REGULARIZATION OF AN UNILATERAL OBSTACLE PROBLEM

Ahmed Addou¹, E. Bekkaye Mermri¹ and Jamal Zahi¹

Abstract. The aim of this article is to give a regularization method for an unilateral obstacle problem with obstacle ψ and second member f, which generalizes the one established by the authors of [4] in case of null obstacle and a second member is equal to constant 1.

Mathematics Subject Classification. 35J85.

Received: May 10, 2001.

1. INTRODUCTION

Let Ω be a bounded domain of \mathbb{R}^n with smooth boundary $\partial\Omega$, $g \in H^{1/2}(\partial\Omega)$ and $\psi \in H^1(\Omega)$. We consider the variational inequality problem - called unilateral obstacle problem -: Find

$$u \in K = \{ v \in H^1(\Omega); v \ge \psi \text{ a.e. in } \Omega, \ v = g \text{ on } \partial\Omega \}$$
(1)

such that

$$\int_{\Omega} \nabla u \nabla (v - u) \mathrm{d}x + \langle f, v - u \rangle \ge 0 \quad \forall v \in K,$$
(2)

where $f \in H^{-1}(\Omega)$. It is well known that Problem (1-2) admits a unique solution (see [5]).

The aim of this article is to develop a regularization method for solving a non differentiable minimization problem which is equivalent to Problem (1-2). The idea of the regularization method is to approximate the non differentiable term by a sequence of differentiable ones depending on $\varepsilon > 0$, $\varepsilon \to 0$. To establish this regularization we give a new formulation of the obstacle problem, which is the subject of Theorem 1. We give three forms of regularization for which we establish the convergence result and *a priori* error estimates. Next by the duality method by conjugate functions (see [2]) we provide *a posteriori* error estimates which is desired for practical implementation for the regularization method.

This study is a generalization of an other one established by the authors of [4], where the obstacle ψ is equal to zero and the second member f is taken equal to the constant 1.

Keywords and phrases. Regularization, obstacle, unilateral.

¹ Department of Mathematics, Faculty of Sciences, University Mohammed I, Oujda, Morocco.

e-mail: zahi-j@sciences.univ-oujda.ac.ma

A. ADDOU ET AL.

2. Formulation and regularization of the problem

Let Ω be a bounded domain of \mathbb{R}^n , with smooth boundary $\partial \Omega$ and $g \in H^{1/2}(\partial \Omega)$, we denote by

$$H^1_q(\Omega) = \{ v \in H^1(\Omega); v = g \text{ on } \partial\Omega \}.$$

For ψ an element of $H^1(\Omega)$ with $\psi \leq g$ on $\partial\Omega$, we set

$$K_{\psi} = \{ v \in H^1_a(\Omega) : v \ge \psi \text{ a.e. on } \Omega \}.$$

Let $f \in H^{-1}(\Omega)$, we assume that f and ψ verify the following hypothesis:

(

$$f - \Delta \psi = F \in L^2(\Omega).$$

We denote by $\langle ., . \rangle$ the duality pairing between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$, and (., .) the inner product of $L^2(\Omega)$. Consider the following variational inequality problem:

$$(P_{\psi}) \begin{cases} \text{Find } u \in K_{\psi} \\ a(u, v - u) + \langle f, v - u \rangle \ge 0 \quad \text{ for all } v \in K_{\psi}, \end{cases}$$

where a(.,.) is defined by

$$u(u,v) = \int_{\Omega} \nabla u . \nabla v, \qquad u, v \in H^1(\Omega)$$

It is well-known that Problem (P_{ψ}) admits a unique solution. For all element $z \in L^2(\Omega)$ we denote

$$z^+ = \max\{z, o\}$$
 and $z^- = \min\{z, o\}$.

If $v \in H^1(\Omega)$, then we have $v^+, v^- \in H^1(\Omega)$ and

$$a(v^+, v^-) = 0. (3)$$

In the sequel we use the same notation g to designate an element of $H^{1/2}(\partial\Omega)$ and an element of $H^1(\Omega)$ which its trace on $\partial\Omega$ is g. We write the obstacle problem (P_{ψ}) on a new form.

