \left\{ \frac{g}{1-f} + \frac{2fh}{(1-f)^2} \right\} \frac{\partial\chi}{\partial n} = 0.$$

Next we form the equation

$$V_4\{V_7(\chi)\} - V_7\{V_4(\chi)\} = 0,$$

which leads to the equation

$$V_8(\chi) \equiv \frac{\partial\chi}{\partial y} + \frac{\partial\chi}{\partial n} = 0;$$

and using this last equation, the equation $V_7(\chi) = 0$ becomes

$$V_9(\chi) \equiv \frac{\partial\chi}{\partial x} + \frac{h}{(1-f)^2} \frac{\partial\chi}{\partial l} - \frac{2h}{1-f} \frac{\partial\chi}{\partial n} = 0:$$

and it may readily be verified that the system of seven equations with twelve independent variables

$$V_2(\chi) = V_3(\chi) = V_4(\chi) = V_5(\chi) = V_6(\chi) = V_8(\chi) = V_9(\chi) = 0$$

is a Jacobian system, which therefore possesses *five* integrals. These integrals are easily found. The equation $V_3(\chi) = 0$, in which only the two variables h and f occur, is easily found to admit the integral $\frac{h}{1-f}$; and clearly this function satisfies all the other equations. Next, the equation $V_4(\chi) = 0$, in which h and f may be treated as constants, admits the integral $g + c\,\frac{h}{1-f}$; and once again this function clearly satisfies all the equations of the system. Thirdly, the equation $V_9(\chi) = 0$ may be treated as an equation with constant coefficients, since x, l, n do not occur in the coefficients, and hence we see that this last equation admits the integrals $2l + n\frac{h}{1-f}$, $n + 2x\frac{h}{1-f}$ and y. Then taking account of the equation $V_8(\chi) = 0$, we see that the equations

$$V_9(\chi) = V_8(\chi) = 0$$

admit the common integrals $2l + (n-y)\frac{h}{1-f}$ and $n - y + 2x\frac{h}{1-f}$;

and in view of the integrals which have already been found, these functions are both seen to satisfy all the equations of the system. Finally it is obvious that the system admits the integral z; and thus we have the seven common integrals, which we now tabulate for convenience as follows :

$$I_1 = \frac{h}{1-f}; \qquad I_2 = g + c\frac{h}{1-f}; \qquad I_3 = 2l + (n-y)\frac{h}{1-f};$$
$$I_4 = n - y + 2x\frac{h}{1-f}; \qquad I_5 = z.$$

Now to establish the *three* equations which the required integral of (48) satisfies, and which can be deduced from these five invariants, we use the given boundary conditions as follows :

When $x = 0$, $u = \frac{1}{2}z^2$ and $l = y^2 + yz$: and therefore

$$m = 0; \qquad n = z; \qquad b = f = 0; \qquad c = 1; \qquad h = 2y + z; \qquad g = y.$$

And thus on the "initial multiplicity", we have

$$\begin{aligned} I_1 &= 2y + z; \\ I_2 &= 3y + z; \\ I_3 &= z(3y + z); \\ I_4 &= z - y; \\ I_5 &= z. \end{aligned}$$

From these expressions we deduce at once the functional relationships

$$\begin{aligned} &I_1 + 2I_4 - 3I_5 = 0, \\ &I_2 + 3I_4 - 4I_5 = 0, \\ &I_3 + I_5(3I_4 - 4I_5) = 0. \end{aligned}$$

These relationships are satisfied throughout the required integral multiplicity : and thus we see that the required integral of (48) satisfies the three equations

$$\frac{h}{1-f} + 2\left\{n - y + 2x\frac{h}{1-f}\right\} - 3z = 0,$$
$$g + c\frac{h}{1-f} + 3\left\{n - y + 2x\frac{h}{1-f}\right\} - 4z = 0,$$
$$2l + (n-y)\frac{h}{1-f} + z\left[3\left\{n - y + 2x\frac{h}{1-f}\right\} - 4z\right] = 0,$$

which after rearranging we write in the form

$$\chi_1 \equiv (1+4x)\frac{h}{1-f} - (2y+3z-2n) = 0,$$
$$\chi_2 \equiv (c+6x)\frac{h}{1-f} + g - (3y+4z-3n) = 0,$$
$$\chi_3 \equiv (6zx - y + n)\frac{h}{1-f} - (3yz + 4z^2 - 3zn - 2l) = 0.$$

the functions χ_1, χ_2, χ_3 being, of course, invariants of the first system of order 2.

