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FASCICULE CXIX

Vorticity and the thermodynamic state in a gas flow

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IN A GAS FLOW

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1. **Introduction** (1). — The *vorticity*

$$\mathbf{w} \equiv \operatorname{curl} \mathbf{v} \tag{1.1}$$

in a gas flow whose velocity is $\mathbf{v}$ is closely connected with the thermodynamic variables — the *entropy* η, the *pressure* p and the *stagnation pressure* p_0, the *temperature* θ and the *stagnation temperature* θ_0, the *enthalpy* h and the *stagnation enthalpy* or *total head* h_0, the *density* ρ and the *stagnation density* ρ_0, and finally the *ultimate speed* v_0. Apparently the discipline of gas dynamics is the only realm of mathematical physics where thermodynamics and mechanics truly cooperate : while the large and fairly rapid deformations experienced by a streaming gas require a genuinely dynamical treatment, forcing complete abandonment of the fictitious and paradoxical "quasi-static process" of classical thermodynamics, yet locally the material is sufficiently near to thermodynamic equili-

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(1) This work was commenced in collaboration with Dr Prim under the Office of Naval Research Contract 53 47; [1947, 3] is a preliminary report. I am obliged to Dr Neményi for criticism, and to Miss Charlotte Brudno for preparation of the MS.

brium that thermodynamical methods based upon the existence of an equation of state for the local state variables remain applicable. Of this fascinating border domain there exists no complete and systematic survey, and in the literature the various quantitative relations are often stated loosely or subject to unnecessary restrictions, and are sometimes deduced by intuitive arguments which serve at best to suggest the plausibility, but fail to establish the truth of the propositions.

In this memoir our interest centers about the vorticity. Our general objective is two-fold : *to characterize irrotational gas flows in thermodynamical terms*, and *in rotational gas flows to search out the relations which bind the vorticity to the thermodynamic variables*. We attempt to give clear, full, and correct statements, substantiated by simple formal proofs, of some known theorems or generalizations of them, to deduce some new theorems, and in particular to present a fundamental simplification of all problems concerning certain types of gases in steady flow which may be thought of as originating in a reservoir at uniform pressure (th. 20). Perhaps more important than any individual theorem, however, is the orderly line of march, in which each new question is naturally suggested by the preceding result.

On the whole, our sequence of presentation is from the general to the particular as far as the physical properties of the *fluid* are concerned, adding new assumptions one by one as necessary to draw increasingly specific conclusions.

2. Some definitions and preliminary lemmas of vector analysis and kinematics. — A vector field $\mathbf{b}$ such that

$$\mathbf{b} = -\operatorname{\mathbf{grad}} \chi, \quad \text{or equivalently,} \quad \operatorname{\mathbf{curl}} \mathbf{b} = 0, \tag{2.1}$$

is a *laminar* field. More generally, any field locally endowed with normal surfaces $\chi = \text{const.}$, i.e.

$$\mathbf{b} = -\upsilon \operatorname{\mathbf{grad}} \chi, \tag{2.2}$$

is a *complex-laminar* field, a laminar field constituting a special case. In introducing these terms Kelvin (²) proved

(²) [1851, § 75]. Kelvin's term is " complex-lamellar "; the term " doubly-laminar " is found in the literature.

LEMMA 1. — *A continuously differentiable field* $\mathbf{b}$ *is complex-laminar if and only if*

$$\mathbf{b}.\mathbf{curl}\,\mathbf{b} = 0. \tag{2.3}$$

A field $\mathbf{b}$ such that

$$\mathbf{b} \times \mathbf{curl}\,\mathbf{b} = 0, \qquad \mathbf{curl}\,\mathbf{b} \neq 0, \tag{2.4}$$

may be called a *Beltrami field*, since Beltrami [3] first exhibited hydrodynamical flows whose velocity vector is of this type. To Neményi we owe the realization of the importance of these fields in gas dynamics, as well as some of the results concerning them which will be developed in this memoir. Notice that as defined here Beltrami fields do *not* contain laminar fields as a special case [4]. The expression of a Beltrami field in terms of scalar functions, analogous to (2.2), is elaborate, and not required in this memoir. On the other hand, the relation between $\mathbf{b}$ and $\mathbf{curl}\,\mathbf{b}$ deserves a nearer analysis. Equivalent to (2.4) is $\mathbf{curl}\,\mathbf{b} = \lambda\mathbf{b}$, $\lambda \neq 0$. Now let σ be a scalar function such that [5] $\operatorname{div}\sigma\mathbf{b} = 0$. Then

$$0 = \operatorname{div}\sigma\mathbf{b} = \operatorname{div}\left(\frac{\sigma}{\lambda}\mathbf{curl}\,\mathbf{b}\right) = \mathbf{grad}\,\frac{\sigma}{\lambda}.\mathbf{curl}\,\mathbf{b} = \lambda\,\mathbf{b}.\mathbf{grad}\,\frac{\sigma}{\lambda}, \tag{2.5}$$

so that $\frac{\sigma}{\lambda}$ is constant upon each vector line, but $\lambda = \frac{|\mathbf{curl}\,\mathbf{b}|}{b}$, and hence we obtain a theorem of Beltrami, as reformulated by Neményi and Prim [6] :

LEMMA 2. — *For a twice continuously differentiable Beltrami field* $\mathbf{b}$, *let* σ *be a scalar function such that*

$$\operatorname{div}\sigma\mathbf{b} = 0; \tag{2.6}$$

then

$$\frac{|\mathbf{curl}\,\mathbf{b}|}{\sigma b} = \text{const.} \tag{2.7}$$

along each vector-line of b.

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[3] [1889]. Such fields occurred earlier in the literature, but only in passing references. Correction added in proof : most of Beltrami's results had been obtained previously by Gromeka [1881, gl. 2, § 9].

[4] This distinction is adopted for later convenience in the statement of theorems.

[5] For any continuously differentiable field $\mathbf{b}$, an infinite number of such scalar functions exist, as was noted by Appell [1897, § 5].

[6] [1949, 4, th. 1].

Let $\mathbf{v}$ be the velocity field of a motion. Then the *continuity* of motion is expressed in part by Euler's equation.

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{v} = 0, \tag{2.8}$$

where ρ is the *density* and t is the *time*. More specifically, in this memoir the term *continuous* flow is to be understood as denoting a flow in a region where $\mathbf{v}$ is single valued and twice continuously differentiable with respect to time and the space variables. Some of our theorems actually hold under less stringent requirements, which we shall not trouble to state except in the few cases when they may be relaxed sufficiently to admit regions of flow in which there are shock waves.

A motion is *steady* if

$$\frac{\partial \rho}{\partial t} = 0, \qquad \frac{\partial \mathbf{v}}{\partial t} = 0.$$

The *vorticity* $\mathbf{w}$ is given by

$$\mathbf{w} \equiv \operatorname{curl} \mathbf{v}; \tag{2.9}$$

it is a measure of the local and instantaneous rate of rotation of the medium ([7]).

A motion whose velocity field $\mathbf{v}$ is laminar, so that

$$\mathbf{w} = 0, \tag{2.10}$$

is an *irrotational motion*. An irrotational motion is characterized by the existence of a *velocity-potential* Φ :

$$\mathbf{v} = -\operatorname{grad} \Phi. \tag{2.11}$$

A motion in which $\mathbf{w} \neq 0$ is *rotational*.

A motion whose velocity field is complex-laminar is a *complex-laminar motion*. By lemma 1 it follows that in continuous complex-laminar rotational motions, and only in such motions, the vorticity and velocity are perpendicular :

$$\mathbf{w} \cdot \mathbf{v} = 0, \qquad \mathbf{w} \neq 0. \tag{2.12}$$

All plane and rotationally-symmetric motions are complex-laminar;

([7]) The several kinematical interpretations of $\mathbf{w}$ are developed in detail in [1952, 1].

since many properties of plane and rotationally-symmetric motions are shared by complex-laminar motions in general, in this memoir we shall rest content in most instances to note the specially simple forms our theorems assume for complex-laminar motions, without further specialization.

A motion whose velocity field is a Beltrami field is a *Beltrami motion*. In Beltrami motions, and only in such motions, the vorticity and velocity are parallel :

$$\mathbf{w} \times \mathbf{v} = 0, \qquad \mathbf{w} \neq 0, \tag{2.13}$$

and the particles rotate about their paths. Complex-laminar motions and Beltrami motions as here defined are mutually exclusive categories, irrotational motions being included in the former but not in the latter. From Euler's continuity equation (2.8) and lemma 2 follows immediately a result of Beltrami (8) :

LEMMA 3. — *In any steady continuous Beltrami motion, upon each stream-line the vorticity is proportional to the momentum* $\rho\mathbf{v}$:

$$\frac{w}{\rho v} = \text{const.} \tag{2.14}$$

Let $\mathbf{a}$ be the *acceleration*. Then it is easy to derive the *acceleration formula of Lagrange* (9),

$$\mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{w} \times \mathbf{v} + \mathbf{grad}\, \frac{1}{2} v^2, \tag{2.15}$$

which is the real starting point of our investigation of vorticity.

By forming the circulation $\oint \mathbf{v}.\, d\mathbf{r}$ about an arbitrary closed circuit $\mathcal{C}$ and by calculating its material derivative we obtain Beltrami's formula (10)

$$\frac{\mathrm{D}}{\mathrm{D}t} \oint_{\mathcal{C}} \mathbf{v} . d\mathbf{r} = \oint_{\mathcal{C}} \mathbf{a} . d\mathbf{r}. \tag{2.16}$$

Hence follows

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(8) [1889].

(9) [1783, § 14].

(10) [1871, § 12]. The material derivative is given by $\frac{\mathrm{D}(\ )}{\mathrm{D}t} = \frac{\partial(\ )}{\partial t} + \mathbf{v}.\, \mathbf{grad}(\ )$.

LEMMA 4. — *A continuous motion is circulation-preserving if and only if*

$$\mathbf{curl\, a} = 0, \tag{2.17}$$

or, equivalently, if and only if there exist an acceleration-potential A :

$$\mathbf{a} = -\,\mathbf{grad}\, A. \tag{2.18}$$

The distribution of vorticity, and indeed the whole dynamics of a circulation-preserving motion, is completely determined by the classical Helmholtz theorems.

By putting (2.11) into Lagrange's formula (2.15) we conclude a result of Vessiot [11] :

LEMMA 5. — *In any continuous irrotational motion there exists an acceleration-potential* A :

$$A = \left[\frac{\partial \Phi}{\partial t} - \frac{1}{2}(\mathbf{grad}\, \Phi)^2\right]. \tag{2.19}$$

Beltrami motions, as we shall see, are not circulation-preserving except in the special case when the vorticity is *steady* :

$$\frac{\partial \mathbf{w}}{\partial t} = 0. \tag{2.20}$$

We require first a result of Masotti [12] :

LEMMA 6. — *The vorticity of a continuous motion is steady if and only if the velocity be the sum of a laminar field and a steady field* :

$$\mathbf{v}(\mathbf{r}, t) = \mathbf{grad}\left[\int U(\mathbf{r}, t)\, dt\right] + \mathbf{u}(\mathbf{r}). \tag{2.21}$$

In particular, in an irrotational motion the vorticity is zero and hence steady *a fortiori*; comparison of (2.11) and (2.21) yields

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(11) [1911, § 4]. This evident result may be proved in many other ways.
(12) [1927, § 2].

LEMMA 7. — *A continuous irrotational motion satisfies* (2.21) *with*

$$U = -\frac{\partial\Phi}{\partial t} + F(t), \qquad \mathbf{u} = 0, \tag{2.22}$$

and the formula (2.19) *becomes*

$$A = -\left(U + \frac{1}{2}\mathbf{v}^2\right) + F(t). \tag{2.23}$$

Now from (2.21) follows

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{grad}[U - F(t)], \tag{2.24}$$

and hence if we put Masotti's result and (2.13) into Lagrange's formula (2.15), by lemma 4 we obtain a generalization of a theorem of Beltrami [13] :

LEMMA 8. — *A continuous Beltrami motion is circulation-preserving if and only if it be a motion with steady vorticity; the acceleration-potential is then*

$$A = -\left(U + \frac{1}{2}v^2\right) + F(t). \tag{2.25}$$

In particular, any steady continuous Beltrami motion is circulation-preserving.

The combined result of lemmas 7 and 8 is the *spatial Bernoulli theorem :*

LEMMA 9. — *In a continuous irrotational motion, or in a continuous Beltrami motion with steady vorticity, we have*

$$U + \frac{1}{2}v^2 + A = F(t); \tag{2.26}$$

In a steady motion,

$$\frac{1}{2}\mathbf{v}^2 + A = C. \tag{2.27}$$

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[13] [1889].

