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# MULTIPLE CRITERIA ASSIGNMENT PROBLEM: COMBINING THE COLLECTIVE CRITERION WITH INDIVIDUAL PREFERENCES

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## 1. INTRODUCTION

In the last few years the problem of determining assignments in pairs of elements of two different sets on the basis of information bearing on the preferences of each term of one set in relation to all the terms of the other has gained currency. For the first time such a model was formulated in a work by D. Gale and L. Shapley (1962). It was here that the term stability of decision was introduced.

The problem was illustrated with two meaningful interpretations: problem of marriages (n bridegrooms and n brides) and problem of admission to colleges (n students and n colleges). The problem formulated in the above work was further enlarged in the works by P.Gärdenfors (1973, 1975).

All works relating to this field examined the problem of assignments as a problem of collective decision-making. These works maintain that a stable decision which is characterised by a large number of pairs with "mutually satisfied elements" is a fair decision.

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The problem can also be posed in a different way so that it would correspond to a number of practical problems. Let us assume that, in addition to the terms of the two sets, there is a person who is responsible for the assignment, i.e. a decision-maker. The example quoted as an illustration states that, in addition to n job seekers and n job possibilities looking for the best executor, there can be an executive who is responsible for the final appointment, It is natural for a reasonable manager to try to ensure assignments that would be characterised by the maximum number of coinciding preferences of jobs and executors. In other words he appraises the quality of an assignment on the basis of a criterion which is typical of problems bearing on collective decision-making.

Here is a practical problem stated along these lines (L. Chernyac, N. Serdetchkina, A. Kozhukharov and T. Patrikeyeva, 1976). The subediting division of a big publishing house receives many manuscripts of books which have to be duly prepared before they go to the printing shop. The manuscripts brought to the division have to be distributed between the subeditors. Every manuscript can be evaluated on the basis of number of criteria, such as the theme, the time allowed for work on it etc. The subeditors in turn may be evaluated on the basis of such criteria as quality of work, individual "capacity", preferable themes etc.

Thus, the statement of the problem is characterised by the following distinguishing features:

1) Presence of a decision-maker (DM).

In the context of the present problem DM plays a different role from the one he plays in ordinary individual decision-making problems. In the proposed statement of the problem a rational-minded decision-maker is bound to proceed from the possibility that the jobs shall be fulfilled by executors fit for them. Indeed, let us assume that all the works(tasks)and executors (persons) can be arranged in pairs so that (a) the abilities of every persons shall enable them to meet the requirements of "his own" task, but not those of the other task, and that (b) every task has "its own" person who is equal to its requirements, whereas the other persons are not. Obviously, in such a situation a rational-minded decision-maker considers such assignments to be the best solution to the problem, though in this case he takes no part in its solution.

However, situations occur when the decision-maker's intervention is necessary.

In a general case of the problem we are examining there are no obvious assignments. In this connection problems of the following type arise: (a) which task out of several is the given concrete person most fit for, according to his characteristics? and (b) which person out of several is the given concrete task most fit for, according to its characteristics? It is possible to obtain the answers to these questions by resorting to special procedures for receiving information from the decision-maker.

2) Presence of multiple criteria evaluations

Every person and every task are characterised by the vector of the evaluations based on the criteria. The criteria for evaluating the person and the task are of the "mirror image" type: one criterion (or several) characterising the abilities of the person correspond to one criterion (or several) which characterise the requirements of the task. The criterion "theme of manuscript" and the criterion "preferred themes of the subeditor" form a pair of criteria in the problem of distribution of manuscripts. The criterion "quality of the work of the "subeditor" corresponds to the criterion "importance of the manuscript".

It should be noted that most of these criteria are characterised by a qualitative subjective nature. The scales of values are usually presented in the form of several verbal formulae. In actual fact the pairs of "mirror image" criteria have a common scale of values although each value of the same scale has two formulae.

