MATHÉMATIQUES ET SCIENCES HUMAINES

P. V. RAMANA MURTY
TERESA ENGELBERT

On valuation in semilattices

Mathématiques et sciences humaines, tome 90 (1985), p. 19-44
<http://www.numdam.org/item?id=MSH_1985__90__19_0>

Math. Sci. hum. (23^e **année, n°90,** *1985*, **p.19-43).**

ON VALUATION IN SEMILATTICES

P.V. RAMANA MURTY
SR. TERESA ENGELBERT *

INTRODUCTION

The purpose of this paper is to extend some of the results obtained in [5] to semilattices. To do so, we need a definition of valuation in semilattices. It is from the survey made by B.Monjardet [4], we take a suitable definition of valuation in semilattices.

Before entering into the matter of this paper, we shall summarize the results we generalize from [5] and [9]. A real valued function v defined on a lattice L is called a valuation on L if for any pair of elements a,b $\in$ L;

$$v(a) + v(b) = v(a \vee b) + v(a \wedge b).$$

A valuation is called isotone if $a \leq b$ implies $v(a) \leq v(b)$ and strictly isotone if $a < b$ implies $v(a) < v(b)$ ([1], Page 231). A real valued function d defined on a set S is called a pseudometric on S if

* Department of Mathematics, Andhra University,Waltair-530 003, India.

We thank Dr. B. Monjardet and Dr. J.P. Barthélemy whose valuable comments have helped us in shaping this paper into the present form.

i) $d(x,x) = 0 \quad ; x \in S$

ii) $d(x,y) = d(y,x); \; x,y \in S$

iii) $d(x,y) + d(y,z) \geq d(x,z); \quad x,y,z \in S$

A pseudo-metric d on S is called a metric on S if $d(x,y) = 0$ implies $x = y$ for $x,y \in S$[1]. G.Birkhoff in [1] has shown that an isotone valuation on a lattice L induces a pseudo-metric on L given by $d(x,y) = v(x \vee y)-v(x \wedge y)$ for all $x,y \in L$ and this pseudo-metric satisfies the inequality

$$d(t \vee x, t \vee y) + d(t \wedge x, t \wedge y) \leq d(x,y) \text{ for all } x,y,t \in L$$

([1], Chapter 10, Theorem 1). In [7] it is proved that if L is a lattice with least element 0 and d is a pseudometric on L such that $d(t \vee x, t \vee y) + d(t \wedge x, t \wedge y) \leq d(x,y)$ for all $x,y,t \in L$; then there exists an isotone valuation v on L such that $d(x,y) = v(x \vee y) - v(x \wedge y)$ for all $x,y \in L$. This v is unique upto a constant additive factor. Moreover if d is induced by an isotone valuation, we have the equations

i) $d(0,a)+d(a,b) = d(0,b)$ whenever $a,b \in L$ and $a \leq b$

ii) $d(a,a \wedge b)+ d(a \wedge b,b) = d(a,b) = d(a,a \vee b)+d(a \vee b,b)$ for all $a,b \in L$ (see [7]).

The first theorem of [5] establishes a one-to-one correspondence between the isotone valuations preserving 0 and the pseudometrics satisfying the above inequality.

Theorem 1 ([5])

In a lattice L with least element 0, there is a one-to-one correspondence between the isotone valuations preserving 0 and the pseudo-metrics satisfying the inequality

$$d(t \vee x, t \vee y) + d(t \wedge x, t \wedge y) \leq d(x,y)$$

A valuation on a lattice L is called distributive if

$$2v(a \vee b \vee c)-2v(a \wedge b \wedge c)= v(a \vee b)+v(b \vee c)+v(c \vee a)-v(a \wedge b)-v(b \wedge c) -v(c \wedge a) \text{ (see [8])}$$

It is known that in a metric distributive lattice,

$$d(a,b)+d(b,c) = d(a,c) \iff b \in [a \wedge c,\ a \vee c]$$

([1], page 234, problem 2). However in theorems 2 and 3 of [5] the implications $d(a,b)+d(b,c)=d(a,c) \Rightarrow b \in [a \wedge c, a \vee c]$ and $b \in [a \wedge c, a \vee c] \Rightarrow d(a,b)+d(b,c)=d(a,c)$ are characterized individually. In fact the first is equivalent to saying that the isotone valuation which induces the pseudo-metric is strictly isotone and the second is equivalent to the distributivity of the valuation. Hence the above result turns out to be a corollary of these two theorems.

<u>Theorem 2([5]):</u>

Let v be an isotone valuation on a lattice L. Then v is strictly isotone if and only if "$d(a,b)+d(b,c)=d(a,c) \Rightarrow b \in [a \wedge c, a \vee c]$" where d is the pseudo-metric induced by v.

<u>Theorem 3 ([5]):</u>

Let v be an isotone valuation on a lattice L. Then v is distributive if and only if "$b \in [a \wedge c, a \vee c] \Rightarrow d(a,b)+d(b,c)=d(a,c)$".

<u>Corollary 1([5]):</u>

In a metric-distributive lattice

$$d(a,b)+d(b,c) = d(a,c) \iff b \in [a \wedge c, a \vee c]$$

Combining Theorems 2 and 3 we get theorem 4 of [5].

<u>Theorem 4 ([5]):</u>

Let v be an isotone valuation on a lattice L. Then for $a,b,c \in L$ $d(a,b)+d(b,c) = d(a,c) \iff b \in [a \wedge c, a \vee c]$" if and only if L is distributive and v is strictly isotone.

In [8] it is shown that if L is a lattice and v is a strictly isotone valuation on L, the distributivity of L is equivalent to the distributivity of the valuation([8],page 109). However, in a lattice L with an isotone valuation, the distributivity of L guarantees the distributivity of the valuation. But the converse need not be true. For example the constant valuation on any lattice L is an isotone valuation which is distributive. But L need not even be modular.

In [9] Wilcox and Smiley have proved that if v is a strictly isotone function defined on a lattice L such that

$v(a \vee b)+v(a \wedge b) \leq v(a) + v(b)$, then

$a \vee (b \wedge c) = (a \vee b) \wedge c$ for $a \leq c$ whenever $v(b \vee c)+v(b \wedge c) = v(b) + v(c)$. It is this theorem of [9] that we extend to join-semilattices. We also give an example to show that the extension is proper.

Theorem 1.2 [9]:

If v is an affine dimension function defined on a lattice L, then $(a \vee b) \wedge c = a \vee (b \wedge c)$ for $a \leq c$ holds for a given pair $b,c \in L$ whenever $v(b \vee c) + v(b \wedge c) = v(b) + v(c)$.