More generally, by putting (2.21) and (2.15) into (2.18) we obtain

$$\mathbf{v} \times \mathbf{w} = \mathbf{grad}\left(\mathrm{U} + \frac{1}{2} v^2 + \mathrm{A} \right), \tag{2.28}$$

and hence follows *Lamb's superficial Bernoulli theorem* [14] :

LEMMA 10. — *In a continuous circulation-preserving motion with steady vorticity there exist surfaces $\mathfrak{S}$ which are simultaneously stream-surfaces and vortex-surfaces, and*

$$\mathrm{U} + \frac{1}{2} v^2 + \mathrm{A} = \mathrm{F}(\mathfrak{S}, t); \tag{2.29}$$

In a steady motion,

$$\frac{1}{2} v^2 + \mathrm{A} = \mathrm{F}(\mathfrak{S}). \tag{2.30}$$

Conversely, if in a continuous circulation-preserving motion there exist a scalar function U *such that* (2.28) *holds* (*and thus* a fortiori *there exist surfaces* $\mathfrak{S}$ *which are simultaneously stream-surfaces and vortex-surfaces*), *then the motion is a motion with steady vorticity, and if further* $\mathrm{U} = \text{const.}$, *then the motion is steady.*

The surfaces $\mathfrak{S}$ are called *Bernoullian surfaces*.

By forming the curl of Lagrange's equation (2.15) we obtain the *kinematical vorticity equation of Lagrange and Beltrami* [15] :

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\mathbf{w}}{\rho}\right) = \frac{1}{\rho}\,\mathbf{curl}\,\mathbf{a} + \frac{\mathbf{w}}{\rho}\cdot\mathbf{grad}\,\mathbf{v}. \tag{2.31}$$

In application to motions of fluids the term *flow* may replace *motion* in the foregoing definitions and lemmas.

3. — **Inviscid fluids. Kelvin's criterion.** — In this memoir we shall deal only with perfect or *inviscid* fluids, which may be defined

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(14) The theorem actually stated by Lamb [1878] concerns steady motion only, and is phrased in dynamical terms.

(15) The special case when $\mathbf{curl}\,\mathbf{a} = 0$ is given in [1762, chap. XLII]; the still more special case when also $\operatorname{div} \mathbf{v} = 0$ is often called "Helmholtz's equation". The general formula is given in [1871, § 6].

as continuous media satisfying *Euler's dynamical equation* :

$$\mathbf{a} = -\frac{1}{\rho}\,\mathbf{grad}\,p + \mathbf{f}, \tag{3.1}$$

where p is the *pressure* and $\mathbf{f}$ is the *extraneous force* per unit mass. When the extraneous force is laminar,

$$\mathbf{f} = -\,\mathbf{grad}\,\upsilon, \tag{3.2}$$

it is said to be *conservative* ([16]).

For a fluid subject to conservative extraneous force we have then

$$\mathbf{curl\,a} = \mathbf{grad}\,p \times \mathbf{grad}\,\frac{1}{\rho}. \tag{3.3}$$

By lemma 4 of § 2 we now conclude that in order for a continuous motion to be circulation-preserving it is necessary and sufficient that there be a relation of the form

$$f(p, \rho, t) = 0; \tag{3.4}$$

that is, either

$$p = p(t), \tag{3.5}$$

in which case the flow is *instantaneously isobaric*, or

$$\rho = \rho(t), \tag{3.6}$$

in which case the flow is *instantaneously isostatic*, or else

$$p = p(\rho, t), \qquad \rho = \rho(p, t), \tag{3.7}$$

i.e. at each instant the surfaces $p =$ const. coincide with the surfaces $\rho =$ const., in which case the flow is *instantaneously barotropic*. Flows not satisfying any of the three conditions (3.5), (3.6), (3.7) are *baroclinic*.

([16]) In mass point dynamics it is customary to require that a force system be steady as well as laminar before the term "conservative" is applied to it. The weaker requirement (3.2) is sufficient for the validity of the curvilinear energy theorems of gas dynamics, although $\frac{\partial \upsilon}{\partial t} = 0$ is necessary for the conservation of *total* energy in barotropic or isochoric motions

If (3.5) reduce to

$$p = \text{const.}, \tag{3.8}$$

the flow is *isobaric*; if (3.6) reduce to

$$\rho = \text{const.}, \tag{3.9}$$

the flow is *isostatic* ([17]); and if (3.7) reduce to

$$p = p(\rho), \qquad \rho = \rho(p), \tag{3.10}$$

the flow is *barotropic* ([18]). Expressing our conclusion (3.4) in the terminology just introduced, we have *Kelvin's criterion* ([19]) : *a continuous flow of an inviscid fluid subject to conservative extraneous force is circulation-preserving if and only if it be locally instantaneously isostatic, isobaric, or barotropic flow.*

By applying lemmas 5 and 8 of § 2, from Kelvin's criterion we conclude that for an inviscid fluid subject to conservative extraneous force a continuous irrotational flow or a Beltrami flow with steady vorticity must be locally instantaneously isochoric, isostatic, or barotropic. From (3.1), (3.2) and (2.18) follows

$$A = \int_0^p \frac{dp}{\rho} + \upsilon, \tag{3.11}$$

and this expression may be put into lemma 9. These results are summarized in ([20])

THEOREM I. — (Basic theorem on irrotational and Beltrami flows.) — *If a continuous flow of an inviscid fluid subject to conservative*

([17]) Instantaneously isostatic flows are to be distinguished from *isochoric* flows, in which $\rho = \text{const.}$ *for each particle*, i.e. $\frac{D\rho}{Dt} = 0$. An isostatic flow is also isochoric, but the converse is false. A flow of an inhomogeneous incompressible fluid is always isochoric, but generally not isostatic nor instantaneously isostatic. A spherically symmetrical oscillation of a gas may be instantaneously isostatic but not isochoric.

([18]) [1933, p. 84 86].

([19]) The actual statement of Kelvin [1869, § 59 (*d*)] is confined to the sufficiency of (3.9) or (3.10).

([20]) The portion of this theorem concerning irrotational flows is common knowledge. A special case of the portion concerning Beltrami flows is given by Neményi and Prim [1949, 4, § 4].

extraneous force be an irrotational *flow or a* Beltrami *flow with* steady vorticity, *then it is locally and instantaneously* isobaric, isostatic, *or* barotropic *flow*, *and the* spatial Bernouilli theorem

$$U + \frac{1}{2} v^2 + \int_0^p \frac{dp}{\rho} + \upsilon = F(t) \tag{3.12}$$

is valid, *where* (3.4) *is to be used in carrying out the quadrature.* All the theorems of this memoir are merely local, and we make no attempt to characterize flows in the large : in the present instance, for example, the flow may well be isobaric in one portion, isostatic in another, and barotropic in a third.

The case of steady motion subject to no extraneous force deserves special attention. Bernoulli's theorem (3.12) now becomes

$$\frac{1}{2} v^2 + \int_0^p \frac{dp}{\rho} = C. \tag{3.13}$$

For an isobaric flow $dp = 0$ and (3.13) shows that the speed v is constant throughout the isobaric region. Neményi and Prim ([21]) have shown that a laminar or Beltrami field of constant magnitude is necessarily a field whose vector-lines are straight. Suppose next that the motion be isostatic or barotropic, and in the case of barotropic motion assume that the function $p = p(\rho)$ be such that $\frac{dp}{d\rho} \geqq 0$, a natural requirement suggested by the physics of the situation. Then (3.13) demonstrates the existence of a *maximum possible speed* or *ultimate speed* v_0 and a definite *stagnation pressure* p_0 attained (if at all) at a stagnation point ($\mathbf{v} = 0$). Both these quantities are *constants of the motion*; they are related by

$$\frac{1}{2} v^2 + \int_0^p \frac{dp}{\rho} = \frac{1}{2} v_0^2 = \int_0^{p_0} \frac{dp}{\rho}. \tag{3.14}$$

Summarizing these results, we have

THEOREM 2. — *In a* steady *continuous* irrotational *or* Beltrami *flow of an inviscid fluid subject to* no extraneous force, *one of the following two conditions prevails locally* :

([21]) [1949, 4, th. 2].

a. The flow is isobaric, the stream-lines are straight, *and* the speed is uniform.

b. The flow is isostatic *or* barotropic, *and is possessed of a definite* ultimate speed v_0 *and* stagnation pressure p_0, *these quantities being related by Bernoulli's theorem* (3.14).

Flows not satisfying the conditions of theorem 2 are not generally possessed of an ultimate speed and a stagnation pressure in this sense. Later (§ 7) we shall see that in an important special case such quantities do indeed exist for a class of rotational flows, but are no longer constants of the flow, being liable to different values upon the different stream-lines.

In any case, we may put (3.3) into the Lagrange-Beltrami equation (2.31), obtaining the *dynamical vorticity equation of Silberstein* (22) :

$$\frac{D}{Dt}\left(\frac{\mathbf{w}}{\rho}\right) = \mathbf{grad\,p} \times \mathbf{grad}\,\frac{1}{\rho} + \frac{\mathbf{w}}{\rho}\cdot\mathbf{grad\,v}, \tag{3.15}$$

whence a portion of theorem 1 is again apparent.

The results of this section are purely dynamical, and are the only such results in this memoir. Without the aid of thermodynamics we can make no further progress in our subject.

4. Thermodynamical assumptions. Classification of fluids. — The basic postulate of Gibbs's (23) theory of equilibrium is the existence of an *equation of state*

$$\mathfrak{E} = f(\mathfrak{V}, \mathfrak{H}, C_1, C_2, \ldots, C_k), \tag{4.1}$$

where $\mathfrak{E}$ is the *total internal energy*, $\mathfrak{V}$ is the *volume*, $\mathfrak{H}$ is the *entropy*, and C_i is the *concentration of the substance i*. This postulate is intended to describe only systems in which at each instant all conditions are the same at every point. Gas dynamics is characterized by introducing *local* thermodynamic variables — the *specific internal energy* ε, the *specific entropy* η, and the *specific*

(22) [1896].
(23) [1875, p. 63].

concentration c_i *of the substance* i, which are assumed to be connected by a *local equation of state*

$$(4.2)\qquad \varepsilon = f(\rho, \eta, c_1, c_2, \ldots, c_k);$$

whose form is unaffected by whatever motion may take place. The local variables are related to the system variables by

$$\mathfrak{E} = \int_{\mathcal{V}} \rho\varepsilon\, d\mathfrak{V}, \qquad \mathfrak{H} = \int_{\mathcal{V}} \rho\eta\, d\mathfrak{V},$$

but in general no system equation of state (4.1) exists. Underlying the first basic assumption of our subject is then the premise that in principle *the specific internal energy at a point can be determined by accumulating a certain amount of statical information, regardless of the state of motion or deformation*; in particular, that the internal energy is completely determined by the relative amount of each substance present and *one new parameter* η, called the specific entropy. The substances i are simply any components of the fluid which it seems desirable to distinguish in a particular problem; they may be different phases of the same chemical compound, ions in solution, atoms excited at different energy levels, etc.

If but one substance be present, so that

$$(4.3)\qquad \varepsilon = f(\rho, \eta),$$

the fluid is *homogeneous*. If

$$(4.4)\qquad \varepsilon = f(\eta), \qquad \rho = \text{const.},$$

the fluid is *homogeneous* and *incompressible* (24). If

$$\rho = \rho(c_1, c_2, \ldots, c_k),$$

the fluid is simply *incompressible*. Fluids which are not incompressible are *compressible*.

For the basic postulate (4.2) there is no direct experimental evidence. Like most of the "laws" of mathematical physics it is a

(24) An incompressible *fluid* must not be confused with an isochoric or isostatic *flow* (§ 3). An incompressible fluid is insusceptible of a change in density, and thus *every* possible flow is isochoric; if it be homogeneous as well, *every* possible flow is isostatic. A highly compressible fluid, however, may happen to experience an isochoric flow, as in a simple vortex or Couette or Poiseuille flow.

pure hypothesis of a mathematical nature, very difficult, nay in the nature of things near to impossible of direct and independent experimental test, but clear enough for the imagination and sufficient (along with other assumptions) to predict with fair accuracy a large class of physical phenomena. The strongest realistic statement to be made in favor of (4.2) is that an assumption equivalent to a special case of it is always made in gas dymamics, and that from this discipline has never yet been derived a result in contradiction with physical experience in a situation to which it could reasonably be applied.

Since $\sum c_i = 1$, one of the concentrations, say c_k, may be eliminated, and (4.2) represents a $k+1$-dimensional *energy surface* in the $k+2$-dimensional space of the variables ε, ρ, η, c_1, c_2, ..., c_{k-1}. The basic postulate (4.2) may thus be expressed in geometrical form : *the path of any "particle" P in the mean motion of a given fluid in the physical space is mapped by the equations*

$$\varepsilon = \varepsilon(\mathrm{P}, t), \qquad \eta = \eta(\mathrm{P}, t) \qquad c_i = c_i(\mathrm{P}, t)$$

onto some curve on the energy surface for that fluid.