3) Objective character of evaluations

In the known statement of the problem of assignments (P.Gärdenfors, 1975) every element of one set gives an evaluation of all the elements of the other set. In this given case DM enlists the services of the experts who will evaluate all the elements of the two sets on the basis of many criteria.

Bearing in mind the above provisions it is possible to state the multiple criteria assignment problem in the following terms: there are n persons and n tasks, each being characterised by a set total of estimates on N criteria. These estimates are made by the experts. There is DM who is the executive responsible for solving the problem of assignments.

It is necessary to determine n "task-person" pairs whose characteristics stand closest to one another (an accurate definition of closeness of characteristics is to follow below)

The problem we have formulated occupies an intermediate position between the individual decision-making problem and collective decision-making problem. In this case DM proceeds from a criterion typical of the collective decision-making problem.

## 2. BASIC IDEAS ON THE SOLUTION OF THE MULTIPLE CRITERIA ASSIGNMENT PROBLEM

The main difficulties in the solution of the problem we are examining consist in

1) the availability of numerous criteria for appraisal of tasks and persons;

2) the need to look into problems with a rather large number of persons and tasks;

3) the desire to produce a method of solution which would require minimum use of information from DM.

The basic idea of the approach set forth below consists in the decomposition of the problem in question. Although each person and task is evaluated on criteria, its characteristics are being studied relatively, not absolutely. As far as every task is concerned it is necessary to determine the degree in which the characteristics of all the persons correspond to its requirements. And as far as every person is concerned it is necessary to establish the degree in which it corresponds to the requirements of all the task. Proceeding from an analysis of that correspondence an attempt is made to determine the assignments.

The method which has been developed includes two important stages. The first important stage is that of formal analysis which is conducted without DM. At this stage obvious assignments (if such are possible) are determined on the basis of information on the tasks and persons. The second stage consists in eliciting additional information from DM and establishing on its basis the task-person pairs which stand closest.

3. FORMAL ANALYSIS

Let us introduce the symbols we need.

Let us designate a set of tasks Oi (i = 1, 2, ... n) and a set of persons Ci (i = 1, 2, ... n). Let $O_i^j$ designate the estimate of task i on criterion j and $C_i^j$ designate the estimate of person i on criterion j (j = 1, 2, ... N). It is worth noting that the number of different merit marks on the scale of one criterion does not exceed usually three or five. It is assumed that the merit marks $K^j$ (j = 1,2, ... N; $K^j$ = 1,2, ... K) on the scales are ordered from the best to the worst (from the highest to the lowest).

Let us assume that task Oi has the estimates $Oi^1$, $Oi^2$ ... $Oi^N$ and person Ci has the estimates $Ci^1$, $Ci^2$, ... $Ci^N$ ($O_i^{\nu} = K_{Oi}^{\nu}$; $C_i^{\theta} = K_{Ci}^{\theta}$ where $K_{Oi}^{\nu}$ and $K_{Ci}^{\theta}$ are merit marks on the scales of criteria $\nu$ and $\theta$).

Let us determine component j of the vector showing the correspondence between the characteristics of person S and the requirements of task t in the following form:

$$C^j_{st} = \begin{cases} 0 & \text{if } K^j_t \quad K^j_s \\ R(O^j_t, C^j_s), & \text{if } K^j_t \quad K^j_s \end{cases} \qquad (I)$$

where $R(O^j_t, C^j_S)$ is the number of merit marks on the scale of criterion j by which $O^j_t = K^j_t$ exceeds $C^j_s = K^j_s$. Vector $C_{st}$ determines the degree in which person S fails to meet the requirements presented by task .

Let us determine as $O^{j,}_{ts} = - C^j_{st}$ component j of the vector showing the correspondence between the characteristics of task t and the requirements of person S.

Vector $O_{ts}$ determines the degree in which the task t fails to meet the requirements of person S.