Now coming to this paper, in the first theorem we prove that the inequality $d(t \vee x, t \vee y)+d(t \wedge x, t \wedge y) \leq d(x,y)$ is equivalent to the following three equalities.

i) $d(a,b)+d(b,c) = d(a,c)$ for $a \leq b \leq c$

ii) $d(a,a \vee b) + d(a \vee b,b) = d(a,b)$

iii) $d(a,a \wedge b) +d(a \wedge b,b) = d(a,b)$; thereby making it possible to extend many of the results on valuation in lattices to valuation in semilattices (see theorems 3 4,5,6 of this paper). Also we observe that if L is a join-semilattice with 0, the set of all isotone valuations preserving 0 is a commutative semi-group with 0 isomorphic to the semi-group of all pseudo-metrics satisfying (1), (2) and (3'), [Theorem 7]. Theorem 2 of this paper shows that the theorem of Wilcox and Smiley ([9], Theorem 1.2) is true for join-semilattices, with a strictly isotone upper valuation. If L is a lattice with a strictly isotone valuation, then L is modular ([8]). In theorem 3 we show that if v is a strictly isotone valuation on a join-semilattice L, then L is a lattice and hence modular. In theorem 11 we show that in a distributive semilattice every isotone valuation is distributive; thus extending the result of [8], page 109. In theorem 13 we characterize the implication "$b \in [t,a \vee c] \Rightarrow d(a,b)+d(b,c)=d(a,c)$" for all $t \in (a|c)$. It is equivalent to the distributivity of v. In theorem ^{V}14 we prove that if "$d(a,b)+d(b,c)=d(a,c) \Rightarrow b \in [t,a \vee c]$"

for all $t \in (alc)_v$, then L is a lattice. We also **give** suitable examples wherever necessary.

NOTATIONS AND DEFINITIONS

As far as possible we follow the notations used by B.Monjardet in [4].

Let L be a join-semilattice. We denote the least element by 0. $[x)$ denotes the set of all upper bounds of x in L and $(x]$, the set of all lower bounds.

$xuy = [x) \cap [y)$ and $xly = (x] \cap (y]$

$v^+(x,y) = \min_{z \geq x,y} v(z)$, $v^-(x,y) = \max_{z \leq x,y} v(z)$

$(xuy)_v = \{ z \in xuy \text{ with } v(z) = v^+(x,y) \}$

$(xly)_v = \{ z \in xly \text{ with } v(z) = v^-(x,y) \}$

Definition 1:

A real valued function v defined on a join-semilattice L is called a valuation on L if and only $v(a)+v(b)=v(a \vee b)+ v^-(a,b)$ for all $a,b \in L$. Where for $x,y \in L$, $v^-(x,y)= \max_{z \leq x,y} v(z)$ exists in L.

Definition 2:

A valuation v on a join-semilattice L is said to be distributive if $2v(a \vee b \vee c)-2v^-(a,b,c)=v(a \vee b)+v(b \vee c)+v(c \vee a)-v^-(a,b) -v^-(b,c)-v^-(c,a)$

where for any $x,yz \in L$, $v^-(x,y,z) = \max_{t \leq x,y,z} v(t)$ exists in L.

Definition 3:

A real-valued function v defined on a join-semilattice L is called an upper valuation on L, if for $a,b,c \in L$,

$v(a)+ v(b) \geq v(a \vee b) + v(c)$ for all $c \leq a,b$ [4].

Definition 4:

A join-semilattice L is called distributive if $a \leq b \vee c (a,b,c \in L)$ implies the existence of $b_1, c_1 \in L$ with $b_1 \leq b$ and $c_1 \leq c$ such that $a = b_1 \vee c_1$ ([3], page 117).

Definition 5:

A join-semilattice L is called modular if $a \leq b \vee c (a,b,c \in L)$ implies that there exist $b_1, c_1 \in L$ such that $b_1 \leq b$ and $c_1 \leq c$ such that $a \vee b_1 = a \vee c_1 = b_1 \vee c_1$ ([6]).

RESULTS

THEOREM 1:

Let d be a pseudo-metric on a lattice L with least element 0. Then the inequality $d(t \vee x, t \vee y) + d(t \wedge x, t \wedge y) \leq d(x,y)$ is equivalent to the following three equalities

1) $d(a,b) + d(b,c) = d(a,c)$ for $a \leq b \leq c$

2) $d(a,a \vee b) + d(a \vee b,b) = d(a,b)$

3) $d(a,a \wedge b) + d(a \wedge b,b) = d(a,b)$

Proof:

First suppose that d satisfies the above inequality. Let $a \leq b \leq c$.

$d(a,b) + d(b,c) \geq d(a,c)$ (Triangle inequality)

Putting $x = a$, $y = c$, $t = b$ we have

$d(b,c) + d(a,b) \leq d(a,c)$. So that

$d(a,b) + d(b,c) = d(a,c)$ for $a \leq b \leq c$.

Now by triangle inequality we have

$d(a,a \vee b) + d(a \vee b,b) \geq d(a,b)$ and

$d(a,a \wedge b) + d(a \wedge b,b) \geq d(a,b)$

putting $x = a$, $y = b$, $t = a$ in the above inequality

$d(a,a \vee b) + d(a,a \wedge b) \leq d(a,b)$

Now putting $x = a$, $y = b$, $t = b$ we have

$d(a \vee b,b) + d(a \wedge b,b) \leq d(a,b)$. Adding

$d(a,a \vee b) + d(a \vee b,b) + d(a,a \wedge b) + d(a \wedge b,b) \leq 2d(a,b)$

$\leq d(a,b) + d(a,a \wedge b) + d(a \wedge b,b)$. So that

$d(a,a \vee b) + d(a \vee b,b) \leq d(a,b)$ and hence

$d(a,a \vee b) + d(a \vee b,b) = d(a,b)$ ......(2)

similarly $d(a,a \wedge b) + d(a \wedge b,b) = d(a,b)$ ....(3)

Conversely suppose that d is a pseudo-metric on L satisfying the equalities (1), (2) and (3). We shall show that d satisfies the above inequality. First we shall prove that $d(x,y) = d(x \wedge y, x \vee y)$

$x \wedge y \leq x \leq x \vee y$. Hence by (1)

$d(x \wedge y,x) + d(x,x \vee y) = d(x \wedge y, x \vee y)$

similarly since $x \wedge y \leq y \leq x \vee y$ we get

$d(x \wedge y, y) + d(y, x \vee y) = d(x \wedge y, x \vee y)$. Adding

$d(x,x \wedge y) + d(x \wedge y,y) + d(x,x \vee y) + d(x \vee y,y) = 2d(x \wedge y, x \vee y)$

By (2) and (3) this becomes

$2d(x,y) = 2d(x \wedge y, x \vee y)$ so that

$d(x,y) = d(x \wedge y, x \vee y)$ ...... (i)

Now we shall prove that

$d(0,x) + d(0,y) = d(0,x \wedge y) + d(0,x \vee y)$

We have $d(x \wedge y,x) + d(x,x \vee y) = d(x \wedge y, x \vee y) = d(x,y)$

$= d(x,x \vee y) + d(x \vee y,y)$ so that

$d(x \wedge y,x) = d(x \vee y,y)$

Now $0 \leq x \wedge y \leq x$ and $0 \leq y \leq x \vee y$ so that

$d(0,x \wedge y) + d(x \wedge y,x) = d(0,x)$ and

$d(0,y) + d(y,x \vee y) = d(0,x \vee y)$

Hence $d(0,x) - d(0,x \wedge y) = d(x \wedge y,x)$

$= d(x \vee y,y)$

$= d(0,x \vee y) - d(0,y)$

so that $d(0,x) + d(0,y) = d(0,x \wedge y) + d(0,x \vee y)$ .... (ii)

Now we shall prove that d satisfies the above inequality.

