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Modélisation Mathématique et Analyse Numérique

NON-TRAPPING SETS AND HUYGENS PRINCIPLE *

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Abstract. We consider the evolution of a set $\Lambda \subset \mathbb{R}^2$ according to the Huygens principle: *i.e.* the domain at time $t > 0$, Λ_t, is the set of the points whose distance from Λ is lower than t. We give some general results for this evolution, with particular care given to the behavior of the perimeter of the evoluted set as a function of time. We define a class of sets (non-trapping sets) for which the perimeter is a continuous function of t, and we give an algorithm to approximate the evolution. Finally we restrict our attention to the class of sets for which the turning angle of the boundary is greater than $-\pi$ (see [2]). For this class of sets we prove that the perimeter is a Lipschitz-continuous function of t. This evolution problem is relevant for the applications because it is used as a model for solid fuel combustion.

Résumé. Considérons l'évolution d'un ensemble $\Lambda \subset \mathbb{R}^2$ suivant le principe de Huygens : au temps $t > 0$, cet ensemble est transformé en Λ_t, l'ensemble des points dont la distance à Λ est inférieure à t. Nous prouvons quelques résultats généraux pour cette évolution et nous étudions en détail l'évolution du périmètre de Λ_t. Nous définissons une classe d'ensembles (dits ensembles non-piégeants) pour lesquels le périmètre est une fonction continue de t, et nous donnons un algorithme pour approcher cette solution. Enfin, nous considérons la classe des ensembles pour lesquels l'intégrale de la courbure sur tout sous-arc orienté de la frontière est supérieure à $-\pi$ (voir [2]). Pour cette classe d'ensembles, nous montrons que le périmètre est une fonction lipschitzienne de t. Cette évolution du périmètre est utilisée comme modèle de combustion de propergols solides.

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1. INTRODUCTION

Let $\Lambda \subset \mathbb{R}^2$ be an open and bounded set, and let us define

$$\Lambda_t = \left\{ \mathbf{x} \in \mathbb{R}^2 : d_\Lambda(\mathbf{x}) < t \right\}, \tag{1.1}$$

where $d_\Lambda(\mathbf{x})$ is the distance between $\mathbf{x}$ and Λ, that is

$$d_\Lambda(\mathbf{x}) = \inf_{\mathbf{y} \in \Lambda} |\mathbf{x} - \mathbf{y}|. \tag{1.2}$$

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Let us denote with P_t and A_t the perimeter and the area of Λ_t respectively.

In this paper we are mainly interested in the properties of P_t as a function of t and of the initial datum Λ.

We describe the following physical situation: a solid homogeneous fuel has a cylindrical cavity whose plane section is Λ. Suppose that the whole internal surface of the fuel is lit at time $t = 0$. Because of the homogeneity of the fuel, the fire front moves with constant velocity v (for sake of simplicity we choose $v = 1$), therefore the plane section of the cavity at time t is Λ_t. The energy produced up to time t is proportional to the volume of the burnt fuel, then the power at time t is proportional to the measure (the perimeter) of the boundary of Λ_t.

In the applications the goal is to know the perimeter, as a function of the time, once that it has been given the shape of the fuel (Λ).

The function "distance from a set" is also a subject of study of shape analysis (see [3] and references therein).

The evolution according to the Huygens principle can be considered as a particular case of the motion by mean curvature, but for the fact that generally, in the literature, the velocity is a non constant function of the curvature. Nevertheless the two problems may have similar features: in particular it is interesting to characterize class of sets for which the evolution has good properties (see [2], and references therein).

Let us sketch out the contents of this paper.

In Section 2 we define the mathematical problem and we give some preliminary properties of the function "distance from a set" and of the perimeter as a function of the time. In particular we give an explicit bound on the perimeter: $P_t \leq 2A_t/t$. This is a regularization result, in fact the perimeter of the evolution of any bounded set of the plane is finite at any $t > 0$ and it can diverge at most as $1/t$ as $t \to 0^+$. Then we define a class of sets, *non-trapping sets*, for which the evolution has very good properties. In particular this definition assures us the continuity of the perimeter as a function of the time, the fact that the boundary of Λ_t is a Jordan curve, and the fact that the fuel remains connected for any t. This requirement is necessary in the applications we are interested in, in fact otherwise the fuel would break up.

In Section 3 we construct explicitly the evolution of sets whose boundary is a Jordan curve made of segments and arcs of circle. This evolution can be computed by means of a very simple algorithm. Then we prove that we can approximate the evolution of the perimeter of a non-trapping set Λ with the evolution of the perimeter of polygons that approximate Λ in the Haussdorff distance. Moreover it is possible to compute explicitly an estimate from above and from below of P_t, in terms of the perimeter of the evoluted of the approximating polygons.

In [2], Chow, Liou and Tsai define the class of regular curves whose turning angle is greater than $-\pi$. With this condition, they are able to prove that the expansion of a curve, by a positive strictly decreasing function of the curvature, has good properties. This extends the results, obtained by Tsai, for star-shaped curves.

In Section 3 we generalize this definition to sets with non regular boundary ($\pi-$sets). We characterize this kind of sets: in particular we prove that the boundary of a $\pi-$set is union of a finite number of Lipschitz curves, and that this number is estimated from above only in terms of the turning angle. Then we give also a bound on the perimeter of a $\pi-$set only in terms of the turning angle and on the diameter of the set.

Finally, we prove that if Λ is a $\pi-$set then Λ_t is a $\pi-$set and P_t is a Lipschitz function of t.