Theorem 1. u is solution of Problem (P_{ψ}) if and only if w = u - g is solution of the following problem:

$$(P) \begin{cases} \text{ Find } w \in H^1_0(\Omega), \\ a(w+g-\psi, v-w) + \varphi(v) - \varphi(w) + (F^-, v-w) \ge 0 \text{ for all } v \in H^1_0(\Omega), \end{cases}$$

where φ is the functional defined by

$$\varphi(v) = (F^+, \phi(v + g - \psi)), \quad v \in H^1_0(\Omega),$$

with

$$\phi(t) = t^+, \quad t \in \mathbb{R}.$$

Proof. From the general theory of variational inequalities (see [3]), Problem (P) admits a unique solution, so it is sufficient to show that w + g is a element of K_{ψ} , where w is the solution of Problem (P). Indeed, for $v = (w + g - \psi)^+ - g + \psi \in H^1_0(\Omega)$, the inequality of (P) becomes

$$-a((w+g-\psi)^{-},(w+g-\psi)^{-})+\varphi(w)-\varphi(w)-(F^{-},(v+g-\psi)^{-})\geq 0.$$

Hence, from (3) we obtain

$$a((w+g-\psi)^{-},(w+g-\psi)^{-})=0$$

we deduce that $w + g - \psi \ge 0$, consequently $w + g \in K_{\psi}$.

It is easy to see that u = w + g verify the inequality of Problem (P), hence the proof is complete. The functional φ being non differentiable on $H_0^1(\Omega)$, we approximate it by a sequence of differentiable functionals, $\varphi_{\varepsilon}(v) = \int_{\Omega} F^+ \phi_{\varepsilon}(v + g - \psi) dx$, $(\varepsilon > 0$, tends to 0). The regularized problem is

$$(P_{\varepsilon}) \begin{cases} \text{Find } w_{\varepsilon} \in H_0^1(\Omega), \\ a(w_{\varepsilon}, v - w_{\varepsilon}) + \varphi_{\varepsilon}(v) - \varphi_{\varepsilon}(w_{\varepsilon}) + \langle l, v - w_{\varepsilon} \rangle \ge 0 \text{ for all } v \in H_0^1(\Omega) \end{cases}$$

where

$$\langle l, v \rangle = a(g - \psi, v) + (F^-, v).$$

Problems (P) and (P_{ε}) are, respectively, equivalent to

$$u \in H^1_g(\Omega) : \quad a(u, v - u) + \tilde{\varphi}(v) - \tilde{\varphi}(u) + \int_{\Omega} (F^- + \Delta \psi)(v - u) \mathrm{d}x \ge 0 \quad \forall v \in H^1_g(\Omega), \tag{4}$$

with

$$\tilde{\varphi}(v) = \int_{\Omega} \phi(v - \psi) \mathrm{d}x, \quad v \in H^1_g(\Omega),$$

and

$$u_{\varepsilon} \in H^1_g(\Omega): \quad a(u_{\varepsilon}, v - u_{\varepsilon}) + \tilde{\varphi}_{\varepsilon}(v) - \tilde{\varphi}_{\varepsilon}(u_{\varepsilon}) + \int_{\Omega} (F^- + \Delta \psi)(v - u_{\varepsilon}) \mathrm{d}x \ge 0 \quad \forall v \in H^1_g(\Omega)$$
(5)

with

$$\tilde{\varphi}_{\varepsilon}(v) = \int_{\Omega} F^+ \phi_{\varepsilon}(v - \psi) \mathrm{d}x, \quad v \in H^1_g(\Omega).$$