$d(t \vee x, t \vee y) + d(t \wedge x, t \wedge y)$

$= d(t \vee x \vee y, (t \vee x) \wedge (t \vee y))$

$+ d((t \wedge x) \vee (t \wedge y), t \wedge x \wedge y)$

by (i)

$\leq d(t \vee x \vee y, (t \vee x) \wedge (t \vee y) + d((t \vee x) \wedge (t \vee y), t \vee (x \wedge y))$

$+ d(t \wedge (x \vee y), (t \wedge x) \vee (t \wedge y)) + d((t \wedge x) \vee (t \wedge y), t \wedge x \wedge y)$

$= d(t \vee x \vee y, t \vee (x \wedge y)) + d(t \wedge (x \vee y), t \wedge x \wedge y)$

by (1) since $t \vee x \vee y \geq (t \vee x) \wedge (t \vee y) \geq t \vee (x \wedge y)$ and

$t \wedge (x \vee y) \geq (t \wedge x) \vee (t \wedge y) \geq t \wedge x \wedge y$

$= d(0, t \vee x \vee y) - d(0, t \vee (x \wedge y)) + d(0, t \wedge (x \vee y)) - d(0, t \wedge x \wedge y)$

$= d(0, t \vee x \vee y) + d(0, t \wedge (x \vee y)) - [d(0, t \vee (x \wedge y)) + d(0, t \wedge x \wedge y)]$

$= d(0,t) + d(0, x \vee y) - [d(0,t) + d(0, x \wedge y)]$ by (ii)

$= d(0, x \vee y) - d(0, x \wedge y)$

$= d(x \vee y, x \wedge y) = d(x,y)$ by (i)

Hence the result.

In the following theorem we extend the result of Wilcox and Smiley (obtained for lattices) to semilattices.

<u>THEOREM 2</u>:

v is a strictly isotone upper valuation on a join-semilattice L. Let $b, c \in L$ such that $b \wedge c$ exists and $v(b \vee c) + v(b \wedge c) = v(b) + v(c)$. Then for $a \leq c$, $(a \vee b) \wedge c$ exists and $(a \vee b) \wedge c = a \vee (b \wedge c)$.

<u>Proof</u>:

Since v is an upper valuation

$v(a) + v(b) \geq v(a \vee b) + v(t)$ for $t \leq a, b$. Let $a \leq c$;

Clearly $a \vee (b \wedge c) \leq a \vee b$ and c. Therefore $a \vee (b \wedge c)$ is a lower bound of $a \vee b$ and c. Let x be any other lower bound of $a \vee b$ and c. Then $x \vee a \vee (b \wedge c) \leq a \vee b$ and c. Since v is an upper valuation this implies

$v(a \vee b) + v(c) \geq v(b \vee c) + v(x \vee a \vee (b \wedge c))$

Now we have $v(x \vee a \vee (b \wedge c)) - v(a \vee (b \wedge c))$

$\leq v(a \vee b) + v(c) - v(b \vee c) - v(a \vee (b \wedge c))$

$= v(a \vee b) + v(c) - [v(b) + v(c) - v(b \wedge c)] - v(a \vee (b \wedge c))$

$= v(a \vee b) + v(b \wedge c) - [v(b) + v(a \vee (b \wedge c))]$

$\leq v(b) + v(a \vee (b \wedge c)) - [v(b) + v(a \vee (b \wedge c))] = 0$

since $v(b) + v[a \vee (b \wedge c)] \geq v(a \vee b) + v(b \wedge c)$

Therefore $v(x \vee a \vee (b \wedge c)) \leq v(a \vee (b \wedge c))$

since v is isotone $v(x \vee a \vee (b \wedge c)) \geq v(a \vee (b \wedge c))$

Therefore $v(x \vee a \vee (b \wedge c)) = v(a \vee (b \wedge c))$ and

this implies $x \vee a \vee (b \wedge c) = a \vee (b \wedge c)$ as v is strictly isotone.

Hence $x \leq a \vee (b \wedge c)$. Therefore $a \vee (b \wedge c)$ is the greatest lower bound of $a \vee b$ and c, so that $a \vee (b \wedge c) = (a \vee b) \wedge c$.

EXAMPLE 1

We give an example of a join-semilattice (which is not a lattice) with a strictly isotone upper valuation in which the above theorem holds.

Consider the infinite join-semilattice in Figure.1. Here a and e are greater than every element of the infinite chain of non-negative integers with least element 0 and without a greatest element. v is a strictly isotone upper valuation on L. For any element x, v(x) is given in the bracket. $b \wedge c$ exists and $v(b \vee c) + v(b \wedge c) = v(b) + v(c)$ $a \wedge e$ does not exist. Here $a \vee (b \wedge c) = a \vee d = d$ and $(a \vee b) \wedge c = b \wedge c = d$ and hence $a \vee (b \wedge c) = (a \vee b) \wedge c$. This is

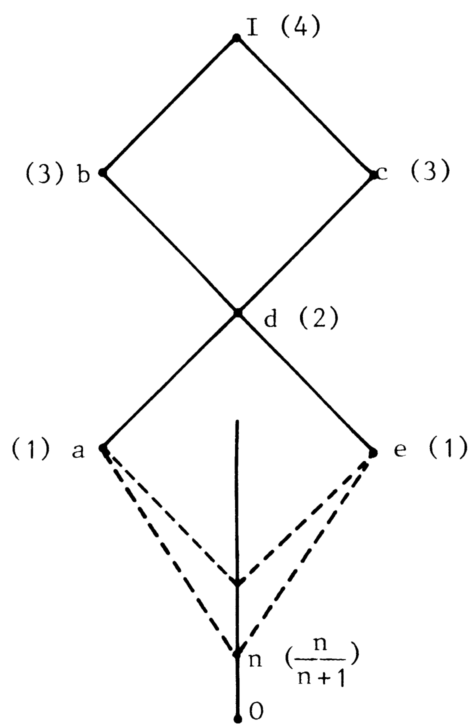

Figure 1

true for all $a,c \in L$ with $a \leq c$. Hence the theorem holds in this example.

EXAMPLE 2

The following example shows that the above result is not true for isotone valuations. Consider the infinite join-semilattice L in Fig.2. Here e and f are greater than every element of the infinite chain of non-negative integers with least element 0 and without a greatest element.

v is an isotone upper valuation on L. For any element $x, v(x)$ is given in the bracket. $b \wedge c$ exists and
$v(b) + v(c) = v(b \vee c) + v(b \wedge c)$
$e \wedge f$ does not exist.
$a \leq c$ and $a \vee (b \wedge c) = a \vee d = a$
$(a \vee b) \wedge c = 1 \wedge c = c$. Hence
$a \vee (b \wedge c) \neq (a \vee b) \wedge c$. Thus
this result is not true for an isotone valuation.

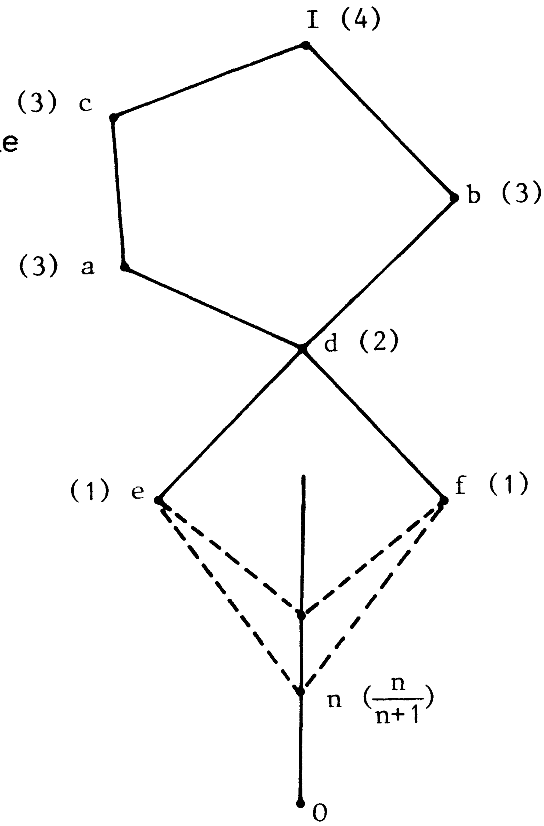

Figure 2

Next theorem shows that no strictly isotone valuation can exist on a proper join-semilattice.