The *temperature* θ, the thermodynamic *pressure* π, and the *potential* μ_i *of the substance i* are defined by

$$\theta \equiv \frac{\partial \varepsilon}{\partial \eta}, \qquad \pi \equiv -\frac{\partial \varepsilon}{\partial \left(\frac{1}{\rho}\right)}, \qquad \mu_i \equiv \frac{\partial \varepsilon}{\partial c_i}. \tag{4.5}$$

In the case of an incompressible substance, the energy surface degenerates by at least one dimension and the above definition of π becomes meaningless. Now the static pressure p has appeared already in Euler's equation (3.1). Characteristic of the discipline of gas dynamics is the further *postulate* that *the thermodynamic pressure is equal to the static pressure* [25].

$$\pi = p. \tag{4.6}$$

[25] That (4.6) is indeed an independent postulate (though rarely mentioned) follows from the difference of the two concepts which p and π represent. The former quantity is a scalar field such that for any imagined closed boundary surface $\mathfrak{S}$ within the fluid, whatever the state of motion, the surface integrals

$$\oint_{\mathfrak{S}} d\mathbf{s}\, p \quad \text{and} \quad \oint_{\mathfrak{S}} d\mathbf{s} \times \mathbf{r} p.$$

In virtue of this postulate the separate symbol π need not be retained and p alone is employed henceforth.

Now in general from $(4.5)_2$ for a compressible fluid we obtain.

$$p = p(\rho, \eta, c_1, c_2, \ldots, c_k). \tag{4.7}$$

The special case in which $\dfrac{\partial p}{\partial \eta} = 0$, $\dfrac{\partial \rho}{\partial p} \neq 0$, so that

$$p = p(\rho, c_1, c_2, \ldots, c_k), \tag{4.8}$$

is called a *piezotropic* fluid ([26]). By Kelvin's criterion (§ 3), all continuous flows of homogeneous inviscid incompressible or piezotropic fluids subject to conservative extraneous force are circulation-preserving. Since such flows form the subject of classical hydrodynamics, in this memoir we have no interest in them *per se*, and thus in general we shall not draw attention to the usually trivial consequences of our theorems which result for fluids *all* of whose possible flows are circulation-preserving. Rather, we shall be interested in characterizing thermodynamically those circulation-preserving flows, especially irrotational flows, which can occur in fluids whose motions in general are not circulation-preserving, as well as in investigating the distribution of vorticity in gas flows in general. We tarry only to notice that for a piezotropic fluid

$$\frac{\partial p}{\partial \eta} = - \frac{\partial^2 \varepsilon}{\partial \eta \, \partial \left(\dfrac{1}{\rho}\right)} = 0, \tag{4.9}$$

and hence

$$\left\{ \begin{aligned} \varepsilon &= \varepsilon_\eta(\eta, c_1, c_2, \ldots c_k) + \varepsilon_\rho(\rho, c_1, c_2, \ldots, c_k), \\ p &= -\frac{\partial \varepsilon_\rho}{\partial \left(\dfrac{1}{\rho}\right)}, \qquad \theta = \frac{\partial \varepsilon}{\partial \eta}; \end{aligned} \right. \tag{4.10}$$

---

are mechanically equivalent respectively to the resultant force and resultant moment exerted upon the fluid inside $\mathcal{S}$ by all the fluid outside $\mathcal{S}$. The latter quantity is the slope of the curve of intersection of the energy surface of the fluid with a $-\rho^{-1} =$ const. plane, taken at a point on the energy surface where ρ, η, and the c_i have appropriate values. This matter is discussed from a more general standpoint in [1952, 2, § 30, § 61, § 61 A].

([26]) [1933, p. 84-86]. A piezotropic *fluid* is not to be confused with a barotropic *flow* (§ 3). Piezotropy is a physical property of a substance, while barotropy is a geometrical property of a particular motion. While indeed every flow of a homogeneous piezotropic fluid is barotropic, the converse statement is false, and also usually a flow of a heterogeneous piezotropic fluid fails to be barotropic, and in general the two concepts are unrelated.

that is, the energy ε may be decomposed into two portions, a thermal energy ε_η depending only upon the entropy and the concentrations, and a volumetric energy ε_ρ depending only upon the density and the concentrations.

If except possibly at certain singular points or curves on the energy surface we have

$$(4.11) \qquad \frac{\partial p}{\partial \eta} \neq 0, \qquad \frac{\partial p}{\partial \rho} \neq 0, \qquad \frac{\partial \theta}{\partial \eta} \neq 0, \qquad \frac{\partial \theta}{\partial \rho} \neq 0,$$

we shall call the fluid *tri-variate* ([27]). In accordance with the remarks of the preceding paragraph, this memoir treats almost exclusively of tri-variate fluids. For a homogeneous tri-variate fluid we have non-degenerate equations of state connecting *any three* ([28]) of the thermodynamic variables η, θ, p, ρ :

$$(4.12) \qquad p = p(\rho, \eta), \qquad \eta = \eta(p, \theta), \qquad \theta = \theta(p, \eta),$$

as well as many others.

For any compressible fluid, by differentiating (4.2) along any curve on the energy surface we obtain

$$(4.13) \qquad d\varepsilon = \theta\, d\eta - p\, d\left(\frac{1}{\rho}\right) + \Sigma \mu^i\, dc_i;$$

in particular, if this curve be the image of the actual motion of some particle P in the physical space, we have

$$(4.14) \qquad \frac{D\varepsilon}{Dt} = \theta \frac{D\eta}{Dt} - p \frac{D\left(\frac{1}{\rho}\right)}{Dt} + \Sigma \mu^i \frac{Dc_i}{Dt}.$$

If throughout a particular motion

$$(4.15) \qquad \frac{Dc_i}{Dt} = 0 \qquad (i = 1, 2, \ldots, k),$$

([27]) Only three of these conditions are independent, in view of the reciprocity relation $\frac{\partial p}{\partial \eta} = -\frac{\partial \theta}{\partial\left(\frac{1}{\rho}\right)}$, which follows from (4.5).

([28]) Some of our theorems on tri-variate fluids require only $p = p(\rho, \eta)$ with $\frac{\partial p}{\partial \eta} \neq 0$, but we adopt the stronger restrictions (4.11) because they lead to somewhat more definite results in some cases and are satisfied by all equations of state proposed for gases.

then the fluid behaves in that motion as an *inert mixture*, the concentrations at each material point remaining constant as that point is carried through the motion. Molecular diffusion, chemical changes, phase changes, etc., do not take place. Mixtures of sea water and fresh water or of air and water vapor in many oceanographical and meteorological investigations are regarded as inert in this sense. A number of theorems of this memoir concern heterogeneous fluids in motion as inert mixtures, but there is no attempt to treat more general types of heterogeneity.

For a homogeneous fluid we have

$$d\varepsilon = \theta\, d\eta - p\, d\left(\frac{1}{\rho}\right) \tag{4.16}$$

for *any* path on the energy surface; hence, in particular,

$$\mathbf{grad}\,\varepsilon = \theta\, \mathbf{grad}\,\eta - p\, \mathbf{grad}\frac{1}{\rho}. \tag{4.17}$$

For a heterogeneous fluid in motion as an inert mixture, neither (4.16) nor (4.17) is generally valid, but nevertheless by (4.14) and (4.15) we have

$$\frac{D\varepsilon}{Dt} = \theta \frac{D\eta}{Dt} - p \frac{D\left(\frac{1}{\rho}\right)}{Dt}. \tag{4.18}$$

The *enthalpy* h is defined by [29]

$$h = \varepsilon + \frac{p}{\rho}. \tag{4.19}$$

Then it is a consequence of (4.17) that for homogeneous fluids we have

$$\theta\, \mathbf{grad}\,\eta = \mathbf{grad}\, h - \frac{1}{\rho}\, \mathbf{grad}\, p. \tag{4.20}$$

5. Homogeneous tri-variate fluids. The Crocco-Vazsonyi relation. By eliminating $\mathbf{grad}\, p$ between (4.20) and Euler's dynamical equation (3.1) we obtain

$$\theta\, \mathbf{grad}\,\eta = \mathbf{grad}\, h + \mathbf{a} - \mathbf{f}; \tag{5.1}$$

[29] For incompressible fluids the thermodynamic pressure is not defined, as we have noted already. The enthalpy h is still to be defined by (4.19), however, with p to be taken as the static pressure which appears in Euler's dynamical equation (3.1).

hence, **f** being supposed conservative,

$$\mathbf{curl\,a} = \mathbf{grad}\,\theta \times \mathbf{grad}\,\eta. \tag{5.2}$$

By Beltrami's criterion (2.17) we now conclude that in order for the motion to be circulation-preserving it is necessary and sufficient that there be a relation of the form

$$f(\theta, \eta, t) = 0, \tag{5.3}$$

as indeed follows equally well from Kelvin's criterion (§ 3) and the various equations of state. The special case

$$\theta = \theta(t) \tag{5.4}$$

is *instantaneously isothermal* flow, while the special case

$$\eta = \eta(t) \tag{5.5}$$

is *instantaneously isentropic* flow. If (5.4) reduce to

$$\theta = \text{const.} \tag{5.6}$$

the flow is *isothermal*, while if (5.5) reduce to

$$\eta = \text{const.} \tag{5.7}$$

the flow is *isentropic* [30]. For a homogeneous fluid any flow in which (5.3) holds is necessarily an isobaric, isostatic, or barotropic flow, and hence is circulation-preserving, and conversely, and thus we have a complementary result concerning the entropy and temperature which is analogous to theorem 1.

Now for a homogeneous incompressible or piezotropic fluid this last result and that of theorem 1 are mere trivialities. For a trivariate fluid, however, most motions are baroclinic, and the analysis yields a thermodynamical characterization of irrotational and Beltrami flows. Recalling that we have equations of state of all the types

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[30] Current usage of this term varies. In this memoir it is applied only to flows of *uniform entropy*, for which the value of η is constant throughout a three-dimensional region, not merely upon a curve or surface. The term *adiabatic* should not be given a local significance, but should be retained in its original sense as applicable to a process taking place within boundaries through which there is no flux of energy.

(4.12) at our disposal, we may consider in turn each possible combination of types of relations

$$f(p, \rho, t) = 0 \quad \text{and} \quad g(\theta, \eta, t) = 0.$$

1° Suppose $p = p(t)$; then by (4.12) if $\theta = \theta(t)$ it follows that $\eta = \eta(t)$, and conversely;

2° If $p = p(t)$, it is possible that $\theta = \theta(\eta, t) \neq \theta(t)$. Then by (4.12) we obtain $\rho = \rho(\eta, t) \neq \rho(t)$;

3° If any one of ρ, θ, η be a function of time only, while the other two be not functions of time only, a parallel argument yields a functional relation connecting the other two and further functional relations connecting either of these with p;

4° Finally, the motion can be barotropic, and simultaneously

$$\theta = \theta(\eta, t) \neq \theta(t).$$

Summarizing these results, we obtain

THEOREM 3. — *If a continuous flow of a* homogeneous *inviscid* tri-variate *fluid subject to conservative extraneous force be an* irrotational *flow or a* Beltrami *flow with* steady vorticity, *then locally the four state variables p, ρ, θ, η are connected in one of the following ways :*

a. All four are functions of time only, *or;*

b. One is a function of time only, *and* the surfaces upon which the other three are constant coincide at each instant, *or ;*

c. At each instant the surfaces of constant density coincide with the surfaces of constant pressure (*instantaneously barotropic flow*), *and* the surfaces of constant temperature coincide with the surfaces of constant entropy.

Returning to the consideration of rotational flows in general, we may put (5.2) into (2.31), obtaining the *vorticity equation of Vazsonyi* (31) ;

$$\frac{D}{Dt}\left(\frac{\mathbf{w}}{\rho}\right) = \frac{1}{\rho}\,\mathbf{grad}\,\theta \times \mathbf{grad}\,\eta + \frac{\mathbf{w}}{\rho}\cdot\mathbf{grad}\,\mathbf{v}, \tag{5.8}$$

whence a portion of theorem 3 is again immediately apparent.

(31) [1945, 1, eq. (5.2)].

We now introduce the *total enthalpy* h_t

$$h_t \equiv h + \frac{1}{2}v^2 = \varepsilon + \frac{p}{\rho} + \frac{1}{2}v^2, \tag{5.9}$$

a variable of particular significance in gas dynamics. From the acceleration formula (2.15) of Lagrange we may then put the dynamical equation (5.1) into the form

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{w} \times \mathbf{v} = \theta\, \mathbf{grad}\, \eta - \mathbf{grad}\, h_t + \mathbf{f}. \tag{5.10}$$

Hence follows

THEOREM 4. — *In any* steady *continuous flow of a* homogeneous *inviscid fluid subject to* no extraneous force, *the vorticity and the thermodynamic variables are connected by the* Crocco-Vazsonyi [32] relation

$$\mathbf{w} \times \mathbf{v} = \theta\, \mathbf{grad}\, \eta - \mathbf{grad}\, h_t. \tag{5.11}$$

Theorem 4 shows that in general an isentropic flow fails to be a Beltrami flow or an irrotational flow. We have, however,

THEOREM 5. — *A* steady *continuous flow of a* homogeneous *inviscid fluid subject to* no extraneous force *which is both* isentropic *and of* constant total enthalpy *is necessarily either a* Beltrami *flow or an* irrotational *flow. In particular, a* complex-laminar *flow satisfying these conditions is always an* irrotational *flow. Conversely, in a* steady *continuous* irrotational *or* Beltrami *flow of a* homogeneous *inviscid fluid subject to no extraneous force*

$$\theta\, \mathbf{grad}\, \eta = \mathbf{grad}\, h_t. \tag{5.12}$$

In particular, a steady *continuous* irrotational *or* Beltrami *flow of a* homogeneous *inviscid fluid subject to* no extraneous force *which is* isentropic *is also a flow of* constant total enthalpy, *and* conversely.

