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FASCICULE XC

Analytic theory of non-linear singular differential equations

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ANALYTIC THEORY OF NON-LINEAR SINGULAR DIFFERENTIAL EQUATIONS

By W. J. TRJITZINSKY,

Professor at the University of Illinois (U. S. A.).

Introduction. — *In this work we consider the non-linear differential equation of order n*

$$(A)\qquad x^p y^{(n)}(x) = a(x, y, y^{(1)}, \ldots, y^{(n-1)}) \qquad (p \text{ a positive integer}),$$

where

$$(1)\qquad a(x, y, y^{(1)}, y^{(n-1)}) = \sum_{i_0, \ldots, i_{n-1} \geqq 0}^{\infty} a_{i_0 \ldots i_{n-1}}(x) y^{i_0} y^{(1)i_1} \ldots y^{(n-1)i_{n-1}}$$

$[a_{0\ldots 0}(x) = 0]$, the $a_{i_0 \ldots i_n}(0)$ are analytic for $|x| \leqq r$ and the series involved in the second member of (1) converges for

$$(1a)\qquad |x| \leqq r, \qquad |y|, |y^{(1)}|, \ldots, |y^{(n-1)}| \leqq \rho \quad (^1).$$

Our present object is to investigate the character of solutions of (A) *in the neighborhood of the singular point* $x = 0$. *This*

(1) Without any loss of generality it may be assumed that not all the numbers $a_{i_0 \ldots i_{n-1}}(0)$ are zero. In fact, if the contrary were the case p could be diminished. Throughout the paper, whenever a statement is made that a power series converges in a closed circular region, it will be understood that the radius of the involved circle is sufficiently small so that the function represented by the series is analytic at every point of the region. That is, all such statements are made for sufficiently small circles. A similar remark is made concerning power series in several variables.

investigation will be given in the complex plane of the variable x. Only those solutions will be considered which vanish at $x = 0$ (1).

It will be convenient to write (A) in the form

$$(A_1) \quad x^p y^{(n)}(x) - a_1(x, y, y^{(1)}, \ldots, y^{(n-1)}) = a_2(x, y, y^{(1)}, \ldots, y^{(n-1)}),$$

where $a_1(x, y, y^{(1)}, \ldots, y^{(n-1)})$ is the part of the second member of (A) linear in $y, y^{(1)}, \ldots, y^{(n-1)}$. Accordingly, $a_2(x, y, y^{(1)}, \ldots, y^{(n-1)})$ is represented by a sum like (1) with $i_0 + \ldots + i_{n-1} \geqq 2$. In the special instance when the second member of (A_1) is identically zero there is at hand a linear homogeneons differential equation of order n

$$(A_2) \qquad x^p y^{(n)}(x) - a_1(x, y, y^{(1)}, \ldots, y^{(n-1)}) = 0$$

which at $x = 0$ possesses a singular point (regular or irregular). Essentially complete developments of the theory of such equations, inasmuch as they relate to the properties of solutions in the neighborhood of the singular point, have been recently given by W. J. Trjitzinsky [*cf.* [19 *a*], in the sequel referred to as (T_1); also, [19 *b*] which will be referred to as (T_2)]. Since some of these results will be needed in the present work it will be assumed that the reader is acquainted with the developments just referred to.

The equation (A_2) possesses n linearly independent formal solutions (2)

$$(2) \qquad \begin{cases} s_i(x) = e^{Q_i(x)} x^{r_i} \sigma_i(x) \\ [Q_i(x) \text{ polynomial in } x^{-1/\alpha_i};\ \text{integer } \alpha_i \geqq 1;\ i = 1, \ldots, n), \end{cases}$$

where

$$(2a) \qquad \sigma_i(x) = {}_0\sigma_i(x) + {}_1\sigma_i(x) \log x + \ldots + {}_{m_i}\sigma_i(x) \log^{m_i} x,$$

with

$$(2b) \qquad {}_j\sigma_i(x) = \sum_{\nu=0}^{\infty} {}_j\sigma_{i:\nu} x^{\frac{\nu}{\alpha_i}} \qquad (j = 0, 1, \ldots, m_i).$$

Let R denote any one of the aggregate of regions (extending to $x = 0$) corresponding to which, according to (T_1), (A_2) possesses a set of n linearly independent solutions $y_i(x)$, analytic in $R(x \neq 0)$

(1) The trivial solution $y = 0$ is to be disregarded of course.

(2) That is, the power series involved in these solutions may diverge for all $x (\neq 0)$.

and such that

$$(3) \qquad y_i(x) \sim s_i(x) \qquad (i = 1, \ldots, n; x \text{ in } R).$$

If nothing is said regarding the number of terms to which an asymptotic relationship holds, such a relationship will be understood to be in the ordinary sense (that is, to infinitely many terms). A relation (3) signifies that $y_i(x)$ is a certain function which can be obtained by replacing in $s_i(x)$ the formal series ${}_j\sigma_i(x)$ [*cf.* (2 *b*)] by certain functions, analytic in $R(x \neq 0)$ and correspondingly asymptotic to the ${}_j\sigma_i(x)$ when x is in R.

In treating the case when $n \geqq 2$ *it will be assumed that not all the polynomials* $Q_i(x)$, *involved in the formal series* (2), *are identically zero.*

In the theory of differential equations (and in the fields of certain other important types of equations) the study of the behaviour of solutions in the neighborhood of a singular point can be best effected on the basis of suitable formal series solutions (the formal series in general involve divergent series). By some analytic process "actual" solutions are found which are functions related in one way or another to the formal solutions. In this connection outstanding are (1) the methods based on what essentially amounts to "exponential summability" of the formal solutions (this involves factorial series and Laplace integrals leading to expressions involving convergent factorial series) and (2) the asymptotic methods. At the basis of the methods of the first type to a large extent lie certain fundamental developments due to N. Nörlund [15]. Whenever methods (1) are applicable the results are superior to those derived by asymptotic methods. Now, as pointed out in (T_2), an equation (A_2) may possess formal solutions to which methods (1) are not applicable. The equation (A_2), however, constitutes a special case of (A_1). Consequently, whith the problem formulated as above, it is observed that asymptotic methods are to be employed in so far as the general problem on and is concerned.

It is essential to note that, generally speaking, a differential system of the form

$$(I) \qquad \frac{dy_i}{dx} = a_i(x, y_1, \ldots, y_n) \qquad (i = 1, \ldots, n)$$

is in a certain sense equivalent to a single ordinary differential equation of finite order. In fact, let $\varphi_0 = \varphi_0(x, y_1, \ldots, y_n)$ be an arbitrary function of the displayed variables. On writing

$$\text{(II)} \qquad y = \varphi_0(x, y_1, \ldots, y_n),$$

by successive differentiations and at each step using the relations of the given differential system we obtain certain expressions

$$y^{(\nu)} \left(= \frac{d^\nu y}{dx^\nu} \right) = \varphi_\nu(x, y_1, \ldots, y_n) \qquad (\nu = 0, 1, \ldots).$$

With a suitable choice of the function φ_0 the Jacobian of the φ_ν $(\nu = 0, 1, \ldots, n-1)$, with respect to $y_1, \ldots, y_n$, will not vanish in some domain $\mathcal{D}$ of the complex variables $x, y_1, \ldots, y_n$. It is then possible to solve the first n equations $y^{(\nu)} = \varphi_\nu$ for $y_1, \ldots, y_n$,

$$\text{(III)} \qquad y_i = g_i(x, y, y^{(1)}, \ldots, y^{(n-1)}) \qquad (i = 1, \ldots, n).$$

Substituting (III) in the relation $y^{(n)} = \varphi_n(x, y_1, \ldots, y_n)$ one obtains an equation of the form

$$\text{(IV)} \qquad y^{(n)} = g(x, y, y^{(1)}, \ldots, y^{(n-1)}).$$

Here the second member depends on the a_i of (I) and on the choice of φ_0. It is clear that, subject to the condition that the Jacobian mentioned above should not vanish in a suitable domain $\mathcal{D}$, the function φ_0 must be chosen as " simple " as possible in order to avoid those difficulties which intrinsically do not belong to the given problem. *The solutions of* (I) *are seen to be expressible with the aid of* (III) *in terms of a solution of* (IV).

In the present work we shall not go any further in the study of the connection between a system (I) and an equation (IV).

Some facts of interest will be pointed out. Suppose the system (I) has a singular point at $x = \infty$. Then one can form the corresponding single equation (IV) so that the latter will posses at $x = \infty$ a singular point of essentially the same type. The particular very important case of (I), namely when the system is of a general type occuring in dynamics (x in the a_i absent; the a_i analytic in $y_1, \ldots, y_n$ at

$$y_1 = \ldots = y_n = 0;$$

the $a_i = 0$ for $y_1 = \ldots = y_n = 0$) leads one to a single equation (IV) with the following property. If in g only the part linear in y, $y^{(1)}, \ldots, y^{(n-1)}$ is retained, there is on hand an ordinary linear differential equation which at $x = \infty$ has an irregular singular point generally of rank one. Analogous statements can be made when (I) is of a more general or different type. For instance, the a_i may be periodic in x, or the system (I) may be of the type considered in the highly significant researches of Bohl [4], Cotton [7] and Perron [16].

If we fix our attention on that very important tradition in the investigation of general problems of dynamics which goes back to the famous memoirs of Liapounoff [12] and Poincaré [18] and is receiving its culminating development in the profound investigations of Birkhoff [3], we observe that it is possible to carry out the developments which are of a purely analytic character (in the small) with the aid of a corresponding equation (IV), provided a suitable analytic theory of the latter equation has been developed.

In a later word the present author intends to present developments of the character just mentioned.

We note that equation (A) does not contain as a special case the equation (IV) corresponding to a system of dynamical type (whether following Birkhoff. Liapounoff and Poincaré or Bohl, Cotton and Perron). In fact, *the present work is not concerned directly with any dynamical aspects of the theory of differential equations.* However, there is no doubt that, *with suitable modifications, analytic methods of the type presented in the subsequent pages are adequate for the treatment of micro-analytic differential problems of dynamical character*. This circumstance adds to the significance of the present work.

The methods of the present author on the whole do not follow any of the earlier patterns. These methods consist in part of the following. *The problem* (A) *is resolved into a succession of linear problems, each with a singular point at $x = 0$. These problems are treated by asymptotic methods with the aid of some earlier results due to Trjitzinsky* [19]. *This is followed by a corresponding transformation. Finally, by a certain limiting process the transformed equation is shown to possess certain suitable solutions.*

First we shall treat the case of the problem (A) when $n = 1$. Then (2) will consist of a single convergent series (not involving logarithms). There will be only one polynomial $Q(x)$. *When* $n=1$ *it will not be neccessarily required that* $Q(x)$ *should be distinct from zero.* The main result for this case is given in the *Existence Theorem I* (§ 6). The treatment of the first order problem is followed by that of the general n-th order problem ($n \geqq 2$). The main result in this connection is embodied in the *Existence Theorem II* (§ 10). The reason for the separate treatment of the two cases is that when $n=1$ results can be obtained which are more specific than those for the higher order problem. Moreover, in developing the first order case one can take advantage of certain previously established results due to Horn [9], Picard [17] and Poincaré [18]. The higher order problem is treated in sections 7, 8, 9, 10.

When $n=1$ equation (A) will be written in the form

$$x^{k+1} y^{(1)}(x) = a(x, y) = \sum_{\nu=1}^{\infty} a_\nu(x) y^\nu, \tag{B}$$

$$a_\nu(x) = \sum_{\iota=0}^{\infty} a_{\nu,\iota} x^\iota \qquad (\nu = 1, 2, \ldots). \tag{5}$$

It will be assumed that the series here involved converge for

$$|x| \leqq r, \qquad |y| \leqq \rho. \tag{5a}$$

For the case when in (B) the integer k is zero essentially complete results have been obtained previously. *Accordingly, in treating this equation it will be assumed that* $k>0$. *With* $k>0$ *the developments of Horn* [9] *would apply only of* $a_{1,0} \neq 0$. *We impose no restrictions on* $a_{1,0}$.

Problem (B) falls in the following two cases.

CASE I. — *In* (B) *we have not all of the numbers*

$$a_{1,0}, \quad a_{1,1}, \quad \ldots, \quad a_{1,k-1} \tag{6}$$

zéro. Thus

$$a_{1,0} = a_{1,1} = \ldots = a_{1,l-1} = 0, \qquad a_{1,l} \neq 0 \qquad (0 \leqq l \leqq k-1). \tag{6a}$$

CASE II. — *In* (B) *all the numbers* (6) *are zero.*

In any case without any loss of generality it may be assumed that in (B)

$$0 = a_{1,k+1} = a_{1,k+2} = \ldots. \tag{7}$$

In fact, the transformation

(8) $$y(x) = g(x)\bar{y}(x),$$

where

(8 a) $$g(x) = 1 + g_1 x + g_2 x^2 + \ldots = e^{\int_0^x (a_{1,k+1} + a_{1,k+2} x + \ldots) dx}$$

will yield the equation

($\bar{\text{B}}$) $$x^{k+1} \bar{y}^{(1)}(x) = \bar{a}(x, \bar{y}) = \sum_{\nu=1}^{\infty} \bar{a}_\nu(x) \bar{y}^\nu,$$

in which

(9) $$\bar{a}_1(x) = a_{1,0} + a_{1,1} x + \ldots + a_{1,k} x^k,$$

(9 a) $$\bar{a}_\nu(x) = a_\nu(x) g^{\nu-1}(x) = \sum_{l=0}^{\infty} \bar{a}_{\nu\,l} x^l \qquad (\nu = 2, 3, \ldots),$$

the series involved in($\bar{\text{B}}$) and (9 a) being convergent for $|x| \leqq r$, $|y| \leqq \bar{\rho}$.

2. **Formal solution (case I).** — Functions $y_j(x)$ $(j = 1, 2, \ldots)$ will be determined so that the formal series

(1) $$s(x) = \sum_{j=1}^{\infty} y_j(x) c^j \qquad (c \text{ an arbitrary constant}).$$

will formally satisfy (B). We note that

(2) $$s^\nu(x) = \sum_{j=1}^{\infty} {}_\nu y_j(x) c^j \qquad ({}_\nu y_j(x) = 0 \text{ for } j < \nu),$$

where for $j \geqq \nu \geqq 2$

(2 a) $$\begin{cases} {}_\nu y_j(x) = \sum y_{n_1}(x)\, y_{n_2}(x) \ldots y_{n_\nu}(x) \\ (n_1 + n_2 + \ldots + n_\nu = j;\ 1 \leqq n_1, n_2, \ldots, n_\nu \leqq j - 1). \end{cases}$$

When $\nu \geqq 2$ the inequalities $n_1, n_2, \ldots, n_\nu \leqq j - 1$ will necessarily hold in view of the following considerations. Suppose one of the numbers $n_1, n_2, \ldots, n_\nu$, say n_1, is $\geqq j$ then, since $n_1 + n_2 + \ldots + n_\nu$ has more than one term (each being not less than unity), we would have $n_1 + \ldots + n_\nu \geqq j + 1$. Thus a contradiction would result.

On substituting (1) in (B) and on using (2) it follows that

$$(3)\quad x^{k+1}s^{(1)}(x) - a[x, s(x)] = \sum_{j=1}^{\infty}[x^{k+1}y_j^{(1)}(x) - a_1(x)y_j(x) - \Psi_j(x)]c^j = 0.$$

Thus, the $y_j(x)$ $(j \geqq 1)$ are to satisfy the equations

$$(4)\qquad x^{k+1}y_j^{(1)}(x) - a_1(x)y_j(x) = \Psi^j(x) \qquad (j = 1, 2, \ldots)$$

where $\psi_1(x) = 0$ and, for $j = 2, 3, \ldots,$

$$(4a)\quad \Psi_j(x) = \Psi_j(x, y_0, \ldots, y_{j-1})$$
$$= \sum_{\nu=2}^{j} a_\nu(x)\,{}_\nu y_j(x) = \sum_{\nu=2}^{j} a_\nu(x) \sum y_{n_1}(x) y_{n_2}(x) \ldots y_{n_\nu}(x)$$
$$(n_1 + \ldots + n_\nu = j;\ 1 \leqq n_1, \ldots, n_\nu \leqq j-1).$$

Accordingly, for $j = 1$, (4) will yield

$$(5)\qquad y_1(x) = t(x) = e^{\int a_1(x)x^{-k-1}\,dx} = e^{q(x)}x^{a_{1\,k}}$$

where

$$(6)\qquad q(x) = q_{k-l}x^{-(k-l)} + \ldots + q_1 x^{-1},$$

$$(6a)\qquad q_\nu = -\frac{1}{\nu}a_{1,k-\nu} \qquad (\nu = 1, 2, k-l),$$

$$(6b)\qquad q_{k-l} \neq 0 \qquad (k - l \geqq 1).$$

Thus, in Case I, the polynomial $q(x)$ is not identically zero.

Definition 1. — *Let* $\mathrm{R}(r_0)$, *where* $0 < r_0 \leqq r$, *denote a region satisfying the following conditions.*

1° *The boundary of* $\mathrm{R}(r_0)$ *consists of an arc of the circle* $|x| = r_0$ *and of curves* B_1, B_2 (*each with a limiting direction at the origin*) *extending from the extremities of this arc to the origin. Except at the origin* B_1 *and* B_2 *have no points in common.*

2° *The real part of* $q(x)$ [*cf.* (6), (6 *a*), (6 *b*)] *does not vanish interior* $\mathrm{R}(r_0)$; *moreover.*

$$(7)\qquad e^{q(x)} \sim 0 \qquad [x \text{ in } \mathrm{R}(r_0)].$$

3° *When x is in* $\mathrm{R}(r_0)$ *every u on the rectilinear segment* $(0, x)$ *is in* $\mathrm{R}(r_0)$.

4° *When x is in* $R(r_0)$ *and u is on the rectilinear segment* $(0, x)$ *the upper bound of*

$$(7a) \qquad |t(u)u^{-k-1}| \qquad [\text{cf. } (5)]$$

is attained at x.

It will be shown that in the Case I regions satisfying the above definition always exist. On writing

$$u = \rho e^{\sqrt{-1}\theta}, \qquad \bar{q}_i = L q_i \qquad (i = 1, \ldots, k-l),$$
$$b = a_{1,k} - k - 1 = b' + \sqrt{-1}\, b'',$$

it follows that

$$(7b) \qquad \begin{aligned} G(\rho, \theta) &= \log |t(u)u^{-k-1}| \\ &= q_{k-l} |\rho^{-(k-l)} \cos[(k-l)\theta - \bar{q}_{k-l}] + \ldots \\ &\quad + |q_1| \rho^{-1} \cos(\theta - \bar{q}_1) + b' \log \rho - b''\theta \end{aligned}$$

and

$$(7c) \qquad \begin{aligned} \rho \frac{\partial G}{\partial \rho} &= -(k-l) |q_{k-l}| \rho^{-(k-l)} \cos[(k-l)\theta - \bar{q}_{k-l}] + \ldots \\ &\quad - |q_1| \rho^{-1} \cos(\theta - \bar{q}_1) + b'. \end{aligned}$$

With $\varepsilon (> 0)$ a fixed number, however small, define sectors $W_m(r_0)$ with the aid of the inequalities

$$(7d) \qquad \begin{aligned} &\left(2m + \frac{1}{2}\right) \frac{\pi}{k-l} + \frac{\bar{q}_{k-l}}{k-l} + \varepsilon \\ &\leqq Lx \leqq \left(2m + \frac{3}{2}\right) \frac{\pi}{k-l} + \frac{\bar{q}_{k-l}}{k-l} - \varepsilon \qquad (m = 0, 1, \ldots; |x| \leqq r_0 \leqq r_0). \end{aligned}$$

For u in $W_m(r_0)$

$$(7e) \qquad |q_{k-l}| \cos[(k-l)\theta - \bar{q}_{k-l}] \leqq -\xi \qquad (< 0),$$

where ξ is independent of u, and $\xi \to 0$ when $\varepsilon \to 0$. Thus, by $(7\,c)$ and since $(k-l)|q_{k-l}| > 0$, it is inferred that

$$\rho \frac{\partial G}{\partial \rho} = -(k-l) |q_{k-l}| \rho^{-(k-l)} \cos[(k-l)\theta - \bar{q}_{k-l}] [1 + \nu(\rho, \theta)]$$

where $|\nu(\rho, \theta)| \leqq 1$ for n in $W_m(r_0)$ (r_0 sufficiently small). Whence on taking account of $(7\,b)$ it is concluded that

$$\frac{\partial}{\partial \rho} \log |t(u)u^{-k-1}| \geqq 0 \qquad [u \text{ in } W_m(r_0)].$$

Accordingly it is seen that $W_m(r_0)$ satisfies conditions 1°, 3°, 4°, of Definition 1. Now

$$(7f)\quad Rq(u) = |q_{k-l}|\,\rho^{-(k-l)}\cos\left[(k-l)\theta - \overline{q}_{k-l}\right] + \ldots + |q_1|\,\rho^{-1}\cos(\theta - \overline{q}_1)$$
$$= |q_{k-l}|\,\rho^{-(k-l)}\cos\left[(k-l)\theta - \overline{q}_{k-l}\right][1 + \nu^1(\rho, \theta)]$$

where, by $(7\,e)$, $|\nu'(\rho, \theta)| \leqq 1/2$ for u in $W_m(r_0)$, provided r_0 is sufficiently small. $Rq(u)$ can not then vanish in $W_m(r_0)$. Moreover, by $(7\,f)$ and $(7e)$

$$|e^{q(u)}| \leqq e^{-\rho^{-(k-l)}\xi.[1+\nu^1(\rho,\theta)]} \leqq e^{-\rho^{-(k-l)}\xi/2},$$

whenever u is in $W_m(r_0)$. Hence, in $W_m(r_0)$, (7) is satisfied. Thus it has been shown that *regions exist, for instance in the form of sectors* $W_m(r_0)$ [*cf.* $(7\,d)$] *which, when* r_0 *is sufficiently small, satisfy all the conditions of Definition* 1. With the aid of more extended developments existence of more general regions, satisfying Definition 1, can be established.

From (4) it follows that

$$(8)\qquad \psi_2(x) = a_2(x)\,y_1^2(x) = t^2(x)\,\varphi_2(x),$$

$$(8a)\qquad \varphi_2(x) = a_2(x) = \sum_{l=0}^{\infty} \varphi_{2,l}\,x^l,$$

the latter series being convergent for $|x| \leqq r$. On writing (4) in the form

$$(9)\qquad y_j(x) = t(x)\int^x u^{-k-1}\,\psi_j(u)\,\frac{du}{t(u)} \qquad (j = 2, 3, \ldots)$$

and on using (8) it is seen that

$$(10)\qquad y_2(x) = t(x)\int^x u^{-k-1}\,t(u)\,\varphi_2(u)\,du$$
$$= t(x)\int^x u^{-k-1+a_{1,k}}\,e^{q(u)}\,\varphi_2(u)\,du.$$

In consequence of the methods of asymptotic integration developed in (T_1) the following statement can be made.

Let

$$(11)\quad Q(x) = Q_\beta x^{-\beta} + Q_{\beta-1}x^{-(\beta-1)} + \ldots + Q_1 x^{-1} \qquad (Q_\beta \neq 0;\ \beta \geqq 1)$$

and let R *be a region of the type specified by Definition* 1 [*with*

$q(x) = Q(x)$, *and the conditions* (4°) *possibly omitted*]. *Suppose* $\varphi(x)$ *is analytic in* $R(x \neq 0)$ *and*

$$\varphi(x) \sim \sum_{i=0}^{\infty} \varphi_i x^i \qquad (x \text{ in } R). \tag{11a}$$

Then the integral

$$\int^x u^a e^{Q(u)} \varphi(u)\, du \tag{12}$$

can be evaluated as a function of the form

$$x^{a+\beta+1} e^{Q(x)} \zeta(x) \tag{12a}$$

where $\zeta(x)$ *is analytic in* $R(x \neq 0)$ *and*

$$\zeta(x) \sim \sum_{i=0}^{\infty} \zeta_i x^i \qquad [x \text{ in } R;\ \zeta_0 = -\varphi_0/(\beta Q_\beta)]. \tag{12b}$$

With the above in view and on taking account of (6), (6 *b*) it si concluded that the function $y_2(x)$ can be evaluated with the aid of (10) as an expression of the form

$$\begin{aligned} y_2(x) &= t(x)\, x^{-k-1+a_{1,k}+k-l+1} e^{q(x)} \eta_2(x) \\ &= x^{-l} t^2(x) \eta_2(x); \end{aligned} \tag{13}$$

here $\eta_2(x)$ is analytic in $R(r)$ $(x \neq 0)$ and

$$\eta_2(x) \sim \sum_{i=0}^{\infty} \eta_{2,i} x^i \qquad [x \text{ in } R(r_0)] \tag{13a}$$ [1].

By (4)

$$\psi_3(x) = a_2(x)\, 2y_1(x) y_2(x) + a_3(x) y_1^3(x). \tag{14}$$

Thus, in consequence of (5) and (13), (13 *a*),

$$\psi_3(x) = x^{-l} t^3(x)\, \varphi_3(x) \tag{14a}$$

where

$$\varphi_3(x) = 2a_2(x)\, \eta_2(x) + a_3(x) x^l$$

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[1] In the case corresponding to that treated by Horn we would have $a_{1,0} \neq 0$ and $l = 0$.

is a function analytic in R(r), such that

$$(14b) \qquad \varphi_3(x) \sim \sum_{\iota=0}^{\infty} \varphi_{3,\iota} x^{\iota} \qquad [x \text{ in } R(r)].$$

Suppose now that

$$(15) \quad \begin{cases} y_{\nu-1}(x) = x^{-(\nu-2)l} t^{\nu-1}(x) \eta_{\nu-1}(x), \\ \psi_{\nu}\ \ (x) = x^{-(\nu-2)l} t^{\nu}\ \ (x) \varphi_{\nu}\ \ (x) \qquad (\nu = 2, \ldots, j-1) \end{cases}$$ (1),

where the functions $\eta_{\nu-1}(x)$, $\varphi_{\nu}(x)$ are analytic in R(r) ($x \neq 0$) and

$$(15a) \quad \begin{cases} \eta_{\nu-1}(x) \sim \sum_{\iota=0}^{\infty} \eta_{\nu-1,\iota} x^{\iota} \\ \varphi_{\nu}\ \ (x) \sim \sum_{\iota=0}^{\infty} \varphi_{\nu,\iota}\ \ x^{\iota} \end{cases} \qquad [x \text{ in } R(r);\ \nu = 2, 3, \ldots, j-1].$$

For x in R(r), $e^{mq} \sim 0$ ($m = 1, 2, \ldots$). Hence application of the statement in italics, following (10), is possible to enable evaluation of the integral

$$\begin{aligned} (16) \qquad y_{j-1}(x) &= t(x)\int^{x} u^{-k-1}\psi_{j-1}(u)\frac{du}{t(u)} \\ &= t(x)\int^{x} u^{-k-1-(j-\nu)l} t^{j-2}(u)\,\varphi_{j-1}(u)\,du \\ &= t(x)\int^{x} u^{-k-1-(j-3)l+(j-2)a_{1,k}} e^{(j-2)q(u)} \varphi_{j-1}(u)\,du \\ &= t(x)\,x^{-l-(j-3)l+(j-2)a_{1,k}} e^{(j-2)q(x)} \eta_{j-1}(x) \\ &= x^{-(j-2)l} t^{j-1}(x) \eta_{j-1}(x) \end{aligned}$$

where $\eta_{j-1}(x)$ is analytic in R(r) ($x \neq 0$) and

$$(16a) \qquad \eta_{j-1}(x) \sim \sum_{\iota=0}^{\infty} \eta_{j-1\ \iota} x^{\iota} \qquad [x \text{ in } R(r)].$$

With the aid of (15), (15 a) (16), (16 a) it follows from (4) that

$$\begin{aligned} (17) \quad \psi_j(x) = \sum_{\nu=2}^{j} a_{\nu}(x) \sum x^{-(n_1-1)l-(n_2-1)l-\ \ -(n\ -1)l}\, t^{n_1+n_2+\ldots+n_{\nu}}(x) \\ \times\, \eta_{n_1}(x)\,\eta_{n_2}(x)\ldots\eta_{n_{\nu}}(x) \qquad [n_1+\ldots+n_{\nu}=j;\ 1 \leq n_1, \ldots, n_{\nu} \leq j-1]. \end{aligned}$$

---

(1) For the present it is assumed that j is a fixed integer ≥ 3.