It would be natural to assume that if two persons meet the level of requirements of the task their estimates are "equally good" for the task and vice versa.

Let us take the vectors $\overline{C}_{1t}$, $\overline{C}_{2t}$, ... $\overline{C}_{nt}$ which are vectors of correspondence to task t and introduce the following binary relation (relation $B_1$):

Vector $\overline{C}_{it}$ will dominate vector $\overline{C}_{pt}$, if

$$C^j_{it} \leqslant C^j_{pt}, \qquad j = 1, 2, \ldots N, \qquad (2)$$

and at least one component is characterised by strict inequality.

Vector $\overline{C}_{it}$ will be equivalent to vector $\overline{C}_{pt}$, if

$$C^j_{it} = C^j_{pt}, \qquad j = 1, 2, \ldots N. \qquad (3)$$

Vectors $\overline{C}_{it}$ and $\overline{C}_{pt}$ will be incomparable, if the conditions of (2) and (3) are not met.

In keeping with binary relation $B_1$ it is possible to construct graph $T_t$ where the arc drawn from $\overline{C}_{it}$ to $\overline{C}_{pt}$ reflects the binary relation of domination (2), the arc with two arrows pointing in opposite directions - the relation of equivalence (3), and the absence of the arc - the relation of incomparability.

In the analogical way it is possible to introduce binary relation $B_2$ between vectors $O_{1m}$, $O_{2m}$,... $O_{nm}$ (for person m).

In keeping with binary relation $B_2$ it is possible to construct graph $S_m$ on the basis of elements $O_{1m}$, $O_{2m}$ ... $O_{nm}$ in the analogical way as the graph $T_t$.

Graphs $T_t$ , $S_m$ ( t, m = 1, 2, ... n ) contain valuable information on the similarity of tasks and persons. To obtain this information it is necessary to analyse the similarity graphs.

The purpose of the analysis is to divide the nodes of the similarity graphs into groups with the help of binary relations between them.

Let us single out in the similarity graph the nodes without arcs directed to them. These nodes either dominate all or part of the other elements, or are incomparable to them, i.e. they form a Pareto set in a space of criteria. Let us name the nodes we have singled out nucleus of the 1st degree.

Now let us remove from the similarity graph the nodes included in the nucleus of the 1st degree and just like in the previous case let us single out of the remaining nodes a nucleus of the 2nd degree.

The nuclei will be singled out until all the nodes in the similarity graph have been exhausted.

If the elements of the nucleus of the Kst degree are incomparable, they shall be indexed $H_k$ and if they are equivalent, they shall be indexed $D_k$.

It is easy to see that index $D_i$ means that the nodes of nucleus i dominate the nodes of nucleus (i + 1).

The similarity graphs can be characterise by probabilistic estimations (A.Kozhukharov, O.Larichev, 1977).

It is possible to find (A.Kozhukharov, O.Larichev, 1977) the evaluation of probabilities that between two nodes in the similarity graph there will be a relation of domination, a relation of domination by all but one or two criteria, assuming that there is an equal probability for obtaining any evaluation on the scales of criteria for persons and tasks. These probabilities are characterised by rather high values. Thus, there being six criteria with three merit marks on the scales the probability of domination of one node over another is 0.52. In this connection it is possible to assume that in real situations the similarity graphs will have a large number of arcs.

The information obtained through the singling out of nuclei in similarity graphs can be well analysed with the help of similarity matrix M.

The columns in the similarity matrix correspond to tasks and the rows to persons. The cell at the intersection of column t and row m should give an evaluation of task t from the standpoint of person m (upper right part of cell) and an evaluation of person m from the standpoint of task t (lower left part of the cell).

The cell in similarity matrix with indices $\boxed{D_1 \diagdown D_1}$ corresponds to what is known as a best assignment. The similarity matrix with indices $\boxed{D_1 \diagdown D_1}$ in all the cells of the main diagonal, corresponds to the best solution.