2. The distance function and the non trapping sets

Let Λ be a bounded set in $\mathbb{R}^2$. Let $d_\Lambda(\mathbf{x}) : \mathbb{R}^2 \to [0, +\infty)$ be the distance between $\mathbf{x}$ and Λ, *i.e.*

$$d_\Lambda(\mathbf{x}) = \inf_{\mathbf{y}\in\Lambda} |\mathbf{x} - \mathbf{y}|. \tag{2.1}$$

We are interested in the properties of the family of sets $\Lambda_t \equiv \{\mathbf{x} : d_\Lambda(\mathbf{x}) < t\}$. In the following we shall denote with $A(D)$ and $P(D)$ the Lebesgue measure and the perimeter of a set D respectively, and with A_t and P_t the Lebesgue measure and the perimeter of Λ_t, where the perimeter of a measurable set D is defined as (see [5]):

$$P(D) = \sup\left\{\int_D \operatorname{div}\mathbf{g} : \ \mathbf{g} \in \mathbf{C}_0^\infty(\mathbb{R}^2; \mathbb{R}),\ \|\mathbf{g}\|_\infty \leq 1\right\}. \tag{2.2}$$

Let us give some preliminary results (for an extensive analysis of the distance function see [3]).

Proposition 2.1.
(i) d_Λ *is a Lipschitz function,* ∇d_Λ *exists a.e. and* $|\nabla d_\Lambda| = 1$ *a.e. in* $d_\Lambda > 0$.
(ii) $|\mathbf{x}|^2 - d_\Lambda^2(\mathbf{x})$ *is a convex function and* d_Λ *is twice-differentiable times a.e. in the set* $d_\Lambda > 0$.
(iii) *For any* $D \subset \Lambda_t$*:*

$$P_t - P(D) \le \frac{2}{t}(A_t - A(D)). \tag{2.3}$$

(iv) *In particular:*

$$P_t \le \frac{2A_t}{t} \cdot \tag{2.4}$$

The first two points follows by the well known Rademacher and Alecksandrov Theorems. The *a priori* bound (2.4) follows from (2.3) by considering $D = \emptyset$. The inequality (2.3) is a consequence of the fact that Λ_t is union of circles of radius t. This fact allows us to deal with very rough sets. The main ingredient of the proof is the following Lemma, which is also the main ingredient of the proof of almost all the results of this paper.

Lemma 2.1 (Addition of disks). *Let* Λ *be a measurable set of area* A *with finite perimeter* P. *Let* B *be a disk of radius* $t > 0$, *and* $\Lambda' = \Lambda \cup B$. *Finally let* A' *and* P' *be the area and the perimeter of* Λ' *respectively. Then*

$$P' - \frac{2A'}{t} \le P - \frac{2A}{t} \cdot \tag{2.5}$$

Proof. First of all, let us notice that if M and N are measurable sets, then

$$P(M \cup N) + P(M \cap N) \le P(M) + P(N), \tag{2.6}$$

which is consequence of the fact that M and N can be approximated in area and perimeter with polygons, and of the lower semi-continuity of the perimeter with respect to the convergence in measure of the sets (see [5]).

Let $C = \Lambda \cap B$, and let us denote with A_C and P_C the area and the perimeter of C respectively. From (2.6) we have

$$\begin{aligned} P' &\le P + 2\pi t - P_C, \\ A' &= A + \pi t^2 - A_C, \end{aligned} \tag{2.7}$$

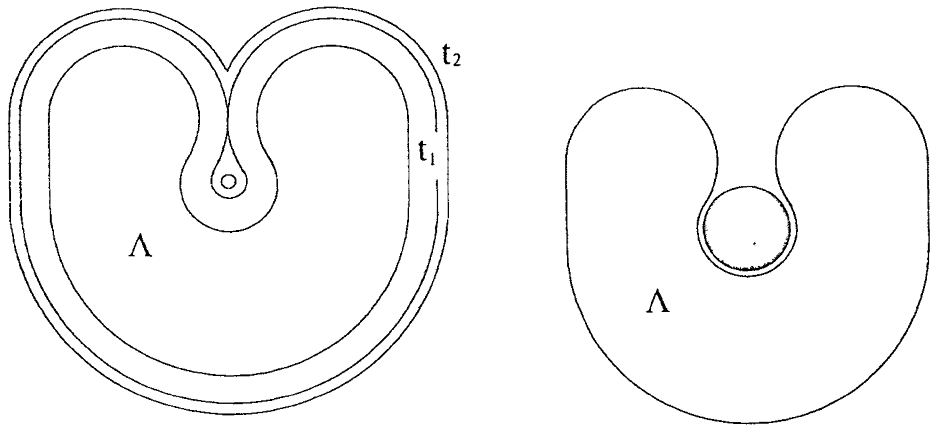
then,

$$P' - \frac{2A'}{t} \le P - \frac{2A}{t} + \frac{2A_C}{t} - P_C. \tag{2.8}$$

By the isoperimetric inequality (see, for instance, [4,5]), $P_C \ge 2\sqrt{\pi}\sqrt{A_C}$, then

$$\frac{2A_C}{t} - P_C \le \frac{2A_C}{t} - 2\sqrt{\pi}\sqrt{A_C} \le 0, \tag{2.9}$$

where in the last inequalities we have used $A_C \le \pi t^2$; (2.7) and (2.9) give (2.5). ☐



(a) (b)

FIGURE 2.1. A trapping set.