Now we observe straight away that in accordance with the result of theorem 6, chapter V, we may eliminate the ratio $\frac{h}{1-f}$ betwen the equations $\chi_1 = 0$ and $\chi_3 = 0$, thereby obtaining an equation of the first order, which the required solution of (48) satisfies. And clearly this is the most direct, and therefore the best, procedure to use for this example. Nevertheless to illustrate the general method, we first of all effect the solution by determining the characteristics of the second system of order 2, by the method of chapter III, using certain of the simplifications discussed in chapter V. We will then derive the same solution by the direct method of theorem 6, which is, of course, only applicable by virtue of the fact that there are, in this particular case, five invariants of order 2 of the first system.

Before going on to complete the solution of the problem, it is worth mentioning that it may easily be shown that the only invariants of the second system of order 2 are y and $(m-z)$. The calculations are similar to those for the first system, and there is no particular point in reproducing them.

Solution by the general method. — We begin by forming, from the equation $\chi_1 = 0$, the two equations corresponding to (21) and (22), chapter III, in which we write $n = 2$. Differentiating χ_1 with respect to y and to z, and omitting the terms involving derivatives of order 3, we see that

$$\left(\frac{d\chi_1}{dy}\right) \equiv -2(1-f); \qquad \left(\frac{d\chi_1}{dz}\right) \equiv -(3-2c);$$

and thus the two equations are

$$\frac{1+4x}{1-f}\,db - 2(1-f)\,dx = 0, \qquad \frac{1+4x}{1-f}\,df - (3-2c)\,dx = 0.$$

Next we form, from the equation $\chi_2 = 0$, the equation corresponding to (22), chapter III. In this case we have

$$\left(\frac{d\chi_2}{dz}\right) = -(4-3c);$$

and thus the equation is

$$\frac{c+6x}{1-f}\,df + dc - (4-3c)\,dx = 0.$$

Then solving these three ordinary differential equations for db, df, dc, and dividing by dx, we see that the characteristics of the second system of order 2, contained in the required integral multiplicity, satisfy the ordinary differential equations

$$(49)\qquad \left\{\begin{aligned} \frac{db}{dx} &= \frac{2(1-f)^2}{1+4x}, \\ \frac{df}{dx} &= \frac{(3-2c)(1-f)}{1+4x}, \\ \frac{dc}{dx} &= \frac{2(c^2-3c+2-x)}{1+4x}. \end{aligned}\right.$$

To these we adjoin the equations (10), (11), of order 2, chapter II, and the equations of contact, which in this case, after putting in the values of μ_1, μ_2, ν_1, ν_2, $\left(\frac{d\Psi}{dy}\right)$, $\left(\frac{d\Psi}{dz}\right)$, are

$$\begin{aligned} & dy = 0, \\ & dz - \frac{h}{1-f}\,dx = 0, \\ & du - \left\{ l + \frac{h}{1-f}\,n \right\} dx = 0, \\ & dl + \left\{ \frac{h}{1-f} \right\}^2 dx = 0, \\ & dm - \frac{h}{1-f}\,dx = 0, \\ & dn - \left\{ g + c\,\frac{h}{1-f} \right\} dx = 0, \\ & dh + \left\{ \frac{g}{1-f} + \frac{2h}{(1-f)^2} \right\} db = 0, \\ & dg + \left\{ \frac{g}{1-f} + \frac{2h}{(1-f)^2} \right\} df = 0, \end{aligned}$$

and thus we have a complete set of ordinary differential equations for the determination of the characteristics of the second system of order 2. As regards the initial values to be used in the integration of the system, let us suppose that on the characteristic in question, when $x = 0$, $z = v$. Since $dy = 0$, y may be treated as a parameter in the integration. And thus, using the conditions written on page 66, we have the initial values :

$$x = 0; \quad z = v; \quad u = \frac{1}{2} v^2; \quad l = y^2 + yv; \quad m = b = f = 0; \quad n = v;$$
$$h = 2y + v; \quad g = y; \quad c = 1.$$

The integration of the system thus expresses all the variables, and in particular, u and z, in terms of x, y and v : and elimination of the parameter v will then express u in terms of x, y, z.