If filling out a similarity matrix the following three cases are possible:

a) there is at least one $\boxed{D_1 \diagdown D_1}$ cell;

b) there are no $\boxed{D_1 \diagdown D_1}$ cells;

c) there are several $\boxed{D_1 \diagdown D_1}$ cells in a column or line;

In case a) it is possible to make the obvious assignments and reduce the dimension of the problem examined. After the dimension has been reduced it is necessary to return to similarity graphs $T_t$ and $S_m$ and find a new similarity matrix. New obvious assignments (if they exist) will be singled out. The probabilistic estimations shows that the probability of appearance of cell with $\boxed{D_1 \diagdown D_1}$ indices is rather high.

Case b) makes it necessary to go over to the next stage of analysis at which the information received from the decicion-maker shall be used.

## 4. THE ELICITATION OF THE INFORMATION FROM DM

The purpose of eliciting additional information from the decision-maker consists in the establishment of new paths in the similarity graphs.

Thus, the decision-maker is confronted with typical problems of comparison of the following type. There are two persons (tasks) and task (person). Which of these persons (tasks) is closest to the task (person) in multiple criteria space?

In the first place, we shall take note of the fact that this class of problems usually has a small number of value points (three-four) on the criterional scales, the number of criteria not exceeding ten. The number of pairs of alternatives in similarity graphs characterised by relations of domination or domination on the basis of all but one or two criteria is 10-20 times greater than in the case of direct comparison of multiple criteria alternatives (B.Berezovski, E.Trachtengerz, 1975).

In solving problems of this kind the approach based on comparison of alternatives by criteria is most expedient. The approach in question does not set itself the aim of producing a general evaluation of the utility of alternatives. Its purpose is to compare one alternative in relation to another by comparing the evaluation on the separate criteria.

Initially the characteristics of the task and each person are examined in pairs.

DM shall put into order the criterional deviations of the evaluations of the person from those of the task beginning from the worst. DM shall cope with this task by taking the corresponding values in pairs (deviations from the evaluation of the object for the worse). With respect to every pair of deviations one of the following decisions is adopted: a) one drop in quality is obviously greater than the other, and b) they are approximately equal.

After that DM executes the operation of comparison OCP-1. In comparing the deviations characterising two persons DM tries to establish whether the maximum deviation for one of the person is so great that one person obviously dominates the other. If the first operation of comparison OCP-1 helps establish that one person dominates the other with respect to closeness to the task the comparison is over. If it has not, DM shall carry out another operation of comparison - the OCP-2.

He compares in turn two drops in quality at a time for two different persons, starting with drops of the greatest magnitude. This comparison may produce two results: a) the drop in quality for person $C_i$ on criterion m is greater than the drop for person $C_j$ on criterion m; b) both drops in quality are equal (or approximately equal).

If the operation of comparison OCP-2 helps establish a relation of domination of a drop in quality for one person over another, the comparison is over. If it does not, DM shall execute another operation of comparisonsthe OCP-3. He compares one of the drops in quality for one person with the result of two drops in quality for the other. In doing so he tries to obtain one of the following results: a) domination or b) equality.

If the operation of comparison OCP-3 helps DM the comparison is over. If after three operations of comparison-OCP-1, OCP-2 and OCP-3 the information received from DM does not help establish who of the persons is closer in characteristics to those of the task, both persons are declared to be of equivalent closeness.

In carrying out the operations of comparison - OCP-1, OCP-2 and OCP-3 the decision-maker actually compares elements which differ in evaluation only on one-three criteria.

The information received from the decision-maker with the help of the above method will help, if necessary, to proceed from the similarity graphs $T_t$ and $S_m$ to linear quasi-orders $\bar{T}_t$ and $\bar{S}_m$.

