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ON BINARY TREES AND PERMUTATIONS

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RÉSUMÉ — Sur les arbres binaires et les permutations.
Chaque arbre binaire est associé à une permutation avec des répétitions qui le détermine uniquement. Deux opérations sont introduites construire l'ensemble de tous les arbres binaires. L'ensemble de toutes les permutations qui correspondent à un arbre binaire donné est déterminé et son nombre cardinal est évalué.

SUMMARY — *Every binary tree is associated to a permutation with repetitions, which determines it uniquely. Two operations are introduced and used for the construction of the set of all binary trees. The set of all permutations which correspond to a given binary tree is determined and its cardinal number is evaluated.*

1. INTRODUCTION

The connection between the set $\mathcal{T}_n$ of rooted unlabelled binary trees with n vertices and the set S_n of permutations on $[n] = \{1,2,...,n\}$ is well known. This connection is based on the labelling of the n vertices of a binary tree T, from the elements of [n] and the various traversals [1], [3], [4].

It is assumed that the labelling is consistent with the partial order of T, the root is labelled by 1, and the corresponding permutation $\sigma = \sigma(1)\sigma(2)...\sigma(n)$ is obtained by the inorder traversal of T (i.e. by visiting the left subtree first, then the root, and then the right subtree, see fig. 1).

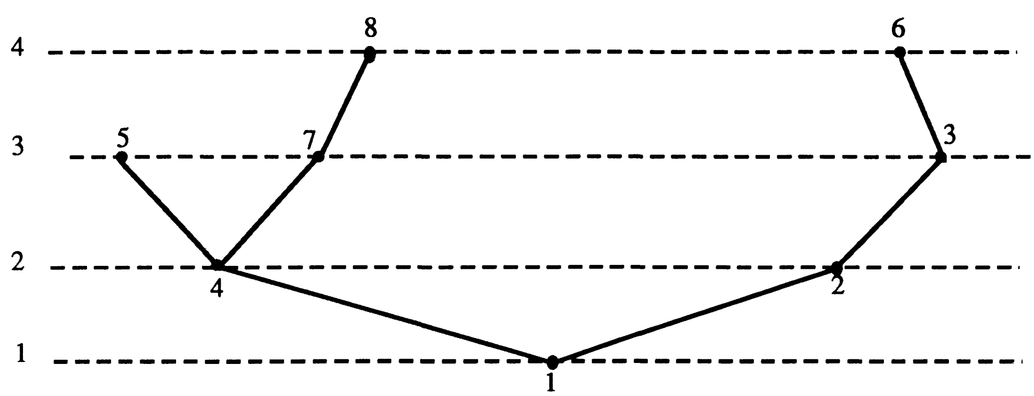

Figure 1. The permutation $\sigma = 54781263$ and its tree

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The set of all permutations σ of S_n obtained by a labelling of the tree T in the inorder traversal is denoted by $S_n(T)$.

In this paper each tree $T \in \mathcal{T}_n$ is associated to a permutation with repetitions (p.r.) $\varphi = \varphi(1)\varphi(2)...\varphi(n)$ which satisfies certain properties and determines the tree T uniquely. These p.r. are used as a main tool for the construction of $\mathcal{T}_n$ and some of its subsets. Further the relation between the sets $\mathcal{T}_n$ and S_n is examined. Given a tree $T \in \mathcal{T}_n$ the associated p.r. is used for the determination of the elements of $S_n(T)$ and the evaluation of its cardinal number.

For the study of the above notions the following symbols are used. For every permutation $\sigma \in S_n$ we denote

$$\lambda_\sigma(i) = \max(\ell_\sigma(i), r_\sigma(i)) \quad , i \in [n]$$

where $\ell_\sigma(i)$ (resp. $r_\sigma(i)$) is the first element on the left (resp. right) of the i^{th} position, which is smaller than $\sigma(i)$. If $\ell_\sigma(i)$ or/and $r_\sigma(i)$ does not exist we assume that $\ell_\sigma(i)$ or/and $r_\sigma(i)$ is equal to zero (see also [3], p. 339).

For every p.r. φ the definitions of $\ell_\varphi(i)$, $r_\varphi(i)$, $\lambda_\varphi(i)$ are given similarly.

For example for the permutations $\sigma = 54781263$ and $\varphi = 32341243$ we have $\lambda_\sigma(6) = \max(1,0) = 1$ and $\lambda_\varphi(7) = \max(2,3) = 3$.