THEOREM 3:

Let L be a join-semilattice with 0. If v is a strictly isotone valuation on L, then L is a lattice and hence modular.

Proof:

We have to show that given any two elements a and b; their glb exists in L. $v^{-}(a,b) = \underset{z \leq a,b}{\text{Max}}\, v(z) = v(z_o)$. $z_o \leq a,b$ implies z_o is a lower bound of both a and b. Let c be any other lower bound. Then $c \vee z_o$ is a lower bound of both a and b which implies $v(c \vee z_o) \leq v(z_o)$ but as v is isotone $v(c \vee z_o) \geq v(z_o)$ and hence $v(c \vee z_o) = v(z_o)$. Again because v is strictly isotone we get $c \vee z_o = z_o$ so that $c \leq z_o$. Hence z_o is the glb of a and b. Therefore L is a lattice. Since v is strictly isotone, L is modular.

EXAMPLE 3

Any number of isotone valuations can exist on a join semi-lattice L; directed below. For example a constant valuation is an isotone valuation on a join semi-lattice. The valuation as shown in Fig.3 is an example of a non-trivial isotone valuation on a join-semilattice L. Where every point x on the infinite chain is a lower bound of b and c.

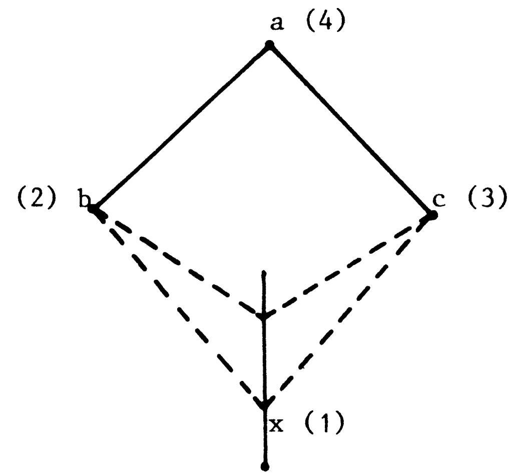

Fig. 3

Although the following theorems 4 and 5 are stated in [4] for completeness we give proofs of the same.

THEOREM 4

An isotone valuation v on a join-semilattice L with O induces a pseudometric on L given by $d(a,b) = v(a \vee b) - v^{-}(a,b)$ which satisfies the following three equalities

1) $d(a,b) + d(b,c) = d(a,c)$ $a \leq b \leq c$

2) $d(a, a \vee b) + d(a \vee b, b) = d(a,b)$ and

3) $d(a,t) + d(t,b) = d(a,b)$ for all $t \in (a l b)_V$.

<u>Proof</u>:

Since an isotone valuation on a join-semilattice L is an isotone upper (increasing) valuation on L; it induces a pseudometric on L given by $d(a,b) = 2v(a \vee b) - v(a) - v(b)$ which satisfies (1) and (2) ([2], Theorem 1 and Note 3).

$$d(a,b) = 2v(a \vee b) - v(a) - v(b)$$

$$= 2v(a \vee b) - [v(a \vee b) + v^{-}(a,b)]$$

$= v(a \vee b) - v^-(a, b)$

Now we have to prove (3') also.

Let $t \in (alb)_v$ so that $v(t) = v^-(a,b)$ and $t \leq a,b$.

Hence $d(a,t) + d(t,b) = v(a) - v(t) + v(b) - v(t)$

$$= v(a) + v(b) - 2v(t)$$

$$= v(a \vee b) + v^-(a,b) - 2v^-(a,b)$$

$$= v(a \vee b) - v^-(a,b) = d(a, b)$$

Hence $d(a,t) + d(t,b) = d(a,b)$ for all $t \in (alb)_v$.

Hence (3') also holds.

THEOREM 5

Let L be a join-semilattice with least element 0 and d be a pseudo-metric on L satisfying

1) $d(a,b) + d(b,c) = d(a,c)$ for $a \leq b \leq c$

2) $d(a, a \vee b) + d(a \vee b, b) = d(a,b)$

3') $d(a,t) + d(t,b) = d(a,b)$ for all $t \in alb$

such that $d(0,t) = \underset{z \leq a,b}{\text{Max}} \; d(0,z)$

Then there exists on isotone valuation v on L preserving 0 such that $d(a,b) = v(a \vee b) - v^-(a,b)$ for all $a,b \in L$.

Proof:

Define $v(x) = d(0,x)$. Then $v(0) = d(0,0) = 0$. If $x \leq y$, we get $0 \leq x \leq y$ so that $d(0,x) + d(x,y) = d(0,y)$ by (i) that is $v(x) + d(x,y) = v(y)$. This implies $v(x) \leq v(y)$ since $d(x,y) \geq 0$. Therefore v is isotone and v preserves 0. $0 \leq a \leq a \vee b$ and $0 \leq b \leq a \vee b$ so that

$d(0,a) + d(a, a \vee b) = d(0, a \vee b)$ and

$d(0,b) + d(b, a \vee b) = d(0, a \vee b)$. Adding

$d(0,a) + d(0,b) + d(a,b) = 2d(0, a \vee b)$ by (2) .

That is $v(a) + v(b) + d(a,b) = 2v(a \vee b)$ ......(i)

Now $0 \leq t \leq a$ and $0 \leq t \leq b$ where t is such that

$$d(0,t) = \max_{z\leq a,b} d(0,z)$$

Then by (1) we have

$d(0,t) + d(t,a) = d(0,a)$

$d(0,t) + d(t,b) = d(0,b)$ Adding

$2d(0,t) + d(a,b) = d(0,a) + d(0,b)$ by (3')

i.e., $v(a) + v(b) - d(a,b) = 2v(t)$ .... (ii)

Adding (i) and (ii) we get $v(a) + v(b) = v(a \vee b) + v(t)$ ... (iii)

Subtracting (ii) from (i)

$$d(a,b) = v(a \vee b) - v(t) \qquad \text{...... (iv)}$$

Now $d(0,t) = \max_{z\leq a,b} d(0,z)$ implies

$v(t) = \max_{z\leq a,b} v(z) = v^{-}(a,b)$ so that

from (iii) and (iv) we get

$v(a) + v(b) = v(a \vee b) + v^{-}(a,b)$ and

$d(a,b) = v(a \vee b) - v^{-}(a,b)$

Thus v is an isotone valuation on L preserving 0.

From the following Lemma we see that isotone valuations on a join-semilattice, L with least element 0 from a semi-group.

Lemma 6:

Let L be a join-semilattice with 0. If v_1 and v_2 are isotone valuations on L. Then v_1+v_2 and αv_1 are isotone valuations on L where α is a non-negative real number and $(v_1+v_2)(x)=v_1(x)+v_2(x)$.

