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ON BINARY TREES AND DYCK PATHS

A. PANAYOTOPOULOS and A. SAPOUNAKIS[1]

RÉSUMÉ — Sur les arbres binaires et les chemins de Dyck.
Une bijection entre les arbres binaires à n sommets et les chemins de Dyck de longeur 2n est obtenue. Deux constructions permettent de passer d'un chemin de Dyck à un arbre binaire et d' un arbre binaire à un chemin de Dyck.

SUMMARY —*A bijection between the set of binary trees with n vertices and the set of Dyck paths of length 2n is obtained. Two constructions are given which enable to pass from a Dyck path to a binary tree and from a binary tree to a Dyck path.*

1. INTRODUCTION

It is well known that there exists a large number of important sets in Combinatorics with the Catalan cardinality [1]. Two of these sets are examined in this paper; the set $\mathcal{T}_n$ of rooted unlabelled binary trees with n vertices and the set $\mathcal{D}_{2n}$ of Dyck paths of length $2n$.

For the study of the above notions we recall several definitions ([2], [3], [4]). For every permutation with repetitions (p.r.) $\varphi=\varphi(1)\varphi(2)...\varphi(n)$ we denote

$$\lambda(i) = \max\,\{l(i), r(i)\}$$

where $l(i)$ (resp. $r(i)$) is the first element on the left (resp. right) of the i^{th} position which is smaller than $\varphi(i)$. If $l(i)$ or/and $r(i)$ does not exist we assume that $l(i)$ or/and $r(i)$ is equal to zero.

For example if φ=32341243 then λ=21230132.

In [3] it is proved that each binary tree T with n vertices is associated to a p.r. $\varphi=\varphi(1)\varphi(2)...\varphi(n)$, which satisfies certain properties and determines T uniquely.

Indeed if the vertices of T are enumerated according to the inorder of T (*i.e.* by visiting the left subtree first, then the root, and then the right subtree) the p.r. φ is defined as follows :

$$\varphi(i)=k \text{ iff the } i^{\text{th}} \text{ vertex of T belongs to the } k^{\text{th}} \text{ level.}$$

For example the corresponding p.r. of the tree T of Figure 1 is φ=32341243.

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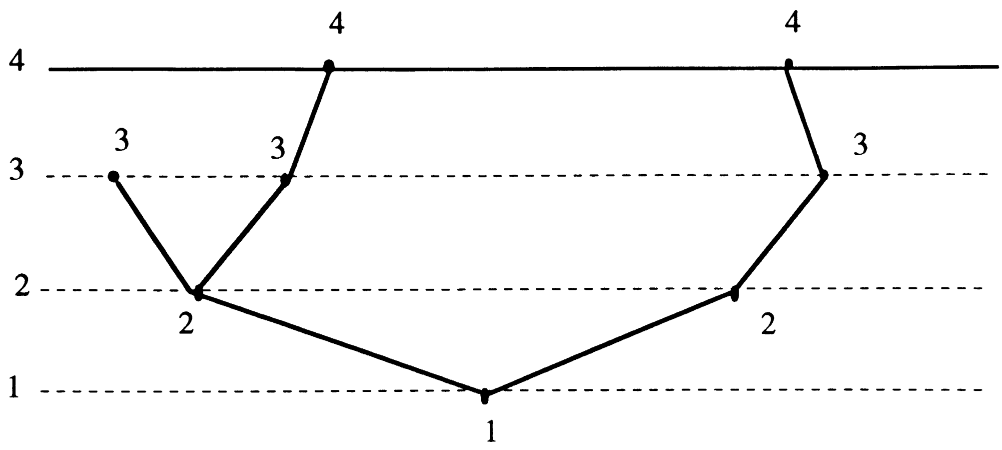

Figure 1. The binary tree of φ=32341243

This p.r. satisfies the following two conditions :

(i) For every $i \neq j \in [n]$, with $\varphi(i)=\varphi(j)=k$, there exists t between i and j such that $\varphi(t)<k$.

(ii) $\lambda(i)=\varphi(i)-1$, for every $i \in [n]$.

On the other hand every p.r. satisfying the above two conditions generates the associated binary tree, so that the set of all binary trees T with n vertices may be identified with the set of all p.r. $\varphi=\varphi(1)\varphi(2)...\varphi(n)$ which satisfy the properties (i) and (ii) (see Proposition 2.1 in [3]).

A Dyck word is a word $w \in \{\alpha,\overline{\alpha}\}^*$ satisfying the following two conditions :

(i) $|w|_\alpha = |w|_{\overline{\alpha}}$

(ii) for every factorization $w=uv$, $|u|_\alpha \geq |u|_\alpha$

where $|w|_\alpha$, $|u|_\alpha$ (resp. $|w|_{\overline{\alpha}}$, $|u|_{\overline{\alpha}}$), denote the number of occurences of the letter α (resp. $\overline{\alpha}$) in the words w,u.

A Dyck path $(s_0,s_1,...,s_{2n})$ of length $2n$, is a minimal path of NxN, lying above the diagonal and joining the points (0,0), (n,n).

It is well known that the Dyck paths of length $2n$ are coded by the Dyck words $w=z_1z_2...z_{2n}$ such that every vertical (resp. horizontal) edge (s_{i-1},s_i) corresponds to the letter $z_i=\alpha$ (resp. $z_i=\overline{\alpha}$).