2. BINARY TREES

Let T be a binary tree with n vertices which belong to m levels ($m \leq n$). If the vertices are enumerated according to the inorder of T then a p.r. $\varphi = \varphi(1)\varphi(2)...\varphi(n)$ is defined as follows : $\varphi(i) = p$ iff the i^{th} vertex of T belongs to the p^{th} level.

For example the corresponding p.r. of the tree T of fig. 2 is $\varphi = 32341243$.

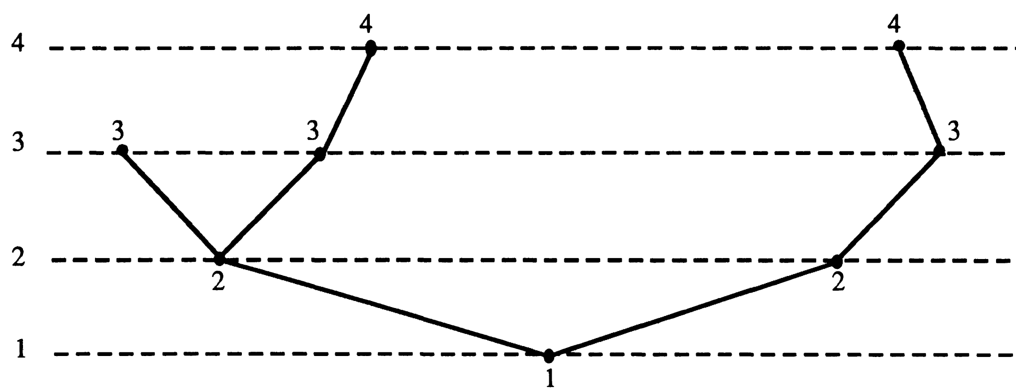

Figure 2. The binary tree of φ=32341243

It follows that φ satisfies the following two properties :

(i) For every $i \neq j \in [n]$ and $p \in [m]$: $\varphi(i) = \varphi(j) = p$ there exists k between i and j such that $\varphi(k) < p$.

(ii) For every $i \in [n]$ we have $\lambda_\varphi(i) = \varphi(i)-1$

The first property is true for p = 1 and it is extended by induction for every $p \in [m]$. Indeed if it is true for p and $i \neq j \in [n]$ with $\varphi(i) = \varphi(j) = p+1$, let $i_1, j_1 \in [n]$ such that the i_1^{th} vertex is the father of the i^{th} vertex and the j_1^{th} vertex is the father of the j^{th} vertex. If $i_1 = j_1$ then the property holds for $k = i_1$, while if $i_1 \neq j_1$ there exists k between i_1 and j_1 such that $\varphi(k) < p$ and the property also holds since k lies also between i and j. For the second property let i >1 and assume that the j^{th} vertex is the father of the i^{th} vertex, then $\varphi(i)-1 = \varphi(j) = \lambda_\varphi(i)$.

From the above discussion we deduce that each binary tree T is associated to a p.r. which satisfies the properties (i) and (ii). The set of these p.r. is denoted by $\mathcal{F}_n$.

Moreover every $\varphi \in \mathcal{F}_n$ generates the associated binary tree T as follows :

If we plot all the points $(i, \varphi(i))$, $i \in [n]$ on the plane we define T to be a tree with these points as vertices. The root of T is defined to be the point (r, 1) where r is the unique element of [n], with $\varphi(r) = 1$. Further the father of each vertex $(i, \varphi(i))$, $i \in [n]\setminus\{r\}$ is defined to be the vertex $(j, \varphi(j))$ where j is the first element of [n], on the left (resp. right) of i such that $\varphi(j) < \varphi(i)$, if $\ell_\varphi(i) > r_\varphi(i)$ (resp. $\ell_\varphi(i) < r_\varphi(i)$).

If we summarize the above results we have the following proposition.

PROPOSITION 2.1. *There exists a one to one correspondence between the sets* $\mathcal{T}_n$ *and* $\mathcal{F}_n$.

The binary tree T may be constructed from the up-down diagram of the associated p.r. φ. For example the binary tree T of fig. 2 is obtained from the up-down diagram of the associated p.r. $\varphi = 32341243$ (see fig. 3) by substituting the line segments with end points lying on levels p,q with $|p-q| > 1$.