Thus

$$(17a) \qquad \psi_j(x) = \sum_{\nu=2}^{j} a_\nu(x) x^{-jl+\nu l} t^j(x) \sum \eta_{n_1}(x)\eta_{n_2}(x)\ldots\eta_{n_\nu}(x)$$
$$= x^{-(j-2)l} t^j(x)\varphi_j(x),$$

where

$$(17b) \qquad \begin{cases} \varphi_j(x) = \sum_{\nu=2}^{j} a_\nu(x) x^{(\nu-2)l} \sum \eta_{n_1}(x)\ldots\eta_{n_\nu}(x) \\ (n_1+\ldots+n_\nu = j;\ 1 \leqq n_1,\ \ldots,\ n_\nu \leqq j-1). \end{cases}$$

Manifestly $\varphi_j(x)$ is analytic in $R(r)$ $(x \neq 0)$; moreover,

$$(17c) \qquad \varphi_j(x) \sim \sum_{i=0}^{\infty} \varphi_{j,i} x^i \qquad [x \text{ in } R(r)].$$

LEMMA I. — *Consider Case I* (§ 1) *of the equation* (B) (§ 1),

$$(B) \qquad x^{k+1} y^{(1)}(x) = a(x, y) \equiv \sum_{\nu=1}^{\infty} a_\nu(x) y^\nu(x).$$

Let $t(x)$ *be defined by* (5) *and let* $R(r)$ *be a region as specified by Definition* 1. *Equation* (B) *possesses a formal solution,*

$$(18) \qquad s(x) = \sum_{j=1}^{\infty} y_j(x) c^j = \sum_{j=1}^{\infty} x^{-(j-1)l} t^j(x)\eta_j(x) c^j.$$

Here c is an arbitrary constant, the $\eta_j(x)$ *are functions analytic in* $R(r)$ $(x \neq 0)$, *such that*

$$(18a) \qquad \eta_j(x) \sim \sum_{i=0}^{\infty} \eta_{j,i} x^i \qquad [j = 1, 2, \ldots;\ x \text{ in } R(r)],$$

$\eta_1(x) \equiv 1$; *moreover, the* $\eta_j(x)$ *are defined in succession with the aid of the relations* (9), (4).

Whenever the series (18) converges, for x in a region

$$R(r_0) 0 < r_0 \leqq r_0$$

and for $|c| \leqq c_0$ (c_0 sufficiently small), it will represent an analytic solution of (B); moreover, te above lemma would give detailed information regarding the behaviour of this solution, for x in $R(r_0)$, in the vicinity of the singular point. When $l=0$, convergence of (18) follows from the developments of Horn in consequence of the consideration of an equation

$$(B^*) \qquad x^{k+1}y^{*(1)}(x) = a^*(x, y^*) \equiv \sum_{\nu=1}^{\infty} a^*_\nu(x) y^{*\nu},$$

which is of the same character as (B) but is so chosen that it has a convergent formal series-solution of type (18); moreover, from the convergence of this series convergence of the original series may be inferred. Proceedings of this type appear to break dow for $l > 0$. However, it is of interest to observe that *the equation*

$$(19) \qquad |x|^{k+1}\frac{dy^*}{d|x|} = a^*_1(|x|, \zeta)y^* + \frac{dy^{*2}}{(1-\beta|x|)(1-\gamma y^*)} \qquad (0<|x|\leqq r_0<r)$$

is " dominant " with respect to (B) *provided* α, β, γ *are suitable positive numbers and provided*

$$(19a) \quad \begin{cases} a^*_1(|x|, \zeta) = \mathrm{R}(a_{1,l}x^l + a_{1,l+1}x^{l+1} + \ldots + a_{1,k-1}x^{k-1}) + |x|^k \mathrm{R} a_{1,k} \\ (\mathrm{R}u = \text{real part of } u;\ \zeta \text{ angle of } x) \quad (^1). \end{cases}$$

This equation, as can be easily observed, is of the same type in $|x|$ as the equation (B), whenever $0<|x|\leqq r_0$. It has a formal solution

$$(20) \qquad s^*(|x|) = \sum_{j=1}^{\infty} |x|^{-(j-1)l} t^{*j}(|x|)\, \eta^*_j(|x|) c^{*j},$$

where c^* is an arbitrary positive constant and

$$(20a) \qquad t^*(|x|) = t^*(|x|, \zeta) = e^{\int^{|x|} |x|^{-k-1} a^*_1(|x|, \zeta)\, d|x|}$$

The following can be demonstrated. The $\eta^*_j(|x|)$ are analytic in $|x|$ for $0<|x|\leqq r_0<r$; moreover, they are *positive* and

$$(20b) \quad \eta^*_j(|x|) \sim \sum_{i=0}^{\infty} \eta^*_{j,i}|x|^i \qquad [0<|x|\leqq r_0;\ \eta^*_0(|x|)\equiv 1;\ j=2, 3, \ldots],$$

(1) When $l=0$ (Horn's case) it is possible to simplify (19 a) by letting $a^*_1 = \mathrm{R}a_{1,0}$. In the general case, however, in (19 a) the numbers $a_{1,l+1}, \ldots, a_{1,k}$ cannot be replaced by zero.

provided (as is assumed throughout) that ζ is allowed to assume only the values of the angle of x, when x is restricted to the region $R(r_0)$ (¹).

Furthermore, there exists a constant η, independent of x and ζ, such that the functions $\eta_j(x)$ occurring in Lemma I, satisfy the inequalities

$$(21)\qquad |\eta_j(x)| < \eta^j \eta_j^*(|x|) \qquad (j = 2, 3, \ldots; \zeta = Lx)$$

for x in $R(r_0)$.

Thus, whenever the formal solution (20), *of* (19), *converges for* $0 < |x| \leqq r_0 \leqq r$ [ζ *restricted as in the statement following* (20 *b*)], *the formal solution* (18) *of* (B) *will converge for* x *in* $R(r_0)$ ($0 < r_0 \leqq r$) *and for* $|c| \leqq c_0$ (r_0 *or* c_0 *sufficiently small*).

When the series (20) diverges the « dominant » equation (19) is still useful, as with the aid of the inequalities (21) and in consequence of the special form of (19) it is always possible to obtain certain inequalities for the absolute values of the $\eta_j(x)$ occurring in (18) (²). But inasmuch as construction of an « actual » solution is concerned we shall have to employ certain asymptotic methods (*cf.* § 4, 6 below).

3. A transformation (Case I). — Let n be a positive integer. In the transformation

$$(1)\qquad y(x) = Y_n(x, c) + c^n \rho_n(x, c)$$

let

$$(1a)\qquad Y_n(x, c) = \sum_{j=1}^{n-1} \rho_j(x) c^j \qquad [\rho_j(x) = y_j(x);\ j = 1, \ldots, n-1;\ cf.\ (18), \S 2],$$

$\rho_n(x, c)$ will be a new variable. As a matter of convenience we shall write

$$(2)\qquad \begin{cases} \rho_j(x) = y_j(x) & (j = 1, 2, \ldots, n-1); \\ \rho_j(x) = 0 & (j = n, n+1, \ldots). \end{cases}$$

. . .

(¹) ζ (= angle of x) plays the role of a parameter of the equation (19).

(²) For the present these details will be omitted.

Before applying (1) to the equation (B) the function

$$(3)\qquad F_n(x, c) \equiv x^{k+1} Y_n^{(1)}(x, c) - a(x, Y_n)$$

will be first considered in some detail.

One taking account of (2) it is noted that $F_n(x, c)$ can be expressed in powers of c by means of an expression analogous to that involved in the second member of (3; § 2),

$$(4)\qquad F_n(x, c) \equiv \sum_{j=1}^{\infty} \left[x^{k+1} \rho_j^{(1)}(x) - a_1(x)\rho_j(x) - \overline{\psi}_j(x)\right] c^j;$$

here

$$(4a)\qquad \overline{\psi}_j(x) = \psi_j(x, \rho_0, \ldots, \rho_{j-1}) \qquad [\text{cf. } (4a), \S\, 2].$$

In consequence of (2) and (4a) by (4; § 2) it follows that

$$(5)\qquad \begin{cases} x^{k+1} \rho_j^{(1)}(x) - a_1(x)\rho_j(x) - \overline{\psi}_j(x) \\ = x^{k+1} y_j^{(1)}(x) - a_1(x) y_j(x) - \psi_j(x) = 0 \\ \qquad (j = 1, 2, \ldots, n-1); \end{cases}$$

$$(5a)\qquad x^{k+1} \rho_n^{(1)}(x) - a_1(x)\rho_n(x) - \overline{\psi}_n(x) = -\psi_n(x);$$

$$(5b)\qquad \begin{cases} x^{k+1} \rho_j^{(1)}(x) - a_1(x)\rho_j(x) - \overline{\psi}_j(x) = -\overline{\psi}_j(x) \\ \qquad (j = n+1, n+2, \ldots). \end{cases}$$

The $\overline{\psi}_j(x)$ are known functions given by relations of the type of (17a; § 2). Thus

$$(6)\qquad \overline{\psi}_j(x) = x^{-(j-2)l}\, t^j(x)\, \overline{\varphi}_j(x) \qquad (j = n, n+1, \ldots),$$

where the $\overline{\varphi}_j(x)$ are analytic in $R(r)$ ($x \neq 0$) and

$$(6a)\qquad \overline{\varphi}_j(x) \sim \sum_{i=0}^{\infty} \overline{\varphi}_{j,i} x^i \qquad [\text{in } R(r);\ \overline{\varphi}_n(x) = \varphi_n(x)].$$

This follows from the fact that the $\overline{\varphi}_j(x)$ are the same functions of the $\overline{\eta}_i(x)$ [the $\overline{\eta}_i(x)$ are the counterpart of the $\eta_i(x)$ of § 2] as the $\varphi_j(x)$ are of the $\eta_i(x)$. while $\overline{\eta}_i(x) = \eta_i(x)$ ($i = 1, 2, \ldots, n-1$) and $\eta_i(x) = 0$ ($i = n, n+1, \ldots$). Hence by virtue of (5), (5a), (5b) and (6), on writing

$$(7)\qquad c x^{-l}\, t(x) = \tau(x),$$

it follows that

$$(8)\qquad -F_n(x, c) = x^{2l}\sum_{j=n}^{\infty}\bar{\varphi}_j(x)\tau^j(x) \qquad [cf.\ (6a)].$$

The series in the second member of (8) converges for $|c|\leqq c_0$. x in $R(r_0)$ ($0 < r_0 \leqq r$; c_0 or r_0 sufficiently small).

Substituting (1) in (B) we get

$$(9)\quad x^{k+1}[Y_n^{(1)}(x, c) + c^n\rho_n^{(1)}(x, c)]$$
$$= a(x, Y_n + c^n\rho_n) = a(x, Y_n) + \alpha_1(x)c^n\rho_n + \alpha_2(x)c^{2n}\rho_n^2 + \ldots,$$

where

$$(9a)\qquad \alpha_m(x) = \frac{1}{m!}\frac{\partial^m a(x, y)}{\partial y^m}\Big]_{y=Y_n}$$
$$= a_m(x) + \sum_{i=1}^{\infty} C_m^{i+m}\, a_{i+m}(x)\, Y_n^i(x, c)$$
$$= a_m(x) + \beta_m(x, c) \qquad (m = 1, 2, \ldots).$$

On writing

$$(10)\qquad c^n\rho_n(x, c) = x^l\tau^n(x)\, z(x, c)$$

and on observing that

$$(10a)\qquad \frac{\tau^{(1)}(x)}{\tau(x)} = \frac{-l}{x} + \frac{t^{(1)}(x)}{t(x)} = \frac{-l}{x} + x^{-k-1}a_1(x)$$

it is concluded that

$$(10b)\quad c^n\rho_n^{(1)}(x, c)$$
$$= x^l\tau^n(x)\left[\left(\frac{l}{x} + n\frac{\tau^{(1)}(x)}{\tau(x)}\right) z(x, c) + z^{(1)}(x, c)\right]$$
$$= x^l\tau^n(x)\left[\left(-(n-1)\frac{l}{x} + n x^{-k-1}a_1(x)\right) z(x, c) + z^{(1)}(x, c)\right].$$

In consequence of (10), (10b), (3) and (8) from (9) we obtain

$$x^{l+k+1}\tau^n(x)\left[\left(-(n-1)\frac{l}{x} + n x^{-k-1}a_1(x)\right) z(x, c) + z^{(1)}(x, c)\right]$$
$$= x^{2l}\tau^n(x)\sum_{j\geqq n}\bar{\varphi}_j(x)\tau^{j-n}(x) + \alpha_1(x)x^l\tau^n(x)\, z(x, c)$$
$$+ \alpha_2(x)x^{2l}\tau^{2n}(x)\, z^2(x, c) + \ldots,$$

so that

$$(11)\qquad z^{(1)}(x, c) = a(x) + q'(x)\, z(x, c) + T[x, z(x, c)].$$

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Here

$$(11a) \qquad a(x) = x^{l-k-1}\tilde{a}(x)$$

where $a(x)$ is analytic in $R(r)$ $(x \neq 0)$ and

$$(11b) \qquad \tilde{a}(x) = \sum_{j \geqq n} \bar{\varphi}_j(x)\,\tau^{j-n}(x) \sim \varphi_n(x);$$

$$(11c) \quad q'(x) = x^{-k-1}[a_1(x) + \beta_1(x, c)] + (n-1)\frac{l}{x} - n x^{-k-1} a_1(x)$$

$$= x^{-k-1}\beta_1(x, c) + (n-1)\left(\frac{l}{x} - x^{-k-1} a_1(x)\right)$$

$$\sim (n-1)\left(\frac{l}{x} - x^{-k-1} a_1(x)\right).$$

Also

$$(12) \quad T[x, z(x, c)] = x^{l-k-1}\tau^n(x)\,[\alpha_2(x)\,z^2(x, c) + \alpha_3(x)x^l\tau^n(x)\,z^3(x, c) + \ldots + \alpha_m(x)x^{(m-2)l}\tau^{(m-2)n}(x)\,z^m(x, c) + \ldots]$$

$$= x^{l-k-1}\tau^n(x)\,W[x, z(x, c)]$$

where, by $(9a)$,

$$(12a) \qquad \alpha_m(x) \sim a_m(x) \qquad (m = 2, 3, \ldots).$$

The asymptotic relations $(11b)$, $(11c)$, $(12a)$ *are inxfor* x *in* $R(r_0)$. *The functions involved in the left members of these relations are analytic in* x *for* x *in* $R(r_0)$ $(x \neq 0)$, *provided* $|c| \leqq c_0$ (c_0 *a fixed number*). *The above asymptotic relationships are in the following sense. Let* $f(x, c)$ *denote any one of the functions*

$$(12b) \qquad \begin{cases} \tilde{a}(x) - \varphi_n(x), \quad q'(x) - (n-1)\left(\dfrac{l}{x} - x^{-k-1} a_1(x)\right), \\ \alpha_m(x) - a_m(x) \qquad (m = 1, 2, \ldots). \end{cases}$$

We have

$$f(x, c) \sim 0 + 0.x + 0.x^2 + \ldots \quad [\text{in } R(r_0)]$$

uniformly with respect to c $(|c| \leqq c_0)$; that is,

$$(13) \qquad |f(x, c)| < |x|^p f_{p,c_0} \qquad [x \text{ in } R(r_0);\ |c| \leqq c_0;\ p = 1, 2, \ldots)$$

where the constants $f_{p,\,e_0}$ are independent of x and c. The rapidity with which the functions $(12b)$ approach zero, as $x \to 0$ within $R(r)$, can be specified by inequalities more accurate than (13). However, such inequalities would not be necessary for our purposes.

On taking account of the way in which the series

$$W(x, z) \equiv W[x, z(x, c)]$$

had been derived *it is seen that* $W(x, z)$ *is analytic in* x *and* z for x *in* $R(r_0)$ ($x \neq 0$; $r_0 \leqq r$) and $|z| \leqq p(r_0)$ (1). Since from (7) it follows that by taking r_u suitably small the upper bound in $R(r_0)$ of $|\tau(x)|$ can be made as small as desired, *it is concluded that* $p(r_0)$ *can be made arbitrarily great by assigning a suitably small value to* r_0. *If* r_0 *is kept fixed the number* $p(r_0)$ *could be made as large as desired by taking* c_0 *sufficiently small.*

LEMMA 2. — *Let n be a fixed posive integer. The transformation*

$$(14) \quad y(x) = \sum_{j=1}^{n-1} x^{-(j-1)l} t^j(x) \eta_j(x) c^j + x^{-(n-1)l} t^n(x) z(x, c) c^n,$$

where $t(x)$ *and the* $\eta_j(x)$ *are functions involved in Lemma* 1, (§ 2), *wheen applied to* (B), *will yield the equation*

$$(15) \quad z^{(1)}(x, c) = a(x) + q'(x) z(x, c) + x^{l-k-1} \tau^n(x) W[x, z(x, c)].$$

The various functions here involved are specifield by (11 a), (11 b), (11 c), (12), (7), (12 a). *Moreover, these functions possess properties indicated in the several italicized statements following* (12 a).

4. Solution of the transformed equation. — We shall now proceed to obtain a solution of (11) bounded in $R(r_0)$ r_0 sufficiently small; $|c| \leqq c_0$; c_0 fixed). In consequence of (11 c; § 3),

$$(1) \quad g(x) = e^{\int^x q'(x)\,dx}$$
$$= e^{(n-1)\int^x [lx^{-1} - x^{-k-1} a_1(x)]\,dx} u(x, c) = x^{(n-1)l} t^{-n+1}(x) u(x, c)$$

where $u(x, c)$ is analytic in x (x in $R(r)$; $|c| \leqq c_0$) and

$$(1\,a) \qquad u(x, c) \sim 1, \quad \frac{1}{u(x, c)} \sim 1 \qquad [x \text{ in } R(r_0);\ |c| \leqq c_0) \quad (2).$$

---

(1) Throughout we keep $|c| \leqq c_0$.

(2) Throughout this section asymptotic relations are uniform [*cf.* the italicized statement in connection with (13; § 3)] with respect to c, provided $|c| \leqq c_0$.

Define $z_0(x)$ by the equation

$$(2) \qquad z_0^{(1)}(x) = a(x) + q'(x)\, z_0(x).$$

In consequence of (1) and (11 a; § 3)

$$(2a) \qquad z_0(x) = x^{(n-1)l}\, t^{-n+1}(x)\, u(x, c) \int^x \frac{t^{n-1}(x)\, x^{-(n-1)l}}{u(x, c)}\, x^{l-k+1}\, \tilde{a}(x)\, dx.$$

By virtue of the statement in italics following (10; § 2) and by (1 a) $z_0(x)$ can be evaluated as a function analytic in x, for x in $R(r)(|c| \leqq c_0)$, and such that

$$(3) \qquad z_0(x) \sim \sum_{i=0}^{\infty} z_{0,i}\, x^i \qquad [\text{in } R(r)].$$

In particular,

$$(3a) \qquad |z_0(x)| \leqq z_0 \qquad [x \text{ in } R(r);\ |c| \leqq c_0].$$

Take c_0 *sufficiently small so that* $p(r) > r_0$) (*cf.* italicized statement preceding Lemma 2 (§ 3). Whence, on writing

$$(4) \qquad z_0 = p(r) - 2\xi \qquad [0 < 2\xi < p(r)],$$

it follows that

$$(4a) \qquad z_0 \leqq p(r_0) - 2\xi,$$

whenever $0 < r_0 \leqq r$. Thus $W[x, z_0(x)]$ is analytic in x in $R(r_0)$; $|c| \leqq c_0$), the corresponding series being absolutely and uniformly convergent.

There exists a constant M, independent of x, z and c, so that

$$(5) \qquad |W(x, z)| < M \qquad [|z| \leqq p(r);\ x \text{ in } R(r);\ |c| \leqq c_0].$$

In consequence of the Cauchy theorem for analytic functions we have

$$(6) \qquad |W(x, z') - W(x, z'')| < \frac{M\, p(r)}{\xi^2} |z - z''| \qquad [x \text{ in } R(r);\ |c| \leqq c_0],$$

provided

$$(6a) \qquad |z'| < p(r) - \xi, \qquad |z''| < p(r) - \xi \quad (^1).$$

---

(1) The Cauchy theorem is applied to $W(x, z)$, considered as an analytic function of z, while $x[\text{in } R(r)]$ and $c(|c| \leqq c_0)$ are considered as parametric variables. The statement in connection with (5) plays an essenial role.

It will be also noted that

$$(7)\qquad |u(x, c)| < h, \qquad \left|\frac{1}{u(x, c)}\right| < h \qquad [x \text{ in } R(r);\ |c| \leqq c_0]$$

where h is independent of x and c. We choose G so that

$$(7a)\qquad G \geqq \frac{M\, p(r)}{\xi^2}$$

and so that

$$(8)\qquad |W[x, z_0(x)]| < G\, |z_0(x)| \qquad [x \text{ in } R(r);\ |c| \leqq c_0].$$

That such a selection is possible can be inferred from the form of the function W (x, z), as defined by (12; § 3).

Consider the function

$$(9)\qquad l_n(u) = |u^{-2(n-1)l-k-1}\, t^{2n-1}(u)| = |t(u)u^{-l}|^{2n-2}\, |t(u)u^{-k-1}|.$$

Recalling that the last factor above possesses the property (4°) of Def. 1 (§ 2) the same is seen to be true of $|t(u)u^{-l}|$ (1). Thus the following condition is satisfied.

1° *When x is in* R(r_0) *and u is on the rectilinear segment* $(0, x)$ *the upper bound of the function* $l_n(u)$, *defined by* (9), *is attained at x.*

Choose r_0 sufficiently small so that the following will also hold.

2° *With* c_0, G *and h fixed in accordance with previous statements, we have*

$$(10)\qquad c_0^n\, G h^2\, |x^{-(n-1)l-k}\, t^n(x)| \leqq \frac{\xi}{p(r) - \xi},$$

for all x in R(r_0).

It is observed that r_0 can be selected independent of n. It is also to be noted that in consequence of the above condition (1°) it follows that

$$(11)\qquad \int_0^x l_n(u)\, |du| \leqq |x|\, l_n(x) \qquad [x \text{ in } R(r_0)].$$

By virtue of the above choice of c_0 and r_0 it can be shown that the

(1) Since $|t(u)n^{-l}| = |t(u)u^{-k-1}|\, |u|^{k+1-l} (k + 1 - l > 0)$.

equations

$$(12)\quad z_j^{(1)}(x) = a(x) + q'(x)\, z_j(x) + \mathrm{T}[x,\, z_{j-1}(x)] \qquad (j = 1,\, 2,\, \ldots)$$

determine functions $z_j(x)\ j = 1,\, 2, \ldots)$, defined for x in $\mathrm{R}(r_0)$ and for $|c| \leqq c_0$. From (12) we have

$$(12a)\quad z_j(x) = g(x) \int_0^x \{ a(u) + \mathrm{T}[u,\, z_{j-1}(u)] \} \frac{du}{g(u)} \qquad (j \geqq 1).$$

On writing

$$(13)\quad w_0(x) = z_0(x), \qquad w_j(x) = z_j(x) - z_{j-1}(x) \qquad (j = 1,\, 2,\, \ldots)$$

in consequence of (12; § 3) and (1; § 3) it is inferred that

$$(14)\quad w_j(x) = g(x) \int_0^x \{ \mathrm{T}[u,\, z_{j-1}(u)] - \mathrm{T}[u,\, z_{j-2}(u)] \} \frac{du}{g(u)}$$
$$= g(x) \int_0^x u^{l-k-1} c^n u^{-nl} t^n(u) \{ \mathrm{W}[u,\, z_{j-1}(u)] - \mathrm{W}[u, z_{j-2}(u)] \} \frac{du}{g(u)}$$
$$\{ j = 1,\, 2,\, \ldots;\ \mathrm{W}[u,\, z_{-1}(u)] \equiv 0 \}.$$

By (6) and $6a$), provided

$$(15)\quad |z_{j-1}(u)|, \qquad |z_{j-2}(u)| < p(r) - \xi \qquad [u \text{ in } \mathrm{R}(r_0);\ |c| \leqq c_0],$$

it follows that

$$(15a)\quad \begin{cases} |\mathrm{W}[u,\, z_{j-1}(u)] - \mathrm{W}[u,\, z_{j-2}(u)] < \mathrm{G}\, |w_{j-1}(u)| \\ \qquad [u \text{ in } \mathrm{R}(r_0);\ |c| \leqq c_0;\ j = 2,\, 3,\, \ldots]. \end{cases}$$

For $j = 1$ ($15a$) has been previously established in (8).