Proof of point (iii) *of Proposition 2.1.* The set Λ_t is the union of D and a countable set of circles of radius t:

$$\Lambda_t = D \cup \bigcup_{k=1}^{\infty} B_t(\mathbf{x}_k). \tag{2.10}$$

Let $D_n = D \cup (\bigcup_{k=1}^{n} B_t(\mathbf{x}_k))$, A_n and P_n the area and the perimeter of D_n respectively, then, cause (2.5),

$$P_{n+1} - \frac{2A_{n+1}}{t} \leq P_n - \frac{2A_n}{t} \leq P(D) - \frac{2A(D)}{t}. \tag{2.11}$$

As $n \to \infty$, $A(\Lambda_t \backslash D_n) = A_t - A_n \to 0$; then equation (2.3) follows from (2.11), by the lower semi-continuity of the perimeter with respect to the convergence on measure of the sets.

In the following we shall concentrate our attention on the family of sets Λ of the plane for which $\mathcal{C}\Lambda_t = \mathbb{R}^2 \backslash \Lambda_t$ is always connected.

This requirement is necessary for the applications, in fact otherwise the solid fuel would break up.

Definition 2.1. Let $\Lambda \subset \mathbb{R}^2$ be a bounded domain, *i.e.* a bounded, open and connected set: Λ is a *non-trapping set* if $\forall \mathbf{x}$, $\forall t > 0$ such that $B_t(\mathbf{x}) \cap \Lambda = \emptyset$, there exists a continuous curve $\mathbf{x}(s) : [0, +\infty) \to \mathbb{R}^2$, with $\mathbf{x}(0) = \mathbf{x}$ and $\lim_{s \to +\infty} |\mathbf{x}(s)| = \infty$, such that $\overline{B_t(\mathbf{x}(s))} \cap \overline{\Lambda} = \emptyset$ [*i.e.* $d_\Lambda(\mathbf{x}(s)) > t$], for $s > 0$.

In Figure 2.1a we have drawn a non non-trapping set and its evoluted at times $t_1 < t_2$. At time t_1 the complementary of Λ_t becomes disconnected. Notice that if Λ is a trapping set, we can put a ball in Λ trapped by Λ (see Fig. 2.1b).

The condition $d_\Lambda(\mathbf{x}(s)) > t$ (instead of the more natural $d_\Lambda(\mathbf{x}(s)) \geq t$) is necessary to obtain that the boundary of Λ_t is a curve, and then that the perimeter is a continuous function of t (see Fig. 2.2). In Figure 2.3 we have drawn a non-trapping set.

Let us summarize the regularity properties for non trapping sets.

Theorem 2.1 (Regularity). *If Λ is a non-trapping set then:*
(i) *$\mathcal{C}\Lambda_t$ is connected for all t,*
(ii) *$\partial\{\mathbf{x} \,|\, d_\Lambda(\mathbf{x}) > t\} = \partial\Lambda_t = \{\mathbf{x} \,|\, d_\Lambda(\mathbf{x}) = t\}$ is a Jordan curve for $t > 0$;*

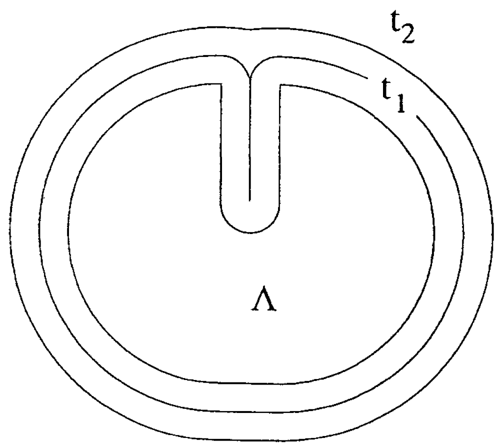


FIGURE 2.2. A trapping set.

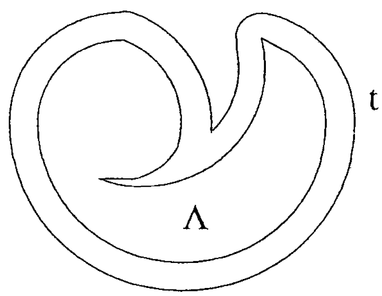


FIGURE 2.3. A non trapping set.

(iii) *$\partial\Lambda_t$ is rectifiable, and it is union of countable set of arcs with continuously turning tangent. In particular, it is possible to parameterize $\partial\Lambda_t$, in the arc length s, as*

$$\mathbf{x}(s) = \mathbf{x}(0) + \int_0^s \mathrm{d}r \begin{pmatrix} \cos\alpha(r) \\ \sin\alpha(r) \end{pmatrix},$$

where $\alpha(s)$ is the turning angle of the tangent in $\mathbf{x}(s)$, and $\alpha(s) - s/t$ is not increasing in s. Furthermore, it exists the curvature $k(s)$ as a measure and $k(s)\mathrm{d}s \leq \frac{1}{t}\mathrm{d}s$;
(iv) *the perimeter $P(t)$ is a continuous function of t, as $t > 0$.*

Proof. The first assertion is quite obvious, while the proof is a little involved. We omit it, addressing the reader to reference [7] pages 160–167, where are discussed some conditions for the connection of sets. Assertion (ii) follows from direct inspection.