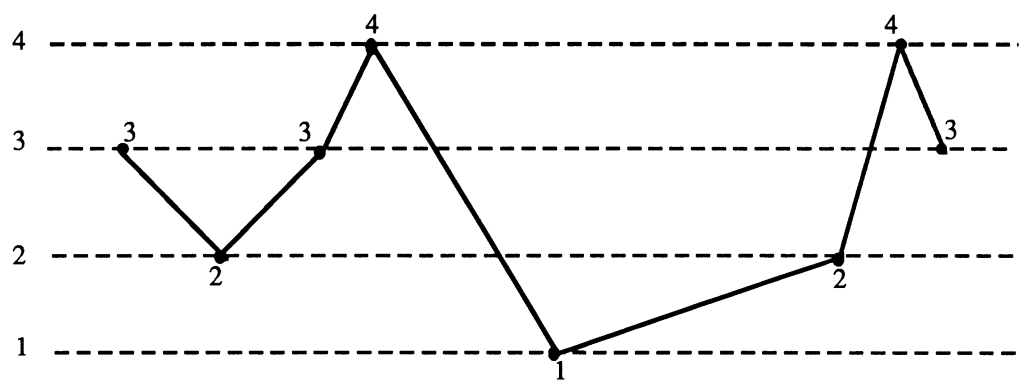

Figure 3. The up-down diagram of φ=32341243

The above correspondence is used for the definition of two operations in $\mathcal{T}_n$ which will be applied for the determination of its elements.

Operation mirror. If $T \in \mathcal{T}_n$ and φ is the associated p.r. we denote by $\bar{\varphi}$ the transpose of φ derived by $\bar{\varphi}(i) = \varphi(n+1-i)$, $i \in [n]$.

It is easy to check that $\bar{\varphi} \in \mathcal{F}_n$ and by proposition 2.1 it is associated to a binary tree $\bar{T}$, which we call the mirror of T. The mirror operation is used for simplifying the construction of $\mathcal{T}_n$. Indeed it is enough to construct the set $\mathcal{A}$ of all binary trees T rooted at (r,1) with $r \leq \frac{n}{2}$, because then $\mathcal{T}_n = \mathcal{A} \cup \bar{\mathcal{A}}$ where $\bar{\mathcal{A}}$ denotes the set of all mirror trees $\bar{T}$, for $T \in \mathcal{A}$.

Operation join. Given two binary trees T_1, T_2 with n_1,n_2 vertices respectively we construct a binary tree T with $n = n_1+n_2+1$ vertices as follows :

If φ_1,φ_2 are the p.r. associated to T_1,T_2 respectively, we define $\overset{+}{\varphi}_1$, $\overset{+}{\varphi}_2$ by $\overset{+}{\varphi}_1(i) = 1+\varphi_1(i)$, $\overset{+}{\varphi}_2(i) = 1+\varphi_2(i)$. It is easy to check that the concatenation $\varphi = \overset{+}{\varphi}_1 1 \overset{+}{\varphi}_2$ of $\overset{+}{\varphi}_1$, 1, $\overset{+}{\varphi}_2$ is a p.r. in $\mathcal{F}_n$ and by proposition 2.1 it is associated to a binary tree $T = T_1 \vee T_2$ which we will call the join of T_1 and T_2. We remark that the construction of $T_1 \vee T_2$ remains valid when either of n_1 or n_2 is equal to zero. In this case we use the concatenations $1\overset{+}{\varphi}_2$ or $\overset{+}{\varphi}_1 1$.

The join operation is used for the generation of every element of $\mathcal{T}_n$ from binary trees with less vertices. Indeed if T is any binary tree with n vertices rooted at the vertex (r,i), n_1=r-1 and n_2=n-r we write the associated p.r. $\varphi=\overset{+}{\varphi}_1 1 \overset{+}{\varphi}_2$ and we denote by T_1 and T_2 the binary trees associated to φ_1 and φ_2 respectively. It is easy to check that $T_1 \in \mathcal{T}_{n_1}$, $T_2 \in \mathcal{T}_{n_2}$ and $T=T_1 \vee T_2$.

The above discussion suggests the following result.

PROPOSITION 2.2. *For the set $\mathcal{T}_n$ we have*

$$\mathcal{T}_n = \bigcup_{k=0}^{n-1} \{T_1 \vee T_2 : T_1 \in \mathcal{T}_k , T_2 \in \mathcal{T}_{n-k-1}\}$$

From the previous propositions we may construct the set $\mathcal{T}_n$ from the elements of the sets $\mathcal{T}_k$, $0 \le k < n$. Indeed if

$$B_k = \{T_1 \vee T_2 : T_1 \in \mathcal{T}_k , T_2 \in \mathcal{T}_{n-k-1}\} , k = 0, 1,..., n-1$$

we first use the join operation for the construction of the sets $B_0,B_1,...B_t$, where $\frac{n}{2} - 1 < t \le \frac{n}{2}$ and then we construct the sets $B_{t+1},B_{t+2},...,B_n$ by applying the mirror operation to each element of the above sets.

