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FURTHER RESULTS ON NEUTRAL CONSENSUS FUNCTIONS

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RÉSUMÉ — Nouveaux résultats sur les fonctions de consensus neutres.
Nous abordons le problème du consensus par une voie ensembliste, en considérant un objet comme un assemblage de "briques" élémentaires. Une fonction de consensus est neutre s'il existe une famille $\mathcal{D}$ d'ensembles telle qu'une brique s appartient au consensus d'un profil si et seulement si l'ensemble des coordonnées des objets contenant s appartient à $\mathcal{D}$. Nous donnons des conditions suffisantes pour que $\mathcal{D}$ soit un filtre de treillis. Dans le cas d'un treillis fini, ces conditions s'avèrent être aussi suffisantes. Notre résultat final porte sur le cas d'un sup-demi-treillis distributif fini, dans lequel nous donnons des conditions nécessaires et suffisantes pour que $\mathcal{D}$ soit un ultrafiltre.

ABSTRACT — *We use a set theoretic approach to consensus by viewing an object as a set of smaller pieces called "bricks". A consensus function is neutral if there exists a family $\mathcal{D}$ of sets such that a brick s is in the output of a profile if and only if the set of positions with objects that contain s belongs to $\mathcal{D}$. We give sufficient set theoretic conditions for $\mathcal{D}$ to be a lattice filter and, in the case of a finite lattice, these conditions turn out to be necessary. Our final result, which involves a finite distributive join semilattice, provides necessary and sufficient conditions for $\mathcal{D}$ to be an ultrafilter.*

1. INTRODUCTION

One of the most influential results in the modern theory of social choice is Arrow's Impossibility Theorem (see Arrow, 1962). A small sample of results that follow Arrow's axiomatic approach are found in: (Aizermann and Aleskerov, 1986); (Barthélemy, 1982); (Brown, 1975); (Leclerc, 1984) and (Mirkin, 1975). The "Arrow-like" results found in these papers deal with choice functions, partial orders, valued preorders, and partitions. In recent years, the above results have been unified using an order theoretic approach to consensus (see Barthélemy et al., 1986), (Monjardet, 1990), and (Leclerc and Monjardet, 1994). The idea behind this approach is to view a mapping $F : L^k \to L$, where L is a finite

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lattice or semilattice, as a consensus function and to require F to satisfy axioms akin to unanimity and independence of irrelevant alternatives. Depending on the structure of L, these axioms force F to be an oligarchy or even a dictatorship (see Theorem 1, Leclerc and Monjardet, 1994). The latest work on this ordinal approach to consensus is found in (Monjardet, 1995).

In two earlier papers, (Crown et al., 1993 and 1994), we proved "Arrow-like" results using neutrality in place of the axiom of independence of irrelevant alternatives. These results are quite similar to ones obtained by Leclerc and Monjardet but our point of view is different. We use a set theoretic approach to consensus by thinking of an object as a set of smaller pieces called "bricks". For example, the ordered pairs that belong to a binary relation are the bricks that make up this binary relation. It is possible to restrict the set theoretic approach to an ordered set by viewing each element as the set of sup-irreducibles it dominates. This is done in Theorems 8 and 9 in the sequel. If there are only a finite number of voters, then these two theorems give necessary and sufficient conditions for a neutral consensus function to be an oligarchy on a finite lattice and a dictatorship on a finite distributive join-semilattice. Our first result, Theorem 7, gives sufficient set theoretic conditions for a neutral consensus function to be an oligarchy. Theorems 7, 8, and 9 are the main results of this paper and they can be found in Section 3. In order to keep this note self contained, the next section is devoted to preliminaries as well as the statements, and some proofs, of results from (Crown et al., 1993).