By (1) and (7) from (14) we obtain

$$(16)\quad |w_j(x)| = |x^{(n-1)l} t^{-n+1}(x)\, u(x,\, c)|$$
$$\times \left| \int_0^x c^n u_1^{-2(n-1)l-k-1} t^{2n-1}(u_1) \frac{1}{u(u_1,\, c)} \right.$$
$$\left. \times \{ \mathrm{W}[u_1,\, z_{j-1}(u_1)] - \mathrm{W}[u_1,\, z_{j-2}(u_1)] \} du_1 \right|$$
$$< c_0^n\, |x^{(n-1)l} t^{-n+1}(x)|\, h^2 \mathrm{G} \int_0^x l_n(u)\, |w_{j-1}(u)|\, |du|$$
$$[\text{cf. } (9);\ x \text{ in } \mathrm{R}(r_0);\ |c| \leqq c_0],$$

provided (15) holds. Suppose (15) holds and assume that we have previously shown that

$$(17)\quad |w_{j-1}(u)| < \beta_{j-1} \qquad [u \text{ in } \mathrm{R}(r_0);\ |c| \leqq c_0].$$

Then by (16) and (11) it would follow that

$$|w_j(x)| < c_0'' \, |x^{(n-1)l} t^{-n+1}(x)| \, h^2 G \, |x| \, l_n(x) \, \beta_{j-1};$$

so that, by (9),

$$|w_j(x)| < c_0'' G h^2 \, |x^{-(n-1)l-k} t^n(x)| \, \beta_{j-1}.$$

Furthermore, in consequence of (10) it would follow that

$$(18) \qquad |w_j(x)| < \frac{\xi}{p(r)-\xi} \beta_{j-1} \qquad [x \text{ in } R(r_0);\ |c| \leqq c_0].$$

The above developments signify that *if for a fixed j inequalities* (15) and (17) are satisfied then (17) *will necessarily hold also for j increased by unity. Moreover, one may take*

$$(19) \qquad \beta_j = \frac{\xi}{p(r)-\xi} \beta_{j-1}.$$

Since

$$|w_0(u)| = |z_0(u)| \leqq z_0 = p(r) - 2\xi < p(r) - \xi$$

it follows that, for $j = 1$, inequalities (15) and (17) are satisfied with $\beta_0 = p(r) - 2\xi$.

Therefore (18) holds for $j = 1$. We have

$$(20) \qquad |w_1(x)| < \beta_1 = \frac{\xi}{p(r)-\xi} \beta_0.$$

Accordingly

$$|z_1(x)| = |w_0(x) + w_1(x)| < \beta_0 + \beta_1 < p(r) - \xi \qquad [x \text{ in } R(r_0);\ |c| \leqq c_0].$$

Whence it is seen that (15) and (17) are satisfied for $j = 2$ with β_1 defined by (20). By the above italicized statement it follows that

$$|w_2(x)| < \beta_2 = \frac{\xi}{p(r)-\xi} \beta_1 = \left(\frac{\xi}{p(r)-\xi}\right)^2 \beta_0 \qquad [x \text{ in } R(r_0)];$$

whence

$$|z_2(x)| = |w_0(x) + w_1(x) + w_2(x)|$$
$$< \beta_0 \left(1 + \frac{\xi}{p(r)-\xi} + \frac{\xi^2}{[p(r)-\xi]^2}\right) < p(r) - \xi \qquad [x \text{ in } R(r_0);\ |c| \leqq c].$$

By induction, in view of the statement subsequent to (18), it is inferred that

$$(21)\qquad |w_j(x)| < \beta_j = [p(r) - 2\xi]\frac{\xi^j}{[p(r)-\xi]^j},$$

$$(21a)\qquad |z_j(x)| \leqq |w_0(x) + w_1(x) + \ldots + w_j(x)| < \beta_0 + \beta_1 + \ldots + \beta_j$$
$$= [p(r) - 2\xi]\left[1 + \frac{\xi}{p(r)-\xi} + \ldots + \frac{\xi^j}{[p(r)-\xi]^j}\right] < p(r) - \xi$$
$$[j = 0, 1, 2, \ldots;\ x \text{ in } R(r_0);\ |c| \leqq c_0].$$

The series

$$(22)\qquad \lim_{j\to\infty} z_j(x) = \sum_{l=0}^{\infty} w_l(x) = z(x, c)$$

is absolutely and uniformly convergent for x in $R(r_0)$. The constituent terms of the series being analytic in $R(r_0)$ $(x \neq 0)$ the same will be true of the limiting function $z(x)$. Moreover, by $(21a)$ it follows that

$$(23)\qquad |z(x, c)| \leqq p(r) - \xi \qquad [x \text{ in } R(r_0);\ |c| \leqq c_0].$$

By (12)

$$w_j^{(1)}(x) = q(x)\, w_j(x) + x^{l-k-1} c^n x^{-nl} t^n(x) \{ W[x, z_{j-1}(x)] - W[x, z_{j-2}(x)]\}.$$

Accordingly, by $(21a)$, (15) and $(15a)$,

$$(24)\qquad \begin{cases} |w_j^{(1)}(x)| < |q(x)|\,|w_j(x)| + |x^{-(n-1)l-k-1} G c^n t^n(x)|\,|w_{j-1}(x)| \\ \qquad [x \text{ in } R(r_0);\ |c| \leqq c_0;\ j = 1, 2, \ldots]. \end{cases}$$

Whence in consequence of the absolute and uniform convergence of the series involved in (22) it is concluded that the series

$$\sum w_j^{(1)}(x)$$

possesses the same property for x in $R(r_0)$ $(|c| \leqq c_0)$. Hence

$$(25)\qquad \lim_j z_j^{(1)}(x) = \frac{d}{dx}\left[\lim_j z_j(x)\right] = z^{(1)}(x, c).$$

Furthermore, it follows without difficulty that

$$(25a)\qquad \lim_j T[x, z_{j-1}(x)] = T\left[x, \lim_j z_{j-1}(x)\right]$$
$$= T[x, z(x)] \qquad [x \text{ in } R(r_0);\ |c| \leqq c_0].$$

Application of (25) and $(25a)$ to (12) makes it évident that the function $z(x, c)$ defined by (22) satisfies equation (15; § 3).

LEMMA 3. — *Suppose that* $r_0 (0 < r_0 \leqq r)$ *is a number* (*independent of* n) *sufficiently small so that the condition of the italicized statement in connection with* (10) *holds. Let* c_0 *satisfy the statement subsequent to* (3a). *The equation* (15; § 3) *of Lemma* 2 (§ 3) *will then possess a solution* $z(x, c)$ *with the following properties.*

1° *The solution is analytic in* x *for* x *in* $R(r_0)$ $(x \neq 0\,;\,|c| \leqq c_0)$, *where* $R(r_0)$ *is a region as specified by Def.* 1 (§ 2).

2° *The solution is bounded uniformly with respect to* x *and* c *when* x *is in* $R(r_0)$ *and* $|c| \leqq c_0$ [*cf.* (23)].

3° *The solution is defined by the series* (22) [*cf.* (13), (12) *and* (14)], *which converges absolutely and uniformly for* x *in* $R(r_0)$ *and for* $|c| \leqq c_0$.

5. A reduction for the case II. — Turning our attention to Case II (§ 1) of the equation (B) (§ 1) we have

$$(B) \quad x^{k+1} y^{(1)}(x) = a(x, y) \equiv a_{1,k} x^k y + \sum_{\nu=2}^{\infty} a_\nu(x) y^\nu$$

$$= a_{1\,k} x^k y + a_{\nu_1}(x) y^{\nu_1} + a_{\nu_2}(x) y^{\nu_2} + \ldots$$

where

$$(1) \quad \begin{cases} a_{\nu_i}(x) = x^{m_i} \bar{a}_i(x); \qquad \bar{a}_i = \gamma_i + \ldots \\ [(\gamma_i = a_{\nu_i, m_i} \neq 0);\ (i = 1, 2, \ldots;\ 2 \leqq \nu_1 < \nu_2 < \ldots)] \end{cases} \quad (^1).$$

At least one of the functions $a_{\nu_i}(x)$ must contain a constant term. Thus

$$(1a) \qquad m_1, m_2, \ldots, m_{\alpha-1} \geqq 1, \qquad m_\alpha = 0 \qquad (\alpha \geqq 1).$$

Consider expressions

$$(2) \qquad \varphi_i(\beta) = -k + m_i + \beta(\nu_i - 1)$$

and define numbers $\beta(i)$ by the equations $\varphi_i[\beta(i)] = 0$; thus

$$(2a) \qquad \beta(i) = \frac{k - m_i}{\nu_i - 1} \qquad (i = 1, 2, \ldots).$$

By (1′a) in particular it follows that

$$(2b) \qquad \beta(\alpha) = \frac{k}{\nu_\alpha - 1} = \frac{l_1}{p_1} \qquad \left(\frac{l_1}{p} \text{ in its lowest terms}\right).$$

(1) In (1)... denotes positive integral powers of x.

Define β' as the greatest one of the numbers $\beta(1), \beta(2), \ldots, \beta(\alpha)$. Accordingly,

(2c) $\beta' = \beta(i_1) = \beta(i_2) = \ldots = \beta(i_H) > \beta(i) \qquad (i_1 < i_2 < \ldots < i_H \leqq \alpha)$

for $i \neq i_1, i_2, \ldots, i_h (i \leqq \alpha)$. It then follow that

$$(3) \qquad \varphi_{i_1}(\beta') = \varphi_{i_2}(\beta') = \ldots = \varphi_{i_H}(\beta') = 0, \qquad \varphi_i(\beta') > 0$$

for $i = 1, 2, \ldots (i \neq i_1, i_2, \ldots, i_H)$.

To prove this statement we not that the equalities of (3) hold in consequence of (2c). Suppose the inequalities of (3) do not all hold as stated. Then for some $i' (i' \neq i_1, i_2, \ldots, i_H)$ we would have

$$(3a) \qquad \varphi_{i'}(\beta') \leqq 0.$$

Case (1) $(i' > \alpha)$. — From the latter inequality it follows that $\beta' \leqq (k - m_{i'})/(\nu_{i'} - 1)$. But $\nu_i > \nu_\alpha$ so that $1/(\nu_{i'} - 1) < 1/(\nu_\alpha - 1)$; moreover, $m_{i'} \geqq 0$. Hence

$$\beta' < \frac{k}{\nu_\alpha - 1} = \beta(\alpha).$$

A contradiction arises since by definition β' is at least equal to $\beta(\alpha)$.

Case (2) $(i' \leqq \alpha)$. — The inequality $\beta' \leqq (K - m_{i'})/(\nu_{i'} - 1)$ would hold as above in consequence of (3a). By (2a) it would follow that

$$(4) \qquad \beta' \leqq \beta(i').$$

On noting that i' has the same properties as indicated in the statement in connection with (2c), it is observed that (4) is in contradiction to (2c) (with $i = i'$).

Consequently the italicized statement in connection with (|3) is seen to be true.

Application of the transformation

$$(5) \qquad y(x) = x^\beta \eta(x) \qquad \left(\beta = \beta' = \frac{l}{p}\right)$$

to (B) will result in

$$(6) \qquad x\,\eta^{(1)}(x) = (a_{1,k} - \beta)\,\eta(x) + \sum_{r=1}^{\infty} a_{\nu_r}(x)\, x^{(\nu_r - 1)\beta - k}\, \eta^{\nu_r}(x).$$

By (1) and (2)

$$(6a) \qquad a_{\nu_r}(x)\, x^{(\nu_r - 1)\beta - k} = x^{\varphi_r(\beta)}\, \overline{a}_r(x) \qquad (r = 1, 2, \ldots).$$

On writing

$$(6b)\quad \varphi_r(\beta) = -k + m_r + \frac{l}{p}(\nu_{l-1}) = \frac{N_r}{p} \qquad \text{(integers } N_r;\ r = 1, 2, \ldots)$$

it is observed that, in view of (3),

$$(6c)\qquad N_{\iota_1} = N_{\iota_2} = \ldots = N_{\iota_H} = 0, \qquad N_r > 0 \qquad (r \neq i_1, i_2, \ldots, \iota_H).$$

Hence (6) may be written in the form

$$(7)\qquad x\,\eta^{(1)}(x) = (a_{1,k} - \beta)\,\eta(x) + \sum_{\nu=2}^{\infty} b_\nu(x)\,\eta^\nu(x),$$

$$(7a)\qquad b_\nu(x) = \sum_{\iota=0}^{\infty} b_{\nu\,\iota}\,x^{\frac{\iota}{p}}$$

the numbers

$$(7b)\qquad b_{\nu_r,0} \qquad (r = \iota_1, \iota_2, \ldots, \iota_H)$$

being the only constant terms [in the various series (7a)] which are distinct from zero ([1]).

With the aid of the further transformation

$$(8)\qquad x = z^p$$

equation (7) assumes the form

$$(9)\qquad z\frac{d\eta}{dz} = p(a_{1,k} - \beta)\eta + \sum_{\nu=2}^{\infty} c_\nu(z)\eta^\nu,$$

$$(9a)\qquad c_\nu(z) = p b_\nu(z^p) = \sum_{\iota=0}^{\infty} c_{\nu\,\iota}\,z_\iota \qquad (\nu = 2, 3, \ldots)$$

where

$$c_{\nu\,0}\begin{cases} \neq 0 & (\nu = \nu_{\iota_1}, \nu_{\iota_2}, \ldots, \nu_{\iota_H}), \\ = 0 & (\nu \neq \nu_{\iota_1}, \nu_{\iota_2}, \ldots, \nu_{\iota_H}), \end{cases}$$

moreover, the series involved in (9) and (9a) converge for

$$(9b)\qquad |z| \leqq r_1, \qquad |\eta| \leqq \rho_1 \qquad (r_1 > 0;\ \rho_1 > 0).$$

Recalling certain developments due to Picard ([2]) and Poincaré ([3]),

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([1]) $b_{\nu_{\iota_1}} = \gamma_1$ and so on [*cf.* (1)].

([2]) PICARD, *Comptes rendus*, vol. 87, 1878, p. 430 and 743.

([3]) POINCARÉ, *Journal de l'Ecole polytechnique*, 1878, p. 13.

which are applicable to equations of the form (9), the following is inferred.

Case (1°). — *The real part of* $p(a_{1,k}-\beta)$ *is positive but* $p(a_{1,k}-\beta)$ *is not a positive integer.* Equation (9) has then a solution

$$(10) \qquad \eta = \sum_{i,j=0}^{\infty} \eta_{i,j} z^i (c z^q)^j \qquad [\eta_{0,0}=0;\ q=p(a_{1,k}-\beta)].$$

Here c is an arbitrary constant and the involved series converged for $|z| \leqq r_0$, $|c| \leqq c_0$ ($r_0, c_0 > 0$; r_0, c_0 sufficiently small).

Case (2°). — *The real part of* $p(a_{1,k}-\beta)$ *is* $\leqq 0$. There is then a solution

$$(10a) \qquad \eta = \sum_{i=1}^{\infty} \eta_i z_i \qquad (\text{convergent for } |z| \leqq r_0).$$

In the next section we shall consider the remaining case of (9):

Case (3°). — $q=p(a_{1,k}-\beta)$ *is a positive integer.*

Lemma 4. — *Consider Case II* (§ 1) *of the equation* (B) [*cf.* (1), (1 *a*)]. *Define* $\beta(i)$ ($i=1, 2, \ldots$) *by* (2*a*) *and let* $\beta=l/p$ *be the number specified by the italicized statement subsequent to* (2*b*). *The transformation*

$$y = x^{\beta}\eta, \qquad x = z^p$$

will yield equation (9), *which in the Case* (1°) *has a convergent solution* (10) *and which in the Case* (2°) *has a convergent solution* (10*a*).

6. The existence theorem (first order problem). — It will be now shown that in case (3°) (§ 5) équation (9; § 5) possesses a solution of the same type as in Case (1°; § 5). Put

$$(1) \qquad \eta = \sum_{j=1}^{\infty} y_j(z) c^j \qquad (c \text{ an arbitrary constant})$$

so that

$$(2) \qquad \eta^{\nu} = \sum_{j=1}^{\sigma} {}_{\nu}y_j(z) c^j \qquad [{}_{\nu}y_j(z) = 0 \text{ for } j < \nu),$$

$$(2a) \qquad \begin{cases} {}_{\nu}y_j(z) = \sum y_{n_1}(z)\ldots y_{n_\nu}(z) \\ [n_1+\ldots+n_\nu = j;\ 1 \leqq n_1, \ldots, n_\nu \leqq j-1;\ j \geqq \nu \leqq 2]. \end{cases}$$

Substituting in (9; § 5) we get

$$z\eta^{(1)} - q\eta - \sum_{\nu=0}^{\infty} c_\nu(z)\eta^\nu \equiv \sum_{j=1}^{\infty} f_j c^j = 0.$$

Thus, equating the f_j ($j = 1, 2, \ldots$) to zero, it follow that

(3) $$zy_j^{(1)}(z) - qy_j(z) = \psi_j(z) \qquad [\psi_1(z) = 0;\ j = 1, 2, \ldots]$$

where

(3a) $$\begin{cases} \psi_j(z) = \sum_{\nu=2}^{j} c_\nu(z) \sum y_{n_1}(z) \ldots y_{n_\nu}(z) \\ (n_1 + \ldots + n_\nu = j;\ 1 \leqq n_1, \ldots, n_\nu \leqq j-1;\ j = 2, 3, \ldots). \end{cases}$$

Whence

(4) $$y_1(z) = t(z) = z^q.$$

Also

(4a) $$\psi_2(z) = z^{2q}\varphi_2(z) \qquad [\varphi_2(z) = c_2(z) = a \text{ c. p. s.}] \ (^1).$$

The $y_j(z)$ ($j = 2, 3, \ldots$) are determined in succession, with the aid of (3a), by the relations

(5) $$y_j(z) = z^q \int_0^z u^{-q-1}\psi_j(u)\,du \qquad (j = 2, 3, \ldots).$$

Thus, by (4a),

(5a) $$y_2(z) = z^{2q}\eta_2(z) \qquad [\eta_2(z) = a \text{ c. p. s.}].$$

Let us assume that, for $\nu = 2, 3, \ldots, j-1$ (fixed $j \geqq 3$),

(6) $$y_{\nu-1}(z) = z^{(\nu-1)q}\eta_{\nu-1}(z) \qquad [\eta_{\nu-1}(z) a \text{ c. p. s.}],$$

(6a) $$\psi_\nu(z) = z^{\nu q}\varphi_\nu(z) \qquad [\varphi_\nu(z) a \text{ c. p. s.}].$$

In consequence of (5) (with j replaced by $j-1$) it is then inferred that

(6b) $$y_{j-1}(z) = z^q \int_0^z u^{(j-2)q-1}\varphi_{j-1}(u)\,du = z^{(j-1)q}\eta_{j-1}(z)$$

(1) The term « a c.p.s. » is to denote in a generic sense a power series in z convergent for $|z| \leqq r$, [*cf.* (9b; § 5)].

where $\eta_{j-1}(z) = a.\mathrm{c.p.s.}$ On the other hand, by (6*b*) and (6) from (3*a*) we would obtain

$$(7)\qquad \psi_j(z) = \sum_{\nu=2}^{j} c_\nu(z) \sum \eta_{n_1}(z)\ldots\eta_{n_\nu}(z)\, z^{(n_1+\ldots+n_\nu)q} = z^{jq}\varphi_j(z)$$

where $\varphi_j(z) = a.\mathrm{c.p.s.}$ By induction it therefore follows that *formulas* (6), (6*a*) hold for $\nu = 1, 2, \ldots$

The equation

$$(8)\qquad z\frac{d\zeta}{dz} = q\zeta + \alpha^2\zeta^2 + \alpha^3\zeta^3 + \ldots,$$

where α is a positive constant, is a special case of (9; § 5). In view of (4) we get

$$(8a)\qquad t\frac{d\zeta}{dt} = \zeta + \frac{\alpha^2}{q}\zeta^2 + \frac{\alpha^3}{q}\zeta^3 + \ldots.$$

In consequence of certain results of Horn (¹) this equation is seen to possess an absolutely convergent solution

$$(9)\quad \zeta = \zeta_1\, t(z)C + \zeta_2\, t^2(z)C^2 + \ldots \qquad (\text{the } \zeta_i \text{ constants};\ i = 1, 2, \ldots)$$

[C a positive arbitrary constant; $|C| \leqq C_0$; $|z| \leqq r(\alpha)$] where $C_0 > 0$, $r(\alpha) > 0$ and C_0 is sufficiently small. Write

$$(9a)\qquad \zeta_j\, t^j(z) = \bar{y}_j(z) = t^j(z)\bar{\eta}_j(z), \qquad \alpha^\nu = \bar{c}_\nu(z).$$

The functions corresponding to the $\Psi_j(z)$ will be

$$(9b)\quad \begin{cases} \psi_j(z) = \displaystyle\sum_{\nu=2}^{j} \alpha^\nu \sum \bar{\eta}_{n_1}(z)\ldots\bar{\eta}_{n_\nu}(z)\, t^{n_1+\ldots+n_\nu} = t^j \sum_{\nu=2}^{j} \alpha^\nu \sum \bar{\eta}_{n_1}(z)\ldots\bar{\eta}_{n_\nu}(z) \\ (n_1 + \ldots + n_\nu = j;\ 1 \leqq n_1, \ldots, n_\nu \leqq j-1;\ j = 2, 3, \ldots) \end{cases}$$

where

$$(9c)\qquad \sum_{\nu=2}^{j} \alpha^\nu \sum \bar{\eta}_{n_1}(z)\ldots\bar{\eta}_{n_\nu}(z) = \bar{\varphi}_j(z) = \sum_{\nu=2}^{j} \alpha^\nu \zeta_{n_1}\ldots\zeta_{n_\nu} = \text{const}.$$

In particular

$$\bar{\eta}_1(z) = 1 = \eta_1(z) \qquad \text{and} \qquad \bar{\varphi}_2(z) = \bar{c}_2(z) = \alpha^2.$$

(¹) Horn, *Journ. f. Math.*, vol. 119, 1898, p. 287.

Hence in view of $(4a)$, if we let $\alpha^2 \geqq |c_2(z)|$ $(|z| \leqq r_1)$, it will follow that $|\varphi_2(z)| \leqq \bar{\varphi}_2(z)$ so that

$$|\psi_2(z)| \leqq \bar{\psi}_2(z) \qquad (|z| \leqq r_1).$$

Whence by (5)

$$|y_2(z)| \leqq |z^q| \int_0^z |u|^{-q-1} |\psi_2(u)|\, d|u|$$

$$\leqq |z|^q \int_0^{|z|} |u|^{-q-1} \bar{\psi}_2(|u|)\, d|u| = \bar{y}_2(|z|) \quad (^1)$$

for $|z| \leqq r_1$. Suppose that, for $\nu = 2, 3, \ldots, j-1$) (fixed $j \geqq 3$),

$$(10) \qquad |y_{\nu-1}(z)| \leqq |z|^{(\nu-1)q}\, \bar{\eta}_{\nu-1}(|z|) \left[= \bar{y}_{\nu-1}(|z|)\right],$$

$$(10a) \qquad |\psi_\nu(z)| = |z|^{\nu q}\, \bar{\varphi}_\nu(|z|) \left[= \bar{\psi}_\nu(|z|)\right] \qquad (|z| \leqq r_1).$$

By (5) (with j diminished by unity) and by $(10a)$ we get

$$(11) \quad |y_{j-1}(z)| \leqq |z|^q \int_0^z |u|^{-q-1} |\psi_{j-1}(u)|\, d|u|$$

$$\leqq |z|^q \int_0^{|z|} |u|^{-q-1} \bar{\psi}_{j-1}(|u|)\, d|u| = \bar{y}_{j-1}(|z|) \qquad (|z| \leqq r_1).$$

Furthermore in consequence of (10) and (11) it is concluded that

$$(12) \quad |\psi_j(z)| \leqq \sum_{\nu=2}^{j} |c_\nu(z)| \sum |y_{n_1}(z) \ldots y_{n_\nu}(z)|$$

$$\leqq \sum_{\nu=2}^{j} \bar{c}_\nu(|z|)\, \bar{y}_{n_1}(|z|) \ldots \bar{y}_{n_\nu}(|z|) = \bar{\psi}_j(|z|) \qquad (|z| \leqq r_1),$$

provided α is sufficiently great so that

$$(13) \qquad |c_\nu(z)| \leqq \alpha^\nu \qquad (\nu = 2, 3, \ldots;\ |z| \leqq r_1) \quad (^2).$$

Thus by induction it has been shown that the inequalities (10)-$(10a)$ are valid for $\nu = 1, 2, \ldots$ [provided α satisfies (13)]. Conse, quently comparison of general terms in the series (1) and (9) will

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(1) At this step use is made of the fact that $\bar{\varphi}_2(u)$ is independent of u.

(2) Such a number α, independent of ν, can be found on account of the conditions of convergence satisfied by the series of the second member of (9; § 5).

give the inequalities

$$(14) \qquad |y_j(z)c^j| \leqq \bar{y}_j(|z|)C_0^j = \zeta_j t^j(|z|)C_0^j \qquad (|z| \leqq r_1).$$

These will hold for $j = 1, 2, \ldots$ and for $|c| \leqq C_0$. On taking account of the character of convergence of the series (9), inequalities (14) are seen to imply *absolute convergence of the series* (1), *whenever* $|c| \leqq C_0$ *and* $|z| \leqq r'$ [r' = least of the numbers r_1, $r(\alpha)$]. This establishes existence of an " actual " solution (1) [with (6) satisfied for $\nu = 1, 2, \ldots$] for the Case (3°) (§ 5).

The results obtained above, together with the previously obtained Lemmas 1, 2, 3, 4, enable formulation of the following theorem.

EXISTENCE THEOREM I. — *Consider equation* (B) *of* § 1 [*cf.* (5), (5*a*) *and* (7) *of* § 1]. *The problem falls in two cases, Case* I *and Case* II (*cf.* § 1).

CASE I. — *Let n be a fixed positive integer, however large. Let* $s(x)$ *be the formal solution of* (B), *as specified in Lemma* 1 *by* (18; § 2), (18*a*; § 2). *Moreover*, $R(r_0)(0 < r_0 \leqq r)$ *is to denote a region of the character specified in Definition* 1 (§ 2). *Positive numbers* r_0, c_0 (r_0 *independent of n*) *can be found so that there exists a solution* $y(x, c)$ (*c an arbitrary constant*) *of* (B), *satisfying the asymptotic relation*

$$(15) \qquad y(x, c) \sim s(x) \qquad [x \text{ in } R(r_0);\ |c| \leqq c_0]$$

in the following sense, We have

$$(16) \quad y(x, c) = \sum_{j=1}^{n-1} x^{-(j-1)l}\, t^j(x)\, \eta_j(x)\, c^j + x^{-(n-1)l}\, t^n(x)\, {}_n z(x, c)\, c^n.$$

Here the $\eta_j(x)$ *are functions analytic in* $R(r)(x \neq 0)$ *and satisfying asymptotic relations* (18*a*; § 2) *in the ordinary sense; furthermore*

$$(16a) \quad t(x) = e^{\int a_1(x) x^{-k-1} dx}, \qquad |{}_n z(x, c)| \leqq \beta_n \qquad [x \text{ in } R(r_0);\ |c| \leqq c_0].$$

Here the constant β_n *is independent of x and c. Moreover*, $y(x, c)$ *is analytic in x and c for x in* $R(r_0)(x \neq 0)$ *and* $|c| \leqq c_0$ (¹).

(¹) $x = 0$ in general is of course a singular point of $y(x, c)$. The region of analyticity can be shown to be more extensive.