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(32) Crocco [1936, eq. (1)] gave the special case $h_t =$ const for perfect gases. A shorter proof was given by Tollmien [1942] and the restriction to perfect gases was removed by Oswatitsch [1943]. The general result is due to Vazsonyi [1945, 1, eq. (6.1)], who notes also a generalization to arbitrary homogeneous fluids [eq. (M'')].

6. The energy equation of C. Neumann, and its consequences for flows of perfect fluids devoid of heat flux. — For any homogeneous continuous medium, or for a heterogeneous medium in motion as an inert mixture, the *conservation of energy* ([33]) is expressed by the equation of C. Neumann ([34]).

$$\rho \frac{D\varepsilon}{Dt} = \mathbf{T} : \mathbf{\Delta} - \operatorname{div} \mathbf{q}, \tag{6.1}$$

where $\mathbf{T}$ is the symmetric *stress dyadic*, $\mathbf{\Delta}$ is the *rate of deformation* ($2\mathbf{\Delta} \equiv \mathbf{grad\,v} + (\mathbf{grad\,v})_c$), and $\mathbf{q}$ is the *heat flux vector*. From (4.17) it follows that

$$\rho\theta \frac{D\eta}{Dt} = \mathbf{W} : \mathbf{\Delta} - \operatorname{div} \mathbf{q}, \tag{6.2}$$

where $\mathbf{W}$ is the stress in excess of the pressure :

$$\mathbf{W} \equiv p\mathbf{I} + \mathbf{T}.$$

Euler's dynamical equation (3.1) is equivalent to the statement that $\mathbf{W} = 0$ for inviscid fluids, and thus the appropriate energy equation is

$$\rho\theta \frac{D\eta}{Dt} = -\operatorname{div} \mathbf{q}. \tag{6.3}$$

In this memoir we shall not have occasion to specialize the form of $\mathbf{q}$, noting simply that if the heat flux arise solely from thermal conduction then Fourier's law gives $\mathbf{q} = -\varkappa\, \mathbf{grad}\,\theta$. In most pro-

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([33]) For the simple media in equilibrium which are considered in classical thermodynamics, this equation reduces to a form equivalent to (4.1). In a medium suffering deformation, however, (4.1) is not valid and the two equations (6.1) and (4.13) express *different* and *independent* assumptions : the former, that mechanical and thermal energy are interconvertible; the latter, the existence of an energy surface characterizing the fluid. The misleading terminology ("First law", "second law" etc.) of thermodynamics is avoided in this memoir.

([34]) [1894, § 4]. For the special case of an inviscid incompressible fluid the energy equation was given by Fourier [1833, eq. (3)]; for small motions of a viscous ideal gas, by Kirchhoff [1868, § 1]. Several authors have proposed extensions of the energy equation to heterogeneous media, but their results are not in agreement; it seems evident, however, that for an inert mixture the energy equation should not be different in form than for a homogeneous fluid, *cf.* [1940, eq. (19)].

blems of gas dynamics it is assumed that $\mathbf{q} = 0$, so that (6.3) reduces to

$$\frac{D\eta}{Dt} = 0, \tag{6.4}$$

and hence we have

THEOREM 6. (Basic energy theorem). — *If an inviscid fluid be in continuous motion as an inert mixture* devoid of heat flux, *then* the entropy of each particle remains constant. *In particular, if the motion be* steady, *then* the entropy is constant along each streamline.

The condition (6.4) is to be contrasted with (5.6) : in general, these motions are *not* isentropic. The isentropic possibility is expressed in the evident

THEOREM 7. — *In an inviscid fluid in condtinuous motion as an inert mixture* devoid of heat flux, *if there exist a certain* isentropic surface [35] *which is* touched by every particle *at some time, then the flow is* isentropic. *In particular, if in a* steady *flow under these hypotheses there exist* one isentropic surface which is touched by every stream-line, *then the flow is* isentropic.

The principal condition of the theorem is typically illustrated by a flow which may be regarded as originating in a « reservoir » at infinity in which the entropy is uniform. Notice also the requirement of continuity : if a surface of constant entropy of the type specified may be found, it follows (subject, of course, to the remaining conditions of the theorem) that the flow is isentropic up to the first shock front encountered by the particles, after which it may well fail of the isentropic property.

In the special case of irrotational or Beltrami flows we may obtain further information about the thermodynamic state, as expressed in

THEOREM 8. (Characterization of irrotational and Beltrami flow) — *If an inviscid* tri-variate *fluid be in continuous* irrotational *flow or* Beltrami *flow with* steady vorticity, *and if the flow be that of*

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[35] This isentropic surface $\eta = \eta_0$ need not be a steady surface, but η_0 must not vary with time.

an inert mixture devoid of heat flux, *then locally either the flow enjoys one or both of the following properties :*

a. All state variables are constant for each particle :

$$\frac{D\eta}{Dt} = 0, \qquad \frac{Dp}{Dt} = 0, \qquad \frac{D\rho}{Dt} = 0, \qquad \frac{D\theta}{Dt} = 0, \qquad \ldots, \tag{6.5}$$

b. The entropy at any point is a function of the concentrations alone :

$$\eta = \eta(c_1, c_2, \ldots, c_k), \tag{6.6}$$

or else

c. The flow is instantaneously isobaric, instantaneously isostatic, *or* instantaneously barotropic, *but not isobaric, isostatic, nor barotropic* (i. e., *a relation of type* (3.4) *with* $\frac{\partial f}{\partial t} \neq 0$ *holds*).

Proof. — By theorem 1, we have a relation of the form $f(p, \rho, t) = 0$. If $\frac{\partial f}{\partial t} \neq 0$, case *c* of the present theorem follows. Suppose henceforth that $\frac{\partial f}{\partial t} \neq 0$, so that the flow is isobaric, isostatic, or barotropic. For a tri-variate fluid we have

$$p = p(\rho, \eta, c_1, c_2, \ldots, c_k), \tag{6.7}$$

and $\frac{\partial p}{\partial \rho} \neq 0$, $\frac{\partial p}{\partial \eta} \neq 0$ except possibly for certain isolated values of the variables. When the motion is that of an inert mixture (§ 4), we have $\frac{Dc_i}{Dt} = 0$, and when there is no heat flux we obtain $\frac{D\eta}{Dt} = 0$ by theorem 6. Differentiating (6.7) yields then

$$\frac{Dp}{Dt} = \frac{\partial p}{\partial \rho}\frac{D\rho}{Dt} + \frac{\partial p}{\partial \eta}\frac{D\eta}{Dt} + \sum_i \frac{\partial p}{\partial c_i}\frac{Dc_i}{Dt} = \frac{\partial p}{\partial \rho}\frac{D\rho}{Dt}. \tag{6.8}$$

For an isobaric flow, (6.7) yields $\frac{D\rho}{Dt} = 0$, and hence by the tri-variate character of the fluid case *a* of the theorem we are proving then follows; similarly, an isostatic motion also yields case *a*. If the motion be barotropic, we have $p = p(\rho) \neq$ const., and hence if

$\frac{D\rho}{Dt} = 0$ case a follows again. Finally, suppose $\frac{D\rho}{Dt} \neq 0$. From the tri-variate character of the fluid we have

$$\eta = \eta(p, \rho, c_1, c_2, \ldots, c_k) \tag{6.9}$$

as an equation of state, and hence in the present motion

$$\eta = \eta[p(\rho), \rho, c_1, c_2, \ldots, c_k], \tag{6.10}$$

whence

$$\eta = f(\rho, c_1, c_2, \ldots, c_k). \tag{6.10 bis}$$

Hence

$$\frac{D\eta}{Dt} = \frac{\partial f}{\partial \rho} \frac{D\rho}{Dt}. \tag{6.11}$$

By theorem 6, the left side of this equation is zero, and by hypothesis $\frac{D\rho}{Dt} \neq 0$; hence

$$\frac{\partial f}{\partial \rho} = 0, \tag{6.12}$$

and thus (6.10 *bis*) reduces to

$$\eta = g(c_1, c_2, \ldots, c_k), \tag{6.13}$$

which is case b of the theorem to be proved. Q.E.D. Notice that in steady flow case c is impossible.

Writing (6.11) in terms of the equation of state (6.10) we obtain

$$\frac{\partial \eta}{\partial p} \frac{dp}{d\rho} + \frac{\partial \eta}{\partial \rho} = 0, \tag{6.14}$$

where $\frac{dp}{d\rho}$ is to be calculated from the barotropic relation $p = p(\rho)$ which holds in the particular motion. Hence

$$c^2 \equiv \frac{dp}{d\rho} = -\frac{\dfrac{\partial \eta}{\partial \rho}}{\dfrac{\partial \eta}{\partial p}} = \left(\frac{\partial p}{\partial \rho}\right)_{\eta, c_1, c_2, \ldots, c_k}; \tag{6.15}$$

that is, the « speed of sound » c for the inert mixture in irrotational flow devoid of heat flux is given by the ordinary formula valid for a homogeneous fluid in the same circumstances.

The special case of theorem 8 resulting when the fluid is homogeneous is important enough to be written out as

THEOREM 9 (Characterization of irrotational and Beltrami flow of homogeneous fluids). — *If a* homogeneous *inviscid* tri-variate *fluid be in continuous* irrotational *flow or* Beltrami *flow with* steady vorticity, *and if the flow be* devoid of heat flux, *then locally either the flow enjoys one or both of the following properties :*

a. All state variables are constant for each particle, *or*

b. The flow is isentropic,

or else

c. The flow is instantaneously isobaric, instantaneously isostatic, *or* instantaneously barotropic, *but* not *isobaric*, *isostatic*, *or barotropic*.

The conclusions of theorems 8 and 9 no longer hold in flows when there is thermal flux, and then indeed it becomes difficult to characterize even irrotational flows in thermodynamic terms. In particular, the relation (6.15) is no longer satisfied, as is revealed by the following generalization of an analysis of Hicks ([36]). In a barotropic flow let c^2 be given by $(6.15)_1$, so that c is the local speed of propagation of discontinuities in the velocity gradient, irrespective of the thermodynamical properties of the medium ([37]). Let c_η^2 be defined by

$$c_\eta^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_{\eta,\ c_1,\ c_2,\ \ldots,\ c_k} \tag{6.16}$$

Then for barotropic flow as an inert mixture the energy equation (6.3) yields

$$\rho\theta\left[\left(\frac{\partial \eta}{\partial p}\right)_\rho \frac{dp}{d\rho} + \left(\frac{\partial \eta}{\partial \rho}\right)_p\right]\frac{D\rho}{Dt} = -\operatorname{div}\mathbf{q}, \tag{6.17}$$

or

$$\rho\theta\left(\frac{\partial \eta}{\partial p}\right)_\rho [c^2 - c_\eta^2]\frac{D\rho}{Dt} = -\operatorname{div}\mathbf{q}. \tag{6.18}$$

([36]) [1948, 3, § 4].

([37]) The proof of this fact given by Hugoniot [1885], [1888] is valid in any motion of an inviscid fluid such that $\rho = F(p)$, where the functional form of F may vary from one particle to another. *Cf.* [1951, 2].

Let $\mathbf{s}$ be the field of unit tangents to the stream-lines : $\mathbf{s} \equiv \frac{\mathbf{v}}{v}$. Then in a steady flow (6.18) becomes

(6.19) $$[c^2 - c_\eta^2]\,\mathbf{s}.\,\mathbf{grad}\log\rho = Q,$$

where

(6.20) $$Q \equiv \frac{-\operatorname{div}\mathbf{q}}{\rho^2 v\theta\left(\frac{\partial\eta}{\partial p}\right)_\rho}.$$

Now for steady flow Euler's continuity equation (2.8) may be put into the form

(6.21) $$-\operatorname{div}\mathbf{s} = \mathbf{s}.(\mathbf{grad}\log v + \mathbf{grad}\log\rho).$$

For steady irrotational or Beltrami flow Euler's dynamical equation (3.1) becomes

(6.22) $$\mathbf{grad}\,\frac{1}{2}v^2 = -c^2\,\mathbf{grad}\log\rho + \mathbf{f},$$

whence from (6.21) follows

(6.23) $$-\operatorname{div}\mathbf{s} = \mathbf{s}.\left[\left(1 - \frac{c^2}{v^2}\right)\mathbf{grad}\log\rho + \frac{\mathbf{f}}{v^2}\right].$$

We may now eliminate $\mathbf{s}.\mathbf{grad}\log\rho$ betwen (6.19) and (6.23), obtaining a result which when expressed in terms of the *local Mach number* ([38]) M,

(6.24) $$M \equiv \frac{v}{c_\eta} = \frac{v}{\sqrt{\left(\frac{\partial p}{\partial\rho}\right)_{\eta,\, c_1,\, c_2,\, \ldots,\, c_k}}},$$

becomes

(6.25) $$c^2 = c_\eta^2\left[1 + \frac{Q(1-M^2)}{M^2 c_\eta^2 \operatorname{div}\mathbf{s} + \mathbf{s.f} - Q}\right].$$

([38]) Only for the case $\mathbf{q} = 0$ is this Mach number $\frac{v}{c_\eta}$ the ratio of the flow speed to the speed of sound, which in a barotropic motion is given rather by $\frac{v}{c}$. The advantage of this definition of Mach number lies in the fact that c_η exists in any motion, and $\frac{v}{c_\eta}$ can be shown to be a similarity parameter, while a definite speed of propagation for waves bearing discontinuities in the velocity gradient exists only for barotropic motions. *Cf.* [1951, 2].