For point (iii) we proceed as in proving the regularity of the boundary of a convex set [1,8]. We can prove the existence of the right and left tangent using the fact that in any points of $\partial\Lambda_t$ there is an internal tangent ball of radius t. The jumps of the tangent, in terms of the turning angle, must be negative, then the jump set is countable. This implies that we can define the arc length. Moreover, in an interval $[s_1, s_2]$ where $\theta(s)$ is continuous, $\theta(s_2) - \theta(s_1) \geq \dfrac{s_2 - s_1}{t}$, because in any points of $\partial\Lambda_t$ there exists a tangent internal ball of radius t.

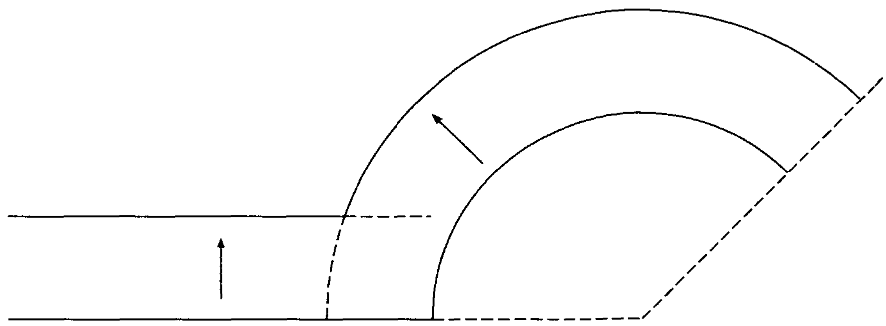


FIGURE 3.1

The continuity of the perimeter, which is consequence of the fact that Λ_t is a non trapping set and $\partial\Lambda_t$ Jordan curve for any t. We omit the proof. □

3. EVOLUTION OF POLYGONS: A NUMERICAL ALGORITHM

In this section we consider the case $\partial\Lambda$ is a curve made of segments and arcs of circle. The main motivation for studying this case, is the fact that the class of this kind of domains is closed for the evolution, and that their evolution is easy to be described and explicitly calculated. Therefore, we have a good tool to obtain a very satisfactory algorithm to approximate the evolution of a generic set.

First of all let us give some general definitions.

Definition 3.1 (Curvilinear polygons). We say that Λ is a curvilinear polygon if $\gamma = \partial\Lambda$ is a Jordan curve union of a finite number of arcs of circle, with external concavity and segments.

Let $\gamma_i : i = 1, 2, ..., n$; be the sides of γ enumerated counterclockwise, including the vertex of local convexity as arcs of circle of radius 0 centered in the vertex itself.

The evolution at time t may be constructed in the following way (see Fig. 3.1):

1. An arc of circle evolves in the arc of circle with the same center, the same angle at the center, and with the radius increased of t. A segment is translated externally for a distance t.
2. Some parts of the arcs constructed in this way fall in the interior of the domain, and they are cut away.

For curvilinear polygons which are non-trapping set, the evolution is particularly simple: before the first time of shock (the time when an arc disappears), we have to compute only the intersections of two consecutive arcs (it is sufficient to solve at most second degree equations).

Another important feature, from the numerical point of view, is the fact that the number of arcs can not increase, and the number of shocks is at most equal to the number of arcs minus two.

The exact computability of the perimeter of the evoluted of a non trapping curvilinear polygon suggest us to approximate the value of $P(\Lambda_t)$ for a generic non trapping set Λ with $P(D_t)$ where D is a polygon which approximate in some sense the set Λ (see Fig. 3.2). This program is successful as stated in the following theorem.

Theorem 3.1 (Approximation). *Let Λ be a non trapping set, and $D \subset \mathbb{R}^2$ be such that $d_H(D, \Lambda) = \inf\{\delta : \Lambda \subset D_\delta,\ D \subset \Lambda_\delta\} = \varepsilon$. Let us indicate with p_t and a_t the perimeter and the area of D_t respectively. Then P_t satisfies*

$$P_t^- \leq P_t \leq P_t^+, \tag{3.1}$$

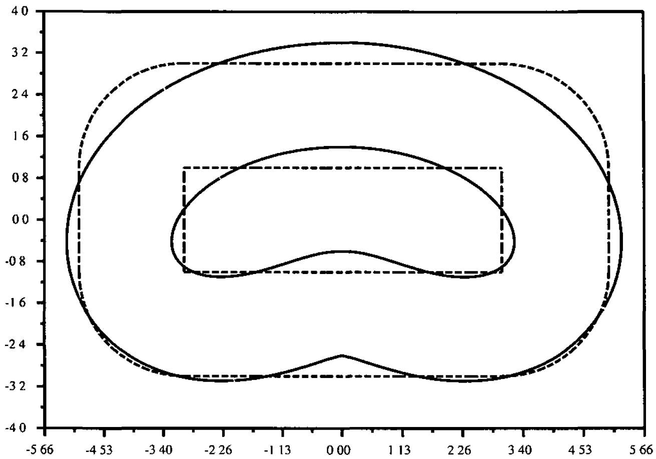


FIGURE 3.2

where

$$P_t^- = p_{t+\varepsilon} - 2\frac{a_{t+\varepsilon} - a_{t-\varepsilon}}{t+\varepsilon}, \qquad P_t^+ = p_{t-\varepsilon} + 2\frac{a_{t+\varepsilon} - a_{t-\varepsilon}}{t}. \tag{3.2}$$

Furthermore $P_t^+ - P_t^- \to 0$ *as* $\varepsilon \to 0$*: in particular*

$$P_t^+ - P_t^- \leq P_{t-\varepsilon} - P_{t+\varepsilon} + \frac{24\varepsilon}{t-\varepsilon} \max_{\tau\in[t-2\varepsilon,t+2\varepsilon]} P_\tau. \tag{3.3}$$

Theorem 3.1 gives us an estimate of the error made approximating the evolution of Λ with the evolution of D. If D is a polygon, p_t, a_t, and therefore P_t^+, P_t^-, can be explicitly calculated. We have seen that the error is estimated by $P_{t-\varepsilon} - P_{t+\varepsilon} + O(\varepsilon)$. Then it is clear that if the perimeter is only continuous, we can only say that the error vanishes as $\varepsilon \to 0$. Below we shall define a class of sets whose perimeter turns out to be a Lipschitz function of t. For this class of sets, the error vanishes linearly in ε.