2. PRELIMINARIES

Let X be an arbitrary nonempty set. Let V be a set with at least two elements, and let X^V denote the set of all functions from V into X. Then $F : X^V \to X$ is called a *consensus function on* X. In addition, we associate with X a nonempty set S (the *bricks*) and a one-to-one function $\gamma : X \to P(S)$. Here $P(S)$ is the power set of S. The idea is that the pair (S, γ), in some way, represents X. Thus we call such a pair a *representation family* (referred to in (Crown et al., 1993) as a *stability family*). As an example, let X be the set of all weak orders on a set E. Then $(E \times E, \gamma)$ is a representation family on X where $\gamma : X \to P(E \times E)$ is given by $\gamma(R) = \{(x, y) : (x, y) \in R\}$.

An element π in X^V is called a *profile* and for each $s \in S$ we let $s_\pi = \{\alpha \in V : s \in \gamma(\pi(\alpha))\}$. A consensus function F is *neutral* with respect to a fixed representation family (S, γ) if, for all $s, s' \in S$ and all profiles π, π', the condition $s_\pi = s'_{\pi'}$ implies $s \in \gamma(F(\pi))$ if and only if $s' \in \gamma(F(\pi'))$. A neutral consensus function is characterized by the fact that there exists a subset $\mathcal{D}$ of $P(V)$ such that $s \in \gamma(F(\pi))$ if and only if $s_\pi \in \mathcal{D}$. In (Crown et al., 1993) we called the set $\mathcal{D}$ a *consensus family on* X.

A nonempty subset $\mathcal{F}$ of $P(V)$ is an *order filter* if the following condition is satisfied: if $A \in \mathcal{F}$ and $A \subseteq B \subseteq V$, then $B \in \mathcal{F}$. A nonempty subset $\mathcal{F}$ of $P(V)$ is a *lattice filter* if it is an order filter such that $A \cap B \in \mathcal{F}$ whenever $A, B \in \mathcal{F}$. An *ultrafilter* of $P(V)$ is a lattice filter $\mathcal{F}$ where either $A \in \mathcal{F}$ or $V - A \in \mathcal{F}$ for any subset A of $P(V)$.

Let F be a neutral consensus function with corresponding consensus family $\mathcal{D}$. We will say that F is a *strong dictatorship* if $\mathcal{D}$ is an ultrafilter. We will call F an *oligarchy* if there is a subset $A \subseteq V$ such that $\mathcal{D} = \{B \subseteq V : A \subseteq B\}$.

For $k \geq 2$, the k-tuple $\{a_1, ..., a_k\}$ of elements from S is *transitive* if, for any $x \in X, \{a_1, ..., a_{k-1}\} \subseteq \gamma(x)$ implies that $a_k \in \gamma(x)$. The element z in X *includes* $\{a_1, ..., a_k\}$

if $\{a_1, ..., a_k\} \subseteq \gamma(z)$, whereas w in X *excludes* $\{a_1, ..., a_k\}$ if $\{a_1, ..., a_k\} \cap \gamma(w) = \emptyset$. For any subsets A, B of S, we say $x \in X$ *separates* A from B if $A \subseteq \gamma(x)$ and $B \cap \gamma(x) = \emptyset$. *From now on, we identify the element $x \in X$ with the set $\gamma(x)$ and any consensus family $\mathcal{D}$ on X is assumed to satisfy: $\emptyset \notin \mathcal{D}$ and $V \in \mathcal{D}$.* To make the proofs in the next section more readable and self-contained we include the statements of some results from (Crown et al., 1993). In some cases we include short proofs.

Lemma 1 *Suppose that for some $k \geq 2$, S contains a transitive k-tuple $\{a_1, ..., a_k\}$. Suppose further that $w, y, z \in X$ can be found such that y separates a_k from $\{a_1, ..., a_{k-1}\}$, z includes $\{a_1, ..., a_k\}$, and w excludes $\{a_1, ..., a_k\}$. Any consensus family $\mathcal{D}$ on X is then an order filter on X.*

Proof. Let $\mathcal{D}$ be a consensus family on X. Let $A \in \mathcal{D}$, and $A \subset B \subset V$. Define a profile π by $\pi(\alpha) = w$ if $\alpha \in A$; $\pi(\alpha) = y$ if $\alpha \in B - A$; $\pi(\alpha) = z$ if $\alpha \notin B$. Then for $a = a_i$ $(1 \leq i < k)$, $a_\pi = A \in \mathcal{D}$. Hence $\{a_1, ..., a_{k-1}\} \subseteq F(\pi)$. By transitivity, $a_k \in F(\pi)$. It follows that $B = (a_k)_\pi \in \mathcal{D}$.