Thus in the sequel the two notions of Dyck paths and Dyck words are identified.

For example the word $w=\alpha\overline{\alpha}\,\alpha\alpha\overline{\alpha}\,\alpha\overline{\alpha}\,\overline{\alpha}\,\alpha\alpha\overline{\alpha}\,\alpha\alpha\overline{\alpha}\,\overline{\alpha}\,\overline{\alpha}$ is identified with the minimal path of NxN described in Figure 2.

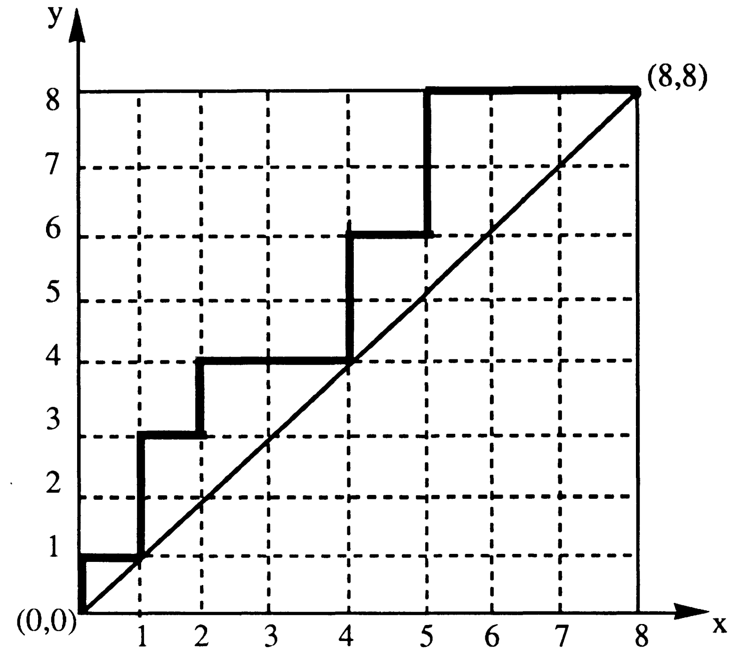

Figure 2. The Dyck word $w=\alpha\bar{\alpha}\,\alpha\alpha\bar{\alpha}\,\alpha\bar{\alpha}\,\bar{\alpha}\,\alpha\alpha\bar{\alpha}\,\alpha\alpha\bar{\alpha}\,\bar{\alpha}\,\bar{\alpha}$

In section 2 we use the associated p.r. φ of a binary tree T for the generation of a Dyck path, which may be obtained from T by a simple practical method.

The converse problem is examined in section 3. Here using well known properties of Dyck words we construct a p.r. φ which satisfies the two conditions which are necessary for the generation of binary trees. Thus, given a Dyck path we generate a binary tree which may be obtained by a simple practical method.

In section 4 we show that each one of the constructions of the previous two sections is the inverse of the other, so that we obtain a new bijection between the sets $\mathcal{T}_n$ and $\mathcal{D}_{2n}$.

2. GENERATION OF DYCK PATHS

For the generation of a Dyck path by a binary tree T we use the associated p.r. $\varphi=\varphi(1)\varphi(2)...\varphi(n)$. We first give a useful property of φ.

LEMMA 2.1. *If* $\varphi(i+1)<\varphi(i)$, $i\in[n\text{-}1]$, *there exists* $\xi\in[i]$ *such that* $\varphi(\xi)=\varphi(i+1)+1$ *and* $\varphi(\xi)<\varphi(\mathrm{t})$, *for every t with* $\xi<t<i+1$.

Proof. Let, $\mathrm{H}=\{p\in[i] : \varphi(i+1)<\varphi(p)<\varphi(t),$ whenever $p<t<i+1\}$. Clearly since $i\in\mathrm{H}$, we have that $\mathrm{H}\neq\varnothing$.

Let $\xi\in\mathrm{H}$ such that $\varphi(\xi)=min\,\{\varphi(p) : p\in\mathrm{H}\}$. If $\lambda(\xi)=l(\xi)$, it follows that there exists $j\in[n]$ such that $j<\xi$, $\varphi(j)=\lambda(\xi)=\varphi(\xi)\text{-}1$ and $\varphi(\xi)<\varphi(t)$ for each t with $j<t<\xi$.

Then since $\xi\in\mathrm{H}$ we have that $\varphi(j)<\varphi(\xi)\leq\varphi(t)$ for every $j<t<i+1$. Further, since $\varphi(i+1)<\varphi(\xi)=\varphi(j)+1$ and $\varphi(i+1)\neq\varphi(j)$ we obtain that $\varphi(i+1)<\varphi(j)$.

This proves that $j\in \mathrm{H}$, though $\varphi(j)<\varphi(\xi)$ which contradicts the minimality property of $\varphi(\xi)$. Thus $\lambda(\xi)=r(\xi)$.

Finally, since $\varphi(t)>\varphi(\xi)$ for each t with $\xi<t<i+1$, we deduce that $\lambda(\xi)=\varphi(i+1)$ and $\varphi(\xi)=\lambda(\xi)+1=\varphi(i+1)+1$. ∎

In the sequel it is assumed that $\varphi(n+1)=0$.

