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Modélisation Mathématique et Analyse Numérique

# HOMOGENIZATION OF A MONOTONE PROBLEM IN A DOMAIN WITH OSCILLATING BOUNDARY

Dominique Blanchard[1], Luciano Carbone[2] and Antonio Gaudiello[3]

**Abstract.** We study the asymptotic behaviour of the following nonlinear problem:

$$\begin{cases} -\operatorname{div}(a(Du_h)) + |u_h|^{p-2}u_h = f & \text{in } \Omega_h, \\ a(Du_h)\cdot\nu = 0 & \text{on } \partial\Omega_h, \end{cases}$$

in a domain $\Omega_h$ of $\mathbb{R}^n$ whose boundary $\partial\Omega_h$ contains an oscillating part with respect to $h$ when $h$ tends to $\infty$. The oscillating boundary is defined by a set of cylinders with axis $0x_n$ that are $h^{-1}$-periodically distributed. We prove that the limit problem in the domain corresponding to the oscillating boundary identifies with a diffusion operator with respect to $x_n$ coupled with an algebraic problem for the limit fluxes.

**Résumé.** Nous étudions le comportement asymptotique du problème non linéaire monotone

$$\begin{cases} -\operatorname{div}(a(Du_h)) + |u_h|^{p-2}u_h = f & \text{dans } \Omega_h, \\ a(Du_h)\cdot\nu = 0 & \text{sur } \partial\Omega_h, \end{cases}$$

posé sur un ouvert $\Omega_h$ de $\mathbb{R}^n$ dont une partie de la frontière oscille avec $h$ lorsque $h$ tend vers $\infty$. Cette partie oscillante est constituée d'un ensemble de cylindres d'axe $Ox_n$ distribués avec la période $h^{-1}$. Nous démontrons que dans le domaine correspondant à la partie oscillante, le problème limite couple un problème de diffusion en $x_n$ et un problème algébrique pour les flux limites.

**AMS Subject Classification.** 35B25, 35J60.

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## Introduction

In this paper we study the asymptotic behaviour, as $h\,(\in \mathbb{N})$ diverges, of a monotone problem defined in a domain $\Omega_h$ of $\mathbb{R}^n$ $(n \geq 2)$, whose boundary contains an oscillating part depending on $h$.

The domain $\Omega_h$ is composed of two parts: a fixed part $\Omega^-$, which is a parallelepiped with sides parallel to the coordinate planes, and a part $\Omega_h^+$ that varies with $h$.

*Keywords and phrases.* Homogenization, nonlinear problem, oscillating boundary.

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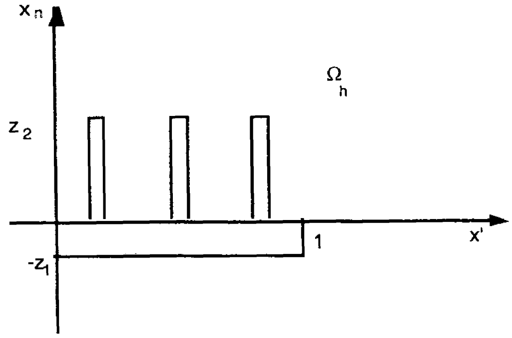


FIGURE 1.

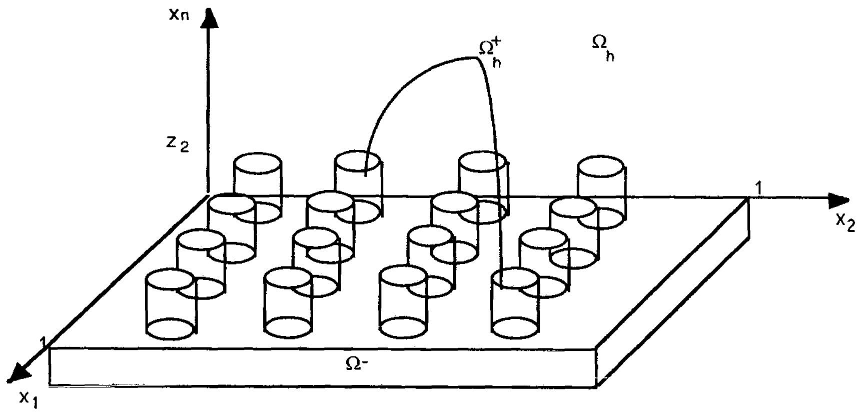


FIGURE 2.

The set $\Omega_h^+$ is defined as follows: let $C_h$ be a cylinder rescaled from a fixed one $C$ by a $h^{-1}$-homothety in the first $n-1$ variables. Then $\Omega_h^+$ is the union of such cylinders distributed with $h^{-1}$-periodicity in the first $n-1$ directions $x_1, \cdots, x_{n-1}$. The lower bases of these cylinders lie on the upper side $\Sigma$ of $\Omega^-$ (see Figs. 1 and 2 for the case $n=2$ and $n=3$ respectively). Observe that the volume of the material included in $\Omega_h^+$ does not converge to zero as $h$ tends to $+\infty$.

We study the asymptotic behaviour of the solution $u_h$, as $h$ diverges, of the following Neumann problem:

$$\begin{cases} -\operatorname{div}(a(Du_h)) + |u_h|^{p-2}u_h = f & \text{in } \Omega_h, \\ a(Du_h)\cdot\nu = 0 & \text{on } \partial\Omega_h, \end{cases} \tag{0.1}$$

where $p$ is a given number in $]1,+\infty[$, $f$ a given function in $L^{\frac{p}{p-1}}(\Omega)$, $a=(a_1,\cdots,a_n)$ a monotone continuous function from $\mathbb{R}^n$ to $\mathbb{R}^n$ satisfying usual growth conditions (see (1.2, 1.3)) and $\nu$ denotes the exterior unit normal to $\Omega_h$.

We denote by $\Omega^+$ the smallest parallelepiped containing the sets $\Omega_h^+$ for every $h$ and set $\Omega = \Omega^+ \cup \Omega^- \cup \Sigma$ (see Fig. 3). Moreover, we denote by $\widetilde{u_h}$ and $\widetilde{\partial u_h/\partial x_i}$ the zero extension to $\Omega$ of $u_h$ and $\partial u_h/\partial x_i$ respectively.

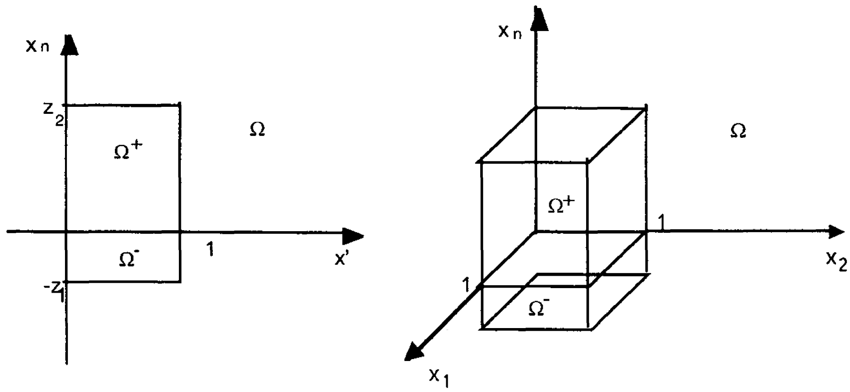


FIGURE 3.