Lemma 2 *Suppose that for some $k \geq 3$, S contains a transitive k-tuple $\{a_1, ..., a_k\}$. Suppose further that $h, t, z \in X$ can be found so that for some i $(1 \leq i \leq k-2)$: h separates $\{a_{i+1}, ..., a_k\}$ from $\{a_1, ..., a_i\}$, t separates $\{a_{i+1}, ..., a_{k-1}\}$ from $\{a_1, ..., a_i, a_k\}$, and z includes $\{a_1, ..., a_k\}$. Any consensus family $\mathcal{D}$ on X is then an order filter on X.*

Proof. Let $\mathcal{D}$ be a consensus family on X. Let $A \in \mathcal{D}$, and $A \subset B \subset V$. Define a profile π by $\pi(\alpha) = z$ if $\alpha \in A$; $\pi(\alpha) = h$ if $\alpha \in B - A$; $\pi(\alpha) = t$ if $\alpha \notin B$. Then for $a = a_j$ $(i+1 \leq j < k)$, $a \in h \cap t \cap z$, so $a_\pi = V$, and $a \in F(\pi)$. If $b = a_j$ $(1 \leq j \leq i)$, then $b \in z - (h \cup t)$, so $b_\pi = A \in \mathcal{D}$ and $b \in F(\pi)$. By transitivity, $a_k \in F(\pi)$. It follows that $B = (a_k)_\pi \in \mathcal{D}$.

Lemma 3 *For $k \geq 3$, let $\{a_1, ..., a_k\}$ be a transitive k-tuple of elements from S. Suppose that there exist elements $g, t, w, z \in X$ and an index $i < k-1$ such that g separates $\{a_1, ..., a_i\}$ from $\{a_{i+1}, ..., a_k\}$, t separates $\{a_{i+1}, ..., a_{k-1}\}$ from $\{a_1, ..., a_i, a_k\}$, z includes $\{a_1, ..., a_k\}$, and w excludes $\{a_1, ..., a_k\}$. Then any consensus family $\mathcal{D}$ on X is closed under finite intersections.*

Proof. Let $\mathcal{D}$ be a consensus family on X. Let $A, B \in \mathcal{D}$. Define a profile π by $\pi(\alpha) = g$ if $\alpha \in A - B$; $\pi(\alpha) = t$ if $\alpha \in B - A$; $\pi(\alpha) = z$ if $\alpha \in A \cap B$; and $\pi(\alpha) = w$ if $\alpha \notin A \cup B$. Then for $a = a_j$ $(1 \leq j \leq i)$, $a_\pi = A \in \mathcal{D}$; for $b = a_j$ $(i+1 \leq j < k)$, $b_\pi = B \in \mathcal{D}$ together imply that $\{a_1, ..., a_{k-1}\} \subseteq F(\pi)$. By transitivity, $a_k \in F(\pi)$. Hence $A \cap B = (a_k)_\pi \in \mathcal{D}$.

A finite lattice X is *distributive* if $s \leq x \vee y$ implies that either $s \leq x$ or $s \leq y$ for any sup-irreducible s and elements x and y in X. It is convenient to represent X by means of the mapping

$$a \mapsto \gamma(a) = \{s : s \leq a \textit{ and } s \textit{ is } sup-irreducible\}.$$

It follows that X is distributive if and only if X is closed under union. The following lemma is a characterization for nondistributivity and it improves statement (2) in Theorem 1 in (Leclerc and Monjardet, 1993)

Lemma 4 *Let X be a finite lattice, and represent X by means of the mapping $a \mapsto \gamma(a) = \{s : s \leq a$ and s is $sup - irreducible\}$. Then the following are equivalent:*
i) X is nondistributive;
ii) For V finite, every consensus family $\mathcal{D}$ on X is a lattice filter.