It is well known that the cardinality of $\mathcal{T}_n$ is equal to the number of Catalan $\frac{1}{n+1}\binom{2n}{n}$. This result is a straightforward corollary of proposition 2.2.

Indeed since the family $\{B_k\}$ k = 0,1,...,n-1 forms a partition of $\mathcal{T}_n$ we have that

$$|\mathcal{T}_n| = \sum_{k=0}^{n-1} |B_k| = \sum_{k=0}^{n-1} |\mathcal{T}_k|\,|\mathcal{T}_{n-k-1}|$$

and using a well known generating function ([2] p. 388) we deduce the desired equality.

We now consider the following problem.

Problem. Given a finite sequence v_1, v_2,...v_m of positive integers such that :

$$v_1=1 , v_1+v_2+...+v_m=n , v_p \le 2v_{p-1} \text{ for } p=2,3,...,m$$

determine the set $\Lambda_n = \Lambda_n(v_1,v_2,...,v_m)$ of all T in $\mathcal{T}_n$ such that the p[th] level of T contains exactly v_p vertices, for every $p \in [m]$.

If we select a finite sequence of sets $Q_1,...,Q_m$ such that :

$$Q_1=\{1\},\ Q_p \subseteq [2v_{p-1}],\ |Q_p| = v_p \quad \text{for } p = 2,3,..,m$$

we construct a p.r. $\varphi = \varphi(1)\varphi(2)...\varphi(n)$ as follows :

First we define by induction a finite sequence of words $u_1,u_2,...,u_m$ of the alphabet $\{0\}\cup[m]$ with the following properties :

(i) $u_1 = 010$
(ii) For every $p \in [m]$ the number of occurences of zero in the word u_p is equal to $2v_p$.
(iii) For every $p \in [m]$ the word u_p is arising from the word u_{p-1} by deleting the k^{th} zero for every $k \in [2v_{p-1}]\setminus Q_p$ and substituting the k^{th} zero by 0p0 for every $k \in Q_p$.

By deleting all the zeros of the word u_m the p.r. φ is obtained. It is shown that $\varphi \in \mathcal{F}_n$.

Indeed we can easily show by induction that for every two nonzero and equal letters of u_p, $p = 2,3,...,m$ there exists a smaller non-zero letter of u_p which lies between them and consequently φ satisfies the first property.

On the other hand if we assume that the second property of φ is false there exists $p \in [m]$ such that $\varphi(i) = p$, $\ell_\varphi(i) < p-1$ and $r_\varphi(i) < p-1$. It follows from the construction of the sequence $u_1,u_2,...u_m$ that $\ell_\varphi(i)$ and $r_\varphi(i)$ belong to u_{p-2} and there exists at least one zero element of u_{p-2} which lies between them. Moreover, since this zero letter is substituted in u_{p-1} by 0(p-1)0 we obtain that the letter p-1 lies in u_{p-1} between $\ell_\varphi(i)$ and $r_\varphi(i)$. Thus if $\ell_\varphi(i) = \varphi(j)$ and $r_\varphi(i) = \varphi(k)$, where $j < i < k$, there exists $t \in [n]$ such that $j < t < k$ and $\varphi(t) = p-1$, which contradicts the definitions of $\ell_\varphi(i)$ and $r_\varphi(i)$.

Now, using proposition 2.1, we obtain a binary tree T with levels $\mathcal{L}_1, \mathcal{L}_2,..., \mathcal{L}_m$ such that $|\mathcal{L}_p| = |\varphi^{-1}(\{p\})| = |Q_p| = v_p$ for each $p \in [m]$. It follows that $T \in \Lambda_n$.