PROPOSITION 2.2. *The word* $\mathrm{w}=\mathrm{z}_1\mathrm{z}_2...\mathrm{z}_{2\mathrm{n}}$ *with*

$$\mathrm{z}_{2i-1}=\begin{cases}\alpha & \text{if } l(i)\leq r(i)\\ \overline{\alpha} & \text{if } l(i)>r(i)\end{cases} \quad \text{and} \quad \mathrm{z}_{2i}=\begin{cases}\alpha & \text{if } \varphi(i)<\varphi(i+1)\\ \overline{\alpha} & \text{if } \varphi(i)>\varphi(i+1)\end{cases}$$

is a Dyck word.

PROOF. Given $p\in[2n]$ we consider the following sets :

$$\mathrm{E}_p=\{2i-1\in[p] : \mathrm{z}_{2i-1}=\alpha\}\ ,\quad \mathrm{F}_p=\{2i\in[p] : \mathrm{z}_{2i}=\alpha\}$$

$$\overline{\mathrm{E}}_p=\{2i-1\in[p] : \mathrm{z}_{2i-1}=\overline{\alpha}\}\ ,\quad \overline{\mathrm{F}}_p=\{2i\in[p] : \mathrm{z}_{2i}=\overline{\alpha}\}.$$

It is clear that these four sets form a partition of $[p]$.

We will define two functions $\mathrm{h}_1 : \overline{\mathrm{F}}_p\rightarrow\mathrm{E}_p$ and $\mathrm{h}_2 : \overline{\mathrm{E}}_p\rightarrow\mathrm{F}_p$.

Indeed, given an element $2i\in\overline{\mathrm{F}}_p$ we have that $\varphi(i+1)<\varphi(i)$ and by Lemma 2.1 there exists unique $\xi\in[i]$ such that $\varphi(\xi)=\varphi(i+1)+1$ and $\varphi(t)>\varphi(\xi)$, for every t with $\xi<t<i+1$.

We define the first function by $\mathrm{h}_1(2i)=2\xi-1$. Clearly, since $l(\xi)<r(\xi)$ we deduce that $2\xi-1\in\mathrm{E}_p$, so that the function h_1 takes its values in E_p.

Now given an element $2i-1\in\overline{\mathrm{E}}_p$ we have that $r(i)<l(i)$, so that there exists unique $j\in[n]$ such that $j<i$, $\varphi(j)=l(i)$ and $\varphi(t)>\varphi(i)$, for every t with $j<t<i$.

We define the second function by $\mathrm{h}_2(2i-1)=2j$. Clearly, since $\varphi(j)<\varphi(j+1)$ we deduce that $2j\in\mathrm{F}_p$ so that the function h_2 takes its values in F_p.

It is easy to check that both these functions are one to one, so that we obtain $|\overline{\mathrm{F}}_p|\leq|\mathrm{E}_p|$ and $|\overline{\mathrm{E}}_p|\leq|\mathrm{F}_p|$ for each $p\in[2n]$.

Further it is easy to see that these two functions are also onto, in the case when $p=2n$, so that we obtain $|\overline{\mathrm{F}}_{2n}|=|\mathrm{E}_{2\mathrm{n}}|$ and $|\overline{\mathrm{E}}_{2n}|=|\mathrm{F}_{2\mathrm{n}}|$.

From the above relations we deduce that

$|\mathrm{z}_1\mathrm{z}_2...\mathrm{z}_p|_{\overline{\alpha}}=|\overline{\mathrm{E}}_p|+|\overline{\mathrm{F}}_p|\leq|\mathrm{F}_p|+|\mathrm{E}_p|=|\mathrm{z}_1\mathrm{z}_2...\mathrm{z}_p|_{\alpha}$ for each $p\in[2n]$, and

$|\mathrm{z}_1\mathrm{z}_2...\mathrm{z}_{2n}|_{\overline{\alpha}}=|\overline{\mathrm{E}}_{2n}|+|\overline{\mathrm{F}}_{2n}|=|\mathrm{F}_{2n}|+|\mathrm{E}_{2n}|=|\mathrm{z}_1\mathrm{z}_2...\mathrm{z}_{2n}|_{\alpha}$.

This shows that $\mathrm{w}=\mathrm{z}_1\mathrm{z}_2...\mathrm{z}_{2n}$ is a Dyck word. ∎

Thus, given a binary tree T with n vertices, a Dyck word of length $2n$ is constructed, which is denoted by w_T.

For example, if T is the binary tree of Figure 1 then w_T is the Dyck word of Figure 2.

REMARK 2.3. A practical method for the determination of the Dyck word w_T is the following :

- We first construct the extended binary tree of T and then we label its root by α and each vertex which is a left (resp. right) child by α (resp. $\overline{\alpha}$).

- The Dyck word w_T is obtained by the inorder traversal of the labelled extended binary tree, starting from the second vertex (see Figure 3).