In a nutshell, we prove the existence of a function $u$ in $L^p(\Omega)\cap W^{1,p}(\Omega^-)$ with derivative with respect to $x_n$ in $L^p(\Omega^+)$ and of $n-1$ functions $(d_1,\cdots,d_{n-1})$ in $(L^p(\Omega^+))^{n-1}$ such that

$$\begin{cases} \widetilde{u_h} \rightharpoonup |\omega| u & \text{weakly in } L^p(\Omega^+), \\ \dfrac{\partial \widetilde{u_h}}{\partial x_n} \rightharpoonup |\omega| \dfrac{\partial u}{\partial x_n} & \text{weakly in } L^p(\Omega^+), \\ u_h \rightharpoonup u & \text{weakly in } W^{1,p}(\Omega^-), \end{cases}$$

as $h$ diverges,

$$\lim_{h\to+\infty} \int_{\Omega_h} (a(Du_h)Du_h + |u_h|^p)\,dx = |\omega| \int_{\Omega^+} (a_n\left(\frac{d_1}{|\omega|},\cdots,\frac{d_{n-1}}{|\omega|},\frac{\partial u}{\partial x_n}\right)\frac{\partial u}{\partial x_n} + |u|^p)\,dx$$
$$+ \int_{\Omega^-} (a(Du)Du + |u|^p)\,dx$$

and $(u,d_1,\cdots,d_{n-1})$ is a weak solution of the following problem:

$$\begin{cases} -\frac{\partial}{\partial x_n} a_n\left(\dfrac{d_1}{|\omega|},\cdots,\dfrac{d_{n-1}}{|\omega|},\dfrac{\partial u}{\partial x_n}\right) + |u|^{p-2}u = f & \text{in } \Omega^+, \\ -\operatorname{div}(a(Du)) + |u|^{p-2}u = f & \text{in } \Omega^-, \\ u^+ = u^-, \quad |\omega| a_n\left(\dfrac{d_1}{|\omega|},\cdots,\dfrac{d_{n-1}}{|\omega|},\dfrac{\partial u^+}{\partial x_n}\right) = a_n(Du^-) & \text{on } \Sigma, \\ a_n\left(\dfrac{d_1}{|\omega|},\cdots,\dfrac{d_{n-1}}{|\omega|},\dfrac{\partial u}{\partial x_n}\right) = 0 & \text{on the upper boundary of } \Omega, \\ a(Du)\cdot\nu = 0 & \text{on } \partial\Omega^- - \Sigma, \\ a_i\left(\dfrac{d_1}{|\omega|},\cdots,\dfrac{d_{n-1}}{|\omega|},\dfrac{\partial u}{\partial x_n}\right) = 0 & \text{in } \Omega^+, \quad \forall i \in 1,\cdots,n-1, \end{cases}$$

where $u^-$ (resp. $u^+$) denotes the restriction of $u$ to $\Omega^-$ (resp. $\Omega^+$) and $|\omega|$ denotes the $(n-1)$-dimensional Lebesgue-measure of the section $\{(x_1,\cdots,x_n)\in C : x_n = 0\}$ of the reference cylinder $C$ (see Th. 1.2 and Cor. 1.3).

The limit behaviour of problem (0.1) with $a(\xi)=\xi$ is studied by Brizzi and Chalot in [5, 6] and, with a non-homogeneous Neumann boundary condition, by Gaudiello in [16].

The limit behaviour of problem (0.1) with $a(\xi) = |\xi|^{p-2}\xi$, $p$ in $[2, +\infty[$, is also obtained, using a few arguments of $\Gamma$-convergence, by Corbo Esposito, Donato, Gaudiello and Picard in [8].

In the context of the asymptotic behaviour of thin plates or cylinders, similar limit problems are obtained in [19, 20].

The goal of the present paper is to achieve the limit process in (0.1) through usual monotonicity methods.

For general references about homogenization, we refer to [2–4, 11, 24]. For the homogenization of quasilinear operators in other periodic frameworks, we refer to [10, 15] for the case of a fixed domain, to [1, 9, 14] for the case of periodically perforated domains and to [7] for reinforcement problems by a layer with oscillating thickness.

If the Neumann boundary condition in Problem (0.1) is replaced by the homogeneous Dirichlet condition $u_h = 0$ on $\partial\Omega_h$, performing the limit process, as $h$ diverges, becomes an easier task that is left to the reader (see *e.g.* [5, 13, 18, 21–23] for similar problems). In this case the limit problem reads as

$$\begin{cases} u \in W_0^{1,p}(\Omega), & \\ u = 0 & \text{in } \Omega^+, \\ -\operatorname{div}(a(Du)) + |u|^{p-2}u = f & \text{in } \Omega^-. \end{cases}$$

As far as this Dirichlet problem is concerned, the lower order term $|u_h|^{p-2}u_h$ may be removed in the whole analysis. By contrast, this term is in general necessary for the Neumann problem in order to derive an estimate on $\|u_h\|_{L^p(\Omega_h)}$ $(p > 1)$ independent of $h$, unless one has a Poincaré -Wirtinger inequality with a constant independent of $h$ in $W^{1,p}(\Omega_h)$. This is still an open problem.

## 1. Statement of the problem and main results

Let $z_1$, $z_2$ be in $]0, +\infty[$, $\omega$ an open smooth subset of $\mathbb{R}^{n-1}$ such that $\omega \subset\subset ]0, 1[^{n-1}$ $(n \geq 2)$. Let us introduce the following domains in $\mathbb{R}^n$:

$$\begin{cases} \Omega = ]0,1[^{n-1} \times ] - z_1, z_2[, \\ \Omega^- = ]0,1[^{n-1} \times ] - z_1, 0[, \quad \Omega^+ = ]0,1[^{n-1} \times ]0, z_2[, \\ \Sigma = ]0,1[^{n-1} \times \{0\}, \\ \Omega_h = \Omega^- \cup \left( \bigcup_{k \in J_h} \left( \dfrac{1}{h}\omega + \dfrac{1}{h}k \right) \times [0, z_2[ \right) \quad h \in \mathbb{N}, \\ J_h = \{ k = (k_1, \cdots, k_{n-1}) \in \mathbb{N}^{n-1} \ : \ 0 \leq k_i \leq h-1, \ i = 1, \cdots, n-1 \} \\ \Omega_h^+ = \Omega^+ \cap \Omega_h \quad h \in \mathbb{N}. \end{cases} \tag{1.1}$$

The generic point of $\mathbb{R}^n$ will be denoted by $x = (x_1, \cdots, x_{n-1}, x_n)$.