Example. Let n = 19, m = 5, $v_1 = 1$, $v_2 = 2$, $v_3 = 3$, $v_4 = 5$ and $v_5 = 8$. We consider the sets $Q_1 = \{1\}$, $Q_2 = \{1,2\}$, $Q_3 = \{1,3,4\}$, $Q_4 = \{1,2,3,5,6\}$ and $Q_5 = \{2,3,4,6,7,8,9,10\}$ then we have

u_1 = 010
u_2 = 0201020
u_3 = 030210302030
u_4 = 04030402104032040304O
u_5 = 4050305040502140503205040503050405O
φ = 4535452145325453545

It is plain to see that each element of Λ_n may be obtained by the above construction from a finite sequence of sets $Q_1,Q_2,...,Q_m$, where $Q_1 = \{1\}$, $Q_p \subseteq [2v_{p-1}]$ and $|Q_p| = v_p$, p = 2,3,...,m.

This suggests the following result.

PROPOSITION 2.3. *For the set* Λ_n *we have*

$$|\Lambda_n| = \prod_{i=2}^{m} \binom{2v_{i-1}}{v_i}$$

3. PERMUTATIONS

Given a binary tree $T \in \mathcal{T}_n$ a relation " $\prec$ " on [n] is defined by $k \prec j$ iff there exists an increasing path of T from the k^{th} vertex to the j^{th} vertex. This relation is characterized with the aid of the associated p.r. φ as follows.

LEMMA 3.1. *If* $j,k \in [n]$ *such that* $\varphi(k) < \varphi(j)$ *then* $k \prec j$ *iff* $\varphi(t) > \varphi(k)$ *for every t which lies between j and k.*

Without loss of generality we may assume that $j < k$. If $k \prec j$ there exists a finite sequence $v_0, v_1, \ldots, v_m$ of vertices of T such $v_o = (k,\varphi(k))$, $v_m = (j,\varphi(j))$ and v_{i-1} is the father of v_i for every $i \in [m]$.

It is shown by induction that $\varphi(t) > \varphi(k)$ for every $j < t < k$. Indeed the result is true for m = 1, since in this case the vertex $(k,\varphi(k))$ is the father of the vertex $(j,\varphi(j))$. Now, if the result is true for m-1 it is also true for m. Indeed if we assume that $(z,\varphi(z))$ is the father of $(j,\varphi(j))$ then by applying the hypothesis of the induction for the sequence $v_o, v_1, \ldots, v_{m-1}$ where $v_o = (k,\varphi(k))$ and $v_{m-1} = (z,\varphi(z))$ we deduce that $\varphi(t) > \varphi(k)$ for every $t \in [n]$ which lies between z and k.

Moreover, since the vertex $(z,\varphi(z))$ is the father of the vertex $(j,\varphi(j))$ we have that

$$\varphi(t) > \varphi(z) = \varphi(j)-1 \geq \varphi(k)$$

for every $t \in [n]$ which lies between z and j. Thus we conclude that $\varphi(t) > \varphi(k)$ for every $t \in [n]$ with $j < t < k$.

Conversely, assuming that $\varphi(t) > \varphi(k)$ for every $t \in [n]$ with $j < t < k$ it is proved that $k \prec j$. Indeed since $\varphi(k) < \varphi(j)$ there exists a number ξ such that $\xi \prec i$ and $\varphi(\xi) = \varphi(k)$. It is enough to show that $\xi = k$. Clearly if $\xi \neq k$ there exists a number η which lies between ξ and k and satisfies the relation $\varphi(\eta) < \varphi(k) = \varphi(\xi)$. If η lies between ξ and j, since $\xi \prec i$, we have that $\varphi(\eta) > \varphi(\xi)$. On the other hand if η lies between k and j, we have by the hypothesis that $\varphi(\eta) > \varphi(k)$. Thus in both cases the desired condradiction is obtained.

The set $S_n(T)$ is defined to be the set of all permutations σ of S_n which satisfy the following two conditions :
(i) $\sigma(r) = 1$, where $(r,1)$ is the root of the tree T.
(ii) $\sigma(k) < \sigma(j)$ for every $k,j \in [n]$ with $k \prec j$.

In other words the set $S_n(T)$ is the set of all different labellings consistent with the partial order of T.

The aim of this section is the determination of the elements of $S_n(T)$ and the evaluation of $|S_n(T)|$.

Before this we give a usefull characterization of the elements of $S_n(T)$.

For every $\sigma \in S_n$ we define a function θ on [n] by the relations $\theta(0) = 0$, $\theta(\sigma(i)) = 1+\theta(\lambda_\sigma(i))$, $i \in [n]$ and by setting the values of θ in their natural positions we obtain a p.r. φ_σ as follows :

$$\varphi_\sigma(i) = \theta(\sigma(i)),\ i \in [n]$$

For example, for the permutation $\sigma = 54781263$ we have $\varphi_\sigma = 32341243$.