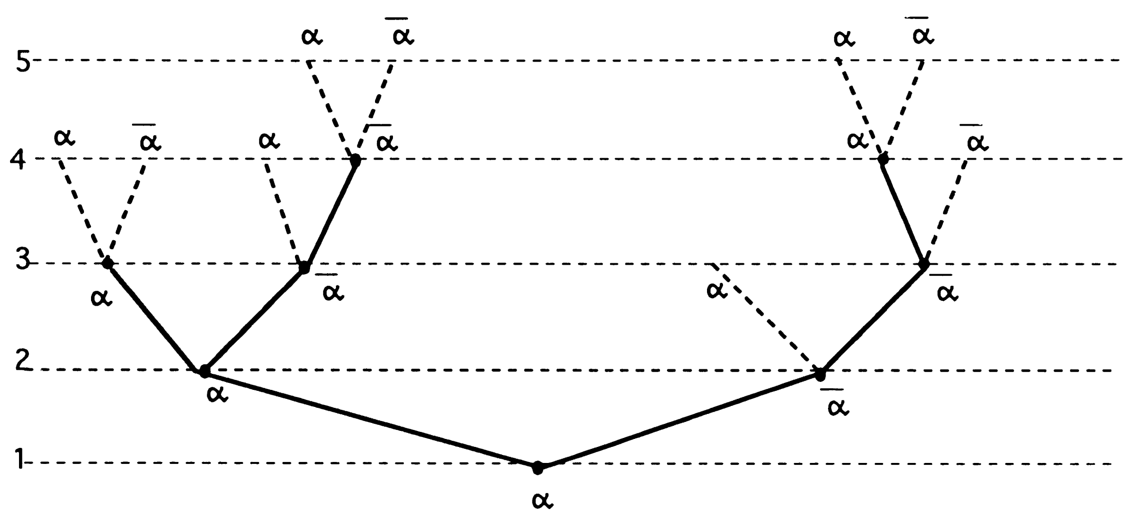

Figure 3. The extended tree of T.

3. GENERATION OF BINARY TREES

For the generation of a binary tree by a Dyck word $w=z_1z_2...z_{2n}$ we consider the following sets : $I=\{i\in[2n] : z_i=\alpha\}$ and $J=\{j\in[2n] : z_j=\overline{\alpha}\}$.

We first summarize the main well known properties of Dyck words, the proof of which is omitted.

LEMMA 3.1. *Let* $i\in I$, $j\in J$, *such that* $i<j$. *Then the following conditions are equivalent :*
(a) The element i is the greatest in $[j\text{-}1]\cap I$ *such that* $|z_iz_{i+1}...z_k|_\alpha \geq |z_iz_{i+1}...z_k|_{\overline{\alpha}}$
for every $k\in[i,j]$.
(b) The element i *is the greatest in* $[j\text{-}1]\cap I$ *such that the subword* $v=z_iz_{i+1}...z_j$ *is a Dyck word .*
(c) The element i *is the greatest in* $[j\text{-}1]\cap I$ *such that* $|z_iz_{i+1}...z_j|_\alpha=|z_iz_{i+1}...z_j|_{\overline{\alpha}}$.

(a') The element j is the least in $[i+1, 2n]\cap J$ such that $|z_e z_{e+1} \dots z_j|_\alpha \le |z_e z_{e+1} \dots z_j|_{\bar{\alpha}}$ *for each* $e \in [i,j]$.

(b') The element j is the least in $[i+1, 2n]\cap J$ such that the subword $v = z_i z_{i+1} \dots z_j$ *is a Dyck word .*

(c') The element j is the least in $[i+1, 2n]\cap J$ such that $|z_i z_{i+1} \dots z_j|_\alpha = |z_i z_{i+1} \dots z_j|_{\bar{\alpha}}$.

We define a map $f : J \to I$ such that for $j \in J$, $f(j)$ is the unique element of $[j-1]\cap I$ satisfying either of the equivalent conditions (a'), (b') and (c') of Lemma 3.1.

Similarly we define a map $g : I \to J$ such that for $i \in I$, $g(i)$ is the unique element of $[i+1,2n]$ satisfying either of the equivalent conditions (a), (b) and (c) of Lemma 3.1.

Clearly, by Lemma 3.1. follows that for $i \in I$ and $j \in J$ we have $f(j)=i$ iff $g(i)=j$, which shows that these two maps are bijections and the one is the inverse of the other.

We recall that a set S of disjoint pairs in $[2n]$ is nested if it satisfies the following condition : For any $\{a,b\},\{c,d\} \in S$ we never have $a<c<b<d$.

LEMMA 3.2. *The set S of the pairs* $\{j, f(j)\}, j \in J$, *is nested.*

PROOF. Assume that there exist j, $m \in J$ such that $i<k<j<m$, where $i=f(j)$ and $k=f(m)$.

Clearly since the subword $z_i z_{i+1} \dots z_j$ is a Dyck word we deduce that,

$$|z_k z_{k+1} \dots z_j|_\alpha \le |z_k z_{k+1} \dots z_j|_{\bar{\alpha}}$$

On the other hand since the subword $z_k z_{k+1} \dots z_m$ is also Dyck we deduce that,

$$|z_k z_{k+1} \dots z_j|_\alpha \ge |z_k z_{k+1} \dots z_j|_{\bar{\alpha}}$$

Thus, from the above two inequalities follows that,

$$|z_k z_{k+1} \dots z_j|_\alpha = |z_k z_{k+1} \dots z_j|_{\bar{\alpha}}$$

which contradicts the maximality property of i in the definition of $i = f(j)$. ∎

REMARK 3.3. A practical method for evaluating the function f and the nested set S from the Dyck path is the following :

To each edge (s_{i-1}, s_i), of the Dyck path we assign the number $i \in [2n]$ and we define the number $p(i)=min\{\tau(s_{i-1}), \tau(s_i)\}$ where $\tau(s_i)=y_i-x_i+1$ is the level of $s_i=(x_i,y_i)$ in the Dyck path.