Let $p$ be a given number in $]1, +\infty[$, $f$ a given function in $L^{\frac{p}{p-1}}(\Omega)$ and $a = (a_1, \cdots, a_n)$ a monotone continuous function from $\mathbb{R}^n$ to $\mathbb{R}^n$ satisfying the following conditions:

$$\exists \alpha \in ]0, +\infty[: \quad \alpha|\xi|^p \leq a(\xi)\xi \qquad \forall \xi \in \mathbb{R}^n, \tag{1.2}$$

$$\exists \beta, \gamma \in ]0, +\infty[: \quad |a(\xi)| \leq \beta + \gamma|\xi|^{p-1} \quad \forall \xi \in \mathbb{R}^n. \tag{1.3}$$

Let us consider the following Neumann problem:

$$\begin{cases} -\operatorname{div}(a(Du_h)) + |u_h|^{p-2}u_h = f & \text{in } \Omega_h, \\ a(Du_h) \cdot \nu = 0 & \text{on } \partial\Omega_h, \end{cases} \tag{1.4}$$

where $\nu$ denotes the exterior unit normal to $\Omega_h$. It is well known (see [17]) that problem (1.4) admits a unique weak solution $u_h$ in $W^{1,p}(\Omega_h)$.

Our aim is to study the asymptotic behaviour of $u_h$ as $h$ diverges.

We recall that a function of $L^p(\Omega^+)$ with derivative with respect to $x_n$ in $L^p(\Omega^+)$ admits a trace on $\Sigma$. Consequently, we introduce the space

$$V^p(\Omega) = \left\{ v \in L^p(\Omega) \ : \ v \in W^{1,p}(\Omega^-),\ \frac{\partial v}{\partial x_n} \in L^p(\Omega^+),\ v^+ = v^- \text{ on } \Sigma \right\}, \tag{1.5}$$

where $v^-$ (resp. $v^+$) denotes the restriction of $v$ to $\Omega^-$ (resp. $\Omega^+$), provided with the norm:

$$\|v\|_{V^p(\Omega)} = \|v\|_{W^{1,p}(\Omega^-)} + \|v\|_{L^p(\Omega^+)} + \left\| \frac{\partial v}{\partial x_n} \right\|_{L^p(\Omega^+)} \qquad v \in V^p(\Omega).$$

We refer to Proposition 4.1 of [8] for the following properties of $V^p(\Omega)$:

**Proposition 1.1.** *$V^p(\Omega)$ is a Banach space and $W^{1,p}(\Omega)$ is dense in $V^p(\Omega)$ with continuous injection.*

Moreover, we recall that

$$\chi_{\Omega_h^+} \rightharpoonup |\omega| \quad \text{in } L^\infty(\Omega^+) \text{ weak } *, \tag{1.6}$$

where $|\omega|$ denotes the $(n-1)$-dimensional Lebesgue measure of $\omega$ and $\chi_A$ denotes the characteristic function of a set $A$.

In the sequel, $\widetilde{v}$ or $[v]^\sim$ denotes the zero-extension to $\Omega$ of any (vector) function $v$ defined on a subset of $\Omega$.

The main result of this paper is given in the following theorem:

**Theorem 1.2.** *Let $u_h$, $h$ in $\mathbb{N}$, be the weak solution of problem (1.4) and $V^p(\Omega)$ the space defined in (1.5). Then, there exists $u$ in $V^p(\Omega)$ such that*

$$\begin{cases} \widetilde{u_h} \rightharpoonup |\omega| u & \text{weakly in } L^p(\Omega^+), \\ \dfrac{\partial \widetilde{u_h}}{\partial x_n} \rightharpoonup |\omega| \dfrac{\partial u}{\partial x_n} & \text{weakly in } L^p(\Omega^+), \\ u_h \rightharpoonup u & \text{weakly in } W^{1,p}(\Omega^-); \end{cases} \tag{1.7}$$

*an increasing sequence of positive integer numbers, still denoted by $\{h\}_{h\in\mathbb{N}}$, and $(d_1, \cdots, d_{n-1})$ in $(L^p(\Omega^+))^{n-1}$, depending possibly on the selected subsequence, such that*

$$\widetilde{\frac{\partial u_h}{\partial x_i}} \rightharpoonup d_i \quad \textit{weakly in } L^p(\Omega^+), \quad \forall i \in \{1, \cdots, n-1\}, \tag{1.8}$$

*as $h$ diverges, where $(u, d_1, \cdots, d_{n-1})$ is a weak solution of the following problem:*

$$\begin{cases} -\dfrac{\partial}{\partial x_n} a_n\left(\dfrac{d_1}{|\omega|}, \cdots, \dfrac{d_{n-1}}{|\omega|}, \dfrac{\partial u}{\partial x_n}\right) + |u|^{p-2}u = f & \text{in } \Omega^+, \\ -\operatorname{div}(a(Du)) + |u|^{p-2}u = f & \text{in } \Omega^-, \\ u^+ = u^-, \quad |\omega| a_n\left(\dfrac{d_1}{|\omega|}, \cdots, \dfrac{d_{n-1}}{|\omega|}, \dfrac{\partial u^+}{\partial x_n}\right) = a_n(Du^-) & \text{on } \Sigma, \\ a_n\left(\dfrac{d_1}{|\omega|}, \cdots, \dfrac{d_{n-1}}{|\omega|}, \dfrac{\partial u}{\partial x_n}\right) = 0 & \text{on } ]0,1[^{n-1} \times \{z_2\}, \\ a(Du) \cdot \nu = 0 & \text{on } \partial\Omega^- - \Sigma, \\ a_i\left(\dfrac{d_1}{|\omega|}, \cdots, \dfrac{d_{n-1}}{|\omega|}, \dfrac{\partial u}{\partial x_n}\right) = 0 & \text{in } \Omega^+, \quad \forall i \in 1, \cdots, n-1 \end{cases} \tag{1.9}$$

*and the function $u$ in $V^p(\Omega)$ satisfying problem (1.9) is unique.*
*Moreover, the energies converge in the sense that:*

$$\lim_{h\to+\infty}\int_{\Omega_h}(a(Du_h)Du_h+|u_h|^p)\,dx$$
$$=|\omega|\int_{\Omega^+}(a_n\left(\frac{d_1}{|\omega|},\cdots,\frac{d_{n-1}}{|\omega|},\frac{\partial u}{\partial x_n}\right)\frac{\partial u}{\partial x_n}+|u|^p)\,dx+\int_{\Omega^-}(a(Du)Du+|u|^p)\,dx$$
$$=\int_\Omega(|\omega|\chi_{\Omega^+}+\chi_{\Omega^-})\,fu\,dx. \tag{1.10}$$