Proof:

First we shall prove that

$$(v_1+v_2)^{-}(x,y) = v_1^{-}(x,y) + v_2^{-}(x,y)$$

Let $v_1^{-}(x,y) = v_1(z_1)$ and $v_2^{-}(x,y) = v_2(z_2)$

Let $t = z_1 \vee z_2$ so that $v_1(t) \geq v_1(z_1)$ and $v_2(t) \geq v_2(z_2)$

But $t \leq x,y$ implies $v_1(t) \leq v_1(z_1)$ and $v_2(t) \leq v_2(z_2)$

Thus $v_1(t) = v_1(z_1)$ and $v_2(t) = v_2(z_2)$

$$v_1(t) + v_2(t) = (v_1 + v_2)(t) \text{ and } t \leq x,y.$$

Now let $z \leq x,y$. Then

$$\begin{aligned}(v_1+v_2)(z) &= v_1(z) + v_2(z) \leq v_1^-(x,y) + v_2^-(x,y) \\ &= v_1(z_1) + v_2(z_2) = v_1(t) + v_2(t) \\ &= (v_1+v_2)(t)\end{aligned}$$

Therefore $(v_1+v_2)^-(x,y)$ exists and is equal to

$v_1(t) + v_2(t) = v_1^-(x,y) + v_2^-(x,y)$. Now

$$\begin{aligned}(v_1+v_2)(x) + (v_1+v_2)(y) &= [v_1(x)+v_1(y)] + [v_2(x) + v_2(y)] \\ &= v_1(x \vee y) + v_1^-(x,y) + v_2(x \vee y) + v_2^-(x,y) \\ &= (v_1+v_2)(x \vee y) + (v_1+v_2)^-(x,y)\end{aligned}$$

so that $v_1 + v_2$ is a valuation on L. Obviously it is isotone.
Next we shall prove that

$$(\alpha v_1)^-(x,y) = \alpha v_1^-(x,y)$$

Let $z \leq x,y$ so that $v_1(z) \leq v_1^-(x,y)$

and hence $\alpha v_1(z) \leq \alpha v_1^-(x,y)$. That is

$(\alpha v_1)(z) \leq \alpha v_1^-(x,y)$ and hence $(\alpha v_1)^-(x,y)$

exists and $(\alpha v_1)^-(x,y) = \alpha v_1^-(x,y)$. Moreover

$$\begin{aligned}(\alpha v_1)(x) + (\alpha v_1)(y) &= \alpha [v_1(x)+v_1(y)] \\ &= \alpha [v_1(x \vee y) + v_1^-(x,y)] \\ &= (\alpha v_1)(x \vee y) + (\alpha v_1)^-(x,y)\end{aligned}$$ so that αv is a

a valuation on L. Obviously it is isotone.

In the following theorem we show that on a join-semilattice L with 0, the isotone valuations preserving 0 correspond one-to-one to the pseudo-metric satisfying (1), (2) and (3') of theorem-5 Moreover it is an isomorphism of semi-groups. Although the first part of the following theorem is stated in [4] we give a proof of the same for the sake of completeness.

THEOREM 7

The set of all isotone valuations preserving 0 on a join-semilattice L with 0 is an additive abelian semigroup with 0 which is isomorphic to the semi-group of all pseudometrics satisfying (1), (2) and (3'). Moreover if σ is the mapping, then $\sigma(\alpha v) = \alpha \ \sigma(v)$ for all $\alpha \geq 0$(α is a real number).

Proof:

Let L be a join-semilattice with least element 0 and d be a pseudometric satisfying (1), (2) and (3') of theorem 5. Let v_d be the isotone valuation induced by d and d_{v_d} be the pseudometric induced by v_d. Then

$$\begin{aligned} d_{v_d}(x,y) = v_d(x \vee y) - v_d^-(x,y) &= 2v_d(x \vee y) - v_d(x) - v_d(y) \\ &= 2d(0,x \vee y) - d(0,x) - d(0,y) \\ &= [d(0,x \vee y)-d(0,x)] \\ &\quad +[d(0,x \vee y)-d(0,y)] \\ &= d(x,x \vee y) + d(x \vee y,y) \quad \text{by (1)} \\ &= d(x,y) \quad \text{by(2)} \end{aligned}$$

Hence $d_{v_d} = d$.

Conversely suppose that v is an isotone valuation on L preserving 0. Let d_v be the pseudometric induced by v. Then d_v satisfies (1), (2) and (3'). Let v_{d_v} be the isotone valuation induced by d_v. Then we claim that $v_{d_v} = v$.

$$\begin{aligned} v_{d_v}(x) = d_v(0,x) &= v(x) - v(0) \\ &= v(x). \end{aligned}$$

Thus $v_{d_v} = v$. Thus the correspondence is one-to-one.

Moreover, by Lemma 1, it is easily seen that the set of all isotone valuations preserving 0 is an additive abelian semigroup. Let $\sigma : v \longrightarrow d_v$ where d_v is the pseudo-metric induced by v be the mapping. Now we claim that σ is a homomorphism.

$$\sigma(v_1+v_2)(x,y) = d_{v_1+v_2}(x,y)$$

$$= (v_1+v_2)(x \vee y) - (v_1+v_2)^-(x,y)$$

$$= v_1(x \vee y) - v_1^-(x,y) + v_2(x \quad y) - v_2^-(x,y) \text{ by Lemma 6}$$

$$= \sigma(v_1)(x,y) + \sigma(v_2)(x,y).$$

More over if $\alpha \geq 0$

$$\sigma(\alpha v)(x,y) = d_{\alpha v}(x,y)$$

$$= (\alpha v)(x \vee y) - (\alpha v)^-(x,y)$$

$$= \alpha[v(x \vee y) - v^-(x,y)] \text{ by Lemma 6.}$$

$$= \alpha \quad \sigma(v)(x,y)$$

Hence the result.

LEMMA 8:

Let L be a join-semilattice with least element 0 and v an isotone valuation on L. Then $x \geq a$, $y \geq b$ imply $v^-(x,y) \geq v^-(a,b)$

Proof:

$$v^-(a,b) = \underset{z \leq a,b}{\text{Max}}\ v(z)$$

$z \leq a,b$ implies $z \leq x,y$ so that $v^-(a,b) \leq v^-(x,y)$

LEMMA 9:

Let L be a join-semilattice with least element 0 and with an isotone valuation v such that $v^-(a,b,c)$ exists for $a,b,c \in L$. Then if $t_1 \in (alb)_v$, $t_2 \in (blc)_v$ then

i) $v^-(a,t_2) = v^-(a,b,c)$

ii) $v^-(t_1,t_2) = v^-(a,b,c)$

Proof:

(i) we have by definition,

$$v^-(a,t_2) = \underset{z \leq a,t_2}{\text{Max}}\ v(z)$$

$z \leq a,t_2$ implies $z \leq a,b,c.$ so that

$v^-(a,t_2) \leq v^-(a,b,c)$

Now $v^-(a,b,c) = \underset{z \leq a,b,c}{\text{Max}}\ v(z) = v(z_1)$ (say)

Putting $t_2' = z_1 \vee t_2$; we have $t_2' \leq b,c$ and hence $v(t_2') \leq v(t_2)$.
But $v(t_2') \geq v(t_2)$ so that $v(t_2') = v(t_2)$.
Also $a \vee t_2' = a \vee z_1 \vee t = a \vee t_2$ since $z_1 \leq a$.
Therefore $v(a \vee t_2') = v(a \vee t_2)$

$$
\begin{aligned}
v^-(a,t_2') &= v(a) + v(t_2') - v(a \vee t_2') \\
&= v(a) + v(t_2) - v(a \vee t_2) \\
&= v^-(a,t_2).
\end{aligned}
$$