In this way we obtain a partition of $[2n]$ into classes C_r, such that :

$$j, k \in C_r \quad \text{iff} \quad p(j) = p(k)$$

If we order the elements of each class, we get the nested set S by choosing pairs of consecutive elements of C_r.

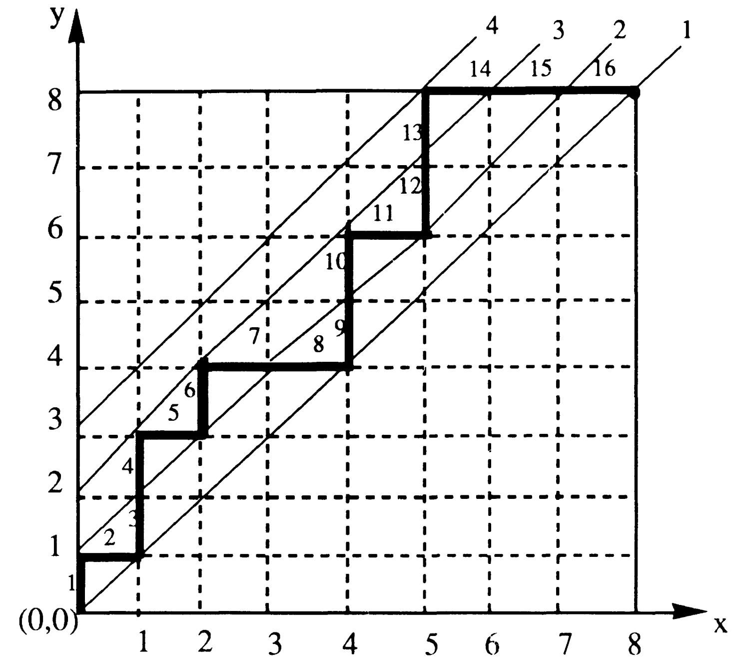

Figure 4. The generation of the nested set.

For example for the Dyck path of Figure 4 we have :

$p(1)=1,\ p(2)=1,\ p(3)=1,\ p(4)=2,\ p(5)=2,\ p(6)=2,\ p(7)=2,\ p(8)=1$
$p(9)=1,\ p(10)=2,\ p(11)=2,\ p(12)=2,\ p(13)=3,\ p(14)=3,\ p(15)=2,\ p(16)=1$

$C_1 = \{1,2,3,8,9,16\}$
$C_2 = \{4,5,6,7,10,11,12,15\}$
$C_3 = \{13,14\}$
$S = \{\{1,2\}, \{3,8\}, \{9,16\}, \{4,5\}, \{6,7\}, \{10,11\}, \{12,15\}, \{13,14\}\}$

We now come to the main construction of the binary tree. Firstly, we will construct by induction two sequences (A_k), (B_k) of subsets of $[2n]$ as follows :

$$B_1 = \{2n\}$$

$$A_k = f(B_k \cap J) \cup g(B_k \cap I)$$

$$B_{k+1} = \{i \in [2n] : i \notin \bigcup_{\lambda=1}^{k} B_\lambda \text{ and } (i-1 \in A_k \text{ or } i+1 \in A_k)\}$$

These two sequences satisfy the following properties:

(i) $|A_k| = |B_k|$.

(ii) $B_k \cap B_v = \varnothing$ and $A_k \cap A_v = \varnothing$ for $k \neq v$.

(iii) If $B_{k+1} \neq \varnothing$ (resp. $A_{k+1} \neq \varnothing$) then $B_k \neq \varnothing$ (resp. $A_k \neq \varnothing$).

(iv) Each element of B_k is an even number, while each element of A_k is an odd number.

The proof of the first three properties is evident, while the fourth is easily proved using the following useful property :

A number $j\in J$ (resp $i\in I$) is even iff the number $f(j)$ (resp. $g(i)$) is odd.

From the above properties follows that there exists $m\in[2n]$ such that $B_k\neq\emptyset$ and $A_k\neq\emptyset$ for every $k\leq m$, while $B_k=A_k=\emptyset$ for $k>m$.

Further we have the following Proposition.

PROPOSITION 3.4. *The family of the sets* B_k *and* A_k, $k\in[m]$ *is a partition of* $[2n]$.

PROOF. We first show that each even number of $[2n]$ is contained in some B_k.

Indeed, if this is not true we consider the non-empty set

$$E=\{2p : p\in[n] \text{ and } 2p\notin \bigcup_{k=1}^{m} B_k\} \text{ and let } M=max\,(E\cap J)\cup g(E\cap I)$$

We consider the following two cases :

1) Let $M=2v$, where $2v\in E\cap J$. Clearly $v<n$, since $2n\notin E$.

We consider two subcases :

1a) If $2v+1\in I$ then $2\mu=g(2v+1)\in J$ and $2\mu>M$; thus by the maximality property of M we deduce that $2\mu\notin E$. It follows that there exists $k\in[m]$ such that $2\mu\in B_k$

Further we have that $2v+1=f(2\mu)\in A_k$ and $2v\in \bigcup_{\lambda=1}^{k+1} B_\lambda$ which is a contradiction.

1b) If $2v+1\in J$, let $2t\in I$ such that $g(2t)=2v+1$. Since $2v+1>M$, by the maximality property of M, follows that $2t\notin E$. Then there exists $k\in[m]$, such that $2t\in B_k$.