A finite join-semilattice X is *distributive* if every principal filter of X is a distributive lattice. We were able to find the following sufficient conditions on X to force a consensus family on X to be an ultrafilter.

Lemma 5 *Let X be a finite distributive join-semilattice that is not a lattice, and represent X as a set of subsets of its sup-irreducibles in the usual manner. If $X \cup \{0\}$ is not distributive, and if X contains at least four atoms, then every consensus family $\mathcal{D}$ on X is an ultrafilter.*

In this note we give necessary conditions on X to force a consensus family on X to be an ultrafilter. Note that a finite distributive join-semilattice that is not a lattice and is represented as a set of subsets of its sup-irreducibles in the usual manner, is not closed under unions or intersections. This leads us to our final lemma.

Lemma 6 *Suppose X is not closed under unions or intersections. If there exist $x, y, z \in X$ such that $x \cup y \notin X$ and $x \cup y \subseteq z$, then any consensus family $\mathcal{D}$ on X which is an order filter is, in fact, an ultrafilter.*

3. RESULTS

If we combine Lemmas 2 and 3, then the existence of a transitive k-tuple along with the five elements g, h, t, w, z from X satisfying the hypotheses of these lemmas forces any consensus family $\mathcal{D}$ on X to be a lattice filter. This is the content of Theorem 29 in (Crown et al., 1993). Our first theorem shows that the element h is unnecessary.

Theorem 7 *For $k \geq 3$, let $(a_1, ..., a_k)$ be a transitive k-tuple of elements of S, and suppose that for some $i < k-1$, there exist elements $g, t, w, z \in X$ such that: z includes $\{a_1, ..., a_k\}$, w excludes $\{a_1, ..., a_k\}$, g separates $\{a_1, ..., a_i\}$ from $\{a_{i+1}, ..., a_k\}$ and t separates $\{a_{i+1}, ..., a_{k-1}\}$ from $\{a_1, ..., a_i, a_k\}$. Then any consensus family $\mathcal{D}$ on X is a lattice filter.*

Proof. Note that the hypothesis of Lemma 3 are satisfied. Therefore, it is only necessary to show that $\mathcal{D}$ is an order filter. Assume that there exists a consensus family $\mathcal{D}$ on X which is not an order filter. Then there exist $A \subset B \subset V$ such that $A \in \mathcal{D}$ and $B \notin \mathcal{D}$. Keep in mind that $V \in \mathcal{D}$ and so $B \neq V$. Define a profile π by $\pi(\alpha) = z$ if $\alpha \in A$; $\pi(\alpha) = g$ if $\alpha \in B - A$; $\pi(\alpha) = t$ if $\alpha \notin B$.
Let $F(\pi) = u$. Assume that $A \cup (V - B) \notin \mathcal{D}$. Then u contains a_k since $(a_k)_\pi = A$. But $a_j \notin u$ for $j = 1, ..., k-1$ since either $(a_j)_\pi = B$ or $(a_j)_\pi = A \cup (V - B)$. Now apply Lemma 1 using the transitive k-tuple $(a_1, ..., a_k)$ and the elements u, z and w. This leads to the conclusion that $\mathcal{D}$ is an order filter contrary to our assumption.

Now, assume that $A \cup (V - B) \in \mathcal{D}$. Then $F(\pi) = u$ contains a_k since $(a_k)_\pi = A$ and contains a_j for $j = i+1, ..., k-1$ since for such j, $(a_j)_\pi = A \cup (V - B)$. Moreover, u does not contain $\{a_1, ..., a_i\}$ because $(a_j)_\pi = B$ when $j = 1, ..., i$. Now apply Lemma 2 using the transitive k-tuple $(a_1, ..., a_k)$ and the elements u, t and z. This again leads to the conclusion that $\mathcal{D}$ is an order filter contrary to our assumption. Hence any consensus family $\mathcal{D}$ on X is an order filter.