To simplify the integration, we use two artifices described in chapter V. First of all, we have mentioned that the second system of order 2 leads to the two invariants y, $(m - z)$, which are thus first integrals of the system written above (which indeed is obvious directly). Therefore, using the boundary conditions we have

$$m - z = - v, \qquad \text{or} \qquad m = z - v.$$

Secondly, we may solve the three equations $\chi_1 = 0$, $\chi_2 = 0$, $\chi_3 = 0$ for h, g, and l in terms of the remaining variables, obtaining thus the three equations

$$\frac{h}{1-f} = \frac{2y + 3z - 2n}{1 + 4x},$$
$$g + c\,\frac{h}{1-f} = \frac{3y + 4z - 3n - 2zx}{1 + 4x},$$
$$l + \frac{h}{1-f} n = \frac{y^2 + 3yz + 2z^2 - xz^2 - n^2}{1 + 4x}.$$

Substituting from these last equations in the second, third and sixth of the equations written on the last page, we have finally a system of *six* ordinary differential equations to express the six variables z, u, n, b, f, c in terms of x, y (which is treated as a parameter throughout) and the parameter v; namely the three equations (49) which contain only the variables x, b, f, c, together with the three equations

$$(50) \qquad \left\{ \begin{aligned} \frac{dz}{dx} &= \frac{2y + 3z - 2n}{1 + 4x}, \\ \frac{dn}{dx} &= \frac{3y + 4z - 3n - 2zx}{1 + 4x}, \\ \frac{du}{dx} &= \frac{y^2 + 3yz + 2z^2 - xz^2 - n^2}{1 + 4x}, \end{aligned} \right.$$

The first two of these equations (which are linear in z and n) contain only x, z, n and (as a parameter) y; and thus these may be integrated to express z and n in terms of x, y and the parameter v introduced by the initial values : after this has been done, the third équation gives u by a quadrature, also in terms of x, y, v. Thus we eventually obtain expressions of the form

$$u = U_1(x, y, v), \qquad z = Z(x, y, v),$$

and the elimination of v leads to the required expression

$$u = U(x, y, z).$$

The first two of the equations (50) cannot be solved in terms of elementary functions; but nevertheless we have reduced the problem to the solution of two simultaneous linear ordinary differential equations with two dependent variables, followed by a quadrature; and thus we regard the problem as solved.

Alternative solution, by the special method of theorem 6. — Reverting to the equations $\chi_1 = 0$, $\chi_2 = 0$, $\chi_3 = 0$ on page 67, we eliminate the ratio $\frac{h}{1-f}$ between the first and last of these equations, obtaining at once the equation

$$(51) \qquad (1 + 4x)\,l - n^2 + (2y + 3z)\,n - y^2 - 3yz - 2z^2 + z^2 x = 0,$$

which the required integral of (48) must satisfy.

Now if we solve this equation of the first order by Cauchy's method, the equations to the Cauchy characteristics are

$$\frac{dx}{1 + 4x} = \frac{dy}{0} = \frac{dz}{2y + 3z - 2n} = \frac{du}{(1 + 4x)\,l - 2n^2 + (2y + 3z)\,n}$$
$$= \frac{dl}{-(4l + z^2)} = \frac{dm}{2y + 3z - 2n} = \frac{dn}{3y + 4z - 2zx - 3n}.$$

The two combinations $dy = 0$, $d(m - z) = 0$ are obvious; and thus taking the initial conditions as before, we have

$$m = z - v,$$

and using the equation (51) to express l in terms of the other variables, and treating y as a parameter, the equations written above reduce to the three equations

$$\frac{dx}{1+4x} = \frac{dz}{2y+3z-2n} = \frac{dn}{3y+4z-2zx-3n}$$
$$= \frac{du}{y^2+3yz+2z^2-xz^2-n^2}.$$

These equations are identical with the equations (50); and with the initial conditions $x=0$, $z=n=v$, $u=\frac{1}{2}v^2$, the solution follows as before.