Case II. — *Let* $\beta = \beta' = l/p$ (*the fraction in its lowest terms be the positive rational number defined at the beginning of* § 5. *There exist positive numbers* r_0, c_0 ($0 < r_0 \leq r$) *such that the following is true. If the real part of* q $[q = p(a_{1,k} - \beta)]$ *is positive* (*including the case when* q *is a positive integer*), *there exists a solution*

$$(17) \qquad y(x, c) = x^{\frac{l}{p}} \sum_{i,j=0}^{\infty} \eta_{i,j} x^{\frac{i}{p}} \left(c x^{\frac{q}{p}}\right)^j \qquad (\eta_{0,0} = 0) \quad (^1).$$

If the real part of q *is not positive there exists a solution*

$$(18) \qquad y(x) = x^{\frac{l}{p}} \sum_{i=1}^{\infty} \eta_i x^{\frac{i}{p}}.$$

In (17) *and* (18) *the* $\eta_{i,j}$ *and* η_i *are constants, and these series converge for* $|x| \leq r_0$, $|c| \leq c_0$.

7. Formal solutions ($n \geq 2$). — Consider the n — th order problem (A) as formulated in § 1. The developments contained in sections 7 and 8 will be given with $R(r_0)$ denoting a region satisfying the definition.

Definition 2. — *Let* $R(r_0)$ *denote any particular one of the set of regions such that the following holds.*

1° *According to the developments given in* (T_1), *the linear equation* (A_2; § 1) *possesses a full set of analytic solutions asymtotic, in* $R(r_0)$, *to the formal series* (2; § 1).

2° *No function of the set*

$$R[Q_i(x) - Q_j(x)] \qquad (i, j = 1, \ldots, n)$$

vanishes interior $R(r_0)$, *unless it is identically zero.*

3° *The boundary of* $R(r_0)$ *consists of an arc of the circle* $|x| = r_0$ *and of curves* B'. B'' *extending from the extremities of this arc to the origin. The curves* B', B'' *are regular in the sense of the term*

(1) When q is a positive integer in (17) we have $\eta_{i,0} = 0$ ($i = 0, 1, 2, \ldots$).

employed in (T_1) (¹). *Moreover, except at the origin,* B' *and* B'' *have no points in common.*

4° *For some of the polynomials* $Q(x)$ *of the set involved in* (2; § 1), *say for the polynomials* $Q_1(x)$, $Q_2(x)$, ..., $Q_m(x)$, *we have*

$$e^{Q_1(x)} \sim 0, \quad e^{Q_2(x)} \sim 0, \quad \ldots, \quad e^{Q_m(x)} \sim 0 \qquad [x \text{ in } R(r_0)]. \tag{1}$$

Existence of regions satisfying the above conditions 1°, 2°, 3° follows directly from the developments given in (T_1). The fact that $R(r_0)$ can be also so chosen that 4° is satisfied is a consequence of the following considerations. If $Q(x)$ is a polynomial in $x^{-\frac{1}{\alpha}}$ (α a positive integer), which is not identically zero, then there exist sectors (²), extending to the origin, in which exp. $Q(x) \sim 0$. On the other hand, by hypothesis *not all* the $Q(x)$ of (2; § 2) are identically zero.

With $R(r_0)$ defined as above first the case will be considered when for x interior $R(r_0)$ and for some δ ($1 \leqq \delta \leqq m$) we have

$$RQ_1(x) = RQ_2(x) = \ldots = RQ_\delta(x) > RQ_i(x) \qquad (i = \delta + 1, \ldots, m). \tag{1a}$$

Every $Q_j(x)$ which is not identically zero can be written in the form

$$Q_j(x) = q_{j:0}\, x^{-\frac{l_j}{\alpha_j}} + \ldots + q_{j:l_j-1}\, x^{-\frac{1}{\alpha_j}} \qquad (q_{j:0} \neq 0;\ l_j \geqq 1) \tag{1b}$$

where α_j, l_j are positive integers. Whenever $Q_j(x) \equiv 0$ we put $l_j/\alpha_j = 0$. *The greatest one of the numbers* l_j/α_j $(j = 1, \ldots, n)$ *will be designated as* l/α (positive integers l, α).

A formal solution of (A) will be found of the form

$$s(x) = \sum_{j=1}^{\infty} y_j(x) c_1^j \qquad (c_1 \text{ an arbitrary constant}). \tag{2}$$

We have

$$s^{(\lambda)\nu}(x) = \sum_{j=0}^{\infty} {}_{\lambda,\nu}y_j(x) c_1^j \qquad [{}_{\lambda,\nu}y_j(x) = 0 \text{ for } j < \nu] \tag{3}$$

(¹) In particular, every such curve would have a limiting direction at the origin.

(²) In fact, regions of a more general character.

where, for $j \geqq \nu \geqq 2$,

$$(3a) \quad \left\{ \begin{array}{l} {}_{\lambda,\nu}y_j(x) = \sum y_{n_1}^{(\lambda)}(x)\, y_{n_2}^{(\lambda)}(x) \ldots y_{n_\nu}^{(\lambda)}(x) \\ (n_1 + n_2 + \ldots + n_\nu = j;\ 1 \leqq n_1,\ n_2,\ \ldots,\ n_\nu \leqq j - 1); \end{array} \right.$$

$$(3b) \quad \left\{ \begin{array}{ll} {}_{\lambda,1}y_j(x) = y_j^{(\lambda)}(x) & (\lambda = 0, 1, \ldots;\ j = 1, 2, \ldots) \\ {}_{\lambda,0}y_j(x) = 0 & (j \geqq 1;\ \lambda \geqq 0;\ {}_{\lambda,0}y_0 = 1) \end{array} \right. \quad (\lambda \geqq 0).$$

Furthermore, for $i_0, i_1, \ldots, i_{n-1} \geqq 0$ $(i_0 + i_1 + \ldots + i_{n-1} \geqq 2)$,

$$(4) \quad \left\{ \begin{array}{l} s^{i_0}(x)\, s^{(1)i_1}(x) \ldots s^{(n-1)i_{n-1}}(x) = \sum_{j=2}^{\infty} y_j^{i_0, i_1, \ldots, i_{n-1}}(x)\, c_1^j \\ \left[y_j^{i_0, i_1, \ldots, i_{n-1}}(x) = 0 \text{ for } j < i_0 + i_1 + \ldots + i_{n-1} \right]. \end{array} \right.$$

Here, for $j \geqq i_0 + \ldots + i_{n-1}$ $(\geqq 2)$,

$$(4a) \quad y_j^{i_0, i_1, \ldots, i_{n-1}}(x) = \sum {}_{0,i_0}y_{j_0}(x)\, {}_{1,i_1}y_{j_1}(x) \ldots {}_{n-1,i_{n-1}}y_{j_{n-1}}(x)$$

where the $j_0, \ldots, j_{n-1}$ assume all the integral values subject to the conditions

$$(4b) \quad j_0 + j_1 + \ldots + j_{n-1} = j, \qquad 1 \leqq j_0,\ j_1,\ \ldots,\ j_{n-1} \leqq j - 1.$$

The inequalities $j_0, \ldots, j_{n-1} \leqq j - 1$ follow from the rest of (4*b*).

On taking (A) in the form (A₁) (§ 1) and on writing

$$(5) \quad a_1(x, y, y^{(1)}, \ldots, y^{(n-1)}) \equiv L_n(x, y) \equiv b_1(x)\, y^{(n-1)}(x) + \ldots + b_n(x)\, y(x)$$

it is noted that the coefficients $b_1(x), \ldots, b_n(x)$, involved in the differential operator L_n, are analytic for $|x| \leqq r$. In consequence of (4) substitution of (2) in (A_1),

$$(A_1) \quad s^{(n)}(x) - x^{-p} L_n(x, s) = a_2(x, s, s^{(1)}, \ldots, s^{(n-1)}),$$

will result, formally, in

$$(6) \quad s^{(n)}(x) - x^{-p} L_\iota(x, s) - x^{-p} a_2(x, s, s^{(1)}, \ldots, s^{(n-1)}) = \sum_{j=1}^{\infty} \Gamma_j(x)\, c_1^j = 0.$$

If (2) is to be a formal solution of (A_1) we must have

$$(7) \quad \Gamma_j(x) \equiv y_j^{(n)}(x) - x^{-p} L_n[x, y_j(x)] - x^{-p} \psi_j(x) = 0 \qquad (j = 1, 2, \ldots)$$

where

$$(7a)\quad \begin{cases} \psi_j(x) = \psi_j(x, y_1, \ldots, y_{j-1}) = \sum a_{i_0, i_1, \ldots, i_{n-1}}(x) y_j^{i_0, i_1, \ldots, i_{n-1}}(x) \\ [2 \leqq i_0 + i_1 + \ldots + i_{n-1} \leqq j;\ j = 2, 3, \ldots;\ cf.\ (4a), (3a), (3b)]. \end{cases}$$

The $\Psi_j(x)(j \geqq 1)$ are obtained with the aid of (4) as the coefficients in the formal expansion

$$a_2(x, s, s^{(1)}, \ldots, s^{(n-1)}) = \sum_{j=1}^{\infty} \psi_j(x) c_1^j.$$

In particular, in consequence of $(7a)$, $(4a)$ and $(3a)$,

$$(8)\quad \begin{cases} \psi_1(x) = 0, \quad \psi_2(x) = \sum a_{i_0, i_1, \ldots, i_{n-1}}(x) y_1^{i_0}(x) y_1^{(1)i_1}(x) \ldots y_1^{(n-1)i_{n-1}}(x) \\ (i_0, i_1, \ldots, i_{n-1} \geqq 0;\ i_0 + i_1 + \ldots + i_{n-1} = 2) \quad (^1) \end{cases}$$

and, in general, for $j \geqq 2$

$$(8a)\quad \psi_j(x) = \sum_{\varphi=2}^{j} \sum{}' \sum{}'' \prod_{\lambda=0}^{n-1} \sum{}''' a_{i_0 \ldots i_{n-1}}(x) y_{n_1}^{(\lambda)}(x) y_{n_2}^{(\lambda)}(x) \ldots y_{n_{i_\lambda}}^{(\lambda)}(x)$$

where

$$(8b)\quad \sum{}' = \sum \qquad (i_0 + i_1 + \ldots + i_{n-1} = \varphi;\ i_0, i_1, \ldots, i_{n-1} \geqq 0),$$

$$(8c)\quad \sum{}'' = \sum \qquad (j_0 + j_1 + \ldots + j_{n-1} = j - \varphi),$$

$$(8d)\quad \sum{}''' = \sum \qquad (n_1 + n_2 + \ldots + n_{i_\lambda} = j_\lambda + i_\lambda;\ n_1, n_2, \ldots, n_{i_\lambda} \geqq 1).$$

Thus $y_1(x)$ must be a solution of

$$(9)\qquad y_1^{(n)}(x) - x^{-p} \mathrm{L}_n[x, y_1(x)] = 0 \qquad [cf.\ (5)] \quad (^2).$$

Now $\mathrm{R}(r_0)$ satisfies condition (1°) of Def. 2 (§ 7). Hence there exists a linearly independent set of solutions of (9), $y_{1:1}(x), \ldots, y_{1:n}(x)$, analytic for x in $\mathrm{R}(r_0)(x \neq 0)$ and of the form

$$(10)\qquad y_{1:i}(x) = e^{Q_i(x)} x^{r_i} \eta_{1.i}(x) \qquad [cf.\ (2);\ \S 1]$$

---

(1) Thus $\Psi_2(x)$ is a homogeneous differential polynomial in $y_1(x)$ of order $(n-1)$ and of degree 2.

(2) This is equation (A_2) of § 1.

where

(10 a) $\eta_{1\,\iota}(x) = [x]_{m_\iota}$ $[m_\iota \geqq 0;\ \iota = 1, \ldots, n;\ x \text{ in } R(r_0)]$.

The symbol $[x]_\nu$ used above and to be employed in the sequel will be specified as follows.

DEFINITION 3. — *Let* R *denote a region extending to the origin. The expression* $[x]_\nu (\nu \geqq 0)$ *will then denote in a generic sense a function of the form*

(11) $$[x]_\nu = {}_0\eta(x) + {}_1\eta(x)\log x + \ldots + {}_\nu\eta(x)\log^\nu x$$

where the ${}_j\eta(x)\ (j = 0, \ldots, \nu)$ *are analytic in* x, *for* x *in* $R(x \neq 0)$, *and*

(11 a) $${}_j\eta(x) \sim {}_j\sigma(x) = \sum_{n=0}^{\infty} {}_j\sigma_n x^{\frac{n}{\alpha}} \qquad (x \text{ in R; integer } \alpha \geqq 1).$$

The formal, possibly divergent series,

(11 b) $${}_0\sigma(x) + {}_1\sigma(x)\log x + \ldots + {}_\nu\sigma(x)\log^\nu x$$

will be generically denoted as $\{x\}_\nu$ [*thus*, $[x]_\nu \sim \{x\}_\nu$ (x *in* R)].

A solution of (9) will be taken in the form

(12) $$\begin{cases} y_1(x) = y_{1\,1}(x) + k_2 y_{1\,2}(x) + \ldots + k_m y_{1\,m}(x) \\ [k_2, \ldots, k_m \text{ arbitrary constants; } |k_\iota| \leqq k \ (\iota = 2, \ldots, m)]. \end{cases}$$

Thus, by (10) and (1 a),

(13) $$y_1(x) = e^{Q_1(x)}[Y_1(x) + o_1(x)],$$

(13 a) $$Y_1(x) = x^{r_1}\eta_{1\,1}(x) + k_2 x^{r_2}\eta_{1\,2}(x) + \ldots + k_\delta x^{r_\delta}\eta_{1\,\delta}(x),$$

(13 b) $$o_1(x) = k_{\delta+1}e^{Q_{\delta+1}(x)-Q_1(x)}x^{r_{\delta+1}}\eta_{1\,\delta+1}(x) + \ldots + k_m e^{Q_m(x)-Q_1(x)}x^{r_m}\eta_{1\,m}(x),$$

where

(14) $$|e^{Q_{\delta+1}(x)-Q_\delta(x)}| \leqq 1, \quad \ldots, \quad |e^{Q_m(x)-Q_1(x)}| \leqq 1 \qquad [x \text{ in } R(r_0)].$$

In (14) the equality sign is possible only along the boundaries of $R(r_0)$.

CASE (A). — $R(r_0)$ contains a subregion $R'(r_0)$ [of the same description as $R(r_0)$] such that

(15) $$e^{Q_{\delta+1}(x)-Q_1(x)} \sim 0, \quad \ldots, \quad e^{Q_m(x)-Q_1(x)} \sim 0 \qquad [x \text{ in } R'(r_0)].$$

We then replace $R(r_0)$ *by* $R'(r_0)$ *but continue to use the symbol* $R(r_0)$. Case (A) is certain to occur, for instance, when the limiting directions at the origin of the two boundaries of the original $R(r_0)$ are distinct. $R'(r_0)$ can then be chosen as a region whose boundaries have at the origin correspondingly the same limiting directions as those of the boundaries of $R(r_0)$. *In Case* (A) *the original region* $R(r_0)$ *could be also used* [*whether* (15) *is or is not satisfied in* $R(r_0)$], *provided in* (12) *we let*

$$k_{\delta+1} = \ldots = k_m = 0.$$

Case (B). — $R(r_0)$ contains no subregion $R'(r_0)$ such that (15) holds. We then continue to use the original region $R(r_0)$; however, the constants $k_{\delta+1}, k_{\delta+2}, \ldots, k_m$ are all put equal to zero [thus $o_1(x)$ would be identically zéro].

In the remainder of this section the developments will be given for the Case (A) with the arbitrary constants $k_{\delta+1}, \ldots, k_m$ present. The corresponding results fro the Case (B) could be immediately inferred from those obtained for the Case (A). It would be necessary only to let $k_{\delta+1} = \ldots = k_m = 0$ and to attribute to $R(r_0)$ its original meaning.

In consequence of (13*b*), (15) and (10*a*)

$$(16) \qquad o_1(x) \sim 0 \qquad [x \text{ in } R(r_0);\ |k_{\delta+1}|, \ldots, |k_m| \leqq k'].$$

Here and in the sequel asymptotic relations (*with respect to* x) *are uniform with respect to the involved arbitrary constants;* that is, the absolute value of an asymptotic remainder is less than a number independent not only of x but also of the arbitrary constants.

It is also to be observed that, throughout, a derivative of a function will be asymptotic to the formal series obtained by differentiating term by term the series to which the given function is known to be asymptotic [1].

Let

$$Q(x) = q_0 x^{-\frac{l}{\alpha}} + \ldots + q_1 x^{-\frac{1}{\alpha}}$$

where $q_0 \neq 0$ and $l \geqq 1$ unless $Q(x) \equiv 0$, when l is defined as zero.

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[1] This is a consequence of the fact that the functions in question are solutions of certain differential equations.

In a generic sense

$$\frac{d}{dx}(e^{Q(x)}x^r[x]_N) = e^{Q(x)}x^{r-\left(1+\frac{l}{\alpha}\right)}[x]_N;$$

thus,

$$(17)\qquad \frac{d^\nu}{dx^\nu}(e^{Q(x)}x^r[x]_N) = e^{Q(x)}x^{r-\nu\left(1+\frac{l}{\alpha}\right)}[x]_N \qquad (\nu = 1, 2, \ldots).$$

Accordingly, by (13), (13*a*), (16), (10*a*) and (17),

$$(18)\qquad y_1^{(\nu)}(x) = e^{Q_1(x)}x^{-\nu\left(1+\frac{l}{\alpha}\right)}y_1{}_1^{[\nu]}(x) \qquad (\nu = 0, 1, \ldots),$$

$$(18a)\qquad y_1^{[\nu]}(x) = x^{r_1}[x]_{m_1} + k_2 x^{r_2}[x]_{m_2} + \ldots + k_\delta x^{r_\delta}[x]_{m_\delta} \qquad [x \text{ in } R(r_0)].$$

The function $[x]_{m_1}$, *involved in the second member of* (18*a*), *is a linear non-homogeneous expression in* $k_{\delta+1}, \ldots, k_m$; however, in the second member of the asymptotic relation satisfied by this function,

$$(18b)\qquad [x]_m \sim \{x\}_{m_1} \qquad [x \text{ in } R(r_0)],$$

the constants $k_{\delta+1}, \ldots. k_m$ do not enter. That is, it can be said that $[x]_{m_1}$ is *asymptotically independent* of $k_{\delta+1}. \ldots, k_m$, provided of course that $|k_{\delta+1}|, \ldots |k_m| \leqq k'$ (k' fixed).

Writing

$$(19)\qquad g_j(x) = k_j x^{r_j} \qquad (j = 1, \ldots, \delta;\ k_0 = 1)$$

we observe that a product of i functions (some of them possibly alike), each of the form

$$(19a)\qquad g_1(x)[x] + g_2(x)[x] + \ldots + g_\delta(x)[x] \qquad [x \text{ in } R(r_0)]$$

and with $\log x$ entering to at most the m — th power, is a function of the form

$$(19b)\qquad \sum_{n_1,\ldots,n_i=1}^{\delta} g_{n_1}(x)\, g_{n_2}(x)\ldots g_{n_i}(x)[x]_{im} \qquad [x \text{ in } R(r_0)]$$

Thus substitution of (12) [*cf*. (13)] in (8) will give in consequence of (18) and (18*a*)

$$(20)\qquad \psi_2(x) = e^{2Q_1(x)}x^{-2(n-1)\left(1+\frac{l}{\alpha}\right)}$$

$$\times \sum x^{\frac{1}{\alpha}A_{i_0\ldots i_{n-1}}} y_1^{[0]i_0}(x)\ldots y_1^{[n-1]i_{n-1}}(x)\, a_{i_0\ldots i_{n-1}}(x)$$
$$(i_0 + \ldots + i_n = 2)$$

where the $A_{i_0}, \ldots, i_{n-1}$ are non-negative integers. Hence, on using the form of the $a_{i_0}, \ldots, i_{n-1}^{(x)}$ it is concluded that

$$(21) \qquad \psi_2(x) = e^{2Q_1(x)} x^{-2(n-1)\left(1+\frac{l}{\alpha}\right)} \varphi_2(x),$$

$$(21a) \qquad \varphi_2(x) = \sum_{n_1, n_2 = 1}^{\delta} g_{n_1}(x) g_{n_2}(x)[x]_{2\overline{m}} \qquad [x \text{ in } R(r_0);\ cf.\ (19)]$$

where $\overline{m}$ is the greatest one of the numbers $m_1, \ldots, m_\delta$.

DEFINITION 4. — *Let $[x]_\nu$ have the significance assigned by Definition 3 and let the $g_i(x)$ be defined by (19). The symbol $[x]_N^\nu$, where ν is a positive integer, will in a generic sense denote a function of the form*

$$(22) \quad [x]_N^\nu = \sum g_{n_1}(x) g_{n_2}(x) \ldots g_{n_\nu}(x)[x]_N \qquad (1 \leqq n_1, n_2, \ldots, n_\nu \leqq \delta).$$

The symbol $\{x\}_N^\nu$ will denote a formal expression of the type

$$(22a) \qquad \{x\}_N^\nu = \sum g_{n_1}(x) g_{n_2}(x) \ldots g_{n_\nu}(x) \{x\}_N.$$

Using the above notation one may write

$$(23) \qquad \varphi_2(x) = [x]_{2\overline{m}}^2 \sim \{x\}_{2\overline{m}}^2 \qquad [x \text{ in } R(r_0)].$$

By virtue of the statement in connection with (17) from (10) and (10a) it follows that

$$(24) \qquad y_{1:i}^{(j-1)}(x) = e^{Q_i(x)} x^{r_i - (j-1)\left(1+\frac{l}{\alpha}\right)} y_{1:i}^{[j-1]}(x),$$

$$(24a) \qquad y_{1:i}^{[j-1]}(x) = [x]_{m_i} \qquad [i = 1, \ldots, n;\ x \text{ in } R(r_0)].$$

Matrix notation will now be introduced, with

$$(a_{i,j}) \qquad (i, j = 1, \ldots, n)$$

denoting a matrix of n rows and n columns, $a_{i,j}$ being the element in the i — th row and the j — th column. The determinant of $(a_{i,j})$ will be designated by the symbol $|(a_{i,j})|$. Accordingly, by (24), (24a).

$$(25) \quad |(y_{1:i}^{(j-1)}(x))| = e^{Q_1(x)+\ldots+Q_n(x)} x^{r_1+\ldots+r_n} \left|\left(x^{-(j-1)\left(1+\frac{l}{\alpha}\right)} y_{1:i}^{[j-1]}(x)\right)\right|$$

$$= e^{Q_1(x)+\ldots+Q_n(x)} x^{r_1+\ldots+r_n} x^{-r'} |(y_{1:i}^{[j-1]}(x))|$$

$$= e^{Q_1(x)+\ldots+Q_n(x)} x^{r_1+\ldots+r_n-\frac{r''}{\alpha}} [x]_0^*$$

$$([x]_0 = d_0 + d_1 x + \ldots)$$

the latter series being convergent for $|x| \leqq r$ (1). Here

$$(25a) \qquad \frac{r''}{\alpha} = \left(1+\frac{l}{\alpha}\right) + 2\left(1+\frac{l}{\alpha}\right) + \ldots + (n-1)\left(1+\frac{l}{\alpha}\right).$$

Not all the d_j ($j = 0, 1, \ldots$) are zero of course. Thus,

$$(25b) \qquad [x]_0 = x^w[d_w + d_{w+1}x + \ldots] \qquad (d_w \neq 0).$$

Hence

$$(26) \quad |(y_{1:i}^{(j-1)}(x))|^{-1} = e^{-Q_1(x)-\ldots-Q_n(x)} x^{-r_1-\ldots-r_n+\frac{r'}{\alpha}}(\gamma_0 + \gamma_1 x + \ldots) \qquad (\gamma_0 \neq 0),$$

$$(26a) \qquad \frac{r'}{\alpha} = \frac{r''}{\alpha} - w \qquad [\text{integer } w \geqq 0;\ cf.\ (25a)].$$

The series involved here are convergent for $|x| \leqq r$. The determinant of the matrix obtained by deleting the j — th row and the i — th column in the matrix $(y_{i:1}^{(j-1)}(x))$ is seen to be of the form

$$(27) \quad e^{Q_1(x)+\ldots+Q_{j-1}(x)+Q_{j+1}(x)+\ldots+Q_n(x)}\, x^{r_1+\ldots+r_{j-1}+r_{j+1}+\ldots+r_n-\gamma_i/\alpha}[x]_{m'_j},$$

where

$$(27a) \qquad \begin{cases} \dfrac{\gamma_i}{\alpha} = \left(1+\dfrac{l}{\alpha}\right)\left[\dfrac{n(n-1)}{2} - (i-1)\right], \\ m'_j = (m_1 + m_2 + \ldots + m_n) - m_j \qquad [x \text{ in } R(r_0)]. \end{cases}$$

Let $\bar{y}_{1:i,j}(x)$ denote the element in the i — th row and j — th column of the inverse of the matrix $(y_{1:i}^{(j-1)}(x))$; that is,

$$(28) \qquad (\bar{y}_{1:i,j}(x)) = (y_{1:i}^{(j-1)}(x))^{-1} \qquad (i, j = 1, \ldots, n).$$

Except for the sign, $\bar{y}_{1:i,j}(x)$ is given by the product of the functions defined by (26) and (27). Thus

$$(28a) \quad \bar{y}_{1:i,j}(x) = e^{-Q_j(x)} x^{-r_j+\gamma'_i/\alpha}[x]_{m'_j} \qquad [i, j = 1, \ldots, n;\ x \text{ in } R(r_0);\ cf.\ (27a)]$$

where, by (26a), (25a) and (27a),

$$(28b) \qquad \frac{\gamma'_i}{\alpha} = \frac{r'}{\alpha} - \frac{\gamma_i}{\alpha} = (i-1)\left(1+\frac{l}{\alpha}\right) - w.$$

---

(1) This follows by a known theorem regarding the Wronskian of an equation of the form (9).