Theorem 7 above is a generalization of Theorem 4.7 in (Crown et al., 1994). Using the notation of Theorem 4.7 identify the following elements: g and a; t and b; w and 0; z and c. Also note that (a, b, c) is transitive. Thus Theorem 4.7 is an order theoretic corollary of Theorem 7. We also note that in the finite case, $\mathcal{D}$ is a lattice filter if and only if F is an oligarchy. Thus, the conditions of Theorem 7, in the finite case, imply that every consensus function is an oligarchy.
For the remainder of this paper, unless otherwise specified, we shall represent the elements of a semilattice X as sets of sup-irreducibles. Specifically, for an element a in X, $\gamma(a) = \{s : s \leq a \text{ and } s \text{ is } sup - irreducible\}$.
The next result provides a situation for which there is a converse to Theorem 7.

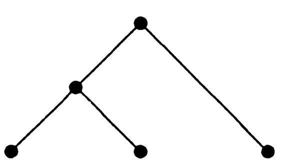
Theorem 8 *Let X be a finite lattice, and represent X by means of the mapping $a \mapsto \gamma(a) = \{s : s \leq a \text{ and } s \text{ is } sup - irreducible\}$. Then the following are equivalent:*
i) X is nondistributive;
ii) For V finite, every consensus family $\mathcal{D}$ on X is a lattice filter;
iii) There exists, for $k \geq 3$, a transitive k-tuple $(a_1, ..., a_k)$ and an index $i < k - 1$ along with elements $g, t \in X$ such that : g separates $\{a_1, ..., a_i\}$ from $\{a_{i+1}, ..., a_k\}$ and t separates $\{a_{i+1}, ..., a_{k-1}\}$ from $\{a_1, ..., a_i, a_k\}$.

Proof. The equivalence of i) and ii) is Lemma 4. Theorem 7 shows that iii) implies ii), keeping in mind that 1 includes and 0 excludes any subset of sup-irreducibles, respectively. We now show that i) implies iii). Assume X is nondistributive. Then there must exist $x, y \in X$ such that $\gamma(x) \cup \gamma(y) \subset \gamma(x \vee y)$. Let $b_1, b_2, ..., b_i$ be maximal elements of $\gamma(x)$, $b_{i+1}, ..., b_{k-1}$ be maximal elements of $\gamma(y)$, and $b_k \in \gamma(x \vee y) - (\gamma(x) \cup \gamma(y))$. Consider the $(k-1)$-tuple $(b_1, ..., b_{k-1})$. If $s < t$ and $b_s \leq b_t$, remove b_s. The remaining elements are pairwise incomparable, and their join still contains b_k. Thus we may assume without loss in generality that the elements of $(b_1, ..., b_{k-1})$ are pairwise incomparable. Furthermore, by the choice of b_k, we can not have $b_k \leq b_s$ for any $s < k$. We may now choose a transitive j-tuple $(a_1, ..., a_j)$ of minimal cardinality such that $(a_1, ..., a_{j-1})$ are pairwise incomparable, and $a_j \not\leq a_i$ for $i < j$. By minimality, we know that $a_1 \not\leq a_2 \vee ... \vee a_{j-1}$. Take $g = a_1$, $t = a_2 \vee ... \vee a_{j-1}$. Then g separates a_1 from $\{a_2, ..., a_j\}$, and t separates $\{a_2, ..., a_{j-1}\}$ from $\{a_1, a_j\}$. To see this, recall that we have already argued that $a_1 \leq t$ fails by minimality. By a similar token, applying minimality, we can not have $a_j \leq t$. This completes the proof.