This relation holds for *an inviscid fluid in any* steady *continuous* irrotational *or* Beltrami *flow as an inert mixture.* As observed by Hicks [39] in his analysis of a special case, it shows that whatever form the heat flux $\mathbf{q}$ may take (so long as Q be finite), at $M = 1$ we shall have $c = c_\eta$, and hence by (6.24) $v = c_\eta = c$, so that $M = 1$ is truly sonic speed even under these rather general circumstances.

7. Consequences of the energy equation in steady flow of fluids devoids of heat flux. — From (5.7) and (6.2) it is easy to derive Vazsonyi's form of the energy equation [40]:

$$(7.1) \qquad \rho \frac{Dh_t}{Dt} = \mathbf{W} : \mathbf{A} - \operatorname{div}\mathbf{q} + \mathbf{v}.(\mathbf{f} + \operatorname{div}\mathbf{W}) + \frac{\partial p}{\partial t}.$$

For inviscid fluids $\mathbf{W} = 0$; when there is no heat flux, $\operatorname{div}\mathbf{q} = 0$; thus if also $\mathbf{v}.\mathbf{f} = 0$ and $\frac{\partial p}{\partial t} = 0$ the right side vanishes, and we obtain

THEOREM 10. — *In an inviscid fluid in continuous motion as an inert mixture* devoid of heat flux, *if* the extraneous force be zero *or normal to the velocity, and if* the pressure field be steady, *then both the* entropy and total enthalpy of each particle remain constant. *In particular, for* steady flow *under these conditions, both* entropy and total enthalpy are constant along each stream-line.

In general the value of the total entahalpy h_t differs from one stream-line to another. The possibility of uniform total enthalpy is expressed by the evident

THEOREM 11. — *In an inviscid fluid in continuous motion as an inert mixture* devoid of heat flux, *if the* extraneous force be zero *or normal to the velocity, if the pressure field be steady, and if moreover there exist a certain (possibly moving, possibly steady)* surface of constant total enthalpy which is touched by every particle *at some time, then the flow is a flow* of uniform total enthalpy. *In particular, if in a* steady flow *under these circumstances there exist* one surface of constant total enthalpy which is touched by every stream-line, *then the flow is a flow of* uniform total enthalpy.

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(39) *Loc. cit.*
(40) [1945, eq. (M‴)].

In any case, for steady flow we have $h_t =$ const. on each stream-line; that is,

$$h_t = \frac{1}{2} v^2 + h = C, \tag{7.2}$$

where the constant C generally has a different value for each stream-line. This statement is the *curvilinear Bernoulli theorem* ([41]) (*cf.* the spatial and superficial Bernoulli theorems in section 2). For homogeneous fluids it has two important consequences.

First, there exists a definite *stagnation enthalpy* h_0 for each stream-line, a unique value which h necessarily assumes at any stagnation point upon that steam-line. Since $\rho = \rho(h, \eta)$ and η is constant along each stream-line, there is a definite value ρ_0 of ρ given by $\rho_0 = \rho(h_0, \eta)$, the *stagnation density*, for each stream-line. Similarly there exists a *stagnation pressure* ([42]) $p_0 = p(\rho_0, \eta)$ and a *stagnation temperature* $\theta_0 = \theta(\rho_0, \eta)$ for each stream-line.

Now $h \equiv \varepsilon + \frac{p}{\rho}$. It is a tacit requirement of thermodynamics that all substances are assumed to have energy surfaces such that ε, $p \equiv -\frac{\partial \varepsilon}{\partial\left(\frac{1}{\rho}\right)}$, and $\theta \equiv \frac{\partial \varepsilon}{\partial \eta}$ have finite lower bounds; by an affine transformation of the energy surface ([43]) we may then choose to measure ε, p, and θ in such a way that $\varepsilon \geqq 0$, $p \geqq 0$, $\theta \geqq 0$. From the hypothesis of continuity of motion it follows that $\rho \geqq 0$. Hence $h \geqq 0$. From (7.2) then follows as a second consequence that there exists *a finite least upper bound* v_0 for the speed, attained (if at all) when $h = 0$. Thus the Bernoulli equation (7.2) becomes

$$h + \frac{1}{2} v^2 = h_0 = \frac{1}{2} v_0^2, \tag{7.3}$$

([41]) From (7.1) it is plain that if $\mathbf{f} = -\rho \operatorname{grad} \upsilon$ then in a steady motion $h_t + \upsilon =$ const. upon each stream line. This more general Bernoulli equation does not appear to lend itself to further developments in gas dynamics.

([42]) For a steady *barotropic* flow subject to no extraneous force the existence of a definite stagnation pressure for each Bernoullian surface is immediate, irrespective of the presence or absence of heat flux (*cf.* § 2).

([43]) Only affine invariants of the energy surface can have physical (i. e. dimensionless) interpretations.

or

(7.4)
$$1-\frac{v^2}{v_0^2}=\frac{h}{h_0}.$$

This *ultimate speed* is analogue to that which exists in a steady irrotational or Beltrami motion (§ 2), except that now it generally has different values upon the different stream-lines. In theorems 10 and 11 the words *ultimate speed* may be substituted for *stagnation enthalpy* in each portion referring to steady flow, and it is in this form that we shall use these results hence forth.

Now by definition $h = h(\eta, \rho)$. If both η and ρ be constant upon each stream-line, so also is h, and thus by (7.3) so also is the speed v. Thus for *a steady irrotational flow satisfying the conditions of theorem 9 and subject to no extraneous force, each region in which the flow is not isentropic is a region in which the speed is constant on each stream-line.* Now it was shown by Caldonazzo ([44]) that *in any steady continuous circulation-preserving complex-laminar flow* (§ 2) *such that the speed is constant upon each stream-line, the normal surfaces are minimal surfaces.* By lemma 5 of section 2 follows *a fortiori* that *the equipotential surfaces are minimal surfaces* in any non-isentropic region of a steady irrotational flow subject to the present assumptions. Hamel ([45]) has proved that consequently any steady irrotational motion in which the speed is constant on each stream-line is locally either a *uniform parallel flow*, a *simple vortex*, or a *helicoidal flow* obtained by superposing upon an irrotational vortex a uniform parallel flow in a direction pependicular to its plane. This class of flows we may call *Hamel flows*. Combining the results of this analysis with theorems 9, 10, and 5 we obtain

THEOREM 12 (Characterization of steady irrotational flows). — *In a* steady *continuous* irrotational *flow of a* homogeneous *inviscid* tri-variate *fluid* devoid of heat flux *and subject to* no extraneous force, *one or both of the following conditions holds locally :*

([44]) [1924, § 6]. *Cf.* [1947, 4]. A shorter proof may be found in [1948, 2, § 3], [1949, 5, chap. V, sect. B].

([45]) [1937].

a. The flow is a Hamel flow, *upon each stream-line of which all the state variables and the speed are constant.*

b. The flow is isentropic *and* all the stagnation quantities v_0, h_0, p_0, ρ_0, θ_0 are uniform.

That a Hamel flow need not be isentropic is illustrated by the evident case of the irrotational vortex, in which the entropy may be arbitrarily distributed from one circular stream-line to another, and some one stagnation quantity such as h_0 may also be assigned arbitrarily.

Gilbarg ([46]) has taken up the difficult question of characterizing irrotational flows in the large, and has succeded in proving that subject to certain specified exceptions a steady plane or rotationally-symmetric flow of a perfect gas under the conditions of theorem 12 if not an isentropic flow in the large is a simple vortex or a parallel flow in the large.

Contrasting theorem 12 and theorem 4, we may say broadly that under the circumstances considered an irrotational flow is isentropic (the Hamel flows constituting a rather degenerate exception), but an isentropic flow will not generally be an irrotational or Beltrami flow unless it be also a flow of uniform total enthalpy. In fact we have a partial converse to theorem 12 in

THEOREM 13. — *Let a* steady *continuous flow of a* homogeneous *inviscid* tri-variate *fluid* devoid of heat flux *and subject to* no extraneous force *satisfy the following conditions :*

a. Every stream-line touches a certain surface of constant entropy, *and*

b. Every stream-line touches a certain suface of constant ultimate speed,

then the flow is an isentropic *flow in which* all stagnation quantities are uniform, *and further it is either an* irrotational *or a* Beltrami *flow. In particular, any* complex-laminar *flow satisfying the conditions of this theorem is* irrotational.

Proof. — By hypothesis *a* and theorem 7 follows the isentropic property. By hypothesis *b* and theorem 11 follows the uniformity

([46]) [1949, 2].

of h_0. By the tri-variate character of the fluid, all stagnation quantities then are uniform. From theorem 4 follows $\mathbf{w} \times \mathbf{v} = o$.

In particular, the foregoing theorem shows that a steady flow satisfying the requirements of the theorem and of such a nature that it may be considered as originating in a reservoir of uniform entropy and stagnation enthalpy is always an irrotational or Beltrami flow (47) at least up the the first shock front.

8. The Crocco vector, and generalized Beltrami flows. — In the foregoing section a definite ultimate speed v_0, constant upon each stream-line, was shown to exist in certain types of flow of inviscid fluids. In this section we consider independently the consequences of the existence of this quantity, developing certain preliminary results which will be put to important applications in section 10.

In reality our results here are purely kinematical : v_0 need not

(47) For barotropic flows of inviscid fluids subject to conservative extraneous force Lamb's superficial Bernoulli theorem (lemma 9 of § 2) holds. Lecornu [1919] attributed to Beltrami the remark that if

$$\frac{1}{2}v^2 + \int \frac{dp}{\rho} + \upsilon = \text{const.}$$

over a three-dimensional region of steady barotropic flow, the motion is necessarily an irrotational or Beltrami motion in that region. While theorem 13 is valid also for barotropic flows, it is not really relevant, and should be replaced by the following sharper statement : in a steady continuous barotropic flow of an inviscid fluid subject to no extraneous force, if it be possible to find a curve along which both density and speed are constant and which touches every Bernoullian surface at least once, then the flow is an irrotational or a Beltrami flow. Thus in particular, as noted by Lecornu, all flows of this class which originate in a quiet reservoir at uniform pressure are necessarily irrotational or Beltrami flows. Now the Lagrange-Cauchy velocity-potential theorem states that under these same circumstances a finite material portion of fluid once in irrotational flow remains ever in irrotational flow. Consequently, a flow of this type starting from rest in a finite vessel remains always irrotational. Not so, however, with a flow such that $\lim v = o$, $\lim w = o$ at ∞, for by lemma 3 of section 2 we have $\frac{w}{\rho v} = \text{const.}$ on each stream-line, so that for these flows it is quite possible that the particles may start from rest in a flow irrotational at ∞ yet acquire rotation, the only condition being that $\lim \frac{w}{\rho v}$ shall exist and have a value other than zero. Thus a flow from a quiet infinite reservoir at uniform pressure may well be a Beltrami flow rather than an irrotational flow. This point was noted by Lecornu, though his discussion was not altogether convincing.

actually be an ultimate speed, but may be merely any quantity satisfying the conditions

$$
(1.8)\quad \begin{cases} \text{I.} & \mathbf{v}.\mathbf{grad}\, \nu_0 = 0; \\ \text{II.} & \text{If } \mathbf{w} \times \mathbf{v} = 0 \quad \text{and} \quad \dfrac{\partial \mathbf{v}}{\partial t} = 0, \quad \text{then} \quad \nu_0 = \text{const.} \end{cases}
$$

The former is a statement that ν_0 is constant on each stream-line. The latter is required for the proof of lemma 2 only; that indeed a uniform ultimate speed ν_0 exists in any steady irrotational or Beltrami flow of an inviscid fluid subject to no extraneous force follows from theorem 2. Our results here are put in terms of an ultimate speed rather than simply an arbitrary function satisfying (8.1) only in view of their applications in section 10.