If $a$ is monotone, there is a unique function $u$ in $V^p(\Omega)$ satisfying problem (1.9) (see Step 10 of Sect. 2). Moreover, if $a$ is strictly monotone, problem (1.9) admits a unique solution $(u,d_1,\cdots,d_{n-1})$ in $V^p(\Omega)\times(L^p(\Omega^+))^{n-1}$ (see Step 11 of Sect. 2). Consequently, convergence (1.8) holds for the whole sequence $\{u_h\}_{h\in\mathbb{N}}$ and Theorem 1.2 yields the following result:

**Corollary 1.3.** *Let $u_h$, $h$ in $\mathbb{N}$, be the weak solution of problem (1.4) with a strictly monotone and $V^p(\Omega)$ the space defined in (1.5). Then,*

$$\begin{cases} |\omega|u & \text{weakly in } L^p(\Omega^+),\\ \dfrac{\partial\widetilde{u_h}}{\partial x_n}\rightharpoonup|\omega|\dfrac{\partial u}{\partial x_n} & \text{weakly in } L^p(\Omega^+),\\ \widetilde{\dfrac{\partial u_h}{\partial x_i}}\rightharpoonup d_i & \text{weakly in } L^p(\Omega^+),\quad \forall i\in\{1,\cdots,n-1\},\\ u_h\rightharpoonup u & \text{weakly in } W^{1,p}(\Omega^-),\end{cases}$$

*as $h$ diverges, where $(u,d_1,\cdots,d_{n-1})$ is the unique weak solution in $V^p(\Omega)\times(L^p(\Omega^+))^{n-1}$ of the problem (1.9). Moreover, the convergence of the energies (1.10) holds.*

**Remark 1.4.** In the case $a(\xi)=\xi$, Corollary 1.3 is proved in [5, 6] by making use of a method introduced by Tartar in [25] (method of oscillating test functions).

The limit behaviour of problem (1.4) with $a(\xi)=\xi$ and with a non-homogeneous Neumann boundary condition is studied in [16]. In this case, an additional term may appear in the limit equation.

In the case $a(\xi)=|\xi|^{p-2}\xi$, with $p$ in $[2,+\infty[$, Corollary 1.3 is also proved in [8] by following a method introduced by De Giorgi and Franzoni in [12] ($\Gamma$-convergence). In this case it results

$$d_1=\cdots=d_{n-1}=0\quad\text{a.e. in }\Omega^+$$

and limit problem (1.9) assumes the following formulation:

$$\begin{cases} -\dfrac{\partial}{\partial x_n}\left(\left|\dfrac{\partial u}{\partial x_n}\right|^{p-2}\dfrac{\partial u}{\partial x_n}\right)+|u|^{p-2}u=f & \text{in } \Omega^+,\\ -\operatorname{div}(|Du|^{p-2}Du)+|u|^{p-2}u=f & \text{in } \Omega^-,\\ u^+=u^-,\quad |\omega|\left|\dfrac{\partial u^+}{\partial x_n}\right|^{p-2}\dfrac{\partial u^+}{\partial x_n}=|Du^-|^{p-2}\dfrac{\partial u^-}{\partial x_n} & \text{on } \Sigma,\\ \dfrac{\partial u}{\partial x_n}=0 & \text{on } ]0,1[^{n-1}\times\{z_2\},\\ |Du|^{p-2}Du\cdot\nu=0 & \text{on } \partial\Omega^--\Sigma. \end{cases}$$ □

The proof of Theorem 1.2 is performed in Section 2 with 12 steps. First, we give *a priori* norm-estimates for $u_h$, $|u_h|^{p-2}u_h$ and $a(Du_h)$. Then, by virtue of the particular shape of $\Omega_h$ and by making use of the method of the oscillating test functions, we identify the limit of $\widetilde{Du_h}$ in $\Omega^-$, $\widetilde{\partial u_h/\partial x_n}$ in $\Omega^+$ and $[a_1(Du_h)]^\sim, \cdots, [a_{n-1}(Du_h)]^\sim$ in $\Omega^+$. Moreover, by a monotonicity argument, we identify the limit of $|\widetilde{u_h}|^{p-2}\widetilde{u_h}$ in $\Omega$, $[a_n(Du_h)]^\sim$ in $\Omega^+$, $a(Du_h)$ in $\Omega^-$ and obtain the last equation in (1.9). Finally, we pass to the limit in (1.4) and we conclude with some results about the uniqueness of the solution of problem (1.9).

## 2. Proof of the results

The proof of Theorem 1.2 will be performed in 12 steps.

*Proof of Theorem 1.2.* The variational formulation of problem (1.4) is given by

$$\begin{cases} \displaystyle\int_{\Omega_h} a(Du_h)Dv + |u_h|^{p-2}u_h v \, dx = \int_{\Omega_h} fv \, dx \quad \forall v \in W^{1,p}(\Omega_h), \\ u_h \in W^{1,p}(\Omega_h). \end{cases} \tag{2.1}$$

In the sequel, $c$ will denote any positive constant independent of $h$.

Step 1. *A priori norm-estimate for $u_h$, $|u_h|^{p-2}u_h$ and $a(Du_h)$*

By choosing $v = u_h$ as test function in (2.1) and by making use of (1.2), it easily results

$$\|u_h\|_{W^{1,p}(\Omega_h)} \le c \quad \forall h \in \mathbb{N}. \tag{2.2}$$

From (2.2) it follows that

$$\| \, |u_h|^{p-2}u_h\|_{L^{\frac{p}{p-1}}(\Omega_h)} \le c \quad \forall h \in \mathbb{N}. \tag{2.3}$$

Moreover, (1.3) and (2.2) provide that

$$\|a(Du_h)\|_{\left(L^{\frac{p}{p-1}}(\Omega_h)\right)^n} \le c \quad \forall h \in \mathbb{N}. \tag{2.4}$$

By virtue of (2.2–2.4), there exists an increasing sequence of positive integer numbers, still denoted by $\{h\}_{h\in\mathbb{N}}$, $u$ in $L^p(\Omega)$, $d = (d_1, \cdots, d_n)$ in $(L^p(\Omega))^n$, $z$ in $L^{\frac{p}{p-1}}(\Omega)$ and $\eta = (\eta_1, \cdots, \eta_n)$ in $\left(L^{\frac{p}{p-1}}(\Omega)\right)^n$ satisfying the following convergences:

$$\widetilde{u_h} \rightharpoonup |\omega| u\chi_{\Omega^+} + u\chi_{\Omega^-} \quad \text{weakly in } L^p(\Omega), \tag{2.5}$$

$$\widetilde{Du_h} \rightharpoonup d \quad \text{weakly in } (L^p(\Omega))^n, \tag{2.6}$$

$$|\widetilde{u_h}|^{p-2}\widetilde{u_h} \rightharpoonup z \quad \text{weakly in } L^{\frac{p}{p-1}}(\Omega), \tag{2.7}$$

$$[a(Du_h)]^\sim \rightharpoonup \eta \quad \text{weakly in } \left(L^{\frac{p}{p-1}}(\Omega)\right)^n, \tag{2.8}$$

as $h$ diverges. *A priori* $u$, $d$, $z$ and $\eta$ could depend on the selected subsequence.