Proof. From (2.3), being $D_{t-\varepsilon} \subset \Lambda_t$:

$$p_{t-\varepsilon} - 2\frac{a_{t-\varepsilon}}{t} \geq P_t - 2\frac{A_t}{t}, \tag{3.4a}$$

which implies, using $A_t \leq a_{t+\varepsilon}$, $P_t \leq P_t^+$. By noticing that $\Lambda_t \subset D_{t+\varepsilon}$:

$$P_t - 2\frac{A_t}{t+\varepsilon} \geq p_{t+\varepsilon} - 2\frac{a_{t+\varepsilon}}{t+\varepsilon}, \tag{3.4b}$$

which implies $P_t \geq P_t^-$. By writing down (3.4a) at time $t + 2\varepsilon$ and (3.4b) at time $t - 2\varepsilon$, we obtain

$$p_{t+\varepsilon} \geq P_{t+2\varepsilon} - 2\frac{A_{t+2\varepsilon} - a_{t+\varepsilon}}{t+2\varepsilon}, \qquad p_{t-\varepsilon} \leq P_{t-2\varepsilon} + 2\frac{a_{t-\varepsilon} - A_{t-2\varepsilon}}{t-\varepsilon}. \tag{3.5}$$

By substituting (3.5) in the expression for $P_t^+ - P_t^-$ and by writing down $A_{t+2\varepsilon} - A_{t-2\varepsilon}$ as $\int_{t-2\varepsilon}^{t+2\varepsilon} \mathrm{d}s P_s$, with a little algebra we find (3.3). □

In the following proposition we summarize the regularity properties of the perimeter along the time evolution.

Proposition 3.1. *Let Λ be a non-trapping curvilinear polygon. If t is not a time of shock, then*

$$\frac{\mathrm{d}}{\mathrm{d}t} P_t = 2\pi - \sum_{i:\alpha_i<0} \left(2\tan\frac{|\alpha_i|}{2} - |\alpha_i| \right), \tag{3.6}$$

where α_i is the turning angle at the vertex between the sides i and $i+1$ of $\partial\Lambda_t$, i.e. the angle between the left and right tangent vectors, measured counterclockwise ($\alpha_i = \pi - \beta_i$, where β_i is the internal angle at the vertex and α_i is positive if and only if the vertex is a vertex of local convexity for Λ_t).

Moreover, for any t and $\tau \geq 0$,

$$P_{t+\tau} \leq P_t + 2\pi\tau. \tag{3.7}$$

The proof of (3.6) follows by direct calculation, by reminding that the total turning angle along $\partial\Lambda_t$ is 2π. The Steiner-type inequality (3.7) is a consequence of the fact that $2\tan\frac{|\alpha|}{2} - |\alpha|$ is non negative. We call equation (3.7) *Steiner-type inequality*, because Steiner proved that for convex sets it is $P_{t+\tau} = P_t + 2\pi\tau$. Steiner type inequalities have been considered, for example, in [6].

In order to give an explicit bound from below of the perimeter, we need to impose more initial regularity for Λ.

Definition 3.2. The *turning angle* for a closed curvilinear polygon $\partial\Lambda$ is

$$\Theta(\partial\Lambda) = \inf_{\Gamma\subset\partial\Lambda} \int_\Gamma k(s)\,\mathrm{d}s, \tag{3.8}$$

where Γ is a connected arc of $\partial\Lambda$, s is the arc length, and $k(s)$ is the curvature with respect to the outer normal. In the vertex of abscissa $\overline{s}$, from the arcs i, $i+1$, the curvature is defined as $k(\overline{s})\mathrm{d}s = \alpha_i \delta(s-\overline{s})\,\mathrm{d}s$.

Definition 3.3. A curvilinear polygon Λ is a $\pi-$*polygon* if $\Theta(\partial\Lambda) > -\pi$.

Let us denote with $\eta(\partial\Lambda)$ the integral on $\partial\Lambda$ of the negative part of the curvature:

$$\eta(\partial\Lambda) = \int \min(k(s),0)\,\mathrm{d}s = \sum_{i:\alpha_i<0} \alpha_i. \tag{3.9}$$

Proposition 3.2 (Some properties of $\pi-$polygons.). *If Λ is a $\pi-$polygon then*

(i) *Λ is a strictly non-trapping set.*

(ii)

$$\eta(\partial\Lambda_t) \geq \eta(\partial\Lambda), \tag{3.10}$$

$$\Theta(\partial\Lambda_t) \geq \Theta(\partial\Lambda), \tag{3.11}$$

(iii) *Λ_t is a $\pi-$polygon.*

(iv)

$$\Theta(\partial\Lambda_t) \geq \eta(\partial\Lambda_t) \geq -\pi\left(\frac{2d}{t} + \frac{d^2}{t^2}\right), \tag{3.12}$$

where d is the diameter of Λ.