It is convenient to introduce the *Crocco* (48) *vector* $\mathbf{v}_C$

$$
(8.2)\qquad \mathbf{v}_C \equiv \frac{\mathbf{v}}{\nu_0}.
$$

The Crocco vector is thus a dimensionless vector tangent to the stream-line at the point in question and of magnitude never exceeding 1. Let $\mathbf{w}_C$ be the curl of the Crocco (48) vector :

$$
(8.3)\qquad \mathbf{w}_C \equiv \mathbf{curl}\, \mathbf{v}_C,
$$

Then

$$
(8.4)\qquad \mathbf{w} = \mathbf{curl}\, \nu_0 \mathbf{v}_C = \nu_0 \mathbf{w}_C + \mathbf{grad}\, \nu_0 \times \mathbf{v}_C.
$$

From (8.2) and (8.3) follows

$$
(8.5)\qquad \mathbf{v}_C . \mathbf{w}_C = \nu_0^2 \mathbf{v} . \mathbf{w},
$$

and hence by lemma 1 of section 2 we obtain

Lemma 1. — *The Crocco vector of a continuous motion is complex-laminar if and only if the flow be complex-laminar.*

Note. — The conditions (8.1) are not used in the proof of this result, which holds for any continuously differentiable scalar ν_0.

---

(48) Crocco [1936] considered only the case when ν_0 is uniform. Hicks [1949, 3] analyses the properties of a general class of dimensionless vector variables of which the Crocco vector is one. A still more general (and purely kinematical) scheme is employed in [1951, 1].

The researches of Neményi and Prim ([49]) have drawn attention to flows such that

$$\mathbf{v}_C \times \mathbf{w}_C = 0, \tag{8.6}$$

the possibility $\mathbf{w}_C = 0$ not being excluded. These flows they call *generalized Beltrami flows;* their dynamical significance will appear in theorem 20 below. The remainder of this section presents some of Neményi and Prim's results in a somewhat broader form ([50]).

We first establish some connections between generalized Beltrami flows and irrotational or ordinary Beltrami flows. Now by Condition II of (8.1), for a steady irrotational or Beltrami flow the ultimate speed v_0 is uniform $\mathbf{grad}\, v_0 = 0$. Hence for these flows (8.4) yields $\mathbf{w} = v_0 \mathbf{w}_C$, so that if $\mathbf{w} = 0$ then $\mathbf{w}_C = 0$, while if $\mathbf{w} \neq 0$ but $\mathbf{v} \times \mathbf{w} = 0$ then $\mathbf{w}_C \neq 0$ but $\mathbf{v}_C \times \mathbf{w}_C = 0$. These results are expressed in

LEMMA 2. — *In a steady irrotational flow the Crocco vector is laminar :*

$$\mathbf{w}_C = 0, \tag{8.7}$$

while a steady Beltrami flow is also a generalized Beltrami flow whose Crocco vector is not laminar :

$$\mathbf{v}_C \times \mathbf{w}_C = 0, \qquad \mathbf{w}_C \neq 0. \tag{8.8}$$

The converse of lemma 2 does not hold, however, for by (8.4), (8.2), and (8.1) follows

$$\mathbf{v} \times \mathbf{w} = v_0^2 \mathbf{v}_C \times \mathbf{w}_C + v^2 \mathbf{grad} \log v_0, \tag{8.9}$$

and hence

LEMMA 3. — *In a flow possessed of a definite ultimate speed, constant on each stream-line, a generalized Beltrami flow is a Beltrami or irrotational flow if and only if the ultimate speed be uniform. Similarly, a flow whose Crocco vector is laminar* ([51])

([49]) [1948, 1, § 5], [1949, 4], [1949, 5].

([50]) [1949, 4, th. 7 and § 7 8].

([51]) Hicks [1948, 3, § 5] discusses flow of this type for a perfect gas in which there is thermal conduction according to Fourier's law.

is always itself a complex-laminar flow, and is furthermore an irrotational flow if and only if the ultimate speed be constant on each of the normal surfaces.

The result of this lemma can be generalized as follows. If $\mathbf{v}_C \times \mathbf{w}_C = 0$, the two summands in $(8.4)_2$ are perpendicular, so that

$$w^2 = v_0^2 w_C^2 + (\mathbf{grad} \log v_0 \times \mathbf{v})^2. \tag{8.10}$$

Simultaneously (8.9) becomes

$$\mathbf{v} \times \mathbf{w} = v^2 \mathbf{grad} \log v_0; \tag{8.11}$$

hence $\mathbf{grad} \log v_0$ is perpendicular to $\mathbf{v}$, so that (8.10) becomes

$$w^2 = v_0^2 w_C^2 + |\mathbf{grad} \log v_0|^2 v^2. \tag{8.12}$$

The angle χ between the stream-line and the vortex-line at each point is given by

$$\cos\chi = \frac{vw}{|\mathbf{v} \times \mathbf{w}|} = \frac{v\sqrt{v_0^2 w_C^2 + |\mathbf{grad} \log v_0|^2 v^2}}{v^2 |\mathbf{grad} \log v_0|}. \tag{8.13}$$

Hence follows an elegant result of Neményi and Prim :

Lemma 4. — *If a rotational flow possessed of a definite ultimate speed, constant on each stream-line, be a generalized Beltrami flow, then the angle χ between the stream-line and the vortex-line is given by*

$$\operatorname{tg}\chi = \frac{v_C}{w_C} |\mathbf{grad} \log v_0|. \tag{8.14}$$

In particular, at a stagnation point where $w_C \neq 0$ the stream-line and vortex-line are tangent.

From (8.11) follows immediately

Lemma 5. — *If a flow possessed of a definite ultimate speed, constant on each stream-line, be a generalized Beltrami flow which is neither an irrotational nor a Beltrami flow* (i. e., $\mathbf{w}_C \times \mathbf{v}_C = 0$, $\mathbf{w} \times \mathbf{v} \neq 0$), *then the surfaces of constant ultimate speed are Bernoullian surfaces* (*cf.* § 2).

In a steady generalized Beltrami flow it follows from (8.21) and Lagrange's formula (2.15) that the acceleration **a** is given by

$$(8.15)\qquad \begin{cases} \mathbf{a} = -v^2\,\mathbf{grad}\log v_0 + \mathbf{grad}\,\frac{1}{2}v^2, \\ \quad = \frac{1}{2}v_0^2\,\mathbf{grad}\,v_C^2, \end{cases}$$

and hence follows

Lemma 6. — *If a flow possessed of a definite ultimate speed, constant on each stream-line, be a steady generalized Beltrami flow, then the acceleration is complex-laminar, its normal surfaces being the surfaces of constant magnitude of the Crocco vector.*

This lemma casts a certain expectation of the dynamical simplicity of a generalized Beltrami flow. The defining characteristic of the theory of barotropic or isochoric flows of perfect fluids subject to conservative extraneous force, whence arise Kelvin's circulation theorem (§ 3), the Helmholtz and Bernoulli theorems (lemma 9 of § 2), and all the main results of classical hydrodynamics, is that the acceleration is *laminar* (lemma 4 of § 2). The case of complex-laminar acceleration may be expected therefore to be next in order of simplicity, and to be distinguished by special dynamical properties. That such is indeed the fact for a certain type of gas will appear in section 10.

We may complete the present kinematicel analysis of generalized Beltrami flows by characterizing the special case when the circulation-preserving property holds. From (8.15) we have

$$(8.16)\qquad \mathbf{curl\,a} = \frac{1}{2}\,\mathbf{grad}\,v_0^2 \times \mathbf{grad}\,v_C^2,$$

and hence by lemma 4 of section 2 follows

Lemma 7. — *If a flow possessed of a definite ultimate speed, constant on each stream-line, be a steady generalized Beltrami flow, then it is circulation-preserving if and only if one or more of the following three conditions be satisfied locally :*

a. The ultimate speed is uniform (and hence by lemma 3 the flow is actually an irrotational or Beltrami flow) ;

b. The magnitude of the Crocco vector is uniform; or

c. Bernoullian surfaces exist and the acceleration is normal to them.

The possibilities a and b follow at once from (8.16). The third possibility is that the surfaces $v_0 =$ const. coincide with the surfaces $v_C =$ const.; by lemma 6, the acceleration is normal to the former, while by lemma 5 these are Bernoullian surfaces, so that the condition c follows.

9. **Prim gases** ([52]). — For steady flow of an inviscid fluid Euler's equations of continuity (2.8) and motion (3.1) become

$$\operatorname{div} \rho \mathbf{v} = 0, \tag{9.1}$$

$$\mathbf{v} . \mathbf{grad}\, \mathbf{v} + \frac{1}{\rho} \mathbf{grad}\, p = \mathbf{f}. \tag{9.2}$$

By inspection of this system we conclude the *invariance theorem :*

THEOREM 14. — *Let p, ρ, $\mathbf{v}$ be the pressure, density, and velocity fields of a* steady *continuous flow of an inviscid fluid subject to the extraneous force* $\mathbf{f}$. *Then if m be any non-vanishing differentiable function which is constant upon each stream-line of this flow* ($\mathbf{v} . \mathbf{grad}\, m = 0$), *the velocity field $\frac{\mathbf{v}}{m}$ and the density field $m^2 \rho$ yield* another flow *having the* same stream-lines *and the* same pressure p, *subject to the extraneous force field $\frac{\mathbf{f}}{m^2}$.*

In general the new flows so obtained will be flows of a different fluid. Starting, for example, with a flow of a homogeneous incompressible fluid of density ρ_0, the invariance theorem yields similar flows of an inhomogeneous incompressible fluid of density $m^2 \rho_0$, where m may be assigned arbitrarily on each stream-line; indeed, the only usefulness of the invariance theorem for incompressible fluids is to show conversely that given a flow of an inhomogeneous incompres-

([52]) The analysis in this and the succeeding section is a modification of that given by Prim [1949, 1].

sible fluid, there exists a flow of a homogeneous incompressible fluid with the same stream-lines and the same pressure field.

In one important special case, however, the class of invariant flows are possible flows for the *same* fluid. Plainly it is necessary to this end that some state variable be constant on each stream-line. When there is no heat flux, it follows from theorem 6 that the entropy η is such a variable, and consequently we may seek to adjust the entropy of the flow whose density is $m^2\rho$ in such a way that it too is an admissible flow for the original fluid. That is, if an equation of state be

$$\rho = f(p, \eta), \tag{9.3}$$

then we shall wish to find an η' such that

$$m^2\rho = f(p, \eta'); \tag{9.4}$$

hence

$$m^2 = \frac{f(p, \eta')}{f(p, \eta)}. \tag{9.5}$$

Now m^2, being simply any function constant on the stream-lines, is independent of p, and hence can be a function of η' only, say $k(\eta')$. From (9.5) it follows then that $f(p, \eta') = f(p, \eta)\, k(\eta')$. Conversely, for an equation of state of this form, viz.

$$\rho = \mathrm{P}(p)\,\mathrm{H}(\eta), \qquad \mathrm{H}'(\eta) \neq 0, \tag{9.6}$$

any similar flow yielded by the invariance theorem is a possible flow of the same fluid. The fluids characterized by this type of invariance, and hence satisfying (9.6), we may call *Prim gases*. Since a Prim gas is a homogeneous tri-variate fluid, the majority of our previously deduced theorems remain valid *a fortiori* for Prim gases. Note that the requirement $\mathrm{H}' \neq 0$ excludes homogeneous incompressible and piezotropic fluids ([53]). Expressing the result of the foregoing analysis we ([54]) obtain the *substitution principle :*

([53]) A number of the theorems in the sequel are deduced without using the requirement $\mathrm{H}' \neq 0$, and hence remain valid in classical hydrodynamics also, where, however, much stronger theorems are available, so we shall not tarry to point out these special cases.

([54]) That this theorem holds for perfect gases (sec below) is indicated by Munk and Prim [1947, 1].

THEOREM 15. — *Let* p, ρ, $\mathbf{v}$ *be the pressure, density, and velocity fields of a* steady *continuous flow of a* homogeneous *inviscid fluid* devoid of heat flux *and subject to the extraneous force* $\mathbf{f}$. *Then if* m *be any non-vanishing differentiable function which is constant upon each stream-line of this flow* ($\mathbf{v}.\mathbf{grad}\, m = 0$), *the velocity field* $\frac{\mathbf{v}}{m}$ *and the density field* $m^2\rho$ *yield* another flow *of* this same fluid *having the same stream-lines and the same pressure field* p, *subject to the extraneous force field* $\frac{\mathbf{f}}{m^2}$, *if and only if the fluid be a* Prim gas.