For $j = 2, 3, \ldots$ equations (7) may be written in the form

$$(29)\qquad y_j(x) = \sum_{\lambda=1}^{n} y_{1:\lambda}(x) \int^x u^{-p}\, \psi_j(u, y_1, \ldots, y_{j-1})\, \bar{y}_{1:n,\lambda}(u)\, du.$$

This, in consequence of (10), (10*a*) and (28*a*) may generically be written as

$$(30)\qquad \begin{cases} y_j(x) = \sum_{\lambda=1}^{n} e^{Q_\lambda(x)} x^{r_\lambda} [x]_{m_\lambda} \int^x e^{-Q_\lambda(u)} u^{-r_\lambda+\gamma'-p} [u]_{m'_\lambda}\, \psi_j(u)\, du \\ \qquad [\gamma' = \gamma'_n/\alpha;\ cf.\ (27a),\ (28a),\ (28b)]. \end{cases}$$

By (21) and (23) the integrand displayed in (30), when $j = 2$, is of the form

$$(31)\qquad e^{2Q_1(u) - Q_\lambda(u)}\, u^{-r_\lambda+\gamma'-p-2(n-1)\left(1+\frac{l}{\alpha}\right)} [u]^2_{2\bar{m}+m'_\lambda} \qquad [u = R(r_0)],$$

since

$$(31a)\qquad [u]_{m_{\lambda'}} [u]^2_{2m} = [u]^2_{2\bar{m}+m_{\lambda'}}.$$

In consequence of the integration methods developed in (T_1) the following is true. *Let* $G(x)$ *be a polynomial in* $x^{-\frac{1}{\alpha}}$ *with the lowest power of* x, $x^{-\frac{\lambda}{\alpha}}$ $(\lambda \geqq 1)$, *actually present unless* $G(x) \equiv 0$, *when we define* λ *as zero* (¹).
Then

$$(32)\qquad \int^x e^{G(u)} u^\rho [u]_N\, du = e^{G(x)} x^{\rho + \left(1+\frac{\lambda}{\alpha}\right)} [x]_{N+1} \qquad [x \text{ in } R(r_0)].$$

Here $[u]_{N+1} = [u]_N$, *unless* $G(u) \equiv 0$ *and* $\rho + 1 = -\frac{\nu}{\alpha}$ (*integer* $\nu \geqq 0$) (²).

Since $g_{n_1}(u) \ldots g_{n_\nu}(u) = cu^q$ (c and q constants) it follows by the definition of $[u]^\nu_N$ [*cf.* (22)] that for x in $R(r_0)$

$$(33)\qquad \int^x e^{G(u)} u^\rho [u]^\nu_N\, du = e^{G(x)} x^{\rho + \left(1+\frac{\lambda}{\alpha}\right)} [x]^\nu_{N+1} = e^{G(x)} x^{\rho+1} [x]^\nu_{N+1}$$

(¹) There are some conditions [*cf.* (T_1)] which $G(x)$ must satisfy with reference to $R(r_0)$. However, in subsequent applications of (32) $G(u)$ is always a function satisfying these conditions.

(²) In the latter case $[x]_{N+1}$ will contain $\log_x^{N+1}$, the coefficient of this power of the logarithm being $cx^{\frac{\nu}{\alpha}}$ (c = constant).

where $G(u)$ has the same meaning as in (32). It will be assumed that no curve $R(jQ_1(x) - Q_\lambda(x)) = 0$ ($j = 2, 3, \ldots; \lambda = \delta + 1, \ldots, n$) is interior $R(r_0)$.

On noting that the integrand displayed iu (30; $j = 2$) is of the form (31) and on using (33) it follows that

$$(34) \qquad y_2(x) = \sum_{\lambda=1}^{n} e^{Q_\lambda(x)} x^{r_\lambda} [x]_{m_\lambda} e^{2Q_1(x) - Q_\lambda(x)} x^{-r_\lambda - \beta} [x]^2_{2\bar{m} + m'_\lambda}$$

so that, in view of (27a) and (28b),

$$(35) \qquad y_2(x) = e^{2Q_1(x)} x^{-\beta} [x]^2_{m(2)} \qquad [x \text{ in } R(r_0)],$$

$$(35a) \qquad \beta = (n-1)\left(1 + \frac{l}{\alpha}\right) + w + p - 1 \qquad (> 0),$$

$$(35b) \qquad m(2) = 2\bar{m} + m_1 + \ldots + m_n^{\circ} \quad (^3).$$

We have previously chosen $y_1(x)$ as a function of the form

$$(36) \qquad y_1(x) = e^{Q_1(x)} [x]^1_{m(1)} \qquad [m(1) = \bar{m};\ x \text{ in } R(r_0)].$$

On the other hand, by (35a)

$$(37) \qquad \psi_2(x) = e^{2Q_1(x)} x^{-2\beta + 2(w+p-1)} [x]^2_{n(2)} \qquad [n(2) = 2\bar{m}].$$

Suppose now that, for x in $R(r_0)$,

$$(38) \quad y_\nu(x) = e^{\nu Q_1(x)} x^{-(\nu-1)\beta} [x]^\nu_{m(\nu)} \qquad (\nu = 1, 2, \ldots, j-1),$$

$$(38a) \quad \psi_\nu(x) = e^{\nu Q_1(x)} x^{-\nu\beta + 2(w+p-1)} [x]^\nu_{n(\nu)} \qquad (\nu = 2, \ldots, j-1) \quad (^1).$$

With the aid of (38) and of (8a) the form of $\Psi_j(x)$ will be determined. In consequence of (17)

$$(39) \quad y_\nu^{(\lambda)}(x) = e^{\nu Q_1(x)} x^{-(\nu-1)\beta - \lambda\left(1 + \frac{l}{\alpha}\right)} [x]^\nu_{m(\nu)} \qquad (\nu = 1, \ldots, j-1;\ \lambda = 0, 1, \ldots).$$

Therefore the product

$$(40) \qquad a_{i_0 \ldots i_{n-1}}(x)\, y_{n_1}^{(\lambda)}(x)\, y_{n_2}^{(\lambda)}(x) \ldots y_{n_{i_\lambda}}^{(\lambda)}(x),$$

---

(3) Use is made of the fact that the functions $2Q_1(u) - Q_\lambda(u)$ ($\lambda = 1, \ldots, n$) are allnot identically zero.

(1) For the present j is a fixed integer $\geqq 3$. We take $m(1) \leqq m(2) \leqq \ldots$ and $n(1) \leqq n(2) \leqq \ldots$ For $j = 2$ formulas (38), (38a) have been established previously.

involved in $(8a)$ and with the subscripts satisfying $(8d)$, is given by

$$(41) \qquad e^{(j\lambda+i\lambda)Q_1(x)}x^{-j\lambda\beta-\lambda i_\lambda\left(1+\frac{l}{\alpha}\right)}[x]^{i_\lambda+i_\lambda}_{M_{j\lambda+i\lambda}}$$

where

$$(41a) \qquad \begin{cases} M_{j\lambda+i\lambda}=\max\{m(n_1)+m(n_2)+\ldots+m(n_{i_\lambda})\} \\ (n_1+n_2+\ldots+n_{i_\lambda}=j_\lambda+i_\lambda;\ n_1,\ n_2,\ \ldots,\ n_{i_\lambda}\geqq 1). \end{cases}$$

Extending the summation symbol $(8d)$ (with respect to $n_1, n_2, \ldots, n_{i_\nu}$) over the terms (40), by virtue of (41) we obtain a function F_λ of the form (41) $(\lambda=0,\ 1,\ \ldots,\ n-1)$. Accordingly, since by $(8c)$

$$j_0+j_1+\ldots+j_{n-1}=j-\varpi$$

and, by $(8b)$.

$$i_0+i_0+\ldots+i_{n-1}=\varpi$$

it follows that

$$(42) \qquad \prod_{\lambda=0}^{n-1}F_\lambda=e^{jQ_1(x)}x^{-(j-\varphi)\beta-[i_1+2i_2+\ldots+(n-1)i_{n-1}](1+l/\alpha)}\times[x]^{j}_{M_{j_0+i_0}+M_{j_1+i_1}+\ldots+M_{j_{n-1}+i_{n-1}}}.$$

Now. under $(8b)$,

$$i_1+2i_2+\ldots+(n-1)i_{n-1}\leqq(n-1)\varphi.$$

Thus, on writing

$$(42a) \qquad \begin{cases} M_{j\varphi}=\max\{M_{j_0+i_0}+M_{j_1+i_1}+\ldots+M_{j_{n-1}+i_{n-1}}\} \\ (j_0+j_1+\ldots+j_{n-1}=j-\varpi;\ i_0+i_1+\ldots+i_{n-1}=\varphi,\ i_0,\ i_1,\ \ldots\ i_{n-1}\geqq 0), \end{cases}$$

from (42) it follows that

$$(42b) \qquad \prod_{\lambda=1}^{n-1}F_\lambda=e^{jQ_1(x)}x^{-(j-\varphi)\beta-(n-1)\varpi\left(1+\frac{l}{\alpha}\right)}[x]^{j}_{M_{j,\varpi}}.$$

Applying the summation symbol $(8c)$ (with respect to $j_0, j_1, \ldots, j_{n-1}$) to the product $(42b)$ we obtain a function ${}_jF_{\varphi\, i_0, i_1, \ldots, i_{n-1}}$ of the same form as the second member of $(42b)$. With the summation with respect to $i_0, i_1, \ldots, i_{n-1}$ extended as specified by $(8b)$ it follows that

$$(43) \qquad \sum{}'\,{}_jF_{\varphi\, i_0, i_1, \ldots, i_{n-1}}={}_jF_\varphi,$$

is a function also of the type of the second member of (42 *b*). Thus, since by (35 *a*)

$$-(j-\varphi)\beta-(n-1)\varphi\left(1+\frac{l}{\alpha}\right)=-j\beta+\varphi(w+p-1),$$

we shall have ${}_j F_\varphi$ of the form

$$(43a)\qquad e^{jQ_1(x)}x^{-j\beta+\varphi(w+p-1)}[x]'_{M_{l,\varphi}}=e^{jQ_1(x)}x^{-j\beta+2(w+p-1)}[x]'_{M_{j,\varphi}},$$

provided $\varphi \geqq 2$ [1]. Hence in consequence of (8 *a*)

$$(44)\qquad \psi_j(x)=\sum_{\varphi=2}^{j}{}_jF_\varphi=e^{jQ_1(x)}x^{-j\beta+2(w+p-1)}[x]'_{n(j)},$$

where

$$(44a)\qquad n(j)=\max M_{j,\varphi}\qquad [\varphi=2,3,\ldots,j,\ cf.\ (42a),(41a)],$$

Thus (38), (38 *a*) imply validity of (38*a*) for $\nu=j$. On making use of (44) it will be proved that (38) holds for $\nu=j$. By (44) the integrand displayed in (30) would be of the form

$$(45)\qquad \left\{\begin{array}{l} e^{jQ_1(u)-Q_\lambda(u)}u^{N_{j,\lambda}}[u]'_{n(j)+m_\lambda} \\ \left(N_{j,\lambda}=-r_\lambda+\dfrac{\gamma'_n}{\alpha}-p-j\beta+2(w+p-1)\right). \end{array}\right.$$

By (28 *b*; $i=n$) and (35 *a*)

$$(45a)\qquad N_{j,\lambda}=-r_\lambda-(j-1)\beta-1.$$

Accordingly, by (33) it follows that the integral displayed in (30) can be evaluated as a function of the form

$$(46)\qquad e^{jQ_1(x)-Q_\lambda(x)}x^{-r_\lambda-(j-1)\beta}[x]'_{n(j)+m'_\lambda}\quad [2].$$

The product of the latter function by

$$e^{Q_\lambda(x)}x^{r_\lambda}[x]_{m_\lambda}$$

is a function ${}_jg_\lambda(x)$ of the form

$$(46a)\qquad e^{jQ_1(x)}x^{-(j-1)\beta}[x]'_{n(j)+m_1+m_2+\ \ +m_n}$$

---

[1] This follows from the inequality $w+p-1 \geqq 0$.

[2] For $j \geqq 2$ [case (A)] no function $jQ_1(u)-Q_\lambda(u)$ ($\lambda=1,\ldots,n$) vanishes identically.

since by $(27\,a)$, $m'_\lambda + m_\lambda = m_1 + m_2 + \ldots + m_n$. Hence

$$(47)\qquad y_j(x) = \sum_{\lambda=1}^{n} {}_j g_\lambda(x) = e^{jQ_1(x)}x^{-(j-1)\beta}[x]^j_{m(j)},$$

$$(47\,a)\qquad m(j) = n(j) + m_1 + m_2 + \ldots + m_n \qquad [cf.\ (44\,a)].$$

Thus by induction it has been proved that for x in $R(r_0)$, relations (38), (38 a) hold for all $\nu = n, 3, \ldots$. The rate at which the numbers $n(\nu)$ may increase with ν can be inferred from (47 a), (44 a), (42 a), (41 a) and from the relations

$$(48)\qquad \begin{cases} m(2) = 2\overline{m} + m_1 + \ldots + m_n \\ n(2) = 2\overline{m} \end{cases} \qquad (\overline{m} = \max[m_1, m_2, \ldots, m_\delta]).$$

In the Case (B) the corresponding relations are slightly modified.

If (1 a) does not hold, so that instead, for some $m^*(\delta \leq m^* < m)$ we have

$$(49)\qquad \begin{cases} RQ_1(x) = \ldots = RQ_\delta(x) > RQ_i(x) \\ [i = \delta+1, \delta+2, \ldots, m^*;\ x \text{ interior } R(r_0)], \end{cases}$$

while

$$(49\,a)\qquad \begin{cases} RQ_1(x) = \ldots = RQ_\delta(x) < RQ_i(x) \\ (i = m^*+1, m^*+2, \ldots, m;\ x \text{ interior } R(r_0)], \end{cases}$$

the preceding developments can be repeated, the only changes being the following. Throughout. m is replaced *by* m^*; moreover, we let

$$(49\,b)\qquad k_{m^*+1} = k_{m^*+2} = \ldots = k_m = 0.$$

The functions $y_\nu(x)$, $\psi_\nu(x)$ $[\nu = 1, 2, \ldots;\ cf.\ (38), (38\,a)]$ are asymptotically independant of some of the arbitrary constants. On noting how these constants enter in the involved functions $[x]^\nu_{m(\nu)}$, $[x]^\nu_{n(\nu)}$ the following lemma can be stated.

LEMMA 4. — *Let the* $Q_j(x)$ $[j = 1, \ldots, n;\ cf.\ (1\,b)]$ *be the polynomials associated with a set* (2; § 1) *of n linearly independent formal solutions of the linear equation* (A_2; § 1) (1). *Let* $R(r_0)$ *be a region satisfying definition* 2. *Unless we have (have* (49), (49 a) *with* $m^* = \delta$, *it is assumed that no curve*

$$R(jQ_1(x) - Q_\lambda(x))$$

is interior $R(r_0)$.

(1) In this section (A_2; § 1) is written in the form (9).

When (la) *holds equation* (A; § 1) *has a formal solution* $s(x)$,

$$(50)\quad s(x)=\sum_{j=1}^{\infty} y_j(x)c_1^j, \qquad y_j(x)=e^{jQ_1(x)}x^{-(j-1)\beta}\eta_j(x) \qquad (j=1, 2, \ldots),$$

where $\beta=(n-1)\left(1+\frac{l}{\alpha}\right)+w+p-1$, c_1 *is an arbitrary constant and, for* $j=1, 2, \ldots$,

$$(50a)\quad \begin{cases} \eta_j(x)=\sum \eta_{j:\alpha_1,\alpha_2,\ldots,\alpha_m}(x)[h_1(x)]^{\alpha_1}[h_2(x)k_2]^{\alpha_2}\ldots[h_m(x)k_m]^{\alpha_m} \\ \qquad (\alpha_1+\alpha_2+\ldots+\alpha_m=j;\ \alpha_1, \alpha_2, \ldots, \alpha_m \geqq 0). \end{cases}$$

In (50 *a*) *the* $h_i(x)$ *are analytic in* $R(r_0)$ $(x\neq 0)$. *Moreover*,

$$(50b)\quad \begin{cases} h_i(x)=x^{r_i} \\ [i=1, 2, \ldots, \delta;\ cf.\ (2), \S 1;\ h_i(x)\sim 0 \text{ in } R(r_0);\ i=\delta+1, \ldots, m)] \end{cases}$$

and $k_2, \ldots, k_m$ *are arbitrary constants. The* $\eta_{j:\alpha_1,\alpha_2,\alpha_m}(x)$ *are functions analytic in* $R(r_0)$ $(x\neq 0)$ *and of the form*

$$(50c)\quad \eta_{j:\alpha_1,\ldots,\alpha_m}(x)=[x]_{m(j)}\sim\{x\}_{m(j)} \qquad [x \text{ in } R(r_0);\ cf.\ (47a)],$$

the involved symbols having the significance indicated in Definition 3. *In the Case* (A) *the region* $R(r_0)$ *is selected so that* (15) *holds for* x *in* $R(r_0)$. *When* (15) *cannot be satisfied or when this condition is deleted the constants* $k_{\delta+1}, k_{\delta+2}, \ldots, k_m$ *are all put equal to zero.*

The alternative of (1 *a*) *is given by* (49), (49 *a*). *We then have a formal solutions as given above, except that in* (50 *a*) *and* (50 *b*) m *is replaced by the smaller number* m^* $(\geqq\delta)$, *involved in* (49), (49 *a*). *Moreover, unless the region* $R(r_0)$ *can be so selected that*

$$(51)\quad e^{Q_{\delta+1}(x)-Q_1(x)}\sim 0, \quad \ldots, \quad e^{Q_{m^*}(x)-Q_1(x)}\sim 0 \qquad [x \text{ in } R(r_0)],$$

the constants $k_{\delta+1}, k_{\delta+2}, \ldots, k_{m^*}$ *are to be all replaced by zero.*

In every case $y_1(x)$ *is a solution of the linear problem* (9) [*cf.* (12), (10)].

NOTE. — The function $e^{Q_1(x)}$, involved in (50), can be any one of the set of functions $e^{Q(x)}$, each of which is asymptotic to zero in $R(r_0)$. The functions $h_i(x)$ $(i>\delta)$ approach zero, as $x\to 0$ within $R(r_0)$, essentially as rapidly as the functions

$$e^{Q_i(x)-Q_1(x)} \qquad (i=\delta+1, \ldots, m'),$$

where m' is m or m^*, as the case may be. The $\psi_j(x)$, occuring in (29), (30), are of the form

$$\psi_j(x) = e^{jQ_1(x)} x^{-j\beta+2(w+p-1)} \varphi_j(x), \tag{52}$$

$$\left\{\begin{array}{l} \varphi_j(x) = \sum \varphi_{j:\alpha_1,\alpha_2,\ldots,\alpha_{m'}}(x)\,[h_1(x)]^{\alpha_1}[h_2(x)k_2]^{\alpha_2}\ldots[h_{m'}(x)k_{m'}]^{\alpha_{m'}} \\ \qquad [\alpha_1+\alpha_2+\ldots+\alpha_{m'}=j;\ \alpha_1, \alpha_2, \ldots, \alpha_{m'} \geqq 0;\ cf.\ (50b)]\ \ (^1). \end{array}\right. \tag{52a}$$

Here the $\varphi_{j:\alpha_1,\alpha_2,\ldots,m'}(x)$ are analytic in $R(r_0)$ $(x \neq 0)$ and are of the form

$$[x]_{n(j)} \sim \{x\}_{n(j)} \qquad [x \text{ in } R(r_0);\ cf. \text{ Def. 3}]. \tag{52b}$$

Mereover, m' is m or m^* as the case may be.

8. A transformation $(n \geqq 2)$. — On the basis of the formal solution (50; § 7) we shall effect the transformation of the equation (A).

$$y(x) = Y(x) + \rho(x), \qquad Y(x) = \sum_{j=1}^{N-1} y_j(x) c_1^j. \tag{1}$$

Here N is a fixed positive integer, however large, and $\rho(x)$ is the new variable. The discussion will be given under the supposition that (1 *a*; § 7) holds. From the results so obtained it would be easy to make inferences regarding the alternative case when the inequalities (49; § 7), (49 *a*; § 7) hold.

We have

$$\begin{array}{l} a_2(x, Y+\rho, \ldots, Y^{(n-1)}+\rho^{(n-1)}) \\ \quad = a_2[x, Y(x), \ldots, Y^{(n-1)}(x)] + \sum \alpha_{i_0,i_1,\ldots,i_{n-1}}(x)\,\rho^{i_0}(x)\ldots\rho^{(n-1)^{i_{n-1}}}(x) \\ \qquad\qquad (i_0+i_1+\ldots+i_{n-1} \geqq 1;\ i_0, \ldots, i_{n-1} \geqq 0), \end{array} \tag{2}$$

$$\left\{\begin{array}{l} \alpha_{i_0,i_1,\ldots,i_{n-1}}(x) = \dfrac{1}{i_0!\ldots i_{n-1}} \dfrac{\partial^{i_0+\ldots+i_{n-1}} a_2}{\partial y^{i_0}\,\partial y^{(1)^{i_1}}\ldots\partial y^{(n-1)^{i_{n-1}}}} \\ \qquad [y(x) = Y(x), \ldots, y^{(n-1)}(x) = Y^{(n-1)}(x)]. \end{array}\right. \tag{2a}$$

---

(1) The functions $h_i(x)$ $(i=\delta+1, \ldots, m')$ may be distinct from the expressions so denoted in (50 *b*).

Taking r_0 sufficiently small so that

(3) $$|Y^{(i)}(x)| \leqq r' < r \qquad [x \text{ in } R(r_0);\ i = 0, 1, \ldots, n-1],$$

the series in the second member of (2) *is observed to be absolutely convergent whenever*

(3a) $$|\rho^{(i)}(x)| \leqq r'' \qquad [r' + r'' \leqq r;\ x \text{ in } R(r_0);\ i = 0, 1, \ldots, n-1].$$

By (2a), for $i_0 + i_1 + \ldots + i_{n-1} \geqq 1$ $(i_0, \ldots, i_{n-1} \geqq 0)$,

(4) $$\begin{cases} \alpha_{i_0,\ldots,i_{n-1}}(x) \\ \quad = \sum a_{i_0+j_0,\ldots,i_{n-1}+j_{n-1}}(x)\, C_{i_0}^{i_0+j_0} \ldots C_{i_{n-1}}^{j_{n-1}+i_{n-1}} Y^{j_0}(x) \ldots Y^{(n-1)j_{n-1}}(x) \\ \qquad [j_0 + \ldots + j_{n-1} \geqq 2 - (i_0 + \ldots + i_{n-1});\ j_0, \ldots, j_{n-1} \geqq 0]. \end{cases}$$

These series converge absolutely and uniformly for x in $R(r_0)$. Now, for x in $R(r_0)$, $Y(x) \sim 0$ $(i = 0, \ldots, n-1)$. Hence from (4) it follows that

(5) $$\begin{cases} \alpha_{i_0,\ldots,i_{n-1}}(x) = a_{i_0,\ldots,i_{n-1}}(x) + \beta_{i_0,\ldots,i_{n-1}}(x) \\ \qquad (i_0, \ldots, i_{n-1} \geqq 0;\ i_0 + \ldots + i_{n-1} \geqq 2), \end{cases}$$

where

(5a) $$a_{i_0,\ldots,i_{n-1}}(x) = 0 \qquad (i_0 + \ldots + i_{n-1} = 1)$$

and the $\beta_{i_0,\ldots,i_{n-1}}(x)$ are analytic in $R(r_0)$ and

(5b) $$\beta_{i_0,\ldots,i_{n-1}}(x) \sim 0 \qquad [x \text{ in } R(r_0);\ i_0 + \ldots + i_{n-1} \geqq 1].$$

The asymptotic relations here and throughout are with respect to x and are uniform with respect to the involved arbitrary constants provided, as we shall indeed assume, the numbers

(6) $$c_1, \qquad c_2 = c_1 k_2, \qquad \ldots, \qquad c_m = c_1 k_m$$

satisfy inequalities

(6a) $$|c_i| \leqq k' \qquad (i = 1, 2, \ldots, m;\ k' \text{ fixed}).$$

With L_n denoting the differential operator of (9; § 7) consider the function

(7) $$-F_N(x) = Y^{(n)}(x) - x^{-p} L_n[x, Y(x)] - x^{-p} a_2[x, Y(x), \ldots, Y^{(n-1)}(x)].$$

Comparison with (6), (7) and (8a) of § 7 enables one to infer that

(7a) $$-F_N(x) = \sum_{j \geqq 1} \overline{\Gamma}_j(x) c_1^j,$$

where

$$(7b) \qquad \bar{\Gamma}_j(x) \equiv y_j^{(n)}(x) - x^{-p}\,\mathrm{L}_n[x,\ y_j(x)] - x^{-p}\,\bar{\psi}_j(x).$$

Here $\bar{\Gamma}_j(x)$ is $\Gamma_j(x)$ and $\bar{\psi}_j(x)$ is $\psi_j(x)$ with $y_{\mathrm{N}}(x)$, $y_{\mathrm{N}+1}(x)$, ..., replaced by zero. Since $\psi_j(x)$ is independant of $y_j(x), y_{j+1}(x), \ldots,$ it follows that

$$(8) \qquad \bar{\psi}_j(x) = \psi_j(x) \qquad (j = 1,\ 2,\ \ldots,\ \mathrm{N}).$$

Thus, by (7), $\bar{\Gamma}_j(x) = \Gamma_j(x) = 0$ $(j = 1,\ 2,\ \ldots,\ \mathrm{N}-1)$ so that

$$(9) \qquad -\,\mathrm{F}_{\mathrm{N}}(x) = -\,x^{-\rho}\sum_{j \geqq \mathrm{N}} \psi_j(x) c_1^j.$$

On using the notation of definition 4 (§ 7) we have

$$(10) \qquad \bar{\psi}_j(x) = e^{j\mathrm{Q}_1(x)}\,x^{-j\beta+2(w+p-1)}\,\bar{\varphi}_j(x),$$

$$(10a) \qquad \bar{\varphi}_j(x) = [x]^j_{h(j)} \sim \{x\}^j_{h(j)} \qquad [x \text{ in } \mathrm{R}(r_0),\ j \geqq \mathrm{N}].$$

Accordingly, the function (9) is of the form

$$(11) \qquad \mathrm{F}_{\mathrm{N}}(x) = x^{-p}\,e^{\mathrm{N}\mathrm{Q}_1(x)}\,x^{-\mathrm{N}\beta+2(w+p-1)}[\,c_1^{\mathrm{N}}\varphi_{\mathrm{N}}(x) + c_1^{\mathrm{N}}\beta_{\mathrm{N}}(x)],$$

$$(11a) \qquad \beta_{\mathrm{N}}(x) \sim 0 \qquad [x \text{ in } \mathrm{R}(r_0)],$$

where $\varphi_{\mathrm{N}}(x)$ *is the function given, for* $j = \mathrm{N}$, *by the formulas* (52 *a*), (52 *b*) *of* § 7.