The following procedure suggests a practical method for the construction of φ_σ.

$$\begin{array}{cccccccccc} & \sigma = & 5 & 4 & 7 & 8 & 1 & 2 & 6 & 3 \\ & \lambda & 4 & 1 & 4 & 7 & 0 & 1 & 3 & 2 \\ \downarrow & \lambda & 1 & 0 & 1 & 4 & 0 & 0 & 2 & 1 \\ & \lambda & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ \hline & \varphi_\sigma = & 3 & 2 & 3 & 4 & 1 & 2 & 4 & 3 \end{array}$$

PROPOSITION 3.2. *We have $\sigma \in S_n(T)$ iff $\varphi_\sigma = \varphi$*

If $\sigma \in S_n(T)$ and $\varphi_\sigma \neq \varphi$, let p be the least positive integer such that $\sigma(i) = p$ and $\varphi_\sigma(i) \neq \varphi(i)$. Let $j,k \in [n]$ such that $\varphi(j) = \ell_\varphi(i)$, $\varphi(k) = r_\varphi(i)$ and $\varphi(t) > \varphi(i)$ for every $t \in [n]\setminus\{i\}$ which lies between j and k. Without loss of generality we may assume that $\ell_\varphi(i) > r_\varphi(i)$.

Then by lemma 3.1 follows that $k \prec j$, $j \prec i$ and $i \prec t$ for every $t \in [n]\setminus\{i\}$ with $j < t < k$.

It follows that $\sigma(k) < \sigma(j) < \sigma(i) < \sigma(t)$ for every $t \in [n]\setminus\{i\}$ with $j < t < k$. This shows that $\sigma(j) = \lambda_\sigma(i)$ and $\varphi_\sigma(j) = \theta(\sigma(j)) = \theta(\sigma(i))-1 = \varphi_\sigma(i)-1 \neq \varphi(i)-1 = \varphi(j)$ through $\sigma(j) < \sigma(i) = p$.

Conversely if $\sigma \in S_n(T)$ and $i,j \in [n] : j \prec i$ it is shown that $\sigma(j) < \sigma(i)$. This is enough to be proved only in the case that $(j,\varphi(j))$ is the father of $(i,\varphi(i))$.
Further without loss of generality we may assume that $\sigma(k) = \ell_\sigma(i) = \lambda_\sigma(i) > r_\sigma(i) = \sigma(z)$.
Then,

$$\varphi(j) = \varphi(i)-1 = \varphi_\sigma(i)-1 = \theta(\sigma(i))-1 = \theta(\sigma(k)) = \varphi(k).$$

It follows from the definition of $\varphi(j) = \ell_\varphi(i)$ that $k < j < z$. Now, if $k \neq j$ we may assume without loss of generality that $k < j < i$.

Then there exists a finite sequence $\sigma(t_1), \sigma(t_2),\ldots,\sigma(t_v)$ of elements of σ such that, $t_o = j$, $t_v = 1$ and $\sigma(t_p) = \lambda_\sigma(t_{p-1})$ for every $p \in [v]$. It follows that

$$\varphi(i) = \theta(\sigma(i)) = \theta(\sigma(t_v)) < \theta(\sigma(t_{v-1})) < \ldots < \theta(\sigma(t_0)) = \theta(\sigma(j)) = \varphi(j)$$

which is a contradiction. Thus $j = k$ and $\sigma(j) = \sigma(k) < \sigma(i)$.

For the determination of the set $S_n(T)$ we consider the following sets :
$L_i = \{j \in \{1,2,\ldots,i-1\} : j \le t < i \Rightarrow \varphi(t) > \varphi(i)\},\ i \in [n]\}$
$R_i = \{j \in \{i+,2,\ldots,n\} : i \le t < j \Rightarrow \varphi(t) > \varphi(i)\},\ i \in [n]\}$
$A_i = L_i \cup R_i,\ i \in [n]$

We note that the elements of each non-empty L_i (resp. R_i) are consecutive integers.

Moreover by Lemma 3.1 we deduce easily the following result.

COROLLARY 3.3. *We have* $\sigma \in S_n(T)$ *iff* $\sigma(i) < \min_{j \in A_i} \sigma(j)$, $\forall\, i \in [n]$

The inequality of the previous proposition shows that the element i precedes j in σ^{-1} for every $j \in A_i$. This enables us to determine the elements of $S_n(T)$ with the aid of their inverses using the sets A_i, $i \in [n]$.