Now in general for any homogeneous fluid it is easy to show from (4.17) and (4.9) that

$$\frac{1}{\rho} = \left(\frac{\partial h}{\partial p}\right)_\eta. \tag{9.7}$$

Hence from (9.6) follows [55]

$$h = \frac{\Pi(p)}{H(\eta)} + F(\eta), \tag{9.8}$$

where $\Pi(p) \equiv \int^p \frac{d\xi}{P(\xi)}$, or

$$P(p) = \frac{1}{\Pi'(p)}. \tag{9.9}$$

Thus the Bernoulli equation (7.3), valid when the extraneous force vanishes, assumes the form

$$\frac{1}{2}v^2 + \frac{\Pi(p)}{H(\eta)} + F(\eta) = \frac{\Pi(p_0)}{H(\eta)} + F(\eta); \tag{9.10}$$

or simply

$$\frac{1}{2}v^2 + \frac{\Pi(p)}{H(\eta)} = \overline{h_0} = \frac{\Pi(p_0)}{H(\eta)} = \frac{1}{2}v_0^2, \tag{9.11}$$

where $\overline{h}_0 = h_0 + F(\eta)$ is constant upon each stream-line. In terms of the Crocco vector (8.2) the Bernoulli equation becomes

$$1 - v_C^2 = \frac{\Pi(p)}{\Pi(p_0)}. \tag{9.12}$$

---

[55] Notice that for homogeneous incompressible or piezotropic fluids the enthalpy is of the form $h = \Pi(p) + H(\eta)$, so that the Prim gas is their multiplicative analogue.

Comparison of this result with (7.4) reveals a characterizing property of Prim gases : *a tri-variate fluid is a Prim gas if and only if the Crocco speed v_C in any steady flow devoid of heat flux and subject to no extraneous force be a function of pressure and of stagnation pressure only.* For the local Mach number (6.24) we have by (9.6) and (9.11)

$$\text{(9.13)} \qquad \dot{M}^2 = \frac{v^2}{\left(\frac{\partial p}{\partial \rho}\right)_\eta} = v^2 P'(p)\,H(\eta) = 2\,P'(p)\,[\Pi(p_0) - \Pi(p)].$$

Thus *for a Prim gas in these circumstances the Mach number is a function only of the local pressure and of the stagnation pressure for the stream-line*, and consequently is not changed in any substitution which leaves the pressure field invariant; hence we have

THEOREM 16. — *Under the conditions of theorem* 15 *if there be* no extraneous force *then all substitute flows have the* same Mach number *field as the original flow.*

As an immediate corollary of the substitution principle follows :

THEOREM 17. — *Corresponding to any* steady *flow of a* Prim gas devoid of heat flux *and subject to* no extraneous force *there exists* another flow *of* the same Prim gas *having* the same stream-lines, the same pressure field, *and* the same Mach number field, *but which is furthermore* a flow of uniform ultimate speed.

Notice that in view of the Rankine-Hugoniot conditions the ultimate speed is continuous across a shock front, so that *the validity of theorem* 17 *is unaffected by the presence of shocks.* As a second corollary of the substitution principle we have

THEOREM 18. — *Corresponding to any* steady *continuous flow of a* Prim gas devoid of heat flux *and subject to* no extraneous force, *there exists another flow of* the same Prim gas, *having* the same stream-lines, the same pressure field, *and* the same Mach number field, *but which is furthermore an* isentropic *flow.*

This theorem cannot be fully extended to flows with shocks, since these commonly are the bearers of discontinuities in the entropy. Although an isentropic flow with the same stream-lines and pressure field can always be found, this flow will generally fail to satisfy the Rankine-Hugoniot conditions at the shock fronts; in other words, if the similar flow is to satisfy the Rankine-Hugoniot conditions, while indeed it can be made isentropic in any one region bounded by shocks, in general it cannot be isentropic outside this region.

A Prim gas for which

$$\Pi(p) = p^{\frac{\gamma-1}{\gamma}}, \qquad \mathrm{H}(\eta) = \mathrm{C}e^{\frac{\eta-\eta_0}{c_p}}, \tag{9.14}$$

where γ, c_p, C, and η_0 are constants, is called a *perfect* or *ideal gas with constant specific heats*. The constant c_p may be shown to be the specific heat at constant pressure, while the specific heat at constant density c_v is given by $c_v = \frac{c_p}{\gamma}$. For a perfect gas (9.6) becomes

$$\rho = \frac{\gamma}{\mathrm{C}(\gamma-1)} p^{\frac{1}{\gamma}} e^{-\frac{\eta-\eta_0}{c_p}}, \tag{9.15}$$

and hence

$$\varepsilon = h - \frac{p}{\rho} = \mathrm{C}'\rho^{\gamma-1} e^{\frac{(\eta-\eta_0)}{c_v}},$$

so that by $(4.5)_1$ we have $\varepsilon = c_v\theta$. Putting $\mathrm{R} \equiv c_p - c_v$, we then obtain $\frac{p}{\theta} = \mathrm{R}\rho$, and hence

$$h = c_p\,\theta = \frac{\gamma}{\gamma-1}\frac{p}{\rho}, \tag{9.16}$$

so that the Bernoulli equation (7.3) becomes

$$\frac{1}{2}v^2 + c_p\theta = c_p\theta_0 = \frac{\gamma}{\gamma-1}\frac{p_0}{\rho_0} = \frac{1}{2}v_0^2. \tag{9.17}$$

10. Vorticity and the thermodynamic state in the steady flow of a Prim gas devoid of heat flux and subject to no extraneous force. — The substitution principle (theorem 15) suggests that the introduction of a modified velocity vector which is invariant under the group of substitutions $\mathbf{v} \leftrightarrow \frac{\mathbf{v}}{m}$, $p \leftrightarrow p$, $\rho \leftrightarrow m^2\rho$, where m is any differentiable function constant upon each stream-line, may serve to eliminate the density and entropy from the equations governing the dynamics of

Prim gases. The most convenient choice for this purpose is the *Crocco vector* $\mathbf{v}_C$, given by (8.2). Now for a Prim gas $\rho \mathbf{v} = P(p) H(\eta) v_0 \mathbf{v}_C$. Since there is no heat flux, by theorem 9 we have $\mathbf{v} . \mathbf{grad}\, \eta = 0$ and $\mathbf{v} . \mathbf{grad}\, v_0 = 0$; hence the continuity equation (9.1) becomes

$$\text{(10.1)} \qquad \operatorname{div} [P(p) \mathbf{v}_C] = 0.$$

Similarly, by using (9.12) we may put the dynamical equation (9.2) into the form

$$\text{(10.2)} \qquad \mathbf{v}_C . \mathbf{grad}\, \mathbf{v}_C + \frac{1}{2}(1 - v_C^2)\, \mathbf{grad} \log \Pi(p) = 0.$$

These basic equations, which for the case of a perfect gas were first given by Hicks, Guenther, and Wasserman ([56]), present in a particularly lucid form the whole dynamics of the steady motion of Prim gases when there is neither extraneous force nor heat flux. They constitute a determinate system for the pressure and the Crocco vector, whence theorem 17 is again apparent. By putting (9.11) into (10.2) and using the identity

$$\mathbf{b} . \mathbf{grad}\, \mathbf{b} = \mathbf{curl}\, \mathbf{b} \times \mathbf{b} + \mathbf{grad} \frac{1}{2} b^2$$

we obtain the central *theorem of Hicks, Guenther, and Wasserman* ([57]) :

THEOREM 19. — *In a* steady *continuous flow of a* Prim gas devoid of heat flux and subject to no extraneous force *we have*

$$\text{(10.3)} \qquad \mathbf{v}_C \times \mathbf{w}_C = \frac{1}{2}(1 - v_C^2)\, \mathbf{grad} \log \Pi(p_0).$$

This theorem shows that a knowledge of the Crocco vector at once yields the distribution of the stagnation pressure p_0 upon the streamlines and hence by the Bernoulli theorem (9.12) the local pressure may be calculated. By formulating a condition of integrability for (10.3) we see that *a vector field* $\mathbf{v}_C$ *may serve as the Crocco*

([56]) [1947, 2, eq. (4.4), (4.2)]. For a perfect gas of uniform stagnation enthalpy (10.1) had been given earlier by Crocco [1936, eq. (6)].

([57]) The case of this theorem valid for a perfect gas is given in [1947, 2, eq. (4.2)].

vector of a steady flow of a Prim gas devoid of heat flux and subiect to no extraneous force if and only if

(10.4) $$\mathbf{curl}\left(\frac{\mathbf{v}_C \times \mathbf{w}_C}{1 - v_C^2}\right) = 0;$$

in particular, any Beltrami field yields an infinite number of dynamically possible flows.

By comparing (10.3) with (8.6) we obtain :

THEOREM 20. — *A* steady *continuous flow of a* Prim gas devoid of heat flux and subject to no extraneous force *is a flow of* uniform stagnation pressure *if and only if it be a* generalized Beltrami flow.

Any flow which may be regarded as originating from a container at uniform pressure must therefore be a generalized Beltrami flow. The seven kinematical lemmas of section 8 on generalized Beltrami flows now assume a definite physical interest.

By applying theorem 20 to a complex-laminar flow and employing lemma 1 of section 8 we obtain.

THEOREM 21. — *A* steady *continuous* complex-laminar *flow of a* Prim gas devoid of heat flux *and subject to* no extraneous force *is a flow of* uniform stagnation pressure *if and only if* its Crocco vector be laminar ($\mathbf{w}_C = 0$).

From theorem 20, (10.1), and lemma 2 of section 2 follows at once the elegant pressure theorem of Neményi and Prim ([58]) :

THEOREM 22. — *In a* steady *continuous flow of a* Prim gas devoid of heat flux *and subject to* no extraneous force, *if the* stagnation pressure be uniform *then*

(10.5) $$\frac{w_C}{P(p)v_C} = \text{const.}$$

upon each stream-line; equivalently

(10.6) $$\frac{w_C}{P(p)\sqrt{\Pi(p_0) - \Pi(p)}} = \text{const.}$$

([58]) A special case is given in [1949, 4, th. 6].

The formula (10.6) giving the Crocco vorticity magnitude as an explicit function of pressure has a counterpart in the case of a complex-laminar flow, as the following analysis demonstrates. By vectorial transformations it is easy to shows that

$$(10.7)\quad \mathbf{curl}\left[\frac{\mathbf{w}_C\times\mathbf{v}_C}{1-v_C^2}\right]=\mathbf{v}_C\,.\,\mathbf{grad}\,\frac{\mathbf{w}_C}{1-v_C^2}-\frac{\mathbf{w}_C}{1-v_C^2}\,.\,\mathbf{grad}\,\mathbf{v}_C+\frac{\mathbf{w}_C}{1-v_C^2}\operatorname{div}\mathbf{v}_C-\mathbf{v}_C\operatorname{div}\frac{\mathbf{w}_C}{1-v_C^2}.$$

Hence by (10.1) and (10.4) follows

$$(10.8)\qquad \frac{1}{v_0}\frac{D}{Dt}\left[\frac{\mathbf{w}_C}{P(p)(1-v_C^2)}\right]=\frac{\mathbf{w}_C}{P(p)(1-v_C^2)}\,.\,\mathbf{grad}\,\mathbf{v}_C+\frac{\mathbf{v}_C}{P(p)}\operatorname{div}\frac{\mathbf{w}_C}{1-v_C^2}.$$

Let the dot product of this equation by $\frac{\mathbf{w}_C}{P(p)(1-v_C^2)}$ be formed; in the case of a *complex-laminar* flow it follows by application of lemma 1 of section 8 that the resulting expression becomes

$$(10.9)\quad \frac{1}{2v_0}\frac{D}{Dt}\left[\frac{\mathbf{w}_C}{P(p)(1-v_C^2)}\right]^2=\frac{\mathbf{w}_C}{P(p)(1-v_C^2)}\,.\,\mathbf{grad}\,\mathbf{v}_C\,.\,\frac{\mathbf{w}_C}{P(p)(1-v_C^2)}.$$

Let x_w be a co-ordinate along the vortex-line, and let x_1 and x_2 be any other co-ordinates such that a triply orthogonal system is obtained :

$$(10.10)\qquad ds^2=h^2\,dx_w^2+h_1^2\,dx_1^2+h_2^2\,dx_2^2.$$

Then (10.9) expressed in this system becomes

$$(10.11)\qquad \frac{1}{v_0}\frac{D}{Dt}\log\frac{w_C}{P(p)(1-v_C^2)}=\frac{\delta\left(\frac{v_w}{v_0}\right)}{\delta x_w},$$

where v_w is the component of $\mathbf{v}$ in the direction of the vortex-line, and the symbol $\frac{\delta}{\delta x_w}$ denotes the physical component of the intrinsic (covariant) derivative in the direction of the vortex-line. Thus

$$(10.12)\quad \frac{1}{v_0}\frac{D}{Dt}\log\frac{w_C}{P(p)(1-v_C^2)}=\frac{1}{h}\frac{\partial\left(\frac{v_w}{v_0}\right)}{\partial x_w}+\frac{v_1}{hh_1v_0}\frac{\partial h}{\partial x_1}+\frac{v_2}{hh_2v_0}\frac{\partial h}{\partial x_2}.$$

Since the flow is complex-laminar, $\dot{v}_w = 0$. Since

$$(10.13) \qquad v_1 = h_1 \frac{Dx_1}{Dt}, \qquad v_2 = h_2 \frac{Dx_2}{Dt}, \qquad v_w = h \frac{Dx_w}{Dt} = 0,$$

(10.12) assumes the form

$$(10.14) \qquad \frac{D}{Dt} \log \left[\frac{w_C}{h P(p)(1 - v_C^2)} \right] = 0.$$

Hence the quantity in brackets is constant on each stream-line. By (9.12) and (9.9) we thus obtain the *generalized Crocco pressure theorem* (59) :

THEOREM 23. — *In a* steady *continuous* complex-laminar *flow of a* Prim gas devoid of heat flux *and subject to* no extraneous force. *let the stream-surfaces* $x_w =$ const. *and* $x_w + dx_w =$ const. *normal to the vortex-lines be distant* $h dx_w$ *from one another, i. e. let* h *be defined by the element of arc-length*

$$(10.15) \qquad ds^2 = h^2 dx_w^2 + \ldots,$$

where x_w *is a co-ordinate along the vortex-line. Then*

$$(10.16) \qquad \frac{w_C}{h} \frac{d \log \Pi(p)}{dp} = \text{const.}$$

upon each stream-line.