There are many methods to construct sequences of differentiable approximations. In this article we take the sequence ϕ_{ε} verifying one of the following choices:

$$\begin{aligned} \mathrm{c1}: \qquad \phi_{\varepsilon}^{1}(t) &= \left\{ \begin{array}{ll} t - \frac{\varepsilon}{2} & \mathrm{if} \ t \geq \varepsilon \\ \frac{t^{2}}{2\varepsilon} & \mathrm{if} \ 0 \leq t \leq \varepsilon \\ 0 & \mathrm{if} \ t \leq 0. \end{array} \right. \\ \mathrm{c2}: \qquad \phi_{\varepsilon}^{2}(t) &= \left\{ \begin{array}{ll} t & \mathrm{if} \ t \geq \varepsilon \\ \frac{1}{2}(\frac{t^{2}}{\varepsilon} + \varepsilon) & \mathrm{if} \ 0 \leq t \leq \varepsilon \\ \frac{\varepsilon}{2} & \mathrm{if} \ t \leq 0. \end{array} \right. \\ \mathrm{c3}: \qquad \phi_{\varepsilon}^{3}(t) &= \left\{ \begin{array}{ll} \sqrt{t^{2} + \varepsilon^{2}} & \mathrm{if} \ t \geq 0 \\ \varepsilon & \mathrm{if} \ t \leq 0. \end{array} \right. \end{aligned}$$

With these choices Problem (P_{ε}) admits a unique solution. To establish the convergence of Sequence u_{ε} we need the following results (see [3]).

Lemma 1. Let V be a Hilbert space, $a : V \times V \to \mathbb{R}$ a continuous, V-elliptic bilinear form, $j : V \to \mathbb{R}$ proper, non negative, convex, weakly continuous function and f is a linear continuous form on V. Assume that $j_{\varepsilon}: V \to \mathbb{R}$, ($\varepsilon > 0$), is a family of non negative convex weakly lower semi-continuous (l.s.c.) functions verifying

$$j_{\varepsilon}(v) \to j(v) \quad \forall v \in V,$$
 (6)

A. ADDOU ET AL.

If
$$u_{\varepsilon} \to u$$
 weakly in V then we have $j(u) \leq \liminf_{\varepsilon \to 0} j_{\varepsilon}(u_{\varepsilon})$. (7)

Let $u, u_{\varepsilon} \in V$ be the solutions of the following variational inequalities:

$$\begin{aligned} a(u,v-u) + j(v) - j(u) + \langle f, v - u \rangle &\geq 0, \quad \forall v \in V, \\ a(u_{\varepsilon},v-u_{\varepsilon}) + j_{\varepsilon}(v) - j_{\varepsilon}(u_{\varepsilon}) + \langle f, v - u_{\varepsilon} \rangle &\geq 0, \quad \forall v \in V, \end{aligned}$$

respectively. Then we have $u_{\varepsilon} \to u$ in V when $\varepsilon \to 0$.

Lemma 2. Assume that

$$j(v) = \int_{\Omega} \phi(v) \mathrm{d}x, \ j_{\varepsilon}(v) = \int_{\Omega} \phi_{\varepsilon}(v) \mathrm{d}x$$

and j is weakly l.s.c. If

$$\phi_{\varepsilon}(t) \to \phi(t) \text{ uniformly in } t, \text{ as } \varepsilon \to 0,$$
(8)

then (6) and (7) are verified.

We notice that if

$$|\phi_{\varepsilon}(t) - \phi(t)| \le c\varepsilon \quad \forall t \in \mathbb{R},\tag{9}$$

then (8) is verified. Since the functions ϕ_{ε}^{j} , j = 1, 2, 3 verify the inequality (9), then we have the convergence $w_{\varepsilon} \to w$ in $H_{0}^{1}(\Omega)$.