Further we have that $2v+1=g(2t)\in A_k$ and $2v\in \bigcup_{\lambda=1}^{k+1} B_\lambda$ which is a contradiction.

2) Let $M=g(2\xi)$, where $2\xi\in E\cap I$.

We consider two subcases :

2a) If $2\xi-1\in I$, then $2\tau=g(2\xi-1)\in J$ and using Lemma 3.2. we have $2\tau>M$.

It follows by the maximality property of M that $2\tau\notin E$ and there exists $k\in[m]$, such that $2\tau\in B_k$. Further we have that $2\xi-1=f(2\tau)\in A_k$ and $2\xi\in \bigcup_{\lambda=1}^{k+1} B_\lambda$ which is a contradiction.

2b) If $2\xi\text{-}1\in J$, we first observe that since

$$|z_1z_2\ldots z_{2\xi\text{-}1}|_\alpha > |z_1z_2\ldots z_{2\xi\text{-}1}|_{\overline{\alpha}}$$

there exists a greatest number $\beta\in[2\xi\text{-}1]\cap I$ such that,

$$|z_\beta z_{\beta+1}\ldots z_{2\xi\text{-}1}|_\alpha > |z_\beta z_{\beta+1}\ldots z_{2\xi\text{-}1}|_{\overline{\alpha}} .$$

Clearly by the maximality property of β follows that $\beta+1\in I$ and $|z_{\beta+1}\ldots z_{2\xi\text{-}1}|_\alpha=|z_{\beta+1}\ldots z_{2\xi\text{-}1}|_{\overline{\alpha}}$. Moreover, from the last equality follows that the number $\beta+1$ is even ; let $\delta\in[\xi\text{-}1]$ such that $\beta=2\delta\text{-}1$. From the maximality property of $2\delta\text{-}1$ follows easily that $g(2\delta\text{-}1)>2\xi$ and, using Lemma 3.2, $g(2\delta\text{-}1) > M$. Then by the maximality property of M we obtain that $g(2\delta\text{-}1)\notin E$ and hence there exists $k\in[m]$ such that $g(2\delta\text{-}1)\in B_k$.

This shows that $2\delta\text{-}1\in f(B_k\cap J)\subseteq A_k$ for some $k\in[m]$ and $2\delta\in\bigcup_{\lambda=1}^{k+1} B_\lambda$.

We choose the greatest number γ with $2\delta\leq 2\gamma<2\xi\text{-}1$, which satisfies the following two conditions :

(i) $|z_{2\gamma}\ldots z_{2\xi\text{-}1}|_\alpha = |z_{2\gamma}\ldots z_{2\xi\text{-}1}|_{\overline{\alpha}}$

(ii) $2\gamma\in B_k\cap I$ for some $k\in[m]$.

We will show that $g(2\gamma)=2\xi\text{-}1$. Clearly, from the first of the above conditions follows that $g(2\gamma)\leq 2\xi\text{-}1$. If $g(2\gamma)=2\varepsilon\text{-}1<2\xi\text{-}1$, then from the maximality property of $2\delta\text{-}1$ follows that $2\varepsilon\in I$.

Moreover if $k\in[m]$ with $2\gamma\in B_k$ we obtain that $2\varepsilon\text{-}1\in A_k$ and $2\varepsilon\in\bigcup_{\lambda=1}^{k+1} B_\lambda$

Finally, since $|z_{2\varepsilon}\ldots z_{2\xi\text{-}1}|_\alpha = |z_{2\varepsilon}\ldots z_{2\xi\text{-}1}|_{\overline{\alpha}}$, it follows that the number 2ε satisfies both conditions (i) and (ii) which contradicts the maximality property of 2γ.

Thus $g(2\gamma)=2\xi\text{-}1$, and since $2\gamma\in B_k\cap I$ for some $k\in[m]$, we obtain that $2\xi\text{-}1\in A_k$ and $2\xi\in\bigcup_{\lambda=1}^{k+1} B_\lambda$ which is a contradiction. This shows that $2p\in\bigcup_{k=1}^{m} B_k$ for each $p\in[n]$.

It remains to show that, $2p\text{-}1\in\bigcup_{k=1}^{m} A_k$ for each $p\in[n]$. Indeed, if $p\in[n]$, without loss of generality we may assume that $2p\text{-}1\in I$.

Then the number $g(2p\text{-}1)$ is even and $g(2n\text{-}1) \in \bigcup_{k=1}^{m} (B_k \cap J)$.

This shows that $2p\text{-}1 \in f\left[\bigcup_{k=1}^{m} (B_k \cap J)\right] \subseteq \bigcup_{k=1}^{m} A_k$ ∎

LEMMA 3.5. *Let $i \in J$, $j \in J$ with $g(i)=j$. If either $i \in A_k$ or $j \in A_k$ then there is no integer t which lies between i, j and belongs to some A_λ with $\lambda \leq k$.*

PROOF. If the result is not true, we consider λ to be the least integer in $[k]$ such that there exists some integer t which lies between i, j and $t \in A_\lambda$. Without loss of generality we may assume that $t \in I$.

Clearly, by Lemma 3.2, the element $g(t)$ lies between i and j. Moreover, since $g(t) \in B_\lambda$ then either $g(t)\text{-}1 \in A_{\lambda\text{-}1}$ or $g(t)+1 \in A_{\lambda\text{-}1}$. In both cases we obtain an element of $A_{\lambda\text{-}1}$ which lies between i and j.