It is obvious that the nominees for the best possible assignments are cells of the type of $\boxed{H_1 \diagdown H_1}$ or $\boxed{H_1 \diagdown D_1}$ and $\boxed{D_1 \diagdown H_1}$ because only such cells may develop into $\boxed{D_1 \diagdown D_1}$ cells after receiving the information from DM.

In a general case the only information DM is required to supply is that which will help compare the nodes of the first nucleus of similarity graphs $T_t$ and $S_m$. The volume of this information is far less than the volume of information needed to put into order all the nodes of these graphs.

## 5. THE EXISTENCE IN PRINCIPLE OF THE SOLUTION OF A MULTIPLE CRITERIA ASSIGNMENT PROBLEM

The information contained in quasi-orders $\bar{T}_t$ and $S_m$ can be entered into matrix $M_1$ which is similar in construction to matrix M. At the intersection of the line and column in matrix $M_1$ there is a $\boxed{D_j \diagdown D_i}$ cell which indicates rank i of the task in the linear quasi-order constructed for the person and rank j of the person in the quasi-order constructed for the task. After matrix $M_1$ has been filled a question of principle arises: is it always possible to make obvious assignments, do cells $\boxed{D_1 \diagdown D_1}$ always exist? The answer to this question is provided by the results given below.

THEOREM1

If one of the similarity graphs $T_t$ , $S_m$ ( $t$ , $m = 1,2, \dots n$) has a node with best evaluations on all criteria, matrix M will have a $\boxed{D_1 \diagdown D_1}$ cell.

THEOREM 2

If N = 1 in matrix M there will always be an $\boxed{D_1 \diagdown D_1}$ cell.

THEOREM 3

If linear quasi-orders $\overline{T}_t$ and $\overline{S}_m$ are non-contradictory ( $t$ , $m = 1,2, \dots n$) matrix $M_1$ shall always have at least one $\boxed{D_1 \diagdown D_1}$ cell.

The proofs of those theorems are given in the paper of A.Kozhukharov, O.Larichev (1977).

## 6. POSSIBLE GENERALISATIONS

The case of several cells $\boxed{D_1 \diagdown D_1}$ in one row or column of matrix give non-uniqueness of the assignments. In this case it is necessary to utilize the additional algorithm for solving the problem (A.Kozhukharov, O.Larichev, 1977). In the same work the following cases are discussed: 1) The case of non-equal numbers of persons and tasks. 2) The case of non-equal numbers of criteria for the estimation of persons and task.

It should be noted in conclusion that the introduction of multiple criteria factors into typical models of operations research makes these models viable and increases their possibilities for practical use. At the same time the character of these models changes, subjective features appearing in them which are chaŕacteristic of the decision-maker or decision making group. This character of the model makes it more suitable and effective as a mean for the decision-maker in analysis of complicated situations in life.

## R E F E R E N C E S

Berezovsky B., Trachtengerz, E. "Probability Estimation of Most Preferable Requirements of Queue Lenght in Operational Systems", Avtomatika i Telemekhanika (Automation and Remote Control), No.1, 1975, pp.187-195 (in Russian).

Chernyac L., Serdetchkina N., Kozhukharov A., Patrikeyeva T. "Model of Manuscript Preparation Process in a Publishing House", in: Algorithms and Control Models in Technical and Organisational Systems (in Russian), Moscow, Institute of Control Sciences, 1976, pp.52-58.

Gale D., Shaply L. "College Admissions and the Stability of Marriage", Amer. Math.Month., No.69, 1962, pp.9-15.

Gärdenfors P. "Assignment Problem Based on Ordinal Preferences", Mgmt. Sci., 20, 1973, Nov., pp.331-340.

Gärdenfors P. "Match-Making: Assignments Based on Bilateral Preferences", Behav. Sci., 20, 1975, pp.166-173.

Kozhukharov A.N., Larichev O.I. "A multi-criteria assignment problem", Avtomatika i Telemekhanika (Automation and Remote Control, N 7, 1977, pp.71-88, (in Russian)