Taking $v = w_{\varepsilon}$ (resp. v = w) in the inequality of Problem (P) (resp. (P_{ε})), we obtain

$$a(w - w_{\varepsilon}, w - w_{\varepsilon}) \le \varphi(w_{\varepsilon}) - \varphi_{\varepsilon}(w_{\varepsilon}) + \varphi_{\varepsilon}(w) - \varphi(w).$$

Consequently, we obtain the following a priori estimate

$$\| w - w_{\varepsilon} \|_{H^1_0(\Omega)} \le (2c \int_{\Omega} F^+)^{\frac{1}{2}} \sqrt{\varepsilon}$$

3. A-posteriori error estimates

In this section we use the duality method by conjugate functions in order to derive the *a posteriori* error estimates of solutions of approximate problems. We need the following preliminary results (see [2])

Let V and V^{*} (resp. Y and Y^{*}) two topological vector spaces and $\langle ., . \rangle_V$ (resp. $\langle ., . \rangle_Y$) denotes the duality pairing between V^{*} and V (resp. Y^{*} and Y). Let φ be a function from V to $\mathbb{R} = \mathbb{R} \cup \{-\infty, +\infty\}$, its conjugate function is defined by

$$\varphi(v^*) = \sup_{v \in V} \langle v^*, v \rangle_V - \varphi(v), \quad v^* \in V^*$$

Assume there exists a continuous linear operator L from V to Y, $L \in \mathcal{L}(V, Y)$, with transpose $L^* \in \mathcal{L}(Y^*, V^*)$. Let J be a function from $V \times Y$ to $\overline{\mathbb{R}}$. We consider the following minimization problem:

$$u \in V, \quad J(u, Lu) = \inf_{v \in V} J(v, Lv). \tag{10}$$

The conjugate function of J is given by

$$J^*(y^*, v^*) = \sup_{v \in V, y \in Y} \{ \langle v^*, v \rangle_V + \langle y^*, y \rangle_Y - J(v, y) \}$$

Theorem 2. Assume that V is a reflexive Banach space and Y a normed vector space. Let $J: V \times Y \to \overline{\mathbb{R}}$ be a proper l.s.c. strictly convex function verifying:

(i) $\exists u_0 \in V$, such that $J(u_0, Lu_0) < \infty$ and $y \to J(u_0, y)$ is continuous at Lu_0 .

(ii) $J(v, Lv) \to +\infty$, as $||v||_V \to +\infty$, $v \in V$.

Then Problem (10) admits a unique solution and

$$J(u, Lu) = \inf_{v \in V} J(v, Lv) = -\sup_{y^* \in Y^*} J^*(-y^*, L^*y^*).$$

Let Ω be an open subset of \mathbb{R}^N , $g: \Omega \times \mathbb{R}^n \to \mathbb{R}$ be the Carathéodory function *i.e.*, $\forall s \in \mathbb{R}^n$, $x \to g(x, s)$ is a measurable function and for almost all $x \in \Omega$, the function $s \to g(x, s)$ is continuous. Then the conjugate function of

$$G(v) = \int_\Omega g(x,v(x)) \mathrm{d}x$$

(assuming G is well defined over some a function space V) is

$$G^*(v^*) = \int_{\Omega} g^*(x, v^*(x)) \mathrm{d}x, \quad \forall v^* \in V^*,$$

where

$$g^*(x,y) = \sup_{s \in \mathbb{R}^N} \{ys - g(x,s)\}$$

For Problem (P) we take

$$V = H^{1}(\Omega), \quad Y = Y^{*} = (L^{2}(\Omega))^{n} \times L^{2}(\Omega)$$

$$Lv = (\nabla v, v)$$

$$J(v, Lv) = H(v) + G(Lv)$$

$$H(v) = \begin{cases} 0 & \text{if } v = g \text{ on } \partial\Omega \\ +\infty & \text{otherwise} \end{cases}$$

$$G(y) = \int_{\Omega} \frac{1}{2} |y_{1}|^{2} + F^{+}(y_{2} - \psi)^{+} + F^{-}y_{2} + \Delta\psi y_{2}$$

$$(\text{Furthermore we assume that } \psi \geq 0)$$

where $y = (y_1, y_2)$ with $y_1 \in (L^2(\Omega))^n$ and $y_2 \in L^2(\Omega)$. A similar notation is used for $y^* \in Y^*$. So the obstacle problem (P) can be rewritten in the form (10). To apply Theorem 2, we compute the conjugate of the functional J.