Indeed first we construct a finite sequence $\Theta_1, \Theta_2, \ldots \Theta_n$ of sets such that :

(i) $\Theta_1 = \{1\}$

(ii) $\Theta_k \subseteq S_k$, $k \in [n]$

(iii) Each element of Θ_{k+1}, $k \in [n-1]$ is arising from an element of Θ_k by inserting the number k+1 in the following way :

If $(k+1) \in A_v$ (resp. $v \in A_{k+1}$), $v \in [k]$ then v (resp. k+1) precedes k+1 (resp. v).

It follows that

$$S_n(T) = \{\sigma \in S_n : \sigma^{-1} \in \Theta_n\}$$

Example. For the binary tree T with associated p.r. $\psi = 231342$ we have

$L_1=\emptyset$	$R_1=\{2\}$	$A_1=\{2\}$
$L_2=\emptyset$	$R_2=\emptyset$	$A_2=\emptyset$
$L_3=\{1,2\}$	$R_3=\{4,5,6\}$	$A_3=\{1,2,4,5,6\}$
$L_4=\emptyset$	$R_4=\{5\}$	$A_4=\{5\}$
$L_5=\emptyset$	$R_5=\emptyset$	$A_5=\emptyset$
$L_6=\{4,5\}$	$R_6=\emptyset$	$A_6=\{4,5\}$

$\Theta_1 = \{1\}$

$\Theta_2 = \{12\}$ because 1 precedes 2

$\Theta_3 = \{312\}$ because 3 precedes 1 and 2

$\Theta_4 = \{3412, 3142, 3124\}$ because 3 precedes 4

$\Theta_5 = \{34512, 34152, 34125, 31452, 31425, 31245)$ because 3 and 4 precede 5

$\Theta_6 = \{364512, 364152, 364125, 361452, 316452, 316425, 361425, 361245, 316245, 312645\}$ because 3 precedes 6 and 6 precedes 4 and 5.

Then $S_6(T) = \{231564, 241563, 251463, 261453, 341562, 351462, 361452, 451362, 461352, 561342\}$

The elements of $S_n(T)$ are also generated by the associated p.r. ψ using a quicksort type procedure. Indeed firstly a finite sequence $M_{k1}, M_{k2}, \ldots, M_{k2^{k-1}}$, $k \in [m]$ of sets is constructed such that each of them contains consecutive elements of [n] and satisfy the following properties :

$M_{11} = [n]$

$M_{k\lambda} = M_{(k+1)(2\lambda-1)} \cup \{m_{k\lambda}\} \cup M_{(k+1)(2\lambda)}$ and $i < m_{k\lambda} < j$ for every $i \in M_{(k+1)(2\lambda-1)}$, $j \in M_{(k+1)(2\lambda)}$

where $m_{k\lambda}$ is the unique element of $M_{k\lambda}$ such that $\psi(m_{k\lambda}) = \min\{\psi(i) : i \in M_{k\lambda}\}$

It follows easily that

$$M_{(k+1)(2\lambda-1)} = L_{m_{k\lambda}} \quad \text{and} \quad M_{(k+1)(2\lambda)} = R_{m_{k\lambda}}$$

Next by selecting a finite sequence $P_{k1}, P_{k2}, \ldots P_{k2^{k-1}}$, $k \in [m]$ of subsets of [n] with

$P_{11} = [n]$

$P_{k\lambda} = P_{(k+1)(2\lambda-1)} \cup \{p_{k\lambda}\} \cup P_{(k+1)(2\lambda)}$ and $|P_{k\lambda}| = |M_{k\lambda}|$ where $p_{k\lambda} = \min P_{k\lambda}$,

we define a permutation $\sigma \in S_n$ by $\sigma(m_{k\lambda}) = p_{k\lambda}$.

It follows easily that $\sigma \in S_n(T)$.