Now $z_1 \leq a, t_2'$ implies $v(z_1) \leq v^-(a,t_2') = v^-(a,t_2)$
So that $v(z_1) = v^-(a,b,c) = v^-(a,t_2)$

(ii) $v^-(t_1,t_2) = \max_{z \leq t_1,t_2} v(z)$

$z \leq t_1,t_2$ implies $z \leq a,b,c$ so that

$$v^-(t_1,t_2) \leq v^-(a,b,c)$$

Now $v^-(a,b,c) = \max_{z \leq a,b,c} v(z) = v(z_1)$ (say)

As above putting $t_1' = z_1 \vee t_1$ and $t_2' = z_1 \vee t_2$ we have

$$v(t_1') = v(t_1) \text{ and } v(t_2') = v(t_2)$$

Also $t_1' \vee t_2' = z_1 \vee t_1 \vee t_2 \geq t_1 \vee t_2$. so that
$v(t_1' \vee t_2') \geq v(t_1 \vee t_2)$

$$
\begin{aligned}
v^-(t_1',t_2') &= v(t_1') + v(t_2') - v(t_1' \vee t_2') \\
&\leq v(t_1) + v(t_2) - v(t_1 \vee t_2) \\
&= v^-(t_1,t_2)
\end{aligned}
$$

$z_1 \leq t_1',t_2'$ implies $v(z_1) \leq v^-(t_1',t_2') \leq v^-(t_1,t_2)$
Hence $v(z_1) = v^-(a,b,c) \leq v^-(t_1,t_2) \leq v^-(a,b,c)$
Therefore $v^-(t_1,t_2) = v^-(a,b,c)$

EXAMPLE 4:

We shall give an example of a distributive join-semilattice.

The lattice L in Fig.4 represents a distributive join-semilattice with 0 and 1. $a \vee b = 1$, $a \wedge b$ does not exist, since any point on the infinite chain which has no upper bound is a lower bound of both a and b
$a \leq a \vee b$ implies there exists $a \leq a, x \leq b$ such that $a = a \vee x$.
$x \leq a \vee b$ implies there exist $x \leq a$, $x \leq b$ such that $x = x \vee x$. Then L is distributive.

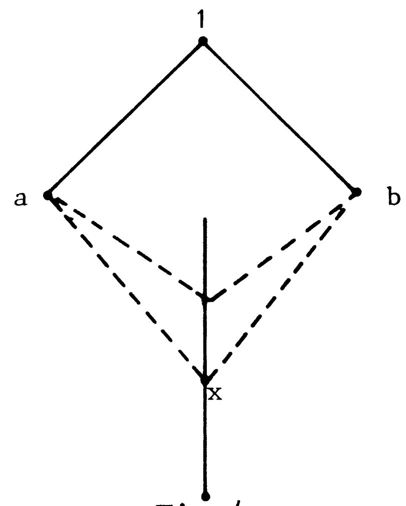

Fig.4

LEMMA 10:
If L is a distributive join semilattice with least element 0 then

i) $v(t_1 \vee t_3) = v^-(a, b \vee c)$

ii) $v(a \vee t_2) = v^-(a \vee b, a \vee c)$

where $t_1 \in (alb)_v$, $t_2 \in (blc)_v$ and $t_3 \in (alc)_v$

Proof:

i) $t_1 \leq a,b$; $t_3 \leq a,c$ implies $t_1 \vee t_3 \leq a, b \vee c$
so that $v(t_1 \vee t_3) \leq v^-(a, b \vee c)$ ...... (1)

Now $v^-(a, b \vee c) = \underset{z \leq a, b \vee c}{\text{Max}} \; v(z) = v(z_1)$ (say)

Since L is distributive $z_1 \leq b \vee c$ implies there exist $b_1, c_1 \in L$ such that $b_1 \leq b$, $c_1 \leq c$ and $z_1 = b_1 \vee c_1$. Putting $t_4 = b_1 \vee t_1$ and $t_5 = c_1 \vee t_3$ we have $b_1 \leq z_1 \leq a$ and $b_1 \leq b$ so that $b_1 \leq a,b$. Also $t_1 \leq a,b$ so that $t_4 = b_1 \vee t_1 \leq a,b$ and hence $v(t_4) \leq v^-(a,b) = v(t_1)$. But $v(t_4) \geq v(t_1)$. Therefore $v(t_4) = v(t_1)$. Similarly $v(t_5) = v(t_3)$.

$$v(t_4 \vee t_5) = v(z_1 \vee t_1 \vee t_3) \geq v(t_1 \vee t_3)$$

$$\begin{aligned} v^-(t_4, t_5) &= v(t_4) + v(t_5) - v(t_4 \vee t_5) \\ &\leq v(t_1) + v(t_3) - v(t_1 \vee t_3) \\ &= v^-(t_1, t_3) \end{aligned}$$

$t_4 \geq t_1$ and $t_5 \geq t_3$ implies

$v^-(t_4,t_5) \geq v^-(t_1,t_3)$ by lemma 8

Hence $v^-(t_4,t_5) = v^-(t_1,t_3)$

$$v(t_4 \vee t_5) = v(t_4) + v(t_5) - v^-(t_4,t_5)$$
$$= v(t_1) + v(t_3) - v^-(t_1,t_3)$$
$$= v(t_1 \vee t_3)$$

$v(t_4 \vee t_5) = v(b_1 \vee t_1 \vee c_1 \vee t_3) = v(z_1 \vee t_1 \vee t_3) \geq v(z_1)$

Therefore $v(t_1 \vee t_3) \geq v(z_1)$ ....(2)

From (1) and (2) $v(t_1 \vee t_3) = v(z_1) = v^-(a,b \vee c)$

ii) $v^-(a \vee b, a \vee c) = v(a \vee b) + v(a \vee c) - v(a \vee b \vee c)$

$$= v(a)+v(b)-v^-(a,b)+v(a)+v(c)-v^-(a,c) - v(a \vee b \vee c)$$
$$= 2v(a)+v(b)+v(c)-v(a \vee b \vee c)-v(t_1)-v(t_3)$$
$$= 2v(a)+v(b \vee c)+v^-(b,c)-v(a \vee b \vee c)-v(t_1)-v(t_3)$$
$$= 2v(a)+v(b \vee c)-v(a \vee b \vee c)+v(t_2) - [v(t_1 \vee t_3)+v^-(t_1,t_3)]$$
$$= v(a)+v\ (a,b \vee c)+v(t_2)-v^-(a,b \vee c)-v^-(a,b,c)$$

by Lemma 10(i) and 9(ii)

$$= v(a)+v(t_2) - v^-(a,b,c)$$
$$= v(a)+v(t_2)-v\ (a,t_2) \text{ by lemma 9(i)}$$
$$= v(a \vee t_2)$$

It is well known that in a distributive lattice every isotone valuation is distributive. In the following theorem it is shown that the same is true even for semilattices the fact of which is not available in the literature.

THEOREM 11:

In a distributive join-semilattice with least element 0, every isotone valuation is distributive.