This contradicts the minimality property of λ. ∎

PROPOSITION 3.6. *The p.r. φ with $\varphi(i)=k$ iff $2i\text{-}1 \in A_k$, generates a binary tree.*

PROOF. It is enough to show that φ satisfies the following two conditions :

(i) For every $i \neq j \in [n]$ with $\varphi(i)=\varphi(j)=k$ there exists t between i and j such that $\varphi(t)<k$.

(ii) $\lambda(i)=\varphi(i)\text{-}1$, for every $i \in [n]$.

We assume that the first condition is not true and let k be the least integer for which there exist $i,j \in [n]$ such that $\varphi(i)=\varphi(j)=k$ and $\varphi(t) \geq k$, for every t between i and j. Without loss of generality we assume that $i<j$.

We consider the case when $2i\text{-}1 \in J$ and $2j\text{-}1 \in I$. (The other three possible cases are proved similarly).

Let $2\beta=f(2i\text{-}1)$ and $2\gamma=g(2j\text{-}1)$. Clearly, since $\varphi(i)=\varphi(j)=k$ we obtain that $2i\text{-}1, 2j\text{-}1 \in A_k$ and $2\beta, 2\gamma \in B_k$. It follows that either $2\beta\text{-}1$ or $2\beta+1 \in A_{k\text{-}1}$, and either $2\gamma\text{-}1$ or $2\gamma+1 \in A_{k\text{-}1}$. But, by Lemma 3.5 we have that $2\beta+1, 2\gamma\text{-}1 \notin A_{k\text{-}1}$, which gives that $2\beta\text{-}1$, $2\gamma+1 \in A_{k\text{-}1}$. It follows that $\varphi(\beta)=\varphi(\gamma+1)=k\text{-}1$.

If either $2\beta<2t\text{-}1<2i\text{-}1$ or $2j\text{-}1<2t\text{-}1<2\gamma$ by Lemma 3.5. follows that $2t\text{-}1 \in A_\lambda$, for some $\lambda<k$, so that $\varphi(t)=\lambda>k$.

On the other hand, if $2i\text{-}1 \leq 2t\text{-}1 \leq 2j\text{-}1$ then by the hypothesis for k follows that $\varphi(t) \geq k$. Thus $\varphi(t) \geq k > k\text{-}1$, for every t which lies between β and $\gamma+1$.

This contradicts the minimality property of k and the proof of the first property is complete.

We now come to the proof of the second property.

We first observe that this is obviously true when $\varphi(i)=1$, so that we may assume that $\varphi(i)>1$. Moreover, without loss of generality, we may assume that $2i\text{-}1 \in \mathrm{I}$.

If we set $2\beta=g(2i\text{-}1)$ and $\varphi(i)=k$, then we have that $2i\text{-}1 \in \mathrm{A}_k$ and $2\beta \in \mathrm{B}_k$. It follows that either $2\beta\text{-}1 \in \mathrm{A}_{k\text{-}1}$ or $2\beta+1 \in \mathrm{A}_{k\text{-}1}$. But, by Lemma 3.5 $2\beta\text{-}1 \notin \mathrm{A}_{k\text{-}1}$, so that $2\beta+1 \in \mathrm{A}_{k\text{-}1}$, and $\varphi(\beta+1) = k\text{-}1 = \varphi(i)\text{-}1$.

On the other hand, from Lemma 3.5 follows that $\varphi(t)>k$ for every $i<t<b$, so that $\varphi(\beta+1)=r(i)$.

Thus we have $\lambda(i) \geq r(i)=\varphi(i)\text{-}1$

Finally since the converse inequality is always true we obtain the desired property of φ. ∎

Given a Dyck word of length $2n$ we have constructed a p.r. $\varphi=\varphi(1)\varphi(2)...\varphi(n)$ which according to proposition 3.6 generates a binary tree with n vertices, which will be denoted by $\mathrm{T_w}$.

For example if $\mathrm{w} = \alpha\bar{\alpha}\ \alpha\alpha\bar{\alpha}\alpha\bar{\alpha}\ \bar{\alpha}\alpha\alpha\bar{\alpha}\alpha\alpha\bar{\alpha}\bar{\alpha}\bar{\alpha}$

then we have :

$\mathrm{B}_1 = \{16\}$, $\mathrm{B}_2 = \{8,10\}$, $\mathrm{B}_3 = \{2,4,12\}$, $\mathrm{B}_4 = \{6,14\}$
$\mathrm{A}_1 = \{9\}$, $\mathrm{A}_2 = \{3,11\}$, $\mathrm{A}_3 = \{1,5,15\}$, $\mathrm{A}_4 = \{7,13\}$

and $\varphi(5)=1$, $\varphi(2)=\varphi(6)=2$, $\varphi(1)=\varphi(3)=\varphi(8)=3$, $\varphi(4)=\varphi(7)=4$.

Thus $\varphi=32341243$ so that the binary tree $\mathrm{T_w}$ is the one of Figure 1.

REMARK 3.7. A practical method for the evaluation of the p.r. φ by the nested set S of the Dyck path is the following.

We first assign the number 0 to the last edge and the number 1 to the k^{th} edge for which $\{k,2n\} \in \mathrm{S}$.