Lemma 3. Let h be a function defined by

$$\begin{array}{rrrr} h: & \mathbb{R} & \to & \mathbb{R} \\ & s & \mapsto & as + b(s-t)^+ \end{array}$$

where a, b and t are constants with $b \ge 0$, then the conjugate function h^* of h is given by

$$h^*(s^*) = \begin{cases} t(s^* - a) & \text{if } a \le s^* \le a + b, \\ +\infty & \text{otherwise.} \end{cases}$$

If $t \geq 0$ then we have

$$0 \le h^*(s^*) \le tb.$$

Proof. We have

$$\begin{array}{lll} h^*(s^*) &=& \sup_{s \in \mathbb{R}} \{ss^* - h(s)\} \\ &=& \sup_{s \in \mathbb{R}} \{ss^* - as - b(s-t)^+\} \\ &=& \max\{\sup_{s \geq t} \{ss^* - as - bs + bt\}, \sup_{s \leq t} \{ss^* - as\}\} \\ &=& \max\{\sup_{s \geq t} \{s(s^* - a - b) + bt\}, \sup_{s < t} \{s(s^* - a)\}\} \cdot \end{array}$$

It is easy to check that

$$h^*(s^*) = \begin{cases} t(s^* - a) & \text{si } a \le s^* \le a + b \\ +\infty & \text{otherwise.} \end{cases}$$

Hence the proof is complete.

We have

$$J^*(-y^*, L^*y^*) = H^*(L^*y^*) + G^*(-y^*)$$

where

$$\begin{array}{lll} H^*(L^*y^*) &=& \sup_{v \in H^1(\Omega)} \{ \langle Lv, y \rangle - H(v) \} \\ &=& \sup_{v \in H^1_g(\Omega)} \int_{\Omega} (\nabla v y_1^* + v y_2^*) \mathrm{d}x \\ &=& \int_{\Omega} (\nabla g y_1^* + g y_2^*) \mathrm{d}x + \sup_{v \in H^1_0(\Omega)} \int_{\Omega} (\nabla v y_1^* + v y_2^*) \mathrm{d}x \\ &=& \begin{cases} \int_{\Omega} (\nabla g y_1^* + g y_2^*) \mathrm{d}x & \text{if } - \mathrm{div} \, y_1^* + y_2^* = 0 \, \mathrm{dans} \, \Omega \\ &\infty & \mathrm{otherwise} \end{cases} \end{array}$$

and

$$G^{*}(-y^{*}) = \sup_{y \in Y} \{ \langle -y^{*}, y \rangle - G(y) \}$$

=
$$\sup_{y \in Y} \int_{\Omega} (-y_{1}^{*}y_{1} - y_{2}^{*}y_{2} - \frac{1}{2} |y_{1}|^{2} - F^{+}(y_{2} - \psi)^{+} - F^{-}y_{2} - \Delta \psi y_{2}) dx$$

and from Lemma 3 we obtain

$$G^{*}(-y^{*}) = \begin{cases} \leq \int_{\Omega} (\frac{1}{2}|y_{1}^{*}|^{2} + F^{+}\psi) dx & \text{if } -f \leq y_{2}^{*} \leq F^{+} - f, \\ \infty & \text{otherwise.} \end{cases}$$

Hence

$$J^{*}(-y^{*}, L^{*}y^{*}) = \begin{cases} \leq \int_{\Omega} (\nabla g y_{1}^{*} + g y_{2}^{*} + \frac{1}{2} |y_{1}^{*}|^{2} + F^{+}\psi) dx & \text{if } \operatorname{div} y_{1}^{*} + y_{2}^{*} = 0\\ \text{and } -f \leq y_{2}^{*} \leq F^{+} - f, \\ \infty & \text{otherwise.} \end{cases}$$
(11)