In the sequel, $\{h\}_{h\in\mathbb{N}}$ will denote the previous selected subsequence of $\mathbb{N}$.

Step 2. *Identification of $d$ on $\Omega^-$ and $d_n$ on $\Omega^+$*

Convergences (2.5, 2.6) provide that

$$d = Du \quad \text{a.e. in } \Omega^- . \tag{2.9}$$

Moreover, by following arguments identical to those used in Proposition 2.2 and Corollary 2.3 of [8], it is easy to prove that

$$d_n = |\omega| \frac{\partial u}{\partial x_n} \quad \text{a.e. in } \Omega^+ \tag{2.10}$$

and

$$u \in V^p(\Omega). \tag{2.11}$$

Step 3. *Identification of $\eta_1, \cdots, \eta_{n-1}$ on $\Omega^+$*

This step is devoted to the proof of

$$\eta_i = 0 \quad \text{a.e. in } \Omega^+, \quad \forall i \in \{1, \cdots, n-1\} \cdot \tag{2.12}$$

For every $i$ in $\{1, \cdots, n-1\}$, let $\{w_h^i\}_{h\in\mathbb{N}}$ be a sequence in $W^{1,\infty}(\Omega^+)$ satisfying the following conditions:

$$w_h^i \to x_i \quad \text{strongly in } L^\infty(\Omega^+) \text{ as } h \to +\infty, \tag{2.13}$$

$$Dw_h^i = 0 \quad \text{a.e. in } \Omega_h^+, \quad \forall h \in \mathbb{N}. \tag{2.14}$$

The existence of such sequences is proved in [8] Lemma 4.3.

By choosing $v = \varphi w_h^i$ and $v = \varphi x_i$, with $\varphi$ in $C_0^\infty(\Omega^+)$, as test functions in (2.1), by virtue of (2.14) we obtain

$$\int_{\Omega^+} ([a(Du_h)]^\sim D\varphi w_h^i + |\widetilde{u_h}|^{p-2}\widetilde{u_h}\varphi w_h^i)\,\mathrm{d}x = \int_{\Omega^+} (\chi_{\Omega_h^+} f\varphi w_h^i)\mathrm{d}x \quad \forall\varphi \in C_0^\infty(\Omega^+), \tag{2.15}$$

$$\int_{\Omega^+} ([a(Du_h)]^\sim D(\varphi x_i) + |\widetilde{u_h}|^{p-2}\widetilde{u_h}\varphi x_i)\,\mathrm{d}x = \int_{\Omega^+} (\chi_{\Omega_h^+} f\varphi x_i)\mathrm{d}x \quad \forall\varphi \in C_0^\infty(\Omega^+), \tag{2.16}$$

for any $h$ in $\mathbb{N}$ and every $i$ in $\{1, \cdots, n-1\}$.

By passing to the limit, as $h$ diverges, in (2.15, 2.16), convergences (1.6, 2.7, 2.8, 2.13) provide that

$$\int_{\Omega^+} (\eta D\varphi x_i + z\varphi x_i)\,\mathrm{d}x = \int_{\Omega^+} |\omega| f\varphi x_i\,\mathrm{d}x \quad \forall\varphi \in C_0^\infty(\Omega^+), \tag{2.17}$$

$$\int_{\Omega^+} (\eta D(\varphi x_i) + z\varphi x_i)\,\mathrm{d}x = \int_{\Omega^+} |\omega| f\varphi x_i\,\mathrm{d}x \quad \forall\varphi \in C_0^\infty(\Omega^+), \tag{2.18}$$

for every $i$ in $\{1, \cdots, n-1\}$.

Statement (2.12) is obtained by subtracting (2.17) from (2.18).

Step 4. *Convergence of the energies*

This step is devoted to the proof of

$$\lim_{h\to+\infty}\int_{\Omega_h}(a(Du_h)Du_h+|u_h|^p)\,dx=\int_{\Omega^+}\eta_n\frac{\partial u}{\partial x_n}\,dx+\int_{\Omega^-}\eta Du\,dx+\int_{\Omega}zu\,dx. \tag{2.19}$$

By passing to the limit, as $h$ diverges, in (2.1) with $v$ in $W^{1,p}(\Omega)$, by virtue of (1.6, 2.7, 2.8, 2.12) we obtain

$$\int_{\Omega^+}\eta_n\frac{\partial v}{\partial x_n}\,dx+\int_{\Omega^-}\eta Dv\,dx+\int_{\Omega}zv\,dx=\int_{\Omega}(|\omega|\chi_{\Omega^+}+\chi_{\Omega^-})\,fv\,dx\quad\forall v\in W^{1,p}(\Omega). \tag{2.20}$$

Since $W^{1,p}(\Omega)$ is dense in $V^p(\Omega)$ (see Prop. 1.1), $v=u$ can be chosen as test function in (2.20). Consequently

$$\int_{\Omega^+}\eta_n\frac{\partial u}{\partial x_n}\,dx+\int_{\Omega^-}\eta Du\,dx+\int_{\Omega}zu\,dx=\int_{\Omega}(|\omega|\chi_{\Omega^+}+\chi_{\Omega^-})\,fu\,dx. \tag{2.21}$$

On the other hand, by choosing $v=u_h$ as test function in (2.1), by virtue of (2.5) we obtain

$$\lim_{h\to+\infty}\int_{\Omega_h}a(Du_h)Du_h+|u_h|^p\,dx=\lim_{h\to+\infty}\int_{\Omega}f\widetilde{u_h}\,dx=\int_{\Omega}f\left(|\omega|u\chi_{\Omega^+}+u\chi_{\Omega^-}\right)\,dx. \tag{2.22}$$

Convergence (2.19) is obtained by comparing (2.21) with (2.22).

Step 5. *Monotone relation*

This step is devoted to the proof of

$$\int_{\Omega^+}\left(\eta_n\left(\frac{\partial u}{\partial x_n}-\tau_n\right)-a(\tau)(d-|\omega|\tau)\right)dx+\int_{\Omega^-}(\eta-a(\tau))(Du-\tau)\,dx+\int_{\Omega^+}\left(z-|\omega||v|^{p-2}v\right)(u-v)\,dx$$
$$+\int_{\Omega^-}\left(z-|v|^{p-2}v\right)(u-v)\,dx\geq 0\quad\forall\tau\in(L^p(\Omega))^n,\quad\forall v\in L^p(\Omega), \tag{2.23}$$

which will enable us to identify $\eta$, $z$ and to derive the equation satisfied by $u$ in $\Omega^+$.