Note that $h = 1$ in a plane flow, $h = r$ in a rotationally-symetric flow, and $\frac{d \log \Pi(p)}{dp} \propto p^{-1}$ for a perfect gas (60).

Returning to the analysis of rotational Prim gas flows in general, by combining theorem 4 and theorem 19 with (8.9) we obtain

(59) This result is derived as a special case of a general theorem of pure kinematics in [1951, 1].

(60) These special cases were first given in [1949, 5, eq. (65), (194)] (for a Prim gas) and in [1948, 4] (for a perfect gas). The original theorems of Crocco [1936] are deduced subject to the restrictive assumption $v_0 =$ const., whose necessity for Crocco's formulation was noticed later by Emmons [1944, App. I] and Vazsonyi [1945, § 8, 10]. Derivations of the original Crocco theorems are given by Tollmien [1942] and Oswatitsch [1943].

THEOREM 24. — *In a* steady *continuous flow of a* Prim gas devoid of heat flux *and subject to* no extraneous force, *we have*

$$h_0(1-v_C^2)[\mathbf{grad}\log h_0 - \mathbf{grad}\log\Pi(p_0)] = \theta\,\mathbf{grad}\,\eta. \tag{10.17}$$

Finally, we eliminate $\mathbf{v}_C \times \mathbf{w}_C$ between (10.3) and (8.9), obtaining

THEOREM 25. — *In a* steady *continuous flow of a* Prim gas devoid of heat flux *and subject to* no extraneous force, *we have*

$$\mathbf{v}\times\mathbf{w} = \frac{1}{2}v_0^2[(1-v_C^2)\,\mathbf{grad}\log\Pi(p_0) + v_C^2\,\mathbf{grad}\log v_0^2]; \tag{10.18}$$

hence

A. *the relation*

$$\mathbf{v}\times\mathbf{w} = \frac{1}{2}v^2\,\mathbf{grad}\log v_0^2 \tag{10.19}$$

holds if and only if the stagnation pressure be uniform, *while*

B. *the relation*

$$\mathbf{v}\times\mathbf{w} = \frac{1}{2}v_0^2(1-v_C^2)\,\mathbf{grad}\log\Pi(p_0) \tag{10.20}$$

holds if and only if the ultimate speed be uniform. *Thus in a flow of* uniform stagnation pressure *which is neither an irrotational nor a Beltrami flow*, Bernoullian surfaces *exist and are* surfaces of constant ultimate speed; *while in a flow of* uniform ultimate speed *which is neither an irrotational nor a Beltrami flow*, Bernoullian surfaces *exist and are surfaces of* constant stagnation pressure. *Finally, in order that the flow be either an* irrotational *or a* Beltrami flow *it is necessary and sufficient that*

$$(1-v_C^2)\,\mathbf{grad}\log\Pi(p_0) = -v_C^2\,\mathbf{grad}\log v_0^2. \tag{10.21}$$

In the special case of a perfect gas, by (9.17) we may put (10.19) into the form

$$\begin{aligned}\mathbf{v}\times\mathbf{w} &= c_p(\theta_0-\theta)\,\mathbf{grad}\log h_0,\\ &= \left(1-\frac{\theta}{\theta_0}\right)\mathbf{grad}\,h_0,\end{aligned} \tag{10.22}$$

a result given by Vazsonyi [61].

[61] [1945, eq. (7.5), (7.5')].

Suppose now the stagnation pressure be uniform and all the other conditions of theorem 25 be satisfied, so that part A of that theorem follows. By theorem 18 we may always find a substitute flow in which v_0 is constant. This substitute flow has the same pressure field, and thus *a fortiori* will again be a flow of uniform stagnation pressure, so that again (10.19) holds, now yielding $\mathbf{v} \times \mathbf{w} = 0$. Thus we have

THEOREM 26. — *Given a* steady *continuous flow of a* Prim gas devoid of heat flux *and subject to* no extraneous force, if the stagnation pressure be uniform, *then there exists another possible flow of the* same Prim gas *satisfying the above conditions, having* the same stream-lines, the same pressure field, *and* the same local Mach number field, *but which is moreover either a* Beltrami flow *or an* irrotational *flow. In particular, if the original flow be* complex-laminar, *the substitute flow is always* irrotational.

Notice that if there be shocks in the flow then in general only one region bounded by shocks can be a region of uniform stagnation presure, since the stagnation pressure is not continuous across a shock, and hence the conditions of the foregoing theorem in general will be satisfied only in this one region.

INDEX OF NAMES.

(The numbers refer to sections).

Beltrami, E., 2, 3, 5, 6, 7, 8, 10.
Bergeron, T., 4.
Bernoulli, D., 2, 3, 7, 8, 9, 10.
Bjerknes, J., 4.
Bjerknes, V., 4.
Brudno, Charlotte, 1.
Caldonazzo, B., 7.
Cauchy, A., 7.
Crocco, L., 5, 8, 9, 10.
Couette, M., 4.
Eckart, C., 6.
Emmons, H., 10.
Euler, L., 2, 3, 4, 6.
Fourier, J., 6, 8.
Gibbs, J. W., 4.
Gilbarg, D., 7.
Gromeka, I., 2.
Guenther, P., 10.
Hamel, G., 7.
Helmholtz, H., 2, 8.
Hicks, B., 6, 8, 10.
Hugoniot, H., 6, 9.
Kelvin, Lord, 2, 3, 4, 5, 8.
Kirchhoff, G., 6.
Lagrange, J. L., 2, 3, 5, 7.
Lamb, H., 2, 7.
Lecornu, L., 7.
Mach, E., 6, 9, 10.
Masotti, A., 2.
Munk, M., 9.
Neményi, P., 1, 2, 3, 8, 10.
Neumann, C., 6.
Oswatitsch, K., 5, 10.
Poiseuille, J., 4.
Prim, R., 1, 2, 3, 7, 8, 9, 10.
Rankine, W., 9.
Silberstein, L., 3.
Solberg, H., 4.
Tollmien, W., 5, 10.
Truesdell, C., 1, 4, 8.
Vazsonyi, A., 5, 7, 10.
Vessiot, E., 2.
Wasserman, R., 10.

INDEX OF DEFINITIONS.

(References are to equation numbers, section numbers, or footnote numbers).

Acceleration-potential A, (2.18).
Adiabatic, ([30]).
Baroclinic, 3.
Barotropic, (3.10).
Barotropic, instantaneously, (3.7).
BELTRAMI field, (2.4).
BELTRAMI flow, generalized, (8.6).
BELTRAMI motion, (2.13).
BERNOULLI theorem, curvilinear, (7.2).
BERNOULLI theorem, spatial, (2.26), (3.12).
BERNOULLI theorem, superficial, (2.29).
BERNOULLIAN surfaces, 2.
Circulation-preserving, 2.
Complex-laminar field, (2.2).
Complex-laminar motion, (2.12).
Compressible fluid, 4.
Concentration c_i, 4.
Conservative force, (3.2).
Continuous flow, 2.
CROCCO vector $\mathbf{V}_C$, (8.2).
Density, stagnation ρ_0, 7.
Energy, specific internal ε, 4.
Energy surface, 4.
Enthalpy h, (4.19).
Enthalpy, stagnation h_0, 7.
Enthalpy, total h_t, (5.9).
Entropy, 4.
Entropy, specific η, 4.
Equation of state, (4.1), (4.2).
Extraneous force $\mathbf{f}$, 3.
Fluid, inviscid, (3.1).
Fluid, perfect, (3.1).
Force, conservative, (3.2).
Force, extraneous $\mathbf{f}$, 3.
Gas, ideal, (9.14).
Gas, perfect, (9.14).
Gas, PRIM, (9.6).
Generalized BELTRAMI flow, (8.6).
HAMEL flow, 7.
Heat flux $\mathbf{q}$, (6.1).
Homogeneous fluid, (4.3).
Ideal gas, (9.14).
Incompressible fluid, (4.4).
Inert mixture, (4.15).
Instantaneously barotropic, (3.7).
Instantaneously isentropic, (5.5).
Instantaneously isobaric, (3.5).
Instantaneously isostatic, (3.6).
Instantaneously isothermal, (5.4).
Internal energy, 4.
Internal energy, specific ε, 4.
Inviscid fluid, (3.1).
Irrotational, (2.10).
Isentropic, (5.7).
Isentropic, instantaneously (5.5).
Isobaric, (3.8).
Isobaric, instantaneously, (3.5).
Isochoric, ([17]).
Isostatic, (3.9).
Isostatic, instantaneously, (3.6).
Isothermal, (5.6).
Isothermal, instantaneously, (5.4).
Laminar field, (2.1).
MACH number M, (6.24).
Mixture, inert, (4.15).
Minimal surfaces, 7.
Perfect fluid, (3.1).
Perfect gas, (9.14).
Piezotropic fluid, (4.7).

Potential, acceleration A, (2.18).
Potential, velocity Φ, (2.11).
Potential of the substance i, (4.5).
Pressure p, 7.
Pressure, stagnation p_0, 3, 7.
Pressure, thermodynamic π, (4.5).
PRIM gas, (9.6).
Sound, speed of c, (6.15).
Specific heats, 9.
Speed of sound c, (6.15).
Stagnation density ρ_0, 7.
Stagnation enthalpy h_0, 7.
Stagnation pressure p_0, 3, 7.
Stagnation temperature θ_0, 7.
State, equation of (4.1), (4.2).
Steady, 2.
Steady vorticity, (2.20).
Stress, (6.1).
Surfaces, minimal, 7.
Temperature θ, (4.5).
Temperature, stagnation θ_0, 7.
Time t, 2.
Total enthalpy h_t, (5.9).
Tri-variate fluid, 4.
Ultimate speed v_0, 3, 7.
Velocity $\mathbf{v}$, 1.
Velocity-potential Φ, (2.11).
Vorticity $\mathbf{w}$, (1.1), (2.9).
Vorticity, steady, (2.20).

INDEX OF SYMBOLS OCCURRING IN MORE THAN ONE SECTION.

(The number given unless otherwise noted is that of the equation in which the symbol first occurs).

$\mathbf{a}$, acceleration, (2.15).

c_i, concentration of the substance i, (4.2).

c_p, specific heat at constant pressure, (9.14).

$\mathbf{f}$, extraneous force, (3.1).

h, enthalpy, (4.19).

h_0, stagnation enthalpy, (7.3).

h_t, total enthalpy, (5.9).

p, pressure, (3.1).

p_0, stagnation pressure, (3.14), (§ 7).

$\mathbf{q}$, heat flux vector, (6.1).

t, time, (2.15).

v_0, ultimate speed, (3.2).

$\mathbf{v}$, velocity, (1.1).

$\mathbf{v}_C$, Crocco vector, (8.2).

$\mathbf{w}$, vorticity, (1.1), (2.9).

$\mathbf{w}_C$, (8.3).

A, acceleration-potential, (2.18).

$\frac{D}{Dt}$, material derivative, (2.16).

H, (9.6).

M, Mach number, (6.24).

P, (9.6).

U, (2.21).

$\mathbf{W}$, stress in excess of pressure, (6.2).

ε, specific internal energy, (4.2).

η, specific entropy, (4.2).

θ, temperature, (4.5).

θ_0, stagnation temperature, § 7.

ρ, density, (2.8).

ρ_0, stagnation density, § 7.

υ, (3.2).

Φ, velocity potential, (2.11).

$\mathbf{\Delta}$, rate of deformation, (6.1).

Π, (9.8).

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TABLE OF CONTENTS.

1. Introduction 1

2. Some definitions and preliminary lemmas of vector analysis and kinematics 2

3. Inviscid fluids. Kelvin's criterion 8

4. Thermodynamical assumptions. Classification of fluids 12

5. Homogeneous tri-variate fluids. The Crocco-Vazsonyi relation 17

6. The energy equation of C. Neumann, and its consequences for flows of perfect fluids devoid of heat flux 21

7. Consequences of the energy equation in steady flows of fluids devoids of heat flux 27

8. The Crocco vector, and generalized Beltrami flows 31

9. Prim gases 36

10. Vorticity and the thermodynamic state in the steady flow of a Prim gas devoid of heat flux and subject to no extraneous force 40

Index of names 47

Index of definitions 48

Index of symbols 50

References 51