We have

$$J(u_{\varepsilon}, Lu_{\varepsilon}) - J(u, Lu) = \int_{\Omega} \frac{1}{2} |\nabla u_{\varepsilon}|^2 - \frac{1}{2} |\nabla u|^2 + F^+ (u_{\varepsilon} - \psi)^+ - F^+ (u - \psi)^+ + F^- u_{\varepsilon} - F^- u + \Delta \psi u_{\varepsilon} - \Delta \psi u.$$

Using (4), with $v = u_{\varepsilon}$, we obtain

$$J(u_{\varepsilon}, Lu_{\varepsilon}) - J(u, Lu) \ge \frac{1}{2} \| \nabla (u_{\varepsilon} - u) \|_{L^{2}(\Omega)}^{2}.$$

940

Applying Theorem 2 and using (11), we have

$$J(u_{\varepsilon}, Lu_{\varepsilon}) - J(u, Lu) \leq \int_{\Omega} (\frac{1}{2} |\nabla u_{\varepsilon}|^{2} + F^{+}(u_{\varepsilon} - \psi)^{+} + F^{-}u_{\varepsilon} + \Delta \psi u_{\varepsilon} + \nabla g y_{1}^{*} + g y_{2}^{*} + \frac{1}{2} |y_{1}^{*}|^{2} + F^{+}\psi) \mathrm{d}x$$

 $\forall y^* = (y_1^*, y_2^*) \in Q^*$, with $-\operatorname{div} y_1^* + y_2^* = 0$ and $-f \leq y_2^* \leq F^+ - f$ a.e. in Ω . Since ϕ_{ε} is differentiable the inequality (5) is equivalent to

$$u_{\varepsilon} \in H^1_g(\Omega) : a(u_{\varepsilon}, v) + \int_{\Omega} (F^+ \phi'_{\varepsilon}(u_{\varepsilon} - \psi) + F^- + \Delta \psi) v \mathrm{d}x = 0. \quad \forall v \in H^1_0(\Omega).$$
(12)

Hence u_{ε} verifies the following Dirichlet problem:

$$\begin{cases} -\Delta u_{\varepsilon} + F^{+}\phi_{\varepsilon}^{'}(u_{\varepsilon} - \psi) + F^{-} + \Delta \psi &= 0 \quad \text{in } \Omega. \\ u_{\varepsilon} &= g \quad \text{on } \partial \Omega. \end{cases}$$

If we take

$$y_1^* = -\nabla u_{\varepsilon}$$
 and $y_2^* = -(F^+ \phi_{\varepsilon}'(u_{\varepsilon} - \psi) + F^- + \Delta \psi).$

Then we have

-div
$$y_1^* + y_2^* = 0$$
 and $-f \le y_2^* \le F^+ - f$.

Therefore, we have the $a \ posteriori$ estimate

$$\frac{1}{2} \| \nabla(u_{\varepsilon} - u) \|_{L^{2}(\Omega)}^{2} \leq \int_{\Omega} (\nabla u_{\varepsilon} \nabla(u_{\varepsilon} - g) + F^{+}(u_{\varepsilon} - \psi)^{+} + F^{-}u_{\varepsilon} + \Delta \psi u_{\varepsilon} - g(F^{+}\phi_{\varepsilon}^{'}(u_{\varepsilon} - \psi) + F^{-} + \Delta \psi) + F^{+}\psi) dx.$$
(13)

Taking $v = u_{\varepsilon} - g \in H^1_0(\Omega)$ in (12), we obtain

$$\int_{\Omega} \nabla u_{\varepsilon} \nabla (u_{\varepsilon} - g) \mathrm{d}x + \int_{\Omega} (F^{+} \phi_{\varepsilon}^{'}(u_{\varepsilon} - \psi) + F^{-} + \Delta \psi)(u_{\varepsilon} - g) \mathrm{d}x = 0.$$