Proof:

Let L be a distributive join-semilattice with least element 0 and with an isotone valuationv. We shall prove that v is distributive.

$$v(a \vee b \vee c)-v^-(a,b,c) = v(a)+v(b \vee c)-v^-(a,b \vee c)-v^-(a,b,c)$$
$$= v(a) +v(b \vee c)-v^-(a,b \vee c)-v^-(a,t_2)$$
by Lemma 9(i)
$$=v(a)+v(b \vee c)-v^-(a,b \vee c)-v(a)-v(t_2)+v(a \vee t_2)$$
$$=v(b \vee c)-v(t_1 \vee t_3)-v(t_2)+v^-(a \vee b,a \vee c)$$
by Lemma (10)
$$=v(b \vee c)-v(t_1)-v(t_3)+v^-(t_1,t_3)-v(t_2)$$
$$+v(a \vee b)+v(a \vee c)-v(a \vee b \vee c)$$

$$2v(a \vee b \vee c)-v^-(a,b,c)=v(a \vee b)+v(b \vee c)+v(c \vee a)-v^-(a,b)-v^-(b,c)$$
$$-v^-(c,a)+v\ (a,b,c)$$ by Lemma 9(ii)

Hence $2v(a \vee b \vee c)-2v^-(a,b,c)=v(a \vee b)+v(b \vee c)+v(c \vee a)-v^-(a,b)$
$$-v^-(b,c)-v^-(c,a)$$

Therefore v is distributive.

LEMMA 12:

Let L be a join-semilattice with least element 0 and with an isotone valuation v such that $v^-(a,b,c)$ exists for all $a,b,c \in L$.

Then if $t_1 \in (alb)_v$, $t_2 \in (blc)_v$, $t_3 \in (cla)_v$ and $t_4 \in (al(b \vee c))_v$ then,

i) $v^-(t_2,t_4) = v^-(a,b,c)$

ii) $v^-(c,t_4) = v^-(c,a)$

Proof:

(i) We have by definition $v^-(t_2,t_4)= \operatorname{Max}_{z \leq t_2,t_4} v(z)$

$z \leq t_2, t_4$ implies $z \leq a,b,c$ so that

$v^-(t_2,t_4) \leq v\ (a,b,c)$

Let $v^-(a,b,c) = v(z_1)$

Putting $t_2' = z_1 \vee t_2, t_4' = z_1 \vee t_4$ we get t_2' and $t_4' \leq b,c$

so that $v(t_2') \leq v(t_2)$ and $v(t_4') \leq v(t_4)$

But $v(t_2') \geq v(t_2)$ and $v(t_4') \geq v(t_4)$ so that

$$v(t_2') = v(t_2) \text{ and } v(t_4') = v(t_4)$$

$v(t_2' \vee t_4') = v(z_1 \vee t_2 \vee t_4) \geq v(t_2 \vee t_4)$

$$\begin{aligned} v^-(t_2',t_4') &= v(t_2') + v(t_4') - v\ (t_2' \vee t_4') \\ &\leq v(t_2) + v(t_4) - v(t_2 \vee t_4) \\ &= v^-(t_2,t_4) \end{aligned}$$

$z_1 \leq t_2', t_4'$ and hence $v(z_1) \leq v\ (t_2',t_4') \leq v\ (t_2,t_4)$

$v^-(t_2,t_4) \leq v^-(a,b,c) = v(z_1) \leq v^-(t_2,t_4)$

Hence $v^-(t_2,t_4) = v^-(a,b,c)$

(ii) By definition $\bar{v}(c,t_4) = \max\limits_{z \leq c,t_4} v(z)$

If $z \leq c,t_4$; then $z \leq c,a$ so that

$$v^-(c,t_4) \leq v^-(c,a) = v(z_1) \qquad \text{(say)}$$

Putting $t_4' = z_1 \vee t_4$ $\quad v(t_4') \geq v(t_4)$

$z_1 \leq c,a$; $t_4 \leq a, b \vee c$ implies $z_1 \vee t_4 \leq a, b \vee c$. So that

$t_4' \leq a, b \vee c$ which implies $v(t_4') \leq v(t_4)$ so that $v(t_4') = v(t_4)$.

$c \vee t_4' = c \vee z_1 \vee t_4 = c \vee t_4$ since $z_1 \leq c$.

$$\begin{aligned} v^-(c,t_4') &= v(c) + v(t_4') - v\ (c \vee t_4') \\ &= v(c) + v(t_4) - v(c \vee t_4) \\ &= v^-(c,t_4) \end{aligned}$$

$z_1 \leq c$ and t_4' . Therefore $v(z_1) \leq v^-(c,t_4') = \bar{v}(c,t_4)$. So that

$v(z_1) = v^-(c,a) = v^-(c,t_4)$.

In [5] we have proved that an isotone valuation v on a lattice L is distributive if and only if

"$b \in [a \wedge c, a \vee c] \Longrightarrow d(a,b)+d(b,c)= d(a,c)$"

However in the following theorem it is extended for semilattices.

THEOREM 13:

Let v be an isotone valuation on a join-semilattice L with 0. Then v is distributive if and only if

"$b \in [t, a \vee c] \Longrightarrow d(a,b)+d(b,c) = d(a,c)$" whenever $t \in (a|c)_v$.

<u>Proof</u>:

First suppose that v is distributive and $b \in [t, a \vee c]$ for some $t \in (a|c)_v$ so that we have $t \leq b \leq a \vee c$ and $a \vee b \vee c = a \vee c$.

$$v^-(a,b,c) \leq v^-(a,c) = v(t)$$

$t \leq a,c$ and b and hence $v(t) \leq v^-(a,b,c)$

Hence $v^-(a,b,c) \leq v(t) \leq v^-(a,b,c)$ implies

$$v^-(a,b,c) = v(t) = v^-(a,c)$$

since v is distributive we have

$$v(a \vee b)+v(b \vee c)+v(c \vee a) - v^-(a,b) - v^-(b,c) - v^-(a,c)$$
$$= 2v(a \vee b \vee c) - 2v^-(a,b,c)$$
$$= 2v(a \vee c) - 2v^-(a,c) \text{ so that}$$

$$v(a \vee b) - v^-(a,b) + v(b \vee c) - v^-(b,c) = v(a \vee c) - v^-(a,c)$$

That is $d(a,b) + d(b,c) = d(a,c)$

Conversely suppose thatt

"$b \in [t, a \vee c] \Longrightarrow d(a,b)+d(b,c) = d(a,c)$" for some $t \in (a|c)_v$

Let $t_1 \in (a|b)_v$, $t_2 \in (b|c)_v$, $t_3 \in (a|c)_v$ and $t_4 \in (a|(b \vee c))_v$

Let $x = t_2 \vee t_3 \vee t_4$ and $y = t_2 \vee z$. where $v(z) = v^-(a \vee t_2, b \vee c)$

First we shall prove that $v(y) = v(z)$

$y = t_2 \vee z \Longrightarrow v(y) \geq v(z)$

$v(z) = v^-(a \vee t_2, b \vee c) \Longrightarrow z \leq a \vee t_2,\ b \vee c$.

$t_2 \leq a \vee t_2$; $t_2 \leq b,c \implies t_2 \leq b \vee c$ so that $t_2 \leq a \vee t_2, b \vee c$.