Substitution of (1) in (A; § 1), with the latter equation in the form (A; § 7), will result in

$$\mathrm{Y}^{(n)}(x) + \rho^{(n)}(x) - x^{-p}\,\mathrm{L}_n[x,\ \mathrm{Y}(x)] - x^{-p}\,\mathrm{L}_n[x,\ \rho(x)]$$
$$-\,x^{-p}\,a_2[x,\ \mathrm{Y}(x) + \rho_n(x),\ \ldots,\ \mathrm{Y}^{(n-1)}(x) + \rho^{(n-1}(x)] = 0.$$

Thus, by (2) and (7),

$$(12) \qquad \rho^{(n)}(x) - x^{-p}\,\mathrm{L}_n[x,\ \rho(x)]$$
$$= x^{-\rho}\sum \alpha_{i_0,\ \ldots,\ i_{n-1}}(x)\,\rho^{i_0}(x)\ldots\rho^{(n-1)^{i_{n-1}}}(x) + \mathrm{F}_{\mathrm{N}}(x)$$
$$[i_0 + \ldots + i_{n-1} \geqq 1;\ i_0,\ \ldots,\ i_{n-1} \geqq 0;\ cf.\ (11),\ (11a),\ (5),\ (5a),\ (5b)]$$

By (5), (5 *a*), (5 *b*) transposition to the left member of the linear part of the second member of (12) will yield

(13) $${}_1\mathrm{L}[\rho(x)] \equiv \rho^{(n)}(x) - x^{-p}\,\mathrm{L}_n^\star[x,\, \rho(x)] = x^{-p}\,{}_1\mathrm{H}[x,\, \rho(x)] + \mathrm{F}_\mathrm{N}(x),$$

(13 a) $$\left\{ {}_1\mathrm{H}[x,\, \rho(x)] = \sum_{(i_0+\ldots+i_{n-1}\geqq 2)} \alpha_{i_0,\ldots,i_{n-1}}(x)\, \rho^{i_0}(x) \ldots \rho^{(n-1)^{i_{n-1}}}(x) \right.$$

where

(13 a′) $$\mathrm{L}_n^\star[x,\, \rho(x)] = b_1^\star(x)\,\rho^{(n-1)}(x) + \ldots + b_n^\star(x)\,\rho(x),$$

the coefficients $b_i^\star(x)$ *being asymptotically the same as the corresponding. ones in* L_n [*cf*, (5; § 7)]. More precisely,

(13 b) $$b_i^\star(x) - b_i(x) \sim 0 \qquad [i = 1,\, \ldots,\, n;\; x \text{ in } \mathrm{R}(r_0)].$$

Equation (13) will be further transformed with the aid of the substitution

(14) $$\rho(x) = e^{\mathrm{G}(x)}\,\zeta(x), \qquad \mathrm{G}(x) = \mathrm{N}\,\mathrm{Q}_1(x) - (\mathrm{N}-1)\beta\log x.$$

This transformation is suggested by the form of $y_\mathrm{N}(x)$, as given by (50; § 7). We have

(14 a) $$\rho^{(\nu)}(x) = \sum_{m=0}^{\nu} \mathrm{C}_m^\nu\, \zeta^{(m)}(x)\, \frac{d^{\nu-m}}{dx^{\nu-m}}\, e^{\mathrm{G}(x)}.$$

Furthermore

(14 b) $$\left\{ \begin{aligned} &\frac{d^j}{dx^j}\, e^{\mathrm{G}(x)} = e^{\mathrm{G}(x)}\,\mathrm{G}_j(x), \qquad \mathrm{G}_j(x) = \mathrm{G}^{(1)}(x)\,\mathrm{G}_{j-1}(x) + \mathrm{G}_{j-1}^{(1)}(x) \\ &\qquad [j = 1,\, 2,\, \ldots;\; \mathrm{G}_0(x) = 1]. \end{aligned} \right.$$

Since, by (14). $\mathrm{G}(x)$ is of the form

(14 c) $$\mathrm{G}(x) = -(\mathrm{N}-1)\beta\log x + g\,x^{-\frac{l}{\alpha}} + \ldots$$

it follows from the recursion relations (14 b) that

(14 d) $$\mathrm{G}_j(x) = x^{-j\left(1+\frac{l}{\alpha}\right)}\, g_j(x) \qquad [j = 0,\, 1,\, \ldots;\; g_0(x) = 1],$$

where the $g_j(x)$ are polynomials in $x^{\frac{1}{\alpha}}$. Thus,

(15) $$\rho^{(\nu)}(x) = e^{\mathrm{G}(x)} \sum_{m=0}^{\nu} \mathrm{C}_m^\nu\, x^{-(\nu-m)\left(1+\frac{l}{\alpha}\right)}\, g_{\nu-m}(x)\,\zeta^{(m)}(x) \qquad (\nu = 0,\, 1,\, \ldots).$$

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(1) Here... stands for a finite number of powers of x higher than that displayed.

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Substitution of (15) in ${}_1L[\rho(x)]$ of (13) will yield, by virtue of (13 *a*), (13 *b*),

$$(16)\qquad {}_1L[\rho(x)] \equiv e^{G(x)} L[\zeta(x)],$$

$$(16a)\qquad L[\zeta(x)] = \zeta^{(n)}(x) - x^{-g} \sum_{\nu=0}^{n-1} \beta_\nu(x)\, \zeta^{(\nu)}(x).$$

Here g is the greatest one of the numbers

$$n\left(1+\frac{l}{\alpha}\right),\quad p+(n-1)\left(1+\frac{l}{\alpha}\right)$$

and the $\beta_\nu(x)$ are analytic in $R(r_0)$ $(x \neq 0)$, asymptotic [in $R(r_0)$] to series of the form

$$(17)\qquad \beta_0 + \beta_1 x^{\frac{1}{\alpha}} + \beta_2 x^{\frac{2}{\alpha}} + \ldots.$$

Substitution of (15) in (13 *a*) will give

$$(18)\qquad {}_1H(x\ \ \rho) = e^{2G(x)} x^{-2(n-1)} H(x, \zeta),$$

where

$$(18a)\quad H(x, \zeta) = \sum_{m=2}^{\infty} \sum_{\iota_0 + \ \ + \iota_{n-1} = m} e^{(m-2)G(x)} x^{-(m-2)(n-1)} h_{\iota_0, \ldots, \iota_{n-1}}(x)\, \zeta^{\iota_0} \ldots \zeta^{(n-1)^{\iota_{n-1}}}$$

Here the $h_{\iota_0,\ \ ,\iota_{n-1}}(x)$ are analytic in $R(r_0)$ and are asymptotic in $R(r_0)$ to series of the form (17). Moreover, as seen from (14) and (3 *a*), the series (18 *a*) is absolutely convergent for

$$(18b)\qquad |\zeta^{(\nu)}| < r''(r_0) \qquad [\nu = 0, 1, \ldots, n-1;\ x \text{ in } R(r_0)],$$

where

$$(18c)\qquad r''(r_0) \to \infty \qquad (\text{when } r_0 \to 0).$$

Thus, in consequence of (16), (16 *a*), (18), (11), (11 *a*), [*cf.* (52 *a*), (52 *b*) of § 7], application of (14) to (13) is seen to result in the equation

$$(19)\quad \left\{ \begin{array}{l} L[\zeta(x)] = e^{G(x)} x^{-n_1} H[x, \zeta(x)] + x^{-n_2} \varphi(x) \\ \left[n_1 = 2(n-1) + p;\ n_2 = -2w - p + 2 + \beta = (n-1)\left(1+\frac{l}{\alpha}\right) + 1 - w \right], \end{array} \right.$$

where

$$(19a)\qquad \varphi(x) = [x]^N_{n(N)} \sim \{x\}^N_{n(N)} \qquad [x \text{ in } R(r_0);\ cf. \text{ Def. 4 (§ 7)}].$$

LEMMA 3. — *Let* N *be a fixed positive integer, however large. Let the functions* $y_1(x), y_2(x), \ldots, y_{N-1}(x)$ *be those involved in* (50; § 7). *Apply the transformations* (1), (14) *to* (A; § 1), [*cf.* (A_1; § 7)]. *The new variable* $\zeta(x)$ *will satisfy equation* (19). *In* (19) L *is given by* (16 *a*) [*cf. the italics following* (16 *a*)]. $G(x)$ *is given by* (14), $\varphi(x)$ *is of the form* (19 *a*) *and* $H(x, \zeta)$ *is of the form* (18 *a*) [*cf. italics after* (18 *a*)]. *Considering the* $\zeta^{(\nu)}(\nu = 0, \ldots, n-1)$ *as variables independant of* x, *the series representing* (18 *a*) *converges absolutely and uniformly in* $R(r_0)$, *provided* (18 *b*) *holds* [*cf.* (18 *c*)]. *Either the number* r_0, *used in the definition of the region* $R(r_0)$, *or the number* k', *involved in* (6 *a*), *must be taken sufficiently small so that* (3)' *is satisfied.*

9. **Existence of « proper » regions.** — Consider now the linear problem (A_2; § 1) with which there are associated formal solutions (2; § 1) [*cf.* (2 *a*) and (2 *b*) of § 1]. We are interested in the case when (A_2; § 1) is formally not of Fuchsian type at $x = 0$; that is, when not all the polynomials $Q(x)$ of (2; § 1) are identically zero. Let the distinct polynomials

$$(1) \qquad P_1(x), \quad P_2(x), \quad \ldots, \quad P_{H_1}(x)$$

constitute the totality of all those $Q(x)$ which are not identically zero. We shall write

$$(1a) \qquad P_i(x) = p_i x^{-\rho_i} + \ldots \qquad (i = 1, 2, \ldots, H_1;\ p_i \neq 0;\ \rho_i > 0),$$

where the ρ_i are rational numbers, the terms displayed in the second member being the leading ones.

DEFINITION 5. — *Let* β *and* H *be positive numbers and let* N *be any integer greater than unity. Let* $P(x)$ *stand for a particular polynomial of the set* (1). *Consider a region* $R(r_0)$ *whose boundary consists of an arc of the circle* $|x| = r_0$ *and of two regular* [1] *curves* B′, B″ *extending from the extremities of this arc to the origin. Such a region will be termed proper with respect to* $P(x)$ *if for some sufficiently small* r_0 (> 0), *independent of* N, *we have*

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(1) The meaning of the term « regular curve » here is the same as in (T_1). Consequently B′B″ have limiting directions at the origin. Mereover, except at the origin B′ and B″ are to have no points in common.

all of the following conditions satisfied when x is any point in $R(r_0)$.

1° *All the points of the rectilinear segment* (o, x) *are in* $R(r_0)$;

2° *The linear equation* $(A_2; \S 1)$ *possesses a full set of analytic solutions which in* $R(r_0)$ *are asymptotic to the series* $(2; \S 1)$.

3° *The real part of* $P(x)$, $RP(x)$, *is the least of the real parts of all those polynomials* $Q(x)$ [*cf.* $(2; \S 1)$] *which are distinct from* $P(x)$.

4° $e^{P(x)} \sim o$.

5° *With* $G(u) = NP(u) - (N-1)\beta \log u$ *and with u on the rectilinear segment* (o, x) *the upper bounds of the functions*

$$(2) \quad f(H, u) = | e^{G(u)} u^{-H}, \qquad f_\lambda(u) = | e^{G(u) - Q_\lambda(u)} u^{-r_\lambda} | \qquad (\lambda = 1, 2, \ldots, n)$$

are attained at x.

It is to be noted that proper regions constitute a particular instance of the regions characterised by Def. 2 (§ 7).

The following lemma regarding proper regions will be now proved.

LEMMA 6. — *Suppose that not all the polynomials* $Q(x)$, *involved in* $(2; \S 1)$ *are identically zero. There exist then regions proper, in the sense of Definition* 5, *with respect to at least some of these polynomials.*

If ε'' is a fixed positive number, however small, it follows from the consideration of $(1a)$ that

$$(3) \qquad e^{P_\iota(x)} \sim o$$

in any region, extending to $x = o$, in which

$$(3a) \qquad \cos(\rho_\iota \bar{x} - \bar{p}_\iota) \leqq -\varepsilon'' \qquad (\bar{p}_\iota = \angle p_\iota;\ \bar{x} = \angle x).$$

This implies that with $\varepsilon > o$, however small, (3) is satisfied in every one of the finite set of sectors $W_{\iota,m}(r_0)$ $(m = o, 1, \ldots)$ characterized by the inequalities

$$(4) \quad \left(2m + \frac{1}{2}\right)\frac{\pi}{\rho_\iota} + \frac{\bar{p}_\iota}{\rho_\iota} + \varepsilon \leqq \bar{x} \leqq \left(2m + \frac{3}{2}\right)\frac{\pi}{\rho_\iota} + \frac{\bar{p}_\iota}{\rho_\iota} - \varepsilon \qquad (|x| \leqq r_0).$$

We select $\varepsilon(>0)$ sufficiently small so that

$$(4a) \qquad \frac{\pi}{\rho_i} - 2\varepsilon > 0 \qquad (i = 1, \ldots, H_1).$$

In consequence of a Fundamental Existence Theorem established by Trjitzinsky [1] the following may be stated. Let $B_{i,j}$ denote a curve along which $R[Q_i(x - Q_j(x)] = 0$, when $Q_i(x)$ is distinct from $Q_j(x)$. Let $R'_1, R'_2, \ldots, R'_{N'}$ be regions separated by the $B_{i,j}$ curves, none of these curves lying interior of an R'_i $(i = 1, \ldots, N')$ [2]. Let it be said that a region R'_i has an angle w_i if the tangents at $x = 0$ to the boundaries of R'_i make an angle w_i [3]. When $w_i \neq 0$, in some cases [for details *cf.* (T_1)]R'_i is replaced by two subregions ${}_lR'_i$, ${}_rR'_i$. The subregion ${}_lR'_i$ has one of the boundaries (extending to $x = 0$) coincident with a boundary of R'_i, while the other boundary (extending to $x = 0$) is a certain regular curve, interior to R'_i, with the same limiting direction at $x = 0$ as that of the other boundary of R'_i. On the other hand, ${}_rR'_i$ is formed similarly with the roles of the two boundaries (extending to $x = 0$) of R'_i interchanged. Thus the angle of ${}_lR'_i$ (and of ${}_rR'_i$) is w_i. Corresponding to a particular region R'_i the linear problem (A_2; § 1) has a full set of analytic solutions which, when $w_i = 0$, are asymptotic to the series (2; § 1) for x in R'_i. When $w_i \neq 0$ the same result holds, unless R''_i is to be replaced by the above regions ${}_lR'_i$, ${}_rR'_i$. When the latter is the case there exists a full set of analytic solutions (for $x \neq 0$) asymptotic in ${}_lR'_i$ to the series (2; § 1); and there also exists another full set of solutions asymptotic to these series in ${}_rR'_i$.

Corresponding to every $Q_i(x)$ which is not identically zero there exists a finite number of curves B_i, defined by the équation $RQ_i(x) = 0$ and extending to the origin. These curves are regular. Interior a circle $|x| = r_0$ (r_0 sufficiently small) the B_i curves have no points in common amongst themselves and with the $B_{i,j}$ curves (except at the origin of course). There is occasion to introduce the B_i curves only if all the $Q_i(x)$ $(i = 1, 2, \ldots, n)$ are distinct from zero.

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(1) *Cf.* (T_1).

(2) For every x in R'_i we have $|x| \leq r_0$. The boundary of R'_i consists of two regular curves and of an arc of the circle $|x| = r_0$ The regular curves extend from the extremities of this arc to the origin; mereover, except at the origin, they have no points in common.

(3) This is the angle corresponding to the interior of R'_i.

Let ξ be a fixed positive number, however small. Take

$$(5) \qquad \xi < \frac{\pi}{\rho_i} - 2\varepsilon \qquad (i = 1, \ldots, H_1).$$

Corresponding to ξ we can take r_0 sufficiently small so that the followings holds. All the curves $B_{i,j}$ and B_i and all the regions R'_i, for which $w_i = 0$, can be enclosed in a set Γ of sectors (bounded by arcs of the circle $|x| = r_0$) the sum of whose angles does not exceed ξ; moreover such a set Γ can be so selected that the limiting directions, at $x = 0$, of the various curves $B_{i,j}$ and B_i are all distinct from those of the boundaries (rays) of Γ. The complete vicinity of $x = 0$ will consist of the sectors Γ and of a certain complementary set of non overlapping and non adjacent sectors T,

$$(6) \qquad T_1, \quad T_2, \quad \ldots, \quad T_N \qquad (|x| \leqq r_0).$$

Corresponding to every T_i the equation (A_2; § 1) has a full set of analytic solutions ${}_iy_j(x)$ such that

$${}_iy_j(x) \sim e^{Q_j(x)} x^{r_j} \sigma_j(x) \qquad (j = 1, \ldots, n; \ x \text{ in } T_i; \ |x| \leqq r_0).$$

Moreover, no curve $B_{\alpha,\beta}$ has at $x = 0$ the same limiting direction as that of any one of the rays bounding the sectors T.

Consider now the sectors $W_{i,m}$ ($i = 1, \ldots, H_1$; $m = 0, 1, \ldots$), as defined by (4). *The angle of* $W_{i,m}$ *is* $\frac{\pi}{\rho_i} - 2\varepsilon$. *All these angles are positive* by (4 *a*). The set of the sectors T has in common with a particular sector $W_{i,m}$ a point set which contains a finite number of non adjacent and non overlapping sectors

$$(7) \qquad T_1^{i,m}, \quad T_2^{i,m}, \quad \ldots, \quad T_{N(i,m)}^{i,m} \qquad (|x| \leqq r_0),$$

each with an angle distinct from zero. Existence of such a set (7) can be proved as follows. Suppose there exists no such set. Then the sector $W_{i,m}$ would be contained in a sector of the set Γ (with some of the boundaries of $W_{i,m}$ and Γ possibly coincident). Now, by construction, the sum of the angles of Γ being equal to or less than than ξ, the angle of $W_{i,m}$ would be $\leqq \xi$. On taking account of (5) and of the above italicized statement, this is seen to be impossible. Hence a set (7) with properties as stated exists.

Let $R(r_0)$ be *any* particular one of the regions

$$T_k^{i,m} \qquad (i = 1, 2, \ldots, H_1; \ m = 0, 1, \ldots; \ k = 1, 2, \ldots, N(i, m)].$$

This region satisfies conditions 1°, 2° of Def. 5. Moreover, no curve $B_{k,j}$ and no curve B_k has at $x = 0$ the limiting direction of a ray bounding $R(r_0)$. *If* $R(r_0)$ *is* $T_j^{l,m}$, in consequence of the fact that $R(r_0)$ is a subset of $W_{l,m}$ *it will follow that* (3) *and* (3 *a*) *hold in* $R(r_0)$. There exist polynomials

$$(8) \qquad Q_{n_1}(x) = Q_{n_2}(x) = \ldots = Q_{n_\delta}(x)$$

such that, for x in $R(r_0)$ and for all j ($\neq n_1, \neq n_2, \ldots, \neq n_\delta$),

$$(8a) \qquad RQ_{n_1}(x) = \ldots = RQ_{n_\delta}(x) < RQ_j(x).$$

Just as a matter of notation, involving no loss of generality, designate the polynomials of (8) as

$$(9) \qquad Q_1(x) = Q_2(x) = \ldots = Q_\delta(x).$$

There are two cases.

CASE I. — $P_l(x) = Q_1(x)$.

CASE II. — $P_l(x) \neq Q_1(x)$ so that

$$(9a) \qquad RQ_1(x) < RP_l(x) \qquad [x \text{ in } R(r_0)].$$

In the Case II in consequence of (3) it follows that

$$(9b) \qquad e^{Q_1(x)} \sim 0 \qquad [x \text{ in } R(r_0)).$$

The relation (9 *b*) will also hold in the case I. This is inferred from the statement in italics preceding (8). *Let* $P(x)$ *denote* $Q_1(x)$. Of course in consequence of (9 *b*) $P(x) \not\equiv 0$. Thus $P(x)$ is a polynomial of the set (1). We have then all the conditions 1°, 2°, 3°, 4° satisfied. In order to demonstrate that $R(r_0)$ is « proper » with respect to $P(x)$ it remains only to prove that the condition (5°) of Def. 5 is satisfied for some sufficiently small r_0 (> 0), independent of N (≥ 2), when in (2) $G(u) = NP(u) - (N-1)\beta \log u$.

With

$$RQ_i(x) = RQ_j(x) < RQ_k(x) \qquad [i, j = 1, \ldots, \delta;\ k = \delta+1, \ldots, n;\ x \text{ in } R(r_0)]$$

it follows that the $f_\lambda(u)$, defined by (2), are of the form

$$(10) \quad f_\lambda(u) = g(\beta + r_\lambda, u) g^{N-2}(\beta, u) \qquad (\lambda = 1, 2, \ldots, \delta),$$

$$(10a) \quad f_\lambda(u) = | e^{Q_1(u) - Q_\lambda(u) - r_\lambda \log u} | g^{N-1}(\beta, u) \qquad (\lambda = \delta+1, \ldots, n)$$

where

$$g(\nu, u) = | e^{Q_1(u) - \nu \log u} |. \tag{11}$$

Let

$$\gamma = \max \{ \beta ; H ; R(\beta + r_\lambda) \} \qquad (\lambda = 1, 2, \ldots, n). \tag{12}$$

Now for $\lambda > \delta$ we have $RQ_1(u) < RQ_\lambda(u)$ [u in $R(r_0)$]. Hence

$$\begin{cases} Q_1(u) - Q_\lambda(u) = a_1 u^{-\alpha_1} + a_2 u^{-\alpha_2} + \ldots + a_k u^{-\alpha_k} \\ (0 < \alpha_k < \ldots < \alpha_2 < \alpha_1 ; \; a_1 \neq 0 ; \; \lambda = \delta + 1, \ldots, n) \end{cases} \tag{13}$$

where the α_i ($i = 1, \ldots, k$) are rational numbers ([1]).
On writing

$$\begin{cases} u = \rho e^{\sqrt{-1}\,\theta}, \quad a_i = | a_i | e^{\sqrt{-1}\,\bar{a}_i}, \quad r_\lambda = r'_\lambda + \sqrt{-1}\, r''_\lambda \\ (i = 1, \ldots, k ; \; \lambda = 1, \ldots, n) \end{cases} \tag{14}$$

it follows that, for $\lambda = \delta + 1, \ldots, n$,

$$\begin{aligned} (15) \quad & R[Q_1(u) - Q_\lambda(u) - r_\lambda \log u] \\ & = G_\lambda(\rho, \theta) \\ & = | a_1 | \rho^{-\alpha_1} \cos(\alpha_1 \theta - \bar{a}_1) + \ldots \\ & \quad + | a_k | \rho^{-\alpha_k} \cos(\alpha_k \theta - \bar{a}_k) - r'_\lambda \log \rho + r''_\lambda \theta \qquad (| a_1 | \neq 0). \end{aligned}$$

For a fixed λ ($\lambda > \delta$) the limiting directions at $x = 0$ of the various curves $B_{1,\lambda}$ (along which $R[Q_1(u) - Q_\lambda(u)] = 0$) are given by the values θ satisfying the equation

$$\cos(\alpha_1 \theta - \bar{a}_1) = 0 \quad ([2]). \tag{16}$$

In consequence of the construction of $R(r_0)$, for no u in $R(r_0)$ (bounding rays included) is θ (= angle of u) coincident with a root of (16). Hence

$$\cos(\alpha_1 \theta - \bar{a}_1) | \geqq \varepsilon' > 0 \qquad [u \text{ in } R(r_0)] \quad ([1]) \tag{17}$$

([1]) The fact that the constants in the second member of (13) depend on λ is not explicitly stated. That is, the involved expression is in a generic sense.

([2]) *Cf.* (T_1).

([1]) Suppose (17) does not hold. Then the lower bound of the continuous function $| \cos(\alpha_1 \theta - \bar{a}_1) |$, for θ on a closed interval Δ, would be zero and would be attained for a particular $\theta = \theta_0$ in Δ. This value of θ would be a root of (16). A contradiction arises since the ray $\theta = \theta_0$ ($| u | \leqq r_0$) is in $R(r_0)$.

where ε' is independent of θ. Since

$$R[Q_1(u) - Q_\lambda(u)] < 0 \qquad [u \text{ in } R(r_0)],$$

(17) *implies*

$$(17a) \qquad \cos(\alpha_1\theta - \bar{a}_1) \leqq -\varepsilon'(<0).$$

Now from (15) it follows that

$$(18) \qquad \rho\frac{\partial G_\lambda}{\partial \rho} = -\alpha_1 \mid a_1 \mid \rho^{-\alpha_1}\cos(\alpha_1\theta - \bar{a}_1) - \ldots$$
$$- \alpha_k \mid a_k \mid \rho^{-\alpha_k}\cos(\alpha_k\theta - \bar{a}_k) - r'_\lambda.$$

By (13) and (17) we have

$$(18a) \qquad \rho\frac{\partial G_\lambda}{\partial \rho} = -\alpha_1 \mid a_1 \mid \rho^{-\alpha_1}\cos(\alpha_1\theta - \bar{a}_1)\,[1 + \nu(\rho, \theta)],$$

$$(18b) \quad \mid \nu(\rho, \theta) \mid \leqq \frac{1}{\alpha_1 \mid a_1 \mid \varepsilon'}[\alpha_2 \mid a_2 \mid \rho^{\alpha_1-\alpha_2} + \ldots$$
$$+ \alpha_k \mid a_k \mid \rho^{\alpha_1-\alpha_k} + \mid r'_\lambda \mid \rho^{\alpha_1}] \leqq \rho^{\alpha_1-\alpha_2}\nu \qquad [u \text{ in } R(r_0)] \quad (^2).$$

Here ν is a constant. For r_0 sufficiently small $\mid \nu(\rho, \theta) \mid \leqq 1$ $[u \text{ in } R(r_0)]$. Hence, on noting that $\nu(\rho, \theta)$ is real, from (18 a) with the aid of (17 a) it is inferred that

$$(19) \qquad \frac{\partial G_\lambda}{\partial \rho} \geqq 0 \qquad [u \text{ in } R(r_0)].$$

It is clear that r_0 can be taken independent of λ so that (19) holds for $\lambda = \delta + 1, \ldots, n$. In consequence of (19), for x in $R(r_0)$ and u on the segment $(0, x)$, the upper bound of $G_\lambda(\rho, \theta)$ and hence of

$$(20) \qquad \mid e^{Q_1(u) - Q_\lambda(u) - r_\lambda \log u} \mid \qquad (\lambda = \delta + 1, \ldots, n)$$

is attained at x.