The estimate (13) becomes

$$\frac{1}{2} \| \nabla (u_{\varepsilon} - u) \|_{L^{2}(\Omega)}^{2} \leq \int_{\Omega} (F^{+}(u_{\varepsilon} - \psi)^{+} + F^{-}u_{\varepsilon} + \Delta \psi u_{\varepsilon} - (F^{+}\phi_{\varepsilon}^{'}(u_{\varepsilon} - \psi) + F^{-} + \Delta \psi)u_{\varepsilon} + F^{+}\psi) \mathrm{d}x.$$

Hence we obtain the a posteriori error estimates. For choices c1 and c2, we have

$$\phi_{\varepsilon}^{'}(t) = \begin{cases} 1 & \text{if } t \geq \varepsilon, \\ \frac{t}{\varepsilon} & \text{if } 0 \leq t \leq \varepsilon, \\ 0 & \text{if } t \leq 0. \end{cases}$$

The *a posteriori* error estimate is

$$\frac{1}{2} \| \nabla(u_{\varepsilon} - u) \|_{L^{2}(\Omega)}^{2} \leq \int_{[0 \leq u_{\varepsilon} - \psi \leq \varepsilon]} F^{+} u_{\varepsilon} (1 - \frac{u_{\varepsilon} - \psi}{\varepsilon}) \mathrm{d}x + \int_{[u_{\varepsilon} - \psi < 0]} F^{+} \psi \mathrm{d}x.$$

For choice c3, we have

$$\phi_{\varepsilon}^{'}(t) = \begin{cases} \frac{t}{\sqrt{t^{2} + \varepsilon^{2}}} & \text{ if } t \geq 0, \\ 0 & \text{ if } t \leq 0. \end{cases}$$

A. ADDOU ET AL.

The *a posteriori* error estimate is

$$\frac{1}{2} \| \nabla (u_{\varepsilon} - u) \|_{L^{2}(\Omega)}^{2} \leq \int_{[u_{\varepsilon} - \psi \geq 0]} F^{+} u_{\varepsilon} (1 - \frac{u_{\varepsilon} - \psi}{\sqrt{(u_{\varepsilon} - \psi)^{2} + \varepsilon^{2}}}) \mathrm{d}x + \int_{[u_{\varepsilon} - \psi < 0]} F^{+} \psi \mathrm{d}x.$$

In particular, when $\psi = 0$ we find

$$\frac{1}{2} \| \nabla(u_{\varepsilon} - u) \|_{L^{2}(\Omega)}^{2} \leq \int_{[0 \leq u_{\varepsilon} \leq \varepsilon]} f^{+} u_{\varepsilon} (1 - \frac{u_{\varepsilon}}{\varepsilon}) \mathrm{d}x,$$
$$\frac{1}{2} \| \nabla(u_{\varepsilon} - u) \|_{L^{2}(\Omega)}^{2} \leq \int_{[u_{\varepsilon} \geq 0]} f^{+} u_{\varepsilon} (1 - \frac{u_{\varepsilon}}{\sqrt{u_{\varepsilon}^{2} + \varepsilon^{2}}}) \mathrm{d}x,$$

respectively.

4. A-posteriori error estimates for regularized discrete problem

Let V_h be a finite element space approximating $H^1(\Omega)$, let V_{0h} be the finite element subspace of V_h consisting of all element of V_h which are zero on the boundary of the domain. We have $V_{0h} \subset H^1_0(\Omega)$. Assume the boundary function g can be represented exactly by a function from V_h . Then, a finite element solution $u_h \in V_h$ for the obstacle problem (P) is determined from the following problem:

$$(P_h) \begin{cases} u_h \in V_h, \ u_h = g \quad \text{on } \partial\Omega\\ a(u_h, v_h - u_h) + (F^+, (u_h - \psi)^+ - (v_h - \psi)^+) + (F^- - \Delta\psi, v_h - u_h) \ge 0\\ \forall v_h \in V_h, \ v_h = g \text{ on } \partial\Omega. \end{cases}$$

If we set $u_{0h} = u_h - g$, then u_{0h} is the solution of the problem

$$(P_{0h}) \begin{cases} u_{0h} \in V_{0h} \\ a(u_{0h}, v_h - u_{0h}) + \varphi(v_h) - \varphi(u_{0h}) + \langle l, v_h - u_{0h} \rangle \ge 0 \quad \forall v_h \in V_{0h}. \end{cases}$$

We can proceed similarly as in [3] to prove the convergence of the finite element approximations and to have *a priori* error estimates.