The proof of (i) follows by geometrical considerations. The inequalities (3.10–3.11) follow by direct inspection. The assertion (iii) is a consequence of the assertion (ii). To prove the inequality (3.12), we need to estimate the curvature of $\partial\Lambda_t$. First of all, by (3.9), $\eta(\partial\Lambda_t) = \int k^-(s)\,\mathrm{d}s$, where k^- is the negative part of the curvature. Let us notice that $\int k(s)\,\mathrm{d}s = \int k^+(s)\,\mathrm{d}s + \int k^-(s)\,\mathrm{d}s$, and $k(s) \leq 1/t$ (see Theorem 2.1). Therefore:

$$\eta(\partial\Lambda_t) = \int k^-(s)\,\mathrm{d}s = 2\pi - \int k^+(s)\,\mathrm{d}s \geq 2\pi - \frac{P_t}{t}\cdot \tag{3.13}$$

By equation (2.4)

$$P_t \leq 2\frac{A_t}{t} \leq \frac{\pi}{2t}(d+2t)^2, \tag{3.14}$$

where d is the diameter of Λ. Using equation (3.14) in (3.13) we obtain equation (3.12). □

As said before, the requirement that Λ is a $\pi-$polygon give more regularity to the function $P(t)$, as stated in the following theorem.

Theorem 3.2. *If Λ is a $\pi-$polygon then P_t is a Lipschitz function of t. In particular, as $\tau > 0$*

$$\left(2\pi - \frac{|\eta(\partial\Lambda_t)|}{|\Theta(\partial\Lambda_t|}f(|\Theta(\partial\Lambda_t)|)\right)\tau \leq P_{t+\tau} - P_t \leq 2\pi\tau, \tag{3.15}$$

where $f(\theta) = 2\tan\frac{\theta}{2} - \theta$, and where, by Proposition 3.1,

$$\begin{aligned} \eta(\partial\Lambda_t) &\geq \max\left\{\eta(\partial\Lambda), -\pi\left(\frac{2d}{t} + \frac{d^2}{t^2}\right)\right\}, \\ \Theta(\partial\Lambda_t) &\geq \max\left\{\Theta(\partial\Lambda), -\pi\left(\frac{2d}{t} + \frac{d^2}{t^2}\right)\right\}\cdot \end{aligned} \tag{3.16}$$

Notice that if $\Theta(\partial\Lambda) = 0$, $P_{t+\tau} = P_t + 2\pi\tau$; in this case the set is convex.

Proof. The right end side of (3.15) is the Steiner type inequality proved in Proposition 3.1.

We have to estimate from above the left end side of (4.2) of Proposition 3.1, with the constraints $\sum_{i:\,\alpha_i<0}|\alpha_i| = |\eta(\partial\Lambda_t)|$ and $|\alpha_i| \leq |\Theta(\partial\Lambda_t)|$. In fact, from Proposition 3.2 $|\eta(\partial\Lambda_t)|$ and $|\Theta(\partial\Lambda_t)|$ are not increasing in t. By observing that $f(a) + f(b) \leq f(a-\varepsilon) + f(b+\varepsilon)$, as $0 < a \leq b < \pi$, $b + \varepsilon < 2\pi$, and $a - \varepsilon \geq 0$, then

$$\sum_{\alpha_i<0} f(|\alpha_i|) \leq nf(|\Theta(\partial\Lambda)|) + f(r|\Theta(\partial\Lambda)|), \tag{3.17}$$

where, by denoting with [ ] the integer part, $n = \left[\frac{|\eta(\partial\Lambda_t)|}{|\theta(\partial\Lambda_t|}\right]$, and $r = \frac{|\eta(\partial\Lambda_t)|}{|\theta(\partial\Lambda_t|} - n$. Equation (3.16) follows from equation (3.17) and form the fact that $f(r\theta) \leq rf(\theta)$, if $r < 1$. □

4. $\pi-$SETS

In Section 2 we have seen how the perimeter is a continuous function of time if Λ is a non-trapping set. As for the $\pi-$polygons, we may obtain stronger regularity for the perimeter function if we make some extra assumptions on the domain Λ.

In a recent paper [2], Chow, Liou and Tsai have defined a class of regular curves, the curves which have *turning angle* greater than $-\pi$. For this class of curves: the expansion by a positive strictly decreasing function of the curvature has good properties: in particular if the curve is initially embedded it remains embedded at all times, eventually becomes convex and tends to a circle in a $\mathbf{C}^2$ norm.

Definition 4.1 (See [2]). A regular curve, γ, satisfies the property $(*)$ if its *turning angle*, $\Theta(\gamma)$, is greater than $-\pi$: the turning angle of a curve is defined as

$$\Theta(\gamma) = \inf_{\Gamma} \int_{\Gamma} k \mathrm{d}s,$$

where $\mathrm{d}s$ is the arc length, k is the curvature, and the inf is taken on all connected arcs, Γ, on γ.

In our case, the evolution is different, in fact the set is evolved by a constant function of the curvature and therefore we cannot, in general, avoid shocks. Nevertheless, for sets whose boundary satisfies the property $(*)$, let say $\pi-$*sets*, we can prove that they remains $\pi-$*sets* at all times and that the perimeter is a Lipschitz function of the time (notice that in Section 3 we have proved these results in the case Λ is a polygon). In this section we extend Definition 4.1 to non-regular sets, and we give a characterization of these class of sets: in particular we prove that the boundary of a $\pi-$*set* is union of a finite number of Lipschitz curves (their number being bounded by a function of the *turning angle*), and that its perimeter is bounded from above by its diameter times a function of the turning angle. Finally, we prove the Lipschitzianity of the perimeter for the evoluted of these sets.