Then we assign the number 0 to the $(k\text{-}1)^{\text{th}}$ and $(k+1)^{\text{th}}$ edges and the number 2 to the λ^{th}, μ^{th} edges for which $\{k\text{-}1,\lambda\} \in \mathrm{S}$ and $\{k+1,\mu\} \in \mathrm{S}$.

Continuing in this way and deleting the zeros we obtain the desired p.r. (see Figure 5).

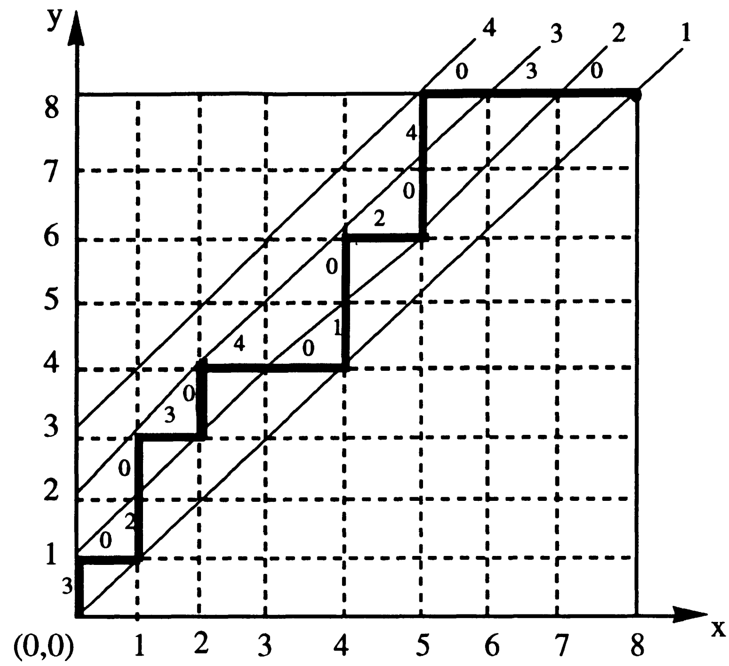

Figure 5. The generation of φ=32341243

4. THE RELATION BETWEEN THE TWO CONSTRUCTIONS

In this section we examine the relation between the two constructions of the previous sections and show that each one of them is the inverse of the other.

PROPOSITION 4.1. *The map* $T \to w_T$ *is a bijection between the set of binary trees with* n *vertices and the set of Dyck words of length* $2n$.

PROOF. Since the sets $\mathcal{T}_n$ and $\mathcal{D}_{2n}$ have the same cardinality it is enough to show that the given map is onto.

For this, let $w=z_1z_2.....z_{2n}$ be a Dyck word and $T= T_w$ the binary tree induced by w according to the construction of section 3.

In order to show that $w=w_T$, we have to prove that :

$$z_{2i-1} = \begin{cases} \alpha, & \text{if } l(i) \le r(i) \\ \bar{\alpha}, & \text{if } l(i) > r(i) \end{cases} \qquad \text{and} \qquad z_{2i} = \begin{cases} \alpha, & \text{if } \varphi(i) < \varphi(i+1) \\ \bar{\alpha}, & \text{if } \varphi(i) > \varphi(i+1) \end{cases}$$

for each $i \in [n]$, where by φ we denote the p.r. associated to T.

For the proof of the first formula we observe that if $\varphi(i)=1$ then $2i-1 \in A_1$ and $z_{2i-1}=\alpha$.

Let $\varphi(i)=k>1$ and $l(i)<r(i)$, we will show that $z_{2i-1}=\alpha$.

Indeed, if $z_{2i-1}=\bar{\alpha}$, we set $2\beta=f(2i-1)$. Clearly since $\varphi(i)=k$ we have that $2i-1 \in A_k$ and $2\beta \in B_k$. It follows that either $2\beta-1 \in A_{k-1}$ or $2\beta+1 \in A_{k-1}$.

Moreover by Lemma 3.5, $2\beta+1\notin A_k$ so that $2\beta-1\in A_k$ and $\varphi(\beta)=k-1$.

Further, by Lemma 3.5 we can easily obtain that $\varphi(t)>k=\varphi(i)$ for each t with $b<t<i$. This shows that, $l(i)=\varphi(\beta)=k-1=\varphi(j)-1=\lambda(i)$.

Thus $l(i)>r(i)$, which is a contradiction.

In the same way it is shown that if $l(i)>r(i)$ then $z_{2i-1}=\overline{\alpha}$, so that the proof of the first formula is complete.

We now come to the proof of the second formula.

Let $i\in[n-1]$, such that $\varphi(i)<\varphi(i+1)$ and $k\in[m]$ with $2i\in B_k$. Clearly, since $2i\in B_k$ we obtain that either $2i-1\in A_{k-1}$ or $2i+1\in A_{k-1}$.

If $z_{2i}=\overline{\alpha}$, there exists $\gamma\leq i$ such that $2\gamma-1=f(2i)$.

If $2i-1\in A_\lambda$, since $2\gamma-1\leq 2i-1<2i$, by Lemma 3.5 we obtain that $\lambda\geq k$, so that $2i+1\in A_{k-1}$. Then we have that $\varphi(i+1)=k-1<\lambda=\varphi(i)$ which is a contradiction.

Thus $z_{2i}=\alpha$.

In the same way, it is shown that if $\varphi(i)>\varphi(i+1)$ then $z_{2i}=\overline{\alpha}$, so that the proof of the proposition is complete. ∎

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