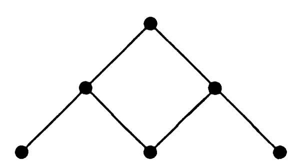
We now consider the case where X is a finite distributive join-semilattice that is not a lattice. If a 0 element is added, then $X \cup \{0\}$ becomes a lattice. If $X \cup \{0\}$ is distributive, then there exists a consensus family on X that contains a pair of complementary subsets of V. In the case where $X \cup \{0\}$ is nondistributive, we have the following result.

Theorem 9 *Let X be a finite distributive join-semilattice that is not a lattice, and represent X as a set of subsets of its sup-irreducibles in the usual manner. Assume $X \cup \{0\}$ is not distributive. Then every consensus family on X is an ultrafilter if and only if $X \neq \{a, b, c, 1\}$ where a, b and c are atoms covered by 1.*

Proof. If $V = \{1, 2, 3\}$, then $\mathcal{D} = \{\{1\}, \{2\}, \{3\}, V\}$ is an example of a consensus family on $X = \{a, b, c, 1\}$ which is not an ultrafilter.
Conversely, if X contains at least four atoms then by Lemma 5 every consensus family on X is an ultrafilter. Since $X \cup \{0\}$ is not distributive it follows that X contains more than one atom. So we can assume that X contains either two or three atoms.
If a is an atom and s is a sup-irreducible such that $a < s$ then the pair (s, a) is transitive. Consider Lemma 1 as applied to the pair (s, a). The element s includes $\{s, a\}$, and a separates $\{a\}$ from $\{s\}$. Let $b \neq a$ be an atom so that $\{s, a\} \cap \{b\} = \emptyset$. Thus b excludes $\{s, a\}$. Hence by Lemma 1, any consensus family is an order filter and, by Lemma 6, must in fact be a ultrafilter.
At this stage, we can assume that X is atomistic and contains 2 or 3 atoms . But if X contains exactly 2 atoms then $X \cup \{0\}$ is distributive. So we can assume that X has exactly 3 atoms denoted by a,b,c. But, by checking cases, one can see that $X \cup \{0\}$ is nondistributive if and only if $X = \{a, b, c, 1\}$ where a,b,c are covered by 1, or $X = \{a, b, c, \sup\{\text{a,c}\}\ , 1\}$ where $\sup\{a, c\}$ and b are coatoms, or $X = \{a, b, c, \sup\{a, b\}, \sup\{b, c\}, 1\}$ where $\sup\{a, b\}$ and $\sup\{b, c\}$ are coatoms as shown below.



Second Case



Third Case

By hypothesis, the first case is not a possibility. For the second case, we can apply Lemma 2. The triple (b, c, a) is transitive, 1 includes $\{b, c, a\}$, c separates $\{c\}$ from $\{b, a\}$ and $\sup\{a, c\}$ separates $\{c, a\}$ from $\{b\}$. By Lemma 2, any consensus family on X is an order filter. For the last possibility, we can again apply Lemma 2. The triple (a, c, b) is transitive, 1 includes $\{b, c, a\}$, c separates $\{c\}$ from $\{a, b\}$ and $\sup\{b, c\}$ separates $\{c, b\}$ from $\{a\}$. Again, by Lemma 2, any consensus family on X is an order filter. Finally, apply Lemma 6 to get that any consensus family on X is an ultrafilter.

Having completed the discussion where X is a distributive semilattice with $X \cup \{0\}$ not distributive, it seems natural to consider the case where X is a finite distributive lattice. In this case, it is not hard to verify that X is not atomistic if and only if the consensus families on X coincide with the order filters of $P(V)$ and X is atomistic if and only if the consensus families on X coincide with all subsets of $P(V)$ that contain V but not $\emptyset$.

It appears that there are many connections between the work done here and in (Crown, et al., 1993, 1994) with the work of (Leclerc and Monjardet, 1994) and (Monjardet, 1995). It would be interesting to see if there was an abstract setting within which both sets of results abide.

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