Consider now the function $g(\gamma, u)$ [*cf.* (11), (12)]. We have $Q_1(u) \not\equiv 0$. Hence $Q_1(u)$ is given by an expression similar to the one in the second member of (13). Furthermore, on taking account of the notation (14), $\log g(\gamma, u)$ would be given by an expression analogous to that in the last member of (15) (with $r'_\lambda = \gamma$, and $r''_\lambda = 0$). The several B_1 curves [*cf.* statement preceding (5)], along which $RQ_1(u) = 0$, possess each a limiting direction at $x = 0$, given by a

(2) Use is made of the fact that the numbers $\alpha_2 - \alpha_3, \ldots, \alpha_2 - \alpha_k, \alpha_2$ are all positive.

root of the equation $\cos\left(\alpha_1\theta - \overline{a_1}\right) = 0$. It is to be recalled that by construction all the B_t curves are exterior to $R(r_0)$ and have limiting directions at $x = 0$ distinct from those of the bounding rays of $R(r_0)$. Accordingly, by a reasoning precisely analogous to that used in proving (17) and (17 a) we again obtain inequalities of similar type. As a consequence $\rho \frac{\partial}{\partial\rho} \log g(\gamma, u)$ is seen to be expressible in the form of the second member of (18 a). Here $|\nu(\rho, \theta)|$ would satisfy (18 b), with r'_λ replaced by γ and r_0 possibly dependent on γ. Hence it is inferred that

$$\frac{\partial g(\gamma, u)}{\partial\rho} \geqq 0 \qquad [u \text{ in } R(r_0)]. \tag{21}$$

Whence it is concluded that, *for x in $R(r_0)$ and for u on the segment* $(0, x)$, *the upper bound of* $g(\gamma, u)$ *is attained at* x Let $\sigma = \sigma' + \sqrt{-1}\,\sigma''$ be a number real or complex, with $\sigma' \leqq \gamma$. Then

$$g(\sigma, u) = g(\gamma, u)\,|u^{\gamma-\sigma}| = g(\gamma, u)\,|u|^{\gamma-\sigma'} e^{\theta\sigma}. \tag{22}$$

With u on a segment $(0, x)$ the upper bound of $|u|^{\gamma-\sigma'} e^{\theta\sigma''}$ will be attained at x. Hence, with x in $R(r_0)$, the same will be true of $g(\sigma, u)$. On taking account of (12) this is seen to imply that the upper bounds of the functions

$$\left\{\begin{array}{l} g(\beta, u), \quad g(H, u), \quad g(\beta + r_\lambda, u) \\ {[u \text{ on } (0, x);\ x \text{ in } R(r_0),\ \lambda = 1, 2, \ldots, n]} \end{array}\right. \tag{23}$$

are attained at x. Hence, by (10) and (10 a) and in consequence of the property, previously stated with respect to (20), it is concluded that the condition (5°) of Def. 5 holds for the functions

$$f_\lambda(u) \qquad (\lambda = 1, \ldots, n)$$

[with $P(u) = Q_1(u)$]. The remaining function (2), $f(H, u)$, is of the form

$$f(H, u) = |e^{Q_1(u)} u^{-H}|\,|e^{Q_1(u)} u^{-\beta}|^{N-1} = g(H, u)\, g^{N-1}(\beta, u). \tag{24}$$

Thus, by virtue of the property proved for the function (23), it is observed that condition 5° holds for the function $f(H, u)$ as well. This establishes lemma 6. Incidentally it has been shown that proper regions can always be constructed in the form of circular sectors. With the aid of a more extented analysis it is possible to obtain proper regions of a more general character.

10. The existence theorem (n-th order problem). — A solution of the equation (19; § 8) will be found for x in a region $R(r_0)$ proper, in the sense of Definition 5 (§ 9), with respect to a non vanishing polynomial $Q(x)$ of the set involved in (2; § 1). As a matter of notation this polynomial will be designated as $Q_1(x)$. We shall have

$$(1) \quad RQ_1(x) = \ldots = RQ_\delta(x) < RQ_i(x) \qquad [i = \delta + 1, \ldots, n;\ x \text{ in } R(r_0)].$$

Moreover, in the sequel, when using the conditions of Def. 5 (§ 9), we shall let $P(x) = Q_1(x)$. It is to be noted that (1; § 7) will be satisfied for some $m \geqq \delta$. The character in $R(r_0)$ of the formal solutions of the non linear problem (A_1; § 1) is specified by Lemma 4 (§ 7). This Lemma is to be applied with the number m^*, involved in (49; § 7) and (49 a; § 7), assigned the value δ. The only arbitrary constants entering in the formal solution will be

$$(2) \qquad c_1, \quad k_2, \quad k_3, \quad \ldots, \quad k_\delta.$$

Now equation (19; § 8) was established under the supposition that (1 a; § 7) holds. For the case under consideration we put

$$(2a) \qquad k_{\delta+1} = k_{\delta+2} = \ldots = k_m = 0,$$

as required by a previous statement [*cf.* (49 b; § 7) with $m^* = \delta$]. For this case equation (19; § 8) will be of the form specified by Lemma 5 (§ 8).

A solution will be found in the form of a series

$$(3) \qquad \zeta(x) = \zeta_0(x) + \zeta_1(x) + \zeta_2(x) + \ldots.$$

Write

$$(3a) \qquad z_j(x) = \zeta_0(x) + \zeta_1(x) + \ldots + \zeta_j(x) \qquad (j = 0, 1, \ldots).$$

The terms of the series will be determined in succession by means of the linear non homogeneous equations

$$(4) \qquad L[\zeta_0(x)] = t_0(x) = x^{-n_1} \varphi(x),$$

$$(4a) \qquad L[\zeta_1(x)] = t_1(x) = e^{G(x)} x^{-n_1} H(x, \zeta_0),$$

$$(4b) \qquad L[\zeta_2(x)] = t_2(x) = e^{G(x)} x^{-n_1} [H(x, z_1) - H(x, z_0)],$$

$$(4c) \qquad \begin{cases} L[\zeta_j(x)] = t_j(x) = e^{G(x)} x^{-n_1} [H(x, z_{j-1}) - H(x, z_{j-2})] \\ \qquad [j = 2, 3, \ldots;\ cf.\ (16a), \S\, 8]. \end{cases}$$

Adding the corresponding members of these equations we obtain, provided certain convergence conditions are satisfied,

$$(4d)\qquad \sum_{l=0}^{\infty} \mathrm{L}[\zeta_l(x)] = \mathrm{L}\left(\sum_j \zeta_j(x)\right)$$
$$= x^{-n_2}\varphi(x) + e^{G(x)} x^{-n_1} \lim_j \mathrm{H}(x, z_{j-1})$$
$$= x^{-n_2}\varphi(x) + e^{G(x)} x^{-n_1} \mathrm{H}\left(x, \lim_j z_{j-1}\right)$$

or

$$(4e)\qquad \mathrm{L}[\zeta(x)] = x^{-n_2}\varphi(x) + e^{G(x)} x^{-n_1} \mathrm{H}(x, \zeta).$$

We shall proceed to construct the $\zeta_j(x)$ $(j = 0, 1, \ldots)$ and to establish appropriate convergence properties of (3).

Consider an equation

$$(5)\qquad \mathrm{L}[\zeta(x)] = t(x).$$

By (16; § 8) and (14; § 8) (5) can be written in the form

$$(5a)\qquad {}_1\mathrm{L}[\rho(x)] = e^{G(x)}\, t(x) \qquad [\rho(x) = e^{G(x)}\zeta(x)],$$

where, L is given by (13; § 8), (13 a; § 8) [*cf.* the statement in italics subsequent to (13 a)]. The solutions of the homogeneous equation, ${}_1\mathrm{L}[\rho(x)] = 0$ are asymptotically the same as those of $(A_2; § 1)$ [1]. Hence a solution of (5 a) can be given in the form

$$(6)\qquad \rho(x) = \sum_{\lambda=1}^{n} e^{Q_\lambda(x)} x^{r_\lambda} \rho_\lambda(x) \int^x e^{-Q_\lambda(u)} u^{-r_\lambda+\gamma'} \bar{\rho}_\lambda(u) e^{G(u)}\, t(u)\, du$$

[*cf.* formulas (28), ..., (30) of § 7], where

$$(6a)\qquad \rho_\lambda(x) = [x]_{m_\lambda}, \qquad \bar{\rho}_\lambda(u) = [u]_{m'_\lambda} \qquad [\lambda = 1, \ldots, n;\ cf.\ \text{Def. 3 of § 7}].$$

Thus, by (5 a) a solution of (5) can be given in the form

$$(7)\qquad \zeta(x) = \sum_{\lambda=1}^{n} e^{Q_\lambda(x) - G(x)} x^{r_\lambda} \rho_\lambda(x)\, \zeta_\lambda^x[t(u)],$$

$$(7a)\qquad \zeta_\lambda^x[t(u)] = \int^x e^{G(u) - Q_\lambda(u)} u^{-r_\lambda} \bar{\rho}_\lambda(u) u^{\gamma'}\, t(u)\, du.$$

---

(1) By (13 b; § 8) the corresponding coefficients of the two equations are asymptotically the same. On the other hand, in consequence of the developments in (T_1) it is observed that the asymptotic form of the solutions is not changed whenever the coefficients of a given equation are replaced by functions which are correspondingly asymptotically identical.

Let ε denote an arbitrarily small positive number. In consequence of (6 a)

(8) $$|\rho_\lambda(x)|, \quad |\bar{\rho}_\lambda(x)| < \rho\,|x|^{-\varepsilon} \qquad [x \text{ in } R(r_0);\ \lambda = 1, \ldots, n].$$

Hence, for x in $R(r_0)$,

$$|\zeta_\lambda^x[t(u)]| < \rho \int_0^x f_\lambda(u)\,|u|^{\gamma'-\varepsilon}\,|t(u)|\,d|u| \qquad (\lambda = 1, \ldots, n),$$

provided the integral in the second member exists. Here $f_\lambda(u)$ is given by (2 ; § 9). By virtue of the satisfied condition 5° of Def. 5 (§ 9) it follows that

(9) $$|\zeta_\lambda^x[t(u)]| < \rho f_\lambda(x) \int_0^x |u|^{\gamma'-\varepsilon}\,|t(u)|\,d|u| \qquad [x \text{ in } R(r_0);\ \lambda = 1, \ldots, n)$$

whenever the integral exists (¹). By (7), (8) and (9) on taking account of the form of $f_\lambda(u)$ it is inferred that

(10) $$|\zeta(x)| < n\rho^2 |x|^{-\varepsilon} \int_0^x |u|^{\gamma'-\varepsilon}\,|t(u)|\,d|u| \qquad [x \text{ in } R(r_0)],$$

if the involved integral converges (²). Let j be a positive integer and assume that, for x in $R(r_0)$,

(11) $$|t(x)| < |e^{jG(x)} x^{-\tau_1 - j\tau'}|\, t_j$$

where τ_1, $\tau'(> 0)$ are *some* real numbers, independent of j. Then, for u in $R(r_0)$,

(11 a) $$\begin{aligned} |u|^{\gamma'-\varepsilon}\,|t(u)| &< t_j\,|e^{jG(u)} u^{-j\tau'} u^{-\tau_1+\gamma'-\varepsilon}| \\ &= t_j\,|e^{G(u)} u^{-\tau'}|^{j-1}\,|e^{G(u)} u^{-\tau'-\tau_1+\gamma'-\varepsilon}| \\ &= t_j\, f^{j-1}(\tau', u)\, f(\tau' + \tau_1 - \gamma' + \varepsilon, u) \end{aligned}$$

[*cf.* (2, § 9)]. *In using the conditions of Def.* 5 (§ 9) β *will be the constant so denoted in Lemma* 4 (§ 7). Write

(12) $$\omega = g + 1 + \frac{1}{2}(n-2)(n-3).$$

(¹) Throughout this section integrals from 0 to x are along a straight line.

(²) It is to be noted that ρ depends only on the character of the linear operator, L and on ε.

Let the number H, involved in Def. 5, be the greatest of the numbers

$$(12a) \qquad \tau', \quad \tau' + \tau_1 + \omega.$$

As $\gamma' > 0$, ε can be so chosen that $\gamma' - \varepsilon > 0$. Thus

$$(12b) \qquad \tau' \leqq H, \qquad \tau' + \tau_1 - \gamma' + \varepsilon < H.$$

With the condition 5° of Def. 5 satisfied for the function $f(H, u)$, the same will be true of the functions

$$f(\tau', u), \quad f(\tau' + \tau_1 - \gamma' + \varepsilon).$$

This fact is a consequence of (12 b) and of the statement in connection with (22, § 9). Therefore the second member of (11 a) attains its upper bound at x, whenever x is in $R(r_0)$ and u is on the rectilinear segment $(0, x)$. Thus, in consequence of (10) we have

$$(13) \qquad \begin{aligned} |\zeta(x)| &< t_j n \rho^2 |x|^{-\varepsilon+1} f^{j-1}(\tau', x) f(\tau' + \tau_1 - \gamma' + \varepsilon, x) \\ &< n\rho^2 t_j \, |e^{jG(x)} x^{-\tau_1 - j\tau'}| \, |x|^{\gamma' - 2\varepsilon + 1} \qquad [x \text{ in } R(r_0)]. \end{aligned}$$

Whence it is observed that (5) *and* (11) *imply* (13) *with* $n\rho^2$ *and* $\gamma' - 2\varepsilon + 1$ *independent of* j.

In view of the purposes on hand it will be essential to obtain certain inequalities for the $|\zeta^{(\nu)}(x)|$ $(\nu = 1, 2, \ldots, n-1)$. On taking account of (16 a, § 8) equation (5) may be written in the form

$$(14) \qquad \zeta^{(n)}(x) = {}_0w(x) + \sum_{i=1}^{n-1} {}_0w_i(x)\,\zeta^{(i)}(x),$$

$$(14a) \quad {}_0w(x) = t(x) + {}_0w_0(x)\zeta(x), \qquad {}_0w_i(x) = x^{-g}\beta_i(x) \qquad (i = 0, 1, \ldots n-1).$$

For convenience of writing some of the integrals in the sequel will be expressed with the aid of negative superscripts; thus,

$$(15) \qquad \begin{cases} w^{(0)}(x) = w(x), \\ w^{(-1)}(x) = \int^{x} w(x_1)\,dx_1, \\ w^{(-2)}(x) = \int^{x} \left(\int^{x_2} w(x_1)\,dx_1 \right) dx_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

Successive integrations by parts applied to (14) will result in

$$(16)\qquad \begin{aligned}\zeta^{(n-1)}(x) = {}_0w^{(-1)}(x) &+ \sum_{\iota=1}^{n-1} {}_0w_\iota(x)\,\zeta^{(\iota-1)}(x)\\ &- \sum_{\iota=1}^{n-1} {}_0w_\iota^{(1)}(x)\,\zeta^{(\iota-2)}(x) + \dots\\ &\pm \sum_{\iota=1}^{n-1} {}_0w_\iota^{(n-2)}(x)\,\zeta^{(\iota-n+1)}(x)\\ &\mp \sum_{\iota=0}^{n-1} \int^x {}_0w_\iota^{(n-1)}(x)\,\zeta^{(\iota-n+1)}(x)\,dx.\end{aligned}$$

Accordingly

$$(16a)\qquad \zeta^{(n-1)}(x) = {}_1w(x) + \sum_{\iota=1}^{n-2} {}_1w_\iota(x)\,\zeta^{(\iota)}(x)$$

where

$$(16b)\qquad \begin{aligned}{}_1w\,(x) = {}_0w^{(-1)}(x) &\pm \sum_{\iota=0}^{n-2} \int^x {}_0w_{n-1-\iota}^{(n-1)}(x)\,\zeta^{(-\iota)}(x)\,dx\\ &\pm \sum_{n-2}^{n-2} [{}_0w_1^{(\iota)}(x) - {}_0w_2^{(\iota)}(x) + \dots \pm {}_0w_{n-1-\iota}^{(n-2)}(x)]\,\zeta^{(-\iota)}(x),\end{aligned}$$

$$(16c)\qquad {}_1w_\iota(x) = {}_0w_{\iota+1}(x) - {}_0w_{\iota+2}^{(1)}(x) + \dots \pm {}_0w_{n-1}^{(n-2-\iota)}(x) \qquad (\iota = 1, \dots, n-2).$$

In general, for $\nu = 1, 2, \dots, n-1$,

$$(17)\qquad \zeta^{(n-\nu)}(x) = {}_\nu w(x) + \sum_{\iota=1}^{n-\nu-1} {}_\nu w_\iota(x)\,\zeta^{(\iota)}(x)$$

where

$$(17a)\qquad \begin{aligned}{}_\nu w\,(x) = {}_{\nu-1}w^{(-1)}(x) &\pm \sum_{\iota=0}^{n-\nu-1} \int {}_{\nu-1}w_{n-\nu-\iota}^{(n-\nu)}(x)\,\zeta^{(-\iota)}(x)\,dx\\ &\pm \sum_{\iota=0}^{n-\nu-1} [{}_{\nu-1}w_1^{(\iota)}(x) - \dots \pm {}_{\nu-1}w_{(n-\nu-1)}^{n-\nu-\iota}(x)]\,\zeta^{(-\iota)}(x),\end{aligned}$$

$$(17b)\qquad \begin{aligned}{}_\nu w_\iota(x) = {}_{\nu-1}w_{\iota+1}(x) &- {}_{\nu-1}w_{\iota+2}^{(1)}(x) + \dots\\ &\pm {}_{\nu-1}w_{n-\nu}^{(n-\nu-1-\iota)}(x) \qquad (\iota = 1, \dots, n-\nu-1).\end{aligned}$$

In particular

(18) $\zeta^{(2)}(x) = {}_{n-2}\dot{w}(x) + {}_{n-2}w_1(x)\,\zeta^{(1)}(x),$

(18 a) $\zeta^{(1)}(x) = {}_{n-1}w(x)\left[= {}_{n-2}w^{(-1)}(x) - \int^{x} {}_{n-2}w_1^{(1)}(x)\zeta(x)\,dx + {}_{n-2}w_1(x)\,\zeta(x)\right].$

The $\beta_i(x)$ of (14 a) are of the form specified in the italicized statement subsequent to (16 a, § 8). We have

$$\beta_i(x) \sim \sum_{j=0}^{\infty} \beta_{i,j}\, x^{\frac{j}{\alpha}} \qquad [x \text{ in } \mathrm{R}(r_0)].$$

It is a consequence of the construction of the operator L that

$$\beta_i^{(\nu)}(x) \sim \sum_{j} \beta_{.,j} \frac{d^\nu}{dx^\nu} x^{\frac{j}{\alpha}} \qquad [x \text{ in } \mathrm{R}(r_0);\ \nu = 1, 2, \ldots].$$

This enables one to assert that the ${}_0w_i(x)$ of (14 a) and the derivatives of these functions satisfy inequalities

(19) $$\left\{\begin{array}{l} |\,{}_0w_i^{(\lambda)}(x)\,| < |\,x\,|^{-g-\lambda}\,{}_0w^{(\lambda)} \\ [i = 0, \ldots, n-1;\ \lambda = 0, 1, \ldots;\ x \text{ in } \mathrm{R}(r_0)]. \end{array}\right.$$

By (19) and (16 c)

(19 a) $$\left\{\begin{array}{l} |\,{}_1w_i^{(\lambda)}(x)\,| < |\,x\,|^{-g-\lambda-(n-2-i)}\,{}_1\overline{w}^{(\lambda)} \leqq |\,x\,|^{-g-\lambda-(n-3)}\,{}_1w^{(\lambda)} \\ [i = 1, \ldots, n-2;\ \lambda = 0, 1, \ldots;\ x \text{ in } \mathrm{R}(r_0)]. \end{array}\right.$$

Similarly from (17 b; $\nu = 2$) it follows that

(19 b) $$\left\{\begin{array}{l} |\,{}_2w_i^{(\lambda)}(x)\,| < |\,x\,|^{-g-\lambda-(n-3)-(n-4)}\,{}_2w^{(\lambda)} \\ [i = 1, \ldots, n-3;\ \lambda = 0, 1, \ldots;\ x \text{ in } \mathrm{R}(r_0)]. \end{array}\right.$$

In general, for $\nu = 0, 1, \ldots, n-1$,

(20) $$\left\{\begin{array}{l} |\,{}_\nu w_i^{(\lambda)}(x)\,| < |\,x\,|^{-g-\lambda-p_\nu}\,w^{(\lambda)} \\ [i = 1, \ldots, n-\nu-1;\ \lambda = 0, 1, \ldots;\ x \text{ in } \mathrm{R}(r_0)]. \end{array}\right.$$

Here

(20 a) $$\left\{\begin{array}{l} p_0 = 0, \\ p_\nu = (n-3)+(n-4)+\ldots+(n-\nu-2) = \frac{\nu}{2}(2n-\nu-5) \qquad (\nu = 0, 1, \ldots). \end{array}\right.$$

Consequently, for $i = 0, \dots, n-\nu-1$,

$$(21)\qquad \begin{aligned} &|{}_{\nu-1}w_{1}^{(i)}(x) - \dots \pm {}_{\nu-1}w_{n-\nu-i}^{(n-\nu-1)}(x)| \\ &\quad < |x|^{-g-p_{\nu-1}-(n-\nu-1)}\,{}_{\nu-1}w \leqq |x|^{-g-p_{\nu}-1}\,w \\ &\quad [x \text{ in } \mathrm{R}(r_0);\ \nu = 1, \dots, n-1]. \end{aligned}$$

By virtue of (14 a), (19; $\lambda = i = 0$) and of (13) from (11) it would follow that

$$(22)\qquad |{}_0w(u)| < {}_0w\,t_j\,|e^{j\mathrm{G}(u)}u^{-\tau_1-j\tau'-g}| \qquad [u \text{ in } \mathrm{R}(r_0)]$$

where ${}_0w$ is independent of j. More precisely, with $r_0 \leqq r$ (r fixed), ${}_0w$ depends only on r, ε and on the operator. L. The second member of (22) can be written as

$$(22a)\qquad {}_0w\,t_j\,|e^{\mathrm{G}(x)}x^{-\tau'}|^{j-1}\,|e^{\mathrm{G}(x)-\tau_1-\tau'-g}|.$$

In consequence of the definition of H and ω [*cf.* (12), (12 a)] it follows that $\tau' \leqq \mathrm{H}$ and $\tau_1 + \tau' + g < \mathrm{H}$. Hence the upper bound of the function (12 a), for u on $(0, x)$ [x in $\mathrm{R}(r_0)$] is attained at x. Whence it is inferred that

$$(23)\qquad |{}_0w^{(-1)}(x)| < {}_0w\,t_j\,|e^{j\mathrm{G}(x)}x^{-\tau_1-j\tau'-g+1}| \qquad [x \text{ in } \mathrm{R}(r_0)].$$

Write (13) in the form

$$|\zeta(u)| < n\rho^2 t_j\,|u|^{\gamma'+2\varepsilon+1} f(\tau_1+\tau', u) f^{j-1}(\tau', u) \qquad [u \text{ in } \mathrm{R}(r_0)].$$

On noting that $\gamma' - 2\varepsilon + 1 > 0$, $\tau_1 + \tau' < \mathrm{H}$ and $\tau' \leqq \mathrm{H}$ it is concluded that the upper bound of the involved second member is attained at x, when u is on $(0, x)$ [x in $\mathrm{R}(r_0)$]. Whence we have

$$(24)\qquad |\zeta^{(-i)}(x)| < {}_0\zeta\,t_j\,|e^{j\mathrm{G}(x)}x^{-\tau_1-j\tau'}| \qquad [i = 0, 1, \dots;\ x \text{ in } \mathrm{R}(r_0)].$$

Here ${}_0\zeta$ is independent of j and depends only on r and on the operator, ${}_1\mathrm{L}$. By (20) and (24), for u in $\mathrm{R}(r_0)$ and $\nu = 1, \dots, n-1$, we have

$$(24a)\qquad |{}_{\nu-1}w_{n-\nu-i}^{(n-\nu)}(u)\,\zeta^{(-i)}(u)| < {}_0\zeta\,w^{(n-\nu)}t_j\,|e^{j\mathrm{G}(u)}u^{-\tau_1-j\tau'-\omega_\nu}|,$$

$$(24b)\qquad \omega_\nu = g + (n-\nu) + p_{\nu-1} \leqq g + p_{n-2} + 1.$$

Except for the constant factor the second member of (24 a) can be written as the product of the functions

$$(24c)\qquad f^{j-1}(\sigma', u), \qquad f(\tau_1 + \tau' + \omega_\nu, u).$$

By (24 b), (20), (12) and by the definition of H it follows that

$$\tau_1 + \tau' + \omega_\nu = \tau_1 + \tau' + g + p_{n-2} + 1 = \tau_1 + \tau' + \omega \leqq H \qquad (\nu = 1, 2, \ldots, n-1).$$

Hence it is inferred that the upper bounds of the functions (24 c) are attained at x when u is on (0, x) [x in $R(r_0)$]. The same will be true of the second member of (24 a). Hence, for x in $R(r_0)$,

$$(25) \quad \left\{ \begin{array}{l} \left| \int^x {}_{\nu-1}w^{(n-\nu)}_{n-\nu-i}(u)\, \zeta^{(-i)}(u)\, du \right| < {}_0\zeta\, w^{(n-\nu)} t_j \mid e^{j G(x)} x^{-\tau_1 - j\tau' - \omega_\nu + 1} \mid \\ \qquad (i = 0, \ldots, n-\nu-1;\ \nu = 1, \ldots, n-1). \end{array} \right.$$

From (16 b), by virtue of (23), (25; $\nu = 1$), (21; $\nu = 1$) and (24), it follows that

$$(26) \qquad \mid {}_1w(x) \mid < {}_1\omega\, t_j \mid e^{j G(x)} x^{-\tau_1 - j\tau' - {}_1\omega_1} \mid,$$

$$(26a) \qquad {}_1\omega_1 = g + n - 2 = g + p_1 + 1 \qquad [x \text{ in } R(r_0)]$$

where ${}_1\omega$ depends *only* on r and ${}_1L$. Since ${}_1w_1 \leqq \omega$ it follows that the upper bound of

$$\mid e^{G(u)} u^{-\tau_1 - \tau' - {}_1\omega_1} \mid \qquad [u \text{ on } (0, x);\ x \text{ in } R(r_0)]$$

is attained at x. Hence the second member of (26) possesses this property. Accordingly

$$(26b) \qquad \mid {}_1w^{(-1)}(x) \mid < {}_1\omega\, t_j \mid e^{j G(x)} x^{-\tau_1 - j\tau' - {}_1\omega_1 + 1} \mid.$$

From (17 a; $\nu = 2$), (26 b), (25; $\nu = 2$), (21; $\nu = 2$) and (24) it is inferred that

$$(27) \qquad \mid {}_2w(x) \mid < {}_2\omega\, t_j \mid e^{j G(x)} x^{-\tau_1 - j\tau' - {}_2\omega_2} \mid,$$

$$(27a) \qquad {}_2\omega_2 = g + p_2 + 1 \leqq \omega \qquad [x \text{ in } R(r_0)]$$

where ${}_2\omega$ depends only on r and ${}_1L$. By induction it can be established that, for $\nu = 1, 2, \ldots, n-1$,

$$(28) \qquad \mid {}_\nu w(x) \mid < {}_\nu\omega\, t_j \mid e^{j G(x)} x^{-\tau_1 - j\tau' - {}_\nu\omega_\nu} \mid,$$

$$(28a) \qquad {}_\nu\omega_\nu = g + p_\nu + 1 \qquad [x \text{ in } R(r_0)]$$

with ${}_\nu\omega$ depending on r and ${}_1L$ only. It is essential to note the following. Suppose (28) had been established for some ν ($\nu < n-1$). We have

$$(28b) \qquad \mid e^{j G(u)} u^{-\tau_1 - j\tau' - {}_\nu\omega_\nu} \mid = f^{j-1}(\tau', u)\, f(\tau_1 + \tau' + \omega, u)]\, u \mid^{\alpha}$$

where $\alpha = \omega - {}_\nu\omega_\nu \geqq 0$ [1], while $\tau_1 + \tau' + \omega \leqq H$. Hence the upper bounds of the three factors in the second member of (28 b) are attained at x when u is on $(0, x)$ [x in $R(r_0)$]. Accordingly, the second member of (28) [with x in (28) replaced by u] would possess the same property. Thus (28) would imply

$$(28c) \qquad |{}_\nu w^{(-1)}(x)| < {}_\nu\omega t_j \, |e^{jG(x)} x^{-\tau_1 - j\tau' - {}_\nu\omega_\nu + 1}| \qquad [x \text{ in } R(r_0)].$$

Relations (28), (28 a), with ν replaced by $\nu + 1$, are established by means of the formulas (17 a), (25), (21) [in (17 a), (25) and (21) ν is to be replaced by $\nu + 1$] and with the aid of (24) and (28 c). This completes the induction.