Example 2. — As another illustration of a solution rapidly effected by the method of theorem 6, we consider (to save repetition of the routine work involved in determining the invariants) once more the same equation (48), which was considered in example 1; but this time we choose boundary conditions for which it is possible to solve the problem explicitly.

Suppose that it is required to find a solution of the equation (48). such that when $x=0$, $u=0$ and $\frac{\partial u}{\partial x}=yz$. Then from these conditions it follows that when $x=0$ we have

$$u=0; \qquad \iota=yz; \qquad m=n=b=f=c=0; \qquad h=z; \qquad g=y.$$

The invariants being, of course, those written on page 66, it is thus clear that when $x=0$,

$$I_1=z; \qquad I_2=y; \qquad I_3=yz; \qquad I_4=-y; \qquad I_5=z.$$

At once we may write down the functional relationships

$$I_1-I_5=0, \qquad I_3+I_4I_5=0,$$

(and of course, the equation $I_2+I_4=0$, which we shall not use in this example), showing that the required integral satisfies the two equations

$$\chi_1 \equiv \frac{h}{1-f}-z=0,$$
$$\chi_3 \equiv 2l+(n-y)\frac{h}{1-f}+z\left\{n-y+2x\frac{h}{1-f}\right\}=0,$$

Eliminating $\frac{h}{1-f}$ between these two equations, in accordance with the result of theorem 6, we obtain the equation

$$l + zn + xz^2 - yz = 0. \tag{52}$$

This is a non-homogeneous *linear* equation of the first order, which the required solution of (48) satisfies : and we solve it in the usual way by considering the ordinary differential equations

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dz}{z} = \frac{du}{yz - xz^2}.$$

From these equations we form the integrable combinations

$$dy = 0; \qquad d(ze^{-x}) = 0; \qquad d\left(u - yz - \frac{z^2}{4} + \frac{z^2 x}{2}\right) = 0;$$

and thus the general integral of (52) is of the form

$$u = yz + \frac{z^2}{4} - \frac{z^2 x}{2} + \psi(y, ze^{-x}).$$

To determine the form of the function ψ appropriate to the particular integral in question, we have from the boundary conditions

$$0 \equiv yz + \frac{z^2}{4} + \psi(y, z);$$

and thus finally we have

$$u = yz + \frac{z^2}{4} - \frac{z^2 x}{2} - yze^{-x} - \left(\frac{z^2}{4}\right) e^{-2x}.$$

It may easily be verified that this is the correct solution, satisfying the equation (48) and the prescribed boundary conditions.

CHAPTER VIII.

General conclusions and remarks.

The whole of the results which we have obtained may be summarised by the statement that all the classical theory of characteristics, of equations in involution, and of Darboux's method of solution, may be generalised to apply to an equation of the second order with three independent variables, provided that it be of rank 2 or 1 (chap. II).

But if the equation be of rank 3, i. e., if the discriminant be different from zero, the classical theory has no counterpart.

And the results which we have obtained may be generalised to deal with equations having any number of independent variables. The extension of the fundamental idea of the rank of an equation of the second order, with m independent variables, is obvious : and it may be shown that, exactly as in the case with which we have dealt, the equation has no characteristics, two distinct systems, or a single system which may be regarded as two confluent systems, according to whether the rank is 3 or greater, 2, or 1 respectively. The extension of Darboux's method may again be made, provided that the equation be of rank 2 or 1. The method may also be extended to suitable equations of higher order.

Throughout all the foregoing work, we have shown that, in order to extend Darboux's method, it is sufficient for the equation to be of rank 2 or 1. But it may also be proved [17] that this condition is also necessary, i. e., that there is no other way in which the method can be extended, and also that there is no possible way of extending it to equations of rank 3.

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