$y = z \vee t_2 \leq a \vee t_2,\ b \vee c$. Hence

$v(y) \leq v^-(a \vee t_2, b \vee c) = v(z)$ so that $v(y) = v(z)$

$$\begin{aligned} v(y) &= v^-(a \vee t_2, b \vee c) \\ &= v(a \vee t_2) + v(b \vee c) - v(a \vee b \vee c) \\ &= v(a) + v(t_2) - v^-(a,t_2) + v(b \vee c) - v(a \vee b \vee c) \\ &= v(a) + v^-(b,c) - v^-(a,b,c) + v(b \vee c) - v(a \vee b \vee c) \end{aligned}$$

by Lemma 9(i)

Next we shall prove that $v(x) = v(y)$

$v(x) = v(t_2 \vee t_3 \vee t_4)$

$v(t_2 \vee t_3 \vee t_4) \geq v(t_2 \vee t_4)$

$t_3 \leq a,c$; $t_4 \leq a, b \vee c$ implies $t_3 \vee t_4 \leq a, b \vee c$

so that $v(t_3 \vee t_4) \leq v(t_4)$

But $v(t_3 \vee t_4) \geq v(t_4)$

$$\begin{aligned} v(t_2 \vee t_3 \vee t_4) &= v(t_2) + v(t_3 \vee t_4) - v^-(t_2, t_3 \vee t_4) \\ &\leq v(t_2) + v(t_4) - v^-(t_2,t_4) \\ &= v(t_2 \vee t_4) \end{aligned}$$

$$\begin{aligned} \text{Hence } v(x) &= v(t_2 \vee t_4) \\ &= v(t_2) + v(t_4) - v^-(t_2,t_4) \\ &= v^-(b,c) + v^-(a, b \vee c) - v^-(a,b,c) \text{ by Lemma 12(i)} \\ &= v^-(b,c) + v(a) + v(b \vee c) - v(a \vee b \vee c) - v^-(a,b,c) \\ &= v(y). \end{aligned}$$

$t_3 \leq x \leq a \vee c$ so that

$d(a,x) + d(x,c) = d(a,c)$. Therefore

$$2v(a \vee x) - v(a) - v(x) + v(c \vee x) - v^-(c,x) = v(a \vee c) - v^-(a,c) \quad \ldots(1)$$

$t_2 \leq y \leq b \vee c$ so that

$d(b,y) + d(y,c) = d(b,c)$. Hence

$$2v(b \vee y)-v(b)-v(y)+v(c \vee y)-v^{-}(c,y) = v(b \vee c)-v^{-}(b,c) \;\ldots\ldots(2)$$

Now we shall prove that $v(c \vee x) = v(c \vee y)$

$$\begin{aligned}
v(c \vee x) &= v(c \vee t_4) \\
&= v(c)+v(t_4) - v^{-}(c,t_4) \\
&= v(c)+v^{-}(a,b \vee c)-v^{-}(a,c) \quad \text{by Lemma 12(ii)} \\
&= v(c)+v(a)+v(b \vee c)-v(a \vee b \vee c)-v^{-}(a,c) \\
&= v(a \vee c)+v(b \vee c)-v(a \vee b \vee c) \\
&= v^{-}(a \vee c, b \vee c)
\end{aligned}$$

$$v(y) = v^{-}(a \vee t_2, b \vee c)$$

$$v^{-}(c,y) = v^{-}(c, a \vee t_2) \quad \text{by Lemma 12(ii)}$$

$$\begin{aligned}
v(c \vee y) &= v(c) + v(y) - v^{-}(c,y) \\
&= v(c) + v^{-}(a \vee t_2, b \vee c)-v^{-}(c, a \vee t_2) \\
&= v(c)+v(a \vee t_2)+v(b \vee c)-v(a \vee b \vee c)-v(c) \\
&\qquad -v(a \vee t_2)+ v(a \vee c) \quad \text{since } t_2 \leq b,c \\
&= v(b \vee c)+v(a \vee c)-v(a \vee b \vee c) \\
&= v^{-}(a \vee c, b \vee c)
\end{aligned}$$

Similarly we get $v(b \vee y) = v^{-}(a \vee b, b \vee c)$ .... (3)

$v(c \vee x) = v(c \vee y)$, $v(x) = v(y)$ implies $v^{-}(c,x) = v^{-}(c,y)$

Subtracting (2) from (1)

$$2v(a \vee x)-v(a)-v(x)-2v(b \vee y)+v(b)+v(y) = v(a \vee c)-v^{-}(a,c)-v(b \vee c)+v^{-}(b,c)$$

$$\begin{aligned}
v(a \vee x) &= v(a \vee t_2) \\
&= v(a)+v(t_2)-v^{-}(a,t_2) \\
&= v(a)+v^{-}(b,c)-v^{-}(a,b,c) \quad \text{by Lemma 9(i)}
\end{aligned}$$

$v(b \vee y) = v^{-}(a \vee b, b \vee c)$ by (3) substituting these values we get

$$2[v(a)+v^{-}(b,c)-v^{-}(a,b,c)]-v(a)-2v^{-}(a \vee b, b \vee c)+v(b) = v(a \vee c)-v^{-}(a,c)-v(b \vee c)+v^{-}(b,c)$$

$$2v(a\vee b\vee c)-2v^{-}(a,b,c)=v(a\vee c)+v(b\vee c)+2v(a\vee b)-v^{-}(b,c)-v^{-}(a,c)-v(a)-v(b)$$

$$=v(a\vee b)+v(b\vee c)+v(c\vee a)-v^{-}(a,b)-v^{-}(b,c)-v^{-}(c,a)$$

Hence v is distributive

In [5] we have proved that if v is an isotone valuation on a lattice L, then v is strictly isotone if and only if

"$d(a,b)+d(b,c) = d(a,c) \Longrightarrow b \in [a\wedge c, a\vee c]$"

But by theorem 3 of this paper if v is strictly isotone, the join semilattice is a lattice. Thus we have the following theorem

THEOREM 14:

Let v be an isotone valuation on a join-semilattice L and d, the induced pseudo-metric. If "$d(a,b)+d(b,c)=d(a,c) \Longrightarrow b \in [t,a\vee c]$" whenever $t \in (alc)_v$; then L is a lattice.

Proof:

First we shall prove that v is strictly isotone. Let $x,y \in L$ and $x < y$. Since v is isotone $v(x)\leq v(y)$. If possible let $v(x)= v(y)$; so that $d(y,x)+d(x,y) = 2d(x,y)$

$=2[v(y)-v(x)]$

$=0$

$=d(y,y)$.

This implies $x \in [t,y]$ for all $t \in (yly)_v$. $y \in (yly)_v \Rightarrow x \in [y,y]$ $\Longrightarrow x = y$. This is a contradiction. Hence $v(x) < v(y)$. So that v is strictly isotone. Since v is strictly isotone by theorem 3, L is a lattice.

REFERENCES

[1] BIRKHOFF G., Lattice Theory, New York, American Mathematical Society, Colloq.Publ., 3rd Edition,1967.

[2] BORDES G., "Métriques bornées définies par des valuations sur un demi-treillis", Math.Sci.Hum., 56(1976),89-95.

[3] GRATZER G., General Lattice Theory, New York, Academic Press Inc., New York 10003,1978.

[4] MONJARDET B., "Metrics on partially ordered sets - a survey" Discrete Mathematics, 35(1981) 173-184.

[5] RAMANA MURTY, P.V., and Sr. TERESA ENGELBERT., "On valuations and standard ideals in Lattices", Mathematics Seminar Notes, vol.9(1982) 371-386.

[6] ROHDES, J.B., "Modular and Distributive semilattices", Trans. Amer. Math. Soc.201(1975)31-41.

[7] SURYA NARAYANA MURTI GANTI., "Characterization of Pseudo-metric Lattices", Mathematics Seminar Notes Vol.6 (1978) 341-344.

[8] SZASZ, G., Introduction to Lattice Theory, New York and London, Academic Press, 3rd edition, 1963.

[9] WILCOX, L.R. and SMILEY, M.F., "Metric Lattices" Ann.Math 40(2)(1939) 309-327.