Definition 4.2 (Regular $\pi-$sets). We say that $\Lambda \subset \mathbb{R}^2$ is a regular $\pi-$*set* if $\gamma = \partial\Lambda$ is a regular Jordan curve that satisfies the property $(*)$.

In Section 3 we have considered the generalization of this definition to a polygon. For the following we need to extend this definition to a more general class of sets:

Definition 4.3 (π-sets). We say that $\Lambda \subset \mathbb{R}^2$ is a $\pi-$set if
1. $\gamma = \partial\Lambda$ is a Jordan curve,
2. there exists $\theta > -\pi$ such that, for any $\varepsilon > 0$ it exists a $\pi-$polygon Λ_ε with $\Theta(\partial\Lambda_\varepsilon) \geq \theta$, such that

$$d_H(\Lambda, \Lambda_\varepsilon) \leq \varepsilon,$$

where d_H is the Haussdorff distance.

We define the *turning angle* of $\partial\Lambda$ as $\Theta(\partial\Lambda) = \sup\{\theta :$ the property 2 holds$\}$.

First of all let us give some general results about $\pi-$sets.

Theorem 4.1 (Characterization of $\pi-$sets). *If Λ is a $\pi-$set then*
1. *Λ is a strictly non-trapping set.*
2. *γ is union of a finite number, let say $n(\gamma)$, of Lipschitz curves; each one is a Lipschitz function with respect to a suitable system of coordinates. The number $n(\gamma)$ is bounded from above only in terms of the turning angle $\Theta(\gamma)$ of γ:*

$$n(\gamma) \leq \left[\frac{2\pi + |\Theta(\partial\Lambda)|}{\pi - |\Theta(\partial\Lambda)|}\right] + 1, \tag{4.1}$$

where $[\,]$ is the integer part.
3. *The perimeter $P(\Lambda)$ of γ is bounded from above only in terms of the diameter of γ and of its turning angle:*

$$P(\Lambda) \leq \frac{\operatorname{diam}(\Lambda)}{\sin\left(\frac{\pi - \Theta(\partial\Lambda)}{4}\right)} \left(\left[2\frac{2\pi + |\Theta(\partial\Lambda)|}{\pi - |\Theta(\partial\Lambda)|}\right] + 1\right) \tag{4.2}$$

(Proof in Appendix).

Remark. In Figure 4.1 we have drawn a strictly non-trapping set which is not a $\pi-$set.

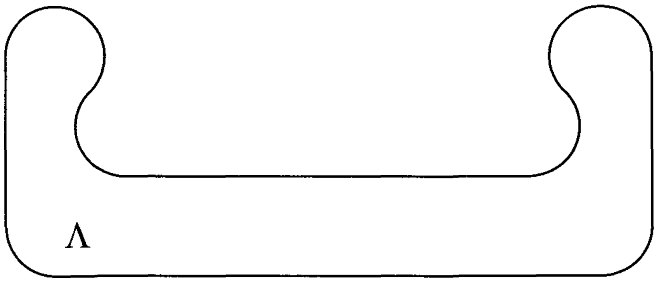


FIGURE 4.1. A non-trapping set that is not a π-set.

Theorem 4.2 (Evolution of a π-set). *Let Λ be a π-set, then:*
1. *Λ_t is a π-set.*
2. *As $t > 0$, the perimeter is a Lipschitz function of t: in particular, as $\varepsilon > 0$,*

$$\left(2\pi - \frac{|\eta_t|}{|\Theta_t|} f(|\Theta_t|)\right) \varepsilon \leq P_{t+\varepsilon} - P_t \leq 2\pi\varepsilon, \tag{4.3}$$

where $f(x) = 2\tan(x/2) - x$, and

$$\begin{aligned} \Theta_t &= \max\left\{\Theta(\gamma), -\pi\left(\frac{2d}{t} + \frac{d^2}{t^2}\right)\right\}, \\ \eta_t &= -\pi\left(\frac{2d}{t} + \frac{d^2}{t^2}\right). \end{aligned} \tag{4.4}$$

Proof. By Definition 4.3 it exists a sequence of π-polygons, $\{\Lambda_k\}_{k\in\mathbb{N}}$, such that $d_H(\Lambda_k, \Lambda) \to 0$, and $\Theta(\partial\Lambda_k) \to \Theta(\Lambda)$. By Proposition 3.2, we know that

$$\begin{aligned} \Theta(\partial\Lambda_{k,t}) &\geq \max\left\{\Theta(\partial\Lambda_k), -\pi\left(\frac{2d}{t} + \frac{d^2}{t^2}\right)\right\}, \\ \eta(\partial\Lambda_{k,t}) &\geq \max\left\{\eta(\partial\Lambda_k), -\pi\left(\frac{2d}{t} + \frac{d^2}{t^2}\right)\right\}, \end{aligned} \tag{4.5}$$

while, by definition of d_H, it is $d_H(\Lambda_{k,t}, \Lambda_t) = d_H(\Lambda_k, \Lambda)$. Therefore $\{\Lambda_{k,t}\}_{k\in\mathbb{N}}$ is a sequence of π-polygons converging to Λ , and Λ_t is a π-set.