Let $\tau$ be in $(L^p(\Omega))^n$ and $v$ in $L^p(\Omega)$.
Since the functions $a(\xi)$ and $|t|^{p-2}t$ are monotone, we obtain

$$(a(Du_h)-a(\tau))\,(Du_h-\tau)+\left(|u_h|^{p-2}u_h-|v|^{p-2}v\right)(u_h-v)\geq 0\quad\text{a.e. in }\Omega_h,\quad\forall h\in\mathbb{N},$$

from which it follows that

$$\int_{\Omega}([a(Du_h)]^{\sim}\widetilde{Du_h}-[a(Du_h)]^{\sim}\tau-a(\tau)\widetilde{Du_h}+a(\tau)\tau\chi_{\Omega_h})\,dx$$
$$+\int_{\Omega}(|\widetilde{u_h}|^p-|\widetilde{u_h}|^{p-2}\widetilde{u_h}v-|v|^{p-2}v\widetilde{u_h}+|v|^p\chi_{\Omega_h})\,dx\geq 0\quad\forall h\in\mathbb{N}. \tag{2.24}$$

By passing to the limit, as $h$ diverges, in (2.24) and by making use of (1.3, 1.6, 2.5–2.9, 2.12, 2.19), we obtain

$$\int_{\Omega^+}\eta_n\frac{\partial u}{\partial x_n}\,dx+\int_{\Omega^-}\eta Du\,dx-\int_{\Omega^+}\eta_n\tau_n\,dx-\int_{\Omega^-}\eta\tau\,dx-\int_{\Omega^+}a(\tau)d\,dx-\int_{\Omega^-}a(\tau)Du\,dx+\int_{\Omega^+}|\omega|a(\tau)\tau\,dx$$
$$+\int_{\Omega^-}a(\tau)\tau\,dx+\int_{\Omega}zu\,dx-\int_{\Omega}zv\,dx-\int_{\Omega^+}|\omega||v|^{p-2}vu\,dx-\int_{\Omega^-}|v|^{p-2}vu\,dx+\int_{\Omega^+}|\omega||v|^p\,dx+\int_{\Omega^-}|v|^p\,dx\geq 0$$

and inequality (2.23) is proved.

Step 6. *Identification of $z$ in $\Omega$*

This step is devoted to the proof of

$$z = |\omega||u|^{p-2}u \quad \text{a.e. in } \Omega^+ \tag{2.25}$$

and

$$z = |u|^{p-2}u \quad \text{a.e. in } \Omega^-. \tag{2.26}$$

Let us remark that a typical nonlinear phenomenon occurs here: (2.7, 2.25) show that the $L^{\frac{p}{p-1}}(\Omega^+)$ - weak limit of $|\widetilde{u_h}|^{p-2}\widetilde{u_h}$ is $|\omega||u|^{p-2}u$ and not, as expected, $|\omega|^{p-1}|u|^{p-2}u$.

By choosing $\tau = \frac{1}{|\omega|}d\chi_{\Omega^+} + Du\chi_{\Omega^-}$ and $v = (u - t\varphi)\chi_{\Omega^+} + u\chi_{\Omega^-}$, with $t$ in $(0, +\infty)$ and $\varphi$ in $C_0^\infty(\Omega^+)$, in (2.23) and by recalling (2.10), we obtain

$$\int_{\Omega^+} \left(z - |\omega||u - t\varphi|^{p-2}(u - t\varphi)\right) t\varphi \,\mathrm{d}x \geq 0 \quad \forall t \in (0, +\infty), \quad \forall \varphi \in C_0^\infty(\Omega^+). \tag{2.27}$$

By dividing (2.27) by $t$ and by passing to the limit as $t$ tends to zero, by virtue of the Lebesgue Theorem it follows that

$$\int_{\Omega^+} \left(z - |\omega||u|^{p-2}u\right) \varphi \,\mathrm{d}x \geq 0 \quad \forall \varphi \in C_0^\infty(\Omega^+),$$

which implies (2.25). Statement (2.26) can be proved in the same way, by choosing $\tau = \frac{1}{|\omega|}d\chi_{\Omega^+} + Du\chi_{\Omega^-}$ and $v = u\chi_{\Omega^+} + (u - t\varphi)\chi_{\Omega^-}$, with $t$ in $(0, +\infty)$ and $\varphi$ in $C_0^\infty(\Omega^-)$, in (2.23).

Step 7. *Equation satisfied by $d$ in $\Omega^+$*

This step is devoted to the proof of

$$a_i\left(\frac{d_1}{|\omega|}, \cdots, \frac{d_{n-1}}{|\omega|}, \frac{\partial u}{\partial x_n}\right) = 0 \quad \text{a.e. in } \Omega^+, \quad \forall i \in \{1, \cdots, n-1\} \cdot \tag{2.28}$$

By choosing $\tau = (\tau_1, \cdots, \tau_{n-1}, \partial u/\partial x_n)\,\chi_{\Omega^+} + Du\chi_{\Omega^-}$ and $v = u$, with $\tau_1, \cdots, \tau_{n-1}$ in $L^p(\Omega^+)$, in (2.23) and by recalling (2.10), we obtain

$$-\int_{\Omega^+} \sum_{j=1}^{n-1} \left(a_j\left(\tau_1, \cdots, \tau_{n-1}, \frac{\partial u}{\partial x_n}\right)(d_j - |\omega|\tau_j)\right) \mathrm{d}x \geq 0 \qquad \forall\,(\tau_1, \cdots, \tau_{n-1}) \in \left(L^p(\Omega^+)\right)^{n-1}. \tag{2.29}$$

Let $i$ be fixed in $\{1, \cdots, n-1\}$. By choosing

$$\begin{cases} \tau_i = \dfrac{d_i - t\varphi}{|\omega|}, \\ \tau_j = \dfrac{d_j}{|\omega|} \qquad \forall j \in \{1, \cdots, n-1\} - \{i\}, \end{cases}$$

with $t$ in $(0, +\infty)$ and $\varphi$ in $C_0^\infty(\Omega^+)$, in (2.29), we obtain

$$\int_{\Omega^+} a_i\left(\frac{d_1}{|\omega|}, \cdots, \frac{d_i - t\varphi}{|\omega|}, \cdots, \frac{d_{n-1}}{|\omega|}, \frac{\partial u}{\partial x_n}\right) t\varphi \,\mathrm{d}x \geq 0 \qquad \forall t \in (0, +\infty), \quad \forall \varphi \in C_0^\infty(\Omega^+). \tag{2.30}$$

By dividing (2.30) by $t$ and by passing to the limit as $t$ tends to zero, by virtue of the assumption on $a$ and the Lebesgue Theorem it follows that

$$\int_{\Omega^+} a_i\left(\frac{d_1}{|\omega|},\dots,\frac{d_i}{|\omega|},\dots,\frac{d_{n-1}}{|\omega|},\frac{\partial u}{\partial x_n}\right)\varphi\, dx \geq 0 \qquad \forall \varphi \in C_0^\infty(\Omega^+),$$

which implies (2.28).