The regularized problem of (P_{0h}) is

$$(P_{0h,\varepsilon}) \begin{cases} u_{0h,\varepsilon} \in V_{0h,\varepsilon} \\ a(u_{0h,\varepsilon}, v_h - u_{0h,\varepsilon}) + \varphi_{\varepsilon}(v_h) - \varphi_{\varepsilon}(u_{0h,\varepsilon}) + \langle l, v_h - u_{0h,\varepsilon} \rangle \ge 0 \quad \forall v_h \in V_{0h}. \end{cases}$$

We can similarly prove that $(P_{0h,\varepsilon})$ have unique solutions and their solution converge to corresponding solution of Problem (P_{0h}) . By the duality theory on the discrete problems we prove the following *a posteriori* error estimates.

For choices c1 and c2, the *a posteriori* error estimate is

$$\frac{1}{2} \| \nabla (u_{h,\varepsilon} - u_h) \|_{L^2(\Omega)}^2 \leq \int_{[0 \leq u_{h,\varepsilon} - \psi \leq \varepsilon]} F^+ u_{h,\varepsilon} (1 - \frac{u_{h,\varepsilon} - \psi}{\varepsilon}) \mathrm{d}x + \int_{[u_{h,\varepsilon} - \psi < 0]} F^+ \psi \mathrm{d}x.$$

For choice c3, the *a posteriori* error estimate is

$$\frac{1}{2} \| \nabla(u_{h,\varepsilon} - u) \|_{L^2(\Omega)}^2 \leq \int_{[u_{h,\varepsilon} - \psi \ge 0]} F^+ u_{\varepsilon,h} (1 - \frac{u_{h,\varepsilon} - \psi}{\sqrt{(u_{h,\varepsilon} - \psi)^2 + \varepsilon^2}}) \mathrm{d}x + \int_{[u_{h,\varepsilon} - \psi < 0]} F^+ \psi \mathrm{d}x.$$

In particular, when $\psi = 0$ we find

$$\frac{1}{2} \| \nabla(u_{h,\varepsilon} - u) \|_{L^{2}(\Omega)}^{2} \leq \int_{[0 \leq u_{h,\varepsilon} \leq \varepsilon]} f^{+} u_{h,\varepsilon} (1 - \frac{u_{h,\varepsilon}}{\varepsilon}) \mathrm{d}x,$$
$$\frac{1}{2} \| \nabla(u_{h,\varepsilon} - u) \|_{L^{2}(\Omega)}^{2} \leq \int_{[u_{h,\varepsilon} \geq 0]} f^{+} u_{h,\varepsilon} (1 - \frac{u_{h,\varepsilon}}{\sqrt{u_{h,\varepsilon}^{2} + \varepsilon^{2}}}) \mathrm{d}x,$$

respectively.

References

- [1] A. Addou and E.B. Mermri, Sur une méthode de résolution d'un problème d'obstacle. *Math-Recherche & Applications* 2 (2000) 59–69.
- [2] I. Ekeland and R. Temam, Analyse convexe et problèmes variationnels. Gauthier-Villars, Eds., Paris, Brussels, Montreal (1974).
- [3] R. Glowinski, J.-L. Lions and R. Trémolières, Numerical Analysis of Variational Inequalities. North-Holland Publishing Company, Amsterdam, New York, Oxford (1981).
- [4] H. Huang, W. Han and J. Zhou, The regularisation method for an obstacle problem. Numer. Math. 69 (1994) 155-166.
- [5] D. Kinderlehrer and G. Stampacchia, An Introduction to Variational Inequalities and their Applications. Academic Press, New York (1980).

To access this journal online: www.edpsciences.org