By (18 a) and (28; $\nu = n - 1$) we obtain

$$(29) \qquad |\zeta^{(1)}(x)| < {}_1\zeta t_j \, |e^{jG(x)} x^{-\tau_1 - j\tau' - {}_1\zeta_1}|,$$

$$(29a) \quad {}_1\zeta = {}_{n-1}\omega, \qquad {}_1\zeta_1 = {}_{n-1}\omega_{n-1} = g + p_{n-1} + 1 \qquad [x \text{ in } R(r_0)].$$

Hence, by (18), [28; $\nu = n - 2$) and (20; $\lambda = 0$],

$$(30) \qquad |\zeta^{(2)}(x)| < {}_2\zeta t_j \, |e^{jG(x)} x^{-\tau_1 - j\tau' - {}_2\zeta_2}|,$$

$$(30a) \qquad {}_2\zeta_2 = {}_1\zeta_1 + g + p_{n-2} > {}_1\zeta_1 \qquad [x \text{ in } R(r_0)]$$

where ${}_2\zeta$ depends on r and ${}_1L$ only. By virtue of (17), (28) and (20; $\lambda = 0$) it follows that, for $\nu = n - 2, n - 3, \ldots, 1$,

$$(31) \quad |\zeta^{(n-\nu)}(x)| < {}_\nu\omega t_j \, |e^{jG(x)} x^{-\tau_1 - j\tau' - {}_\nu\omega_\nu}| + \sum_{i=1}^{n-\nu-1} \omega^{(0)} |x|^{-g-p_\nu} |\zeta^{(i)}(x)|.$$

Suppose that, for $i = 1, 2, \ldots, \alpha - 1$ $(2 \leqq \alpha \leqq n - 1)$,

$$(32) \qquad |\zeta^{(i)}(x)| < {}_i\zeta t_j \, |e^{jG(x)} x^{-\tau_1 - j\tau' - {}_i\zeta_i}|,$$

$$(32a) \qquad {}_i\zeta_i = {}_{i-1}\zeta_{i-1} + g + p_{n-i} \qquad [x \text{ in } R(r_0)]$$

where the ${}_i\zeta$ $(i = 1, 2, \ldots, \alpha - 1)$ depend on r and ${}_1L$ only. By (31; $\nu = n - \alpha$) we then would obtain (32) with i replaced by α and ${}_\alpha\zeta_\alpha$ equal to the greatest of the numbers

$${}_{n-\alpha}\omega_{n-\alpha}, \quad g + p_{n-\alpha} + {}_i\zeta_i \qquad (i = 1, 2, \ldots, \alpha - 1).$$

By (28 a) and since the ${}_i\zeta_i$ increase with i it follows that

$${}_\alpha\zeta_\alpha = {}_{\alpha-1}\zeta_{\alpha-1} + g + p_{n-\alpha}.$$

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[1] This inequality is a consequence of (12), (28 a) and (20 a).

Moreover, the number ${}_\alpha\zeta$ can be chosen depending on r, ε and ${}_1\mathrm{L}$ only. Thus (32) and (32 a) hold for $i = 1, 2, \ldots, n-1$. By (29 a), (28 a) and (32 a)

$$(33)\quad {}_\iota\zeta_\iota = 1 + \iota g + p_{n-1} + p_{n-2} + \ldots + p_{n-\iota} \qquad [\iota = 1, 2, \ldots, n-1;\ cf.\ (20)].$$

The following Lemma has been established.

Lemma 6. — *Let* $\mathrm{R}(r_0)$ $(r_0 \leq r)$ *be a region, as specified in Definition* 5 (§ 9), *proper with respect to* $\mathrm{Q}_1(x)$. *Let* r_0 *be sufficiently small so that all the conditions of Definition* 5 *hold when* H *is assigned the value specified in connection with* (12 a) *and* (12). *Consider an equation* (5), *where* L *is given by* (16 a; § 8) *and where* $t(x)$ *is a function satisfying* (11) (*where* j *is a fixed positive integer*). *There exists a solution of* (5), $\zeta(x)$, *analytic in* $\mathrm{R}(r_0)$ $(x \neq 0)$ *and together with its derivatives satisfying the inequalities*

$$(34)\quad |\zeta^{(\nu)}(x)| < \bar{\zeta} t_j \,|\, e^{j\mathrm{G}(x)} x^{-\tau_1 - j\tau' - {}_\nu\zeta_\nu} | \qquad [\nu = 0, 1, \ldots, n-1;\ x \text{ in } \mathrm{R}(r_0)].$$

Here $0 = {}_0\zeta_0 < {}_1\zeta_1 < \ldots < {}_{n-1}\zeta_{n-1}$ [*cf.* (33), (29 a), (20)]; *moreover,* $\bar{\zeta}$ *is a constant depending only on* r *and on the character of the operator* L [*that is* (A_2; § 1)], (¹).

Let r be a positive number. The transformation

$$(35)\qquad z^{(\nu)} = x^{-\tau} \bar{z}^{[\nu]} \qquad (\nu = 0, 1, \ldots, n-1),$$

applied to $\mathrm{H}(x, z)$ of (18 a; § 8), will result in

$$(36)\qquad \mathrm{H}(x, z) = x^{-2\tau}\, \mathrm{W}(x, \bar{z})$$

where

$$(36a)\quad \mathrm{W}(x, \bar{z}) = \sum_{m=2}^{\infty} \sum_{\iota_0 + \ldots + \iota_{n-1} = m} g^{m-2}(x)\, h_{\iota_0, \ldots, \iota_{n-1}}(x)\, \bar{z}^{[0]\iota_0} \bar{z}^{[1]\iota_1} \ldots \bar{z}^{[n-1]\iota_{n-1}},$$

$$(36b)\qquad g(x) = e^{\mathrm{G}(x)} x^{-(n-1)-\tau}.$$

In consequence of the convergence properties of $\mathrm{H}(x, z)$ [*cf.* § 8; in particular, (18 b; § 8), (18c; § 8)] the following can be stated regarding $\mathrm{W}(x, \bar{z})$, when the $\bar{z}^{[\nu]}$ $(\nu = 0, 1, \ldots, n-1)$ are considered as variables not necessarily depending on x.

(¹) It is essential to not that the ${}_\nu\zeta_\nu$ $(\nu = 1, \ldots, n-1)$ depend only on L.

There exists a positive number $r' = r'(r_0)$ $(r' \to \infty$ *as* $r_0 \to 0)$ (¹) *such that, whenever*

$$(37) \qquad |\bar{z}_1^{[\nu]}|, \quad |\bar{z}_2^{[\nu]}| \leqq 2r'(r_0) \qquad (\nu = 0, 1, \ldots, n-1),$$

we have

$$(37a) \quad |W(x, \bar{z}_2) - W(x, \bar{z}_1)| < M\{|\bar{z}_2^{[0]} - \bar{z}_1^{[0]}| + |\bar{z}_2^{[1]} - \bar{z}_1^{[1]}| + \ldots + |\bar{z}_2^{[n-1]} - \bar{z}_1^{[n-1]}|\}$$

when x is in $R(r_0)$. *Here* M *is independent of* x,

$$\bar{z}_1^{[\nu]}, \quad \bar{z}_2^{[\nu]} \qquad (\nu = 0, 1, \ldots, n-1).$$

The proof of the above may be made on the basis of the Cauchy integral theorem for analytic functions of several variables.

On writing

$$(38) \qquad z_j^{(\nu)}(x) = x^{-\tau} \bar{z}_j^{[\nu]}(x) \qquad (\nu = 0, 1, \ldots, n-1;\ j = 0, 1, \ldots)$$

equations (4), (4*a*), (4*b*) are brought to the form

$$(39) \qquad L[\zeta_0(x)] = t_0(x) = x^{-n_2} \varphi(x) \qquad [\zeta_0^{(\nu)}(x) = x^{-\tau} \bar{z}_0^{[\nu]}(x)],$$

$$(39a) \quad \begin{cases} L[\zeta_j(x)] = t_j(x) = e^{G(x)} x^{-n_1 - 2\tau} [W(x, \bar{z}_{j-1}) - W(x, \bar{z}_{j-2})] \\ [j = 1, 2, \ldots;\ \bar{z}_{-1}^{[\nu]} = 0\ (\nu = 0, \ldots, n-1)] \end{cases} \quad (^2)$$

where, by (3 *a*), (38),

$$(40) \quad \bar{z}_{j-1}^{[\nu]}(x) = \bar{z}_{j-2}^{[\nu]}(x) + x^{\tau} \zeta_{j-1}^{(\nu)}(x) \qquad (\nu = 0, \ldots, n-1;\ j = 1, 2, \ldots).$$

Thus, in view of the above italicized statement, inequalities

$$(41) \qquad \begin{cases} |\bar{z}_{j-1}^{[\nu]}(x)|, \quad |\bar{z}_{j-2}^{[\nu]}(x)| \leqq 2r'(r_0) \\ [\nu = 0, 1, \ldots, n-1;\ x \text{ in } R(r_0);\ \text{fixed } j \geqq 1] \end{cases}$$

would imply that $t_j(x)$, as defined in (39 *a*), satisfies

$$(42) \qquad |t_j(x)| < |e^{G(x)} x^{-n_1 - 2\tau}|\, M \sum_{\nu=0}^{n-1} |\bar{z}_{j-1}^{[\nu]}(x) - \bar{z}_{j-2}^{[\nu]}(x)|$$

$$= |e^{G(x)} x^{-n_1 - \tau}|\, M \sum_{\nu=0}^{n-1} |\zeta_{j-1}^{(\nu)}(x)| \qquad [x \text{ in } R(r_0)].$$

(¹) r' depends on the choice of τ.

(²) It is to be noted that $W(x, 0) = 0$.

Solving equation (39) by asymptotic methods a solution $\zeta_0(x)$ is obtained, analytic in $R(r_0)$ $(x \neq 0)$, together with its derivatives satisfying inequalities

$$(43) \qquad |\zeta_0^{(\nu)}(x)| \leqq |x|^{-\tau}\zeta_0 \qquad [\nu = 0, \ldots, n-1;\ x \text{ in } R(r_0)].$$

Let this value of τ *be used in the transformation* (35). *In the sequel Lemma* 6 *will be applied with*

$$(43a) \qquad \tau_1 = \tau - \zeta, \qquad \tau' = n_1 + \tau + \zeta \qquad (\zeta = {}_{n-1}\zeta_{n-}).$$

Corresponding to this choice of τ_1 *and* τ', *applicability of the Lemma necessitates that* r_0 *be sufficiently small. Choose* r_0 *also so that*

$$(43b) \qquad \zeta_0 \leqq r'(r_0) \quad (^1).$$

It is observed that in consequence of (43), (43 b) and (38; $j=1$) the inequalities (41) hold for $j=1$ $\left(\text{with } \bar{z}_{-1}^{[\nu]}(x) = 0\right)$. Hence by (42; $j=1$) (43) and (43 b)

$$(44) \qquad \begin{cases} |t_1(x)| < |e^{G(x)}x^{-n_1-\tau}|\, M n r' |x|^{-\tau} = |e^{G(x)}x^{-\tau_1-\tau'}|\, t_1 \\ \qquad [x \text{ in } R(r_0);\ t_1 = M n r']. \end{cases}$$

In applying Lemma 6 the inequalities (34) will be used in the simplified form

$$(45) \qquad \begin{cases} |\zeta^{(\nu)}(x)| < \bar{\zeta} t_j\, |e^{jG(x)}x^{-\tau_1 - j\tau' - \zeta}| \\ (\nu = 0, 1, \ldots, n-1;\ x \text{ in } R(r_0);\ \zeta = {}_{n-1}\zeta_{n-1}) \quad (^2). \end{cases}$$

By (44) and Lemma 6 the equation (39 a; $j=1$) possesses a solution $\zeta_1(x)$ such that

$$(46) \quad |\zeta_1^{(\nu)}(x)| < \bar{\zeta} t_1\, |e^{G(x)}x^{-\tau_1-\tau'-\zeta}| \qquad [\nu = 0, \ldots, n-1;\ x \text{ in } R(r_0)].$$

Thus, by (40; $j=2$) we have for x in $R(r_0)$

$$(46a) \quad \left|\bar{z}_1^{[\nu]}(x)\right| \leqq \left|\bar{z}_0^{[\nu]}(x)\right| + |x|^{\tau} |\zeta_1^{(\nu)}(x)|$$
$$< r' + \bar{\zeta} t_1\, |e^{G(x)}x^{-\tau_1-\tau'-\zeta+\tau}| = r' + \bar{\zeta} t_1\, |e^{G(x)}x^{-\tau'}| < r' + \delta$$
$$[\delta = r'/(1+r');\ \nu = 0, 1, \ldots, n-1]$$

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(1) This is possible since $r'(r_0) \to \infty$ as $r_0 \to 0$.

(2) In consequence of a previous remark ζ depends on the linear operator ${}_1L$ only.

provided r_0 *is such that, with* g_1 *denoting the greater one of the numbers* $\bar{\zeta} t_1$, $\mathrm{M} n \bar{\zeta}$, *we have*

$$(47) \qquad \Gamma(x) = |e^{\mathrm{G}(x)} x^{-\tau'} g_1| \leqq \delta \qquad [x \text{ in } \mathrm{R}(r_0)].$$

Since $r' + \delta < 2r'$ it is observed that inequalities (41) are satisfied for $j = 2$ so that, by (42; $j = 2$) and (46),

$$(48) \qquad \begin{aligned} |t_2(x)| &< |e^{\mathrm{G}(x)} x^{-n_1-\tau}| \, \mathrm{M} n \bar{\zeta} t_1 \, |e^{\mathrm{G}(x)} x^{-\tau_1-\tau'-\zeta}| \\ &= |e^{2\mathrm{G}(x)} x^{-\tau_1-2\tau'}| \, t_2 \qquad (t_2 = t_1 \mathrm{M} n \bar{\zeta}). \end{aligned}$$

We now solve (39 a; $j = 2$). By Lemma **6** (with $j = 2$) it follows that

$$(48a) \qquad |\zeta_2^{(\nu)}(x)| < \bar{\zeta} t_2 \, |e^{2\mathrm{G}(x)} x^{-\tau_1-2\tau'-\zeta}| \qquad (\nu = 0, 1, \ldots, n-1).$$

From (40; $j = 3$) it is inferred that

$$(48b) \qquad |\bar{z}_2^{[\nu]}(x)| < |\bar{z}_1^{[\nu]}(x)| + |x|^{\tau} \bar{\zeta} t_2 \, |e^{2\mathrm{G}(x)} x^{-\tau_1-2\tau'-\zeta}|$$

so that in consequence of (43 a) and (46 a)

$$(48c) \qquad |\bar{z}_2^{[\nu]}(x)| < r' + \delta + \bar{\zeta} t_2 \, |e^{2\mathrm{G}(x)} x^{-2\tau'}|.$$

Now since $t_2 = t_1 \mathrm{M} n \bar{\zeta}$ and g_1 is the greater one of the numbers $\bar{\zeta} t_1$, $\mathrm{M} n \bar{\zeta}$ it follows that $\bar{\zeta} t_2 \leqq g_1^2$. Thus by (47) and (48 c)

$$(49) \quad |z_2^{[\nu]}(x)| < r' + \delta + \delta^2 < 2r' \qquad [x \text{ in } \mathrm{R}(r_0);\ \nu = 0, 1, \ldots, n-1).$$

Assume now that for some $j\,(j \geqq 2)$ we have

$$(50) \qquad \left\{ \begin{array}{l} |\bar{z}_{j-1}^{[\nu]}(x)| < r' + \delta + \delta^2 + \ldots + \delta^{j-1} \; (< 2r') \\ \quad [\nu = 0, 1, \ldots, n-1;\ x \text{ in } \mathrm{R}(r_0)] \quad (^1) \end{array} \right.$$

and that, for x in $\mathrm{R}(r_0)$,

$$(51) \qquad |t_j(x)| < |e^{j\mathrm{G}(x)} x^{-\tau_1-j\tau'}| \, t_j \qquad [t_j = t_1 (\mathrm{M} n \bar{\zeta})^{j-1}] \quad (^2).$$

In view of (51) and by Lemma 6 the equation $\mathrm{L}(\zeta_j) = t_j(x)$ possesses a solution $\zeta_j(x)$ for which

$$(52) \quad |\zeta_j^{(\nu)}(x)| < \bar{\zeta} t_j \, |e^{j\mathrm{G}(x)} x^{-\tau_1-j\tau'-\zeta}| \qquad [\nu = 0, \ldots, n-1;\ x \text{ in } \mathrm{R}(r_0)].$$

---

(1) This has been previously established in (43), (43 b), (46 a), (48 c) for $j = 1$, $j = 2$, $j = 3$. For $j = 1$ the second member of (50) is written as r'.

(1) (51) has been proved for $j = 1$ and $j = 2$ in (44) and (48).

By virtue of (40), with j increased by unity, it would follow that

$$(53) \qquad \begin{cases} |\bar{z}_j^{[\nu]}(x)| < r' + \delta + \ldots + \delta^{j-1} + |x|^{\tau} \bar{\zeta} t_j \, | e^{jG(x)} x^{-\tau_1 - j\tau' - \zeta} | \\ [\nu = 0, \ldots, n-1;\ x \text{ in } R(r_0)]. \end{cases}$$

Now, by (43 a). $\tau - \tau_1 - \zeta = 0$. Hence on substituting the expression for t_j and on noting the definition of g_1, as given in connection with (47), it is inferred that

$$(54) \qquad \begin{aligned} |\bar{z}^{[\nu]}(x)| &< r' + \delta + \ldots + \delta^{j-1} + (\bar{\zeta} t_1)(Mn\bar{\zeta})^{j-1} | e^{jG(x)} x^{-j\tau'} | \\ &\leqq r' + \delta + \ldots + \delta^{j-1} + \Gamma^j(x) \leqq r' + \delta + \ldots + \delta^j < 2r' \\ &[x \text{ in } R(r_0);\ \nu = 0, \ldots, n-1]. \end{aligned}$$

Accordingly, by (50) and (54) the inequalities (41) are seen to hold with j replaced by $j+1$. Therefore (42) holds with j increased by unity. With the aid of (52) we obtain

$$|t_{j+1}(x)| < | e^{G(x)} x^{-n_1-\tau} | \, Mn\bar{\zeta} t_j \, | e^{jG(x)} x^{-\tau_1 - j\tau' - \zeta} |.$$

By (43 a) $-n_1 - \tau - \tau_1 - j\tau' - \zeta = -\tau_1 - (j+1)\tau'$. Thus

$$(55) \qquad |t_{j+1}(x)| < | e^{(j+1)G(x)} x^{-\tau_1 - (j+1)\tau'} | \, t_{j+1} \qquad [t_{j+1} = t_1 (Mn\bar{\zeta})^j].$$

Hence it is observed that (50) and (51) imply (54) and (55). It follows by induction that inequalities (50) and (51) hold for all positive j, the same of course being true of the inequalities (52).

In view of the above it is concluded that $\zeta(x)$, as given by the series (3), represents a solution of the equation (19; § 8). Each of the series

$$(56) \qquad \zeta^{(\nu)}(x) = \sum_{j=0}^{\infty} \zeta_j^{(\nu)}(x) \qquad (\nu = 0, 1, \ldots, n-1)$$

will converge absolutely and uniformly when x is in $R(r_0)$. In fact, by (52)

$$(56a) \qquad \begin{aligned} |\zeta^{(\nu)}(x)| &\leqq \sum_{j=0}^{\infty} |\zeta_j^{(\nu)}(x)| < \bar{\zeta}\,(x)^{-\tau_1-\zeta} \frac{t_1}{Mn\bar{\zeta}} \sum_{j=0}^{\infty} (Mn\bar{\zeta})^j \, | e^{jG(x)} x^{-j\tau'} | \\ &\leqq |x|^{-\tau_1-\zeta} r' \sum_{j=0}^{\infty} \Gamma^j(x) \leqq |x|^{-\tau_1-\zeta} r'(1 + \delta + \ldots) = |x|^{-\tau} r'(1 + r') \\ &[x \text{ in } R(r_0);\ \nu = 0, 1, \ldots, n-1]. \end{aligned}$$

Higher ordered derivatives of $\zeta(x)$ will be also represented by absolutely and uniformly convergent series. The constituent elements of the series (56) being analytic in $R(r_0)$. for every x, except $x = 0$, the functions $\zeta^{(\nu)}(x)$ ($\nu = 0, 1, \ldots$) will possess the same property. It is not difficult to see that the heuristically outlined interchanges of limiting processes, involved in (4 d), are legitimate in consequence of (52) and (51).

The developments of sections 7, 8, 9, 10 enable formulation of the following theorem.

Existence Theorem II. — *Consider the non-linear n-th order problem* (A), *as formulated* § 1. *The corresponding linear equation* (A_2; § 1) *has a complete set of formal solutions* (2; § 1). *Assume that not all the polynomials* $Q_\iota(x)$ *involved in* (2; § 1) *are zero. That is.* (A_2) *is to be formally not of Fuchsian type at* $x = 0$. *As stated in Lemma* 5 (§ 9) *there exist regions " proper ", in the sense of Def.* 5 (§ 9), *with respect to some of those* $Q_\iota(x)$ *which are not identically zero. Let* $R(r_0)$ (*cf. Def.* 5) *be such a region and designate the* $Q(x)$ *with respect to which* $R(r_0)$ *is proper as* $Q_1(x)[= Q_2(x) = \ldots = Q_\delta(x)]$.

Consider a formal solution $s(x)$ *satisfying equation* (A) *and specified in Lemma* 4 (§ 7) *under the assumption that* (49; § 7), (49 a; § 7) *hold with* $m^* = \delta$:

$$(57) \quad \left\{ \begin{aligned} & s(x) = s(x, c_1, k_2, k_3, \ldots, k_\delta) = \sum_{j=1}^{\infty} e^{jQ_1(x)} x^{-(j-1)\beta} \eta_j(x) c_1^j \\ & \left[\beta = (n-1)\left(1 + \frac{l}{\alpha}\right) + w + p - 1;\ |c_1|,\ |c_1 k_2|,\ \ldots,\ |c_1 k_\delta| \leqq k' \right]. \end{aligned} \right.$$

Here δ *arbitrary constants.* $c_1, k_2, \ldots, k_\delta$, *are involved. Given* $N(> 1)$, *however large. equation* (A) *has a solution* $y(x)$, *analytic in* $R(r_0)$ ($x \neq 0$), *with a singular point at* $x = 0$ *and such that*

$$(58) \qquad y(x) \sim s(x) \qquad [x \text{ in } R(r_0)].$$

Here r_0 *must be sufficiently small* (*cf. Def.* 5) *but can be taken independent of* N, *whenever* k' (*depending on* N) *is taken sufficiently small* [*cf.* Lemma 5 (§ 8)]. *The asymptotic relation* (58) *is in the following sense.*

The solution $y(x)$ is representable with the aid of the expression

$$(59) \qquad y(x) = \sum_{j=1}^{N-1} e^{jQ_1(x)} x^{-(j-1)\beta} \eta_j(x) c_1^j + e^{NQ_1(x)} x^{-(N-1)\beta} \zeta(x),$$

where $\zeta(x)$ is a function (defined by the convergent limiting process developed in this section), analytic in $R(r_0)$ $(x \neq 0)$. *This function, together with its derivatives, satisfies inequalities*

$$(60) \qquad |\zeta^{(\nu)}(x)| < |x|^{-\tau} k \qquad [x \text{ in } R(r_0);\ \nu = 0, 1, \ldots, n-1]$$

where k and τ are constants, the latter depending only on the character of the linear problem (A; § 1). [*Nothing is assumed regarding the curves* $R(jQ_1 - Q_\lambda) = 0$].

NOTE. — The asymptotic character of $y^{(1)}(x), \ldots, y^{(n-1)}(x)$ can be easily inferred from (59), (60). The asymptotic character of the derivatives $y^{(n)}(x), y^{(n+1)}(x), y^{(n+2)}(x) \ldots$, can be inferred directly with the aid of equation (A; § 1). It is essential to note that the functions $\eta_j(x)$, involved in (57), are well defined by means of the recursion differential equations of § 7. In all cases whatsoever the $\eta_j(x)$ possess certain asymptotic forms specified in Lemma 4 (§ 7). The first term of the formal series (57) is a solution, involving a number of arbitrary constants, of the linear problem corresponding to (A). Under additional hypotheses with respect to the given problem (A; § 1) the method of defining the $\eta_j(x)$ may yield additional information regarding their properties (1). Thus, for instance, under appropriate restrictions the formal series to which the $\eta_j(x)$ are asymptotic may be " summable ", say, with the aid of Laplace integrals leading to convergent factorial series. In the latter case such expressions, involving convergent factorial series, would correspondingly represent the $\eta_j(x)$. We have termed the relationship (59), (60) asymptotic, since this relationship implies that the sum of the first N — 1 terms of the formal series (57) can be used for computation of the " actual " solution $y(x)$ with an error which can be

(1) The propertise of interest are those for the neighborhood [within $R(r_0)$] of $x = 0$.

made as small as desired by letting r_0 be suitably small [while x is restricted to $R(r_0)$].

Similar remarks can be made, of course, regarding the Existence Theorem I (§ 6).

Finally, it is to be noted that a slightly greater generality can be achieved when the previously used conditions of the type

$$e^{Q(x)} \sim 0 \qquad [x \text{ in } R(r_0), \text{ also cf } (7), \S\, 2]$$

are replaced by certain other less stringent relations.

BIBLIOGRAPHY.

A NOTE REGARDING THE BIBLIOGRAPHY.

In order to avoid repetition the authors referred to in the excellent bibliography contained in the work of H. Dulac, Points singuliers des équations différentielles (*Mémorial des Sciences mathématiques*, Paris, 1934), are merely mentioned in [1].

In the subsequent numbers we refer to various contributions (some of them already included in the bibliography of Dulac) which have a more pronounced bearing on the particular aspects of the theory of differential equations considered by the present author.

On the whole, the connections between the contributions referred to below with those of the present author are somewhat remote. This is especially true with regard to the involved methods. There is no pretence for completeness of the bibliography. On the other hand, it is believed that all of the more relevant contributions have been indicated.

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