Step 8. *Identification of $\eta_n$ in $\Omega^+$ and $\eta$ in $\Omega^-$*

This step is devoted to the proof of

$$\eta_n = |\omega| a_n\left(\frac{d_1}{|\omega|},\dots,\frac{d_{n-1}}{|\omega|},\frac{\partial u}{\partial x_n}\right) \quad \text{a.e. in } \Omega^+ \tag{2.31}$$

and

$$\eta = a(Du) \quad \text{a.e. in } \Omega^-. \tag{2.32}$$

By choosing $\tau = \left(\frac{d_1}{|\omega|},\dots,\frac{d_{n-1}}{|\omega|},\frac{d_n}{|\omega|} - t\varphi\right)\chi_{\Omega^+} + Du\chi_{\Omega^-}$ and $v = u$, with $t$ in $(0,+\infty)$ and $\varphi$ in $C_0^\infty(\Omega^+)$, in (2.23) and by recalling (2.10), we obtain

$$\int_{\Omega^+}\left(\eta_n - a_n\left(\frac{d_1}{|\omega|},\dots,\frac{d_{n-1}}{|\omega|},\frac{\partial u}{\partial x_n} - t\varphi\right)|\omega|\right)t\varphi\, dx \geq 0 \qquad \forall t \in (0,+\infty), \quad \forall \varphi \in C_0^\infty(\Omega^+). \tag{2.33}$$

By dividing (2.33) by $t$ and by passing to the limit as $t$ tends to zero, by virtue of the assumption on $a$ and the Lebesgue Theorem it follows that

$$\int_{\Omega^+}\left(\eta_n - a_n\left(\frac{d_1}{|\omega|},\dots,\frac{d_{n-1}}{|\omega|},\frac{\partial u}{\partial x_n}\right)|\omega|\right)\varphi\, dx \geq 0 \qquad \forall \varphi \in C_0^\infty(\Omega^+),$$

which implies (2.31).

On the other hand, by choosing $\tau = \frac{1}{|\omega|}d\chi_{\Omega^+} + (Du - t\varphi)\chi_{\Omega^-}$ and $v = u$, with $t$ in $(0,+\infty)$ and $\varphi$ in $\left(C_0^\infty(\Omega^-)\right)^n$, in (2.23) and by recalling (2.10), it yields

$$\int_{\Omega^-}(\eta - a(Du - t\varphi))\,t\varphi\, dx \geq 0 \qquad \forall t \in (0,+\infty), \quad \forall \varphi \in \left(C_0^\infty(\Omega^-)\right)^n. \tag{2.34}$$

By dividing (2.34) by $t$ and by passing to the limit as $t$ tends to zero, by virtue of the assumption on $a$ and the Lebesgue Theorem it follows that

$$\int_{\Omega^-}(\eta - a(Du))\,\varphi\, dx \geq 0 \quad \forall \varphi \in \left(C_0^\infty(\Omega^-)\right)^n,$$

which implies (2.32).

Step 9. *Equation satisfied by u and d*

By passing to the limit, as $h$ diverges, in (2.1) with $v$ in $W^{1,p}(\Omega)$ and by making use of (1.6, 2.7, 2.8, 2.11, 2.12, 2.25, 2.26, 2.31, 2.32), it follows that

$$\begin{cases} \displaystyle\int_{\Omega^+} |\omega| a_n \left(\frac{d_1}{|\omega|}, \cdots, \frac{d_{n-1}}{|\omega|}, \frac{\partial u}{\partial x_n}\right) \frac{\partial v}{\partial x_n} \, dx + \int_{\Omega^-} a(Du) Dv \, dx + \int_{\Omega^+} |\omega| |u|^{p-2} uv \, dx + \int_{\Omega^-} |u|^{p-2} uv \, dx \\ \displaystyle = \int_\Omega (|\omega| \chi_{\Omega^+} + \chi_{\Omega^-}) fv \, dx \qquad \forall v \in W^{1,p}(\Omega), \quad (u, d_1, \cdots, d_{n-1}) \in V^p(\Omega) \times \left(L^p(\Omega^+)\right)^{n-1}. \end{cases} \tag{2.35}$$

Since $W^{1,p}(\Omega)$ is dense in $V^p(\Omega)$ (see Prop. 1.1), (2.35) implies that

$$\begin{cases} \displaystyle\int_{\Omega^+} |\omega| a_n \left(\frac{d_1}{|\omega|}, \cdots, \frac{d_{n-1}}{|\omega|}, \frac{\partial u}{\partial x_n}\right) \frac{\partial v}{\partial x_n} \, dx + \int_{\Omega^-} a(Du) Dv \, dx + \int_{\Omega^+} |\omega| |u|^{p-2} uv \, dx + \int_{\Omega^-} |u|^{p-2} uv \, dx \\ \displaystyle = \int_\Omega (|\omega| \chi_{\Omega^+} + \chi_{\Omega^-}) fv \, dx \qquad \forall v \in V^p(\Omega), \quad (u, d_1, \cdots, d_{n-1}) \in V^p(\Omega) \times \left(L^p(\Omega^+)\right)^{n-1}. \end{cases} \tag{2.36}$$

Moreover, as proved in (2.28),

$$a_i \left(\frac{d_1}{|\omega|}, \cdots, \frac{d_{n-1}}{|\omega|}, \frac{\partial u}{\partial x_n}\right) = 0 \quad \text{a.e. in } \Omega^+, \quad \forall i \in \{1, \cdots, n-1\} \cdot \tag{2.37}$$

Step 10. *Uniqueness of u*

This step is devoted to prove that there exists a unique function $u$ in $V^p(\Omega)$ satisfying problem (2.36, 2.37).

Let $(u, d_1, \cdots, d_{n-1})$ and $(\overline{u}, \overline{d_1}, \cdots, \overline{d_{n-1}})$ two solutions in $V^p(\Omega) \times (L^p(\Omega^+))^{n-1}$ of problem (2.36, 2.37).