Example. For the p.r. $\varphi = 32341243$ (see fig. 2) we have :
$M_{11} = \{1,2,3,4,5,6,7,8\}$, $m_{11} = 5$
$M_{21} = \{1,2,3,4\}$, $m_{21} = 2$, $M_{22} = \{6,7,8\}$, $m_{22} = 6$
$M_{31} = \{1\}$, $m_{31} = 1$, $M_{32} = \{3,4\}$, $m_{32} = 3$, $M_{33} = \varnothing$, $M_{34} = \{7,8\}$, $m_{34} = 8$
$M_{41} = M_{42} = M_{43} = \varnothing$, $M_{44} = \{4\}$, $m_{44} = 4$, $M_{45} = M_{46} = \varnothing$, $M_{47} = \{7\}$, $m_{47} = 7$, $M_{48} = \varnothing$

If we select the sets $P_{k\lambda}$ as follows :
$P_{11} = \{1,2,3,4,5,6,7,8\}$
$P_{21} = \{4,5,7,8\}$, $P_{22} = \{2,3,6\}$
$P_{31} = \{7\}$, $P_{32} = \{5,8\}$, $P_{33} = \varnothing$, $P_{34} = \{3,6\}$
$P_{41} = P_{42} = P_{43} = \varnothing$, $P_{44} = \{8\}$, $P_{45} = P_{46} = \varnothing$, $P_{47} = \{6\}$, $P_{48} = \varnothing$
we obtain
$\sigma(5) = 1$, $\sigma(2) = 4$, $\sigma(6) = 2$, $\sigma(1) = 7$, $\sigma(3) = 5$, $\sigma(8) = 3$, $\sigma(4) = 8$, $\sigma(7) = 6$ and $\sigma = 54781263$

We note that each $S_n(T)$ contains exactly one scaffold permutation and one Catalan permutation.

According to Rosenstiehl [3], if the vertices of T are labelled in the shelling order, the permutation defined by taking the labels in the inorder of the vertices is called a scaffold permutation of order n.

The scaffold permutation of $S_n(T)$ may be constructed easily by the following formula $\sigma(i) = |\pi(i)| + |\tau(i)|$ where :

$$\pi(i) = \{j \in [n] : \varphi(j) \leq \varphi(i)-1\}$$
$$\tau(i) = \{j \in [i] : \varphi(j) = \varphi(i)\}$$

In other words $\sigma(i)$ is equal to the number of all terms of φ which are either less than $\varphi(i)$, or equal to $\varphi(i)$ but on the left side of $\varphi(i+1)$. For example for the binary tree T with associated p.r $\varphi = 32341243$ (see fig. 2) we have the scaffold permutation $\sigma = 42571386$.

A permutation $\sigma \in S_n$ is called Catalan iff there are no indices $i<j<k$ such that $\sigma(j) < \sigma(k) < \sigma(i)$ (see [2], p. 239). The Catalan permutation of $S_n(T)$ may be constructed easily by selecting the finite sequence $(P_{k\lambda})$ in the previous quicksort procedure with the following property :

$$i < j \text{ for every } i \in P_{(k+1)(2\lambda-1)}, j \in P_{(k+1)(2\lambda)}$$

For example for the binary tree T with associated p.r. $\varphi = 32341243$ (see fig. 2) we have the Catalan permutation $\sigma = 32451687$.

Finally for the cardinal number of $S_n(T)$ we have the following proposition.

PROPOSITION 3.4. *For the set $S_n(T)$ we have*

$$|S_n(T)| = \prod_{k=1}^{n} \binom{|A_k|}{|L_k|}$$

From the quicksort procedure we note that each element of $S_n(T)$ depends on the choice of the finite sequence $(P_{k\lambda})$. Thus the cardinality of $S_n(T)$ is equal to the number of all possible choices of $(P_{k\lambda})$.

Moreover since the sets $A_{(k+1)(2\lambda-1)}$ and $A_{(k+1)(2\lambda)}$ are chosen from the set $A_{k\lambda}$ in

$$\binom{|P_{k\lambda}|-1}{|P_{(k+1)(2\lambda-1)}|}$$

ways we deduce that

$$|S_n(T)| = \prod_{k,\lambda}\binom{|P_{k\lambda}|-1}{|P_{(k+1)(2\lambda-1)}|}.$$

Moreover by the relations

$$|P_{(k+1)(2\lambda-1)}| = |M_{(k+1)(2\lambda-1)}| = |L_{m_{k\lambda}}|$$

and

$$|P_{k\lambda}|-1 = |M_{k\lambda}|-1 = |L_{m_{k\lambda}}| + |R_{m_{k\lambda}}| = |A_{m_{k\lambda}}|$$

we conclude that

$$|S_n(T)| = \prod_{k,\lambda}\binom{|A_{m_{k\lambda}}|}{|L_{m_{k\lambda}}|} = \prod_{i=1}^{n}\binom{|A_i|}{|L_i|}.$$

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