By subtracting the equation satisfied by $(\overline{u}, \overline{d_1}, \cdots, \overline{d_{n-1}})$ from the equation satisfied by $(u, d_1, \cdots, d_{n-1})$, we obtain

$$\begin{aligned} &\int_{\Omega^+} |\omega| \left( a_n \left(\frac{d_1}{|\omega|}, \cdots, \frac{d_{n-1}}{|\omega|}, \frac{\partial u}{\partial x_n}\right) - a_n \left(\frac{\overline{d_1}}{|\omega|}, \cdots, \frac{\overline{d_{n-1}}}{|\omega|}, \frac{\partial \overline{u}}{\partial x_n}\right)\right) \frac{\partial v}{\partial x_n} \, dx + \int_{\Omega^-} (a(Du) - a(D\overline{u})) Dv \, dx \\ &\qquad + \int_{\Omega^+} |\omega| \left(|u|^{p-2}u - |\overline{u}|^{p-2}\overline{u}\right) v \, dx + \int_{\Omega^-} \left(|u|^{p-2}u - |\overline{u}|^{p-2}\overline{u}\right) v \, dx = 0 \qquad \forall v \in V^p(\Omega) \end{aligned} \tag{2.38}$$

and

$$a_i \left(\frac{d_1}{|\omega|}, \cdots, \frac{d_{n-1}}{|\omega|}, \frac{\partial u}{\partial x_n}\right) - a_i \left(\frac{\overline{d_1}}{|\omega|}, \cdots, \frac{\overline{d_{n-1}}}{|\omega|}, \frac{\partial \overline{u}}{\partial x_n}\right) = 0 \quad \text{a.e. in } \Omega^+, \quad \forall i \in \{1, \cdots, n-1\} \cdot \tag{2.39}$$

Equation (2.39) imply that

$$\int_{\Omega^+} |\omega| \left[ \sum_{i=1}^{n-1} \left( \left( a_i \left(\frac{d_1}{|\omega|}, \cdots, \frac{d_{n-1}}{|\omega|}, \frac{\partial u}{\partial x_n}\right) - a_i \left(\frac{\overline{d_1}}{|\omega|}, \cdots, \frac{\overline{d_{n-1}}}{|\omega|}, \frac{\partial \overline{u}}{\partial x_n}\right)\right) \left(\frac{d_i}{|\omega|} - \frac{\overline{d_i}}{|\omega|}\right)\right)\right] dx = 0. \tag{2.40}$$

By adding (2.40) to (2.38) with $v = u - \overline{u}$, it follows that

$$\int_{\Omega^+} |\omega| \Bigg[ \left( a\left(\frac{d_1}{|\omega|}, \cdots, \frac{d_{n-1}}{|\omega|}, \frac{\partial u}{\partial x_n}\right) - a\left(\frac{\overline{d_1}}{|\omega|}, \cdots, \frac{\overline{d_{n-1}}}{|\omega|}, \frac{\partial \overline{u}}{\partial x_n}\right)\right) \left(\left(\frac{d_1}{|\omega|}, \cdots, \frac{d_{n-1}}{|\omega|}, \frac{\partial u}{\partial x_n}\right) - \left(\frac{\overline{d_1}}{|\omega|}, \cdots, \frac{\overline{d_{n-1}}}{|\omega|}, \frac{\partial \overline{u}}{\partial x_n}\right)\right)\Bigg] \mathrm{d}x + \int_{\Omega^-} (a(Du) - a(D\overline{u}))(Du - D\overline{u})\, \mathrm{d}x$$
$$+ \int_{\Omega^+} |\omega| \left(|u|^{p-2}u - |\overline{u}|^{p-2}\overline{u}\right)(u - \overline{u})\, \mathrm{d}x + \int_{\Omega^-} \left(|u|^{p-2}u - |\overline{u}|^{p-2}\overline{u}\right)(u - \overline{u})\, \mathrm{d}x = 0. \quad (2.41)$$

Since $a(\xi)$ and $|t|^{p-2}t$ are monotone functions, (2.41) gives that

$$\int_{\Omega^+} \Bigg[ \left( a\left(\frac{d_1}{|\omega|}, \cdots, \frac{d_{n-1}}{|\omega|}, \frac{\partial u}{\partial x_n}\right) - a\left(\frac{\overline{d_1}}{|\omega|}, \cdots, \frac{\overline{d_{n-1}}}{|\omega|}, \frac{\partial \overline{u}}{\partial x_n}\right)\right) \left(\left(\frac{d_1}{|\omega|}, \cdots, \frac{d_{n-1}}{|\omega|}, \frac{\partial u}{\partial x_n}\right) - \left(\frac{\overline{d_1}}{|\omega|}, \cdots, \frac{\overline{d_{n-1}}}{|\omega|}, \frac{\partial \overline{u}}{\partial x_n}\right)\right)\Bigg] \mathrm{d}x = 0 \quad (2.42)$$

and

$$\int_{\Omega} \left(|u|^{p-2}u - |\overline{u}|^{p-2}\overline{u}\right)(u - \overline{u})\, \mathrm{d}x = 0. \quad (2.43)$$

Since $|t|^{p-2}t$ is strictly monotone, from (2.43) it follows that

$$u = \overline{u} \quad \text{a.e. in } \Omega.$$

Step 11. *Uniqueness of the solution of problem (2.36, 2.37) with a strictly monotone*

This step is devoted to a proof that problem (2.36, 2.37) admits a unique solution, if $a$ is strictly monotone. Let $(u, d_1, \cdots, d_{n-1})$ and $(\overline{u}, \overline{d_1}, \cdots, \overline{d_{n-1}})$ two solutions in $V^p(\Omega) \times \left(L^p(\Omega^+)\right)^{n-1}$ of problem (2.36, 2.37). Step 10 provides that

$$u = \overline{u} \quad \text{a.e. in } \Omega.$$

Moreover, if $a$ is strictly monotone, from (2.42) it follows that

$$d_1 = \overline{d_1}, \cdots, d_{n-1} = \overline{d_{n-1}} \quad \text{a.e. in } \Omega^+.$$

Step 12. *Conclusion: End of proof of Theorem 1.2 and Corollary 1.3*

First, let us observe that the particular shape of $\Omega_h^+$ provides that (see [6, 8])

$$\widetilde{\frac{\partial u_h}{\partial x_n}} = \frac{\partial \widetilde{u_h}}{\partial x_n} \quad \text{a.e. in } \Omega^+, \quad \forall h \in \mathbb{N}. \quad (2.44)$$

Then, convergences (1.7, 1.8) follow from (2.5, 2.6, 2.9–2.11, 2.44).

The limit problem (1.9) is given (in a weak formulation) by (2.36, 2.37) of Step 9.

The convergence of the energies (1.10) is obtained by passing to the limit, as $h$ diverges, in (2.1) with $v = u_h$ as test function, by making use of convergence (2.5) and by choosing $v = u$ as test function in (2.36).

The uniqueness of $u$ proved in Step 10 implies that convergences (1.7, 1.10) are true for all the sequence $\{u_h\}_{h\in\mathbb{N}}$. The proof of Theorem 1.2 is complete. □

If $a$ is strictly monotone, the uniqueness of the solution of problem (1.9) proved in Step 11 implies that convergence (1.8) also holds true for the whole sequence $\{u_h\}_{h\in\mathbb{N}}$. Corollary 1.3 is established. □

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