Equation (4.3) is a direct consequence of the analogous results for polygons (Theorem 3.2), and of the approximation Theorem 3.1. In fact $\{\Lambda_{k,t}\}_{k\in\mathbb{N}}$ is a sequence of polygons which satisfy (4.3), and $P(\Lambda_{k,t}) \to P(\Lambda_t)$ as $k \to \infty$. □

Remarks.

1. Notice that, if $\eta(\partial\Lambda_k)$ does not diverge to $-\infty$ as $k \to \infty$, we can choose η_t as the maximum between the expression in (4.4) and the $\limsup_{k\to+\infty} \eta(\partial\Lambda_k)$.

2. In general, $\eta(\partial\Lambda_k)$ and $\frac{\mathrm{d}}{\mathrm{d}t}P_t$ may diverge, as $t \to 0$. For instance, let us consider the curve γ in Figure 4.2, which is the boundary of a π-set. We choose $\{x_i\}_{1,+\infty} = 1/i^\alpha$, where $\alpha > 0$, and the heights $y_i = x_i - x_{i+1}$. The length of γ is $2l + 2 + \sqrt{2}$. The turning angle of γ is $\Theta(\gamma) = -\frac{3}{4}\pi$, while $\eta(\gamma) = -\infty$. An estimate from

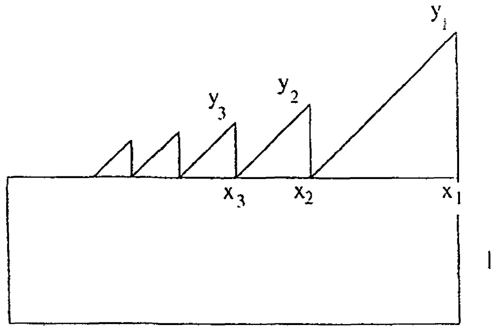


FIGURE 4.2

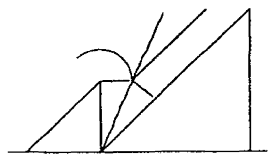


FIGURE 4.3

above of $\frac{\mathrm{d}}{\mathrm{d}t}P_t$, as $t > 0$, is:

$$\frac{\mathrm{d}}{\mathrm{d}t}P_t \leq 2\pi - c\,N(t),$$

where $c = 2\tan\frac{|\Theta(\gamma)|}{2} - |\Theta(\gamma)|$, and $N(t)$ is the number of angles equal to $\Theta(\gamma)$ not yet disappeared at time t. The ith angle, starting from the right, begins to increase when $t = y_{i+1}\tan\frac{\pi}{8}$ (see Fig. 4.3), then $N(t)$ may be easily explicitly estimated from below as

$$N(t) > \frac{\text{const.}}{t^{1+\alpha}}.$$

Therefore $\frac{\mathrm{d}}{\mathrm{d}t}P_t \to -\infty$, as $t \to 0$.

3. Equations (3.10–3.12) of Proposition 3.2 hold also for generic strictly non-trapping sets. Then, if Λ is a strictly non-trapping set it will eventually become a $\pi-$set. More precisely, see equation (3.12), Λ_t is a $\pi-$set if $t > (1+\sqrt{2})\text{diam}(\Lambda)$. After this time, Theorem 4.2 applies to Λ_t.

4. In this section we have extended Definition 4.2 of $\pi-$set to non regular sets, by means of an approximation procedure. We can consider the following geometrical definition, that extends, in a more natural way, the definition of turning angle to non regular sets.

Definition 4.4 (Geometrical π–sets)**.** We say that $\Lambda \subset \mathbb{R}^2$, open and connected, is a geometrical π–set if it exists $\alpha > 0$ such that for any $\mathbf{x}, \mathbf{y} \in \partial\Lambda$ there exist two half-line r_x, r_y, of end points $\mathbf{x}$ and $\mathbf{y}$ respectively, such that $r_x \cap \Lambda = \emptyset$, $r_y \cap \Lambda = \emptyset$, $r_x \cap r_y = \emptyset$, and the angle between r_x and r_y is not lower than α. The turning angle of $\partial\Lambda$ is $\Theta(\partial\Lambda) = \alpha - \pi$.

It is easy to prove that if $\partial\Lambda$ is a regular Jordan curve, then Definition 4.4 is equivalent to Definition 4.2. Moreover we conjecture that it is also equivalent to Definition 4.3.

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Appendix. Proof of Theorem 4.1.

We consider for first the case Λ is a regular π−set, *i.e.* γ is a regular curve that satisfies $(*)$. Let us parameterize γ, in the arc length s, as $\gamma = \{\mathbf{x}(s) : s \in [0, L)\}$, where L is the perimeter of γ, and

$$\mathbf{x}(t) = \mathbf{x}(0) + \int_0^t \mathrm{d}s \begin{pmatrix} \cos\alpha(s) \\ \sin\alpha(s) \end{pmatrix},$$

where $\begin{pmatrix} \cos\alpha(s) \\ \sin\alpha(s) \end{pmatrix}$ is the versor of the tangent to γ in $\mathbf{x}(s)$. Notice that $\alpha(t_2) - \alpha(t_1)$ is the *turning angle* along the piece of curve that goes from $\mathbf{x}(s_1)$ and $\mathbf{x}(s_2)$. Therefore

$$\begin{aligned} \alpha(s_2) - \alpha(s_1) &\geq \Theta(\gamma) \geq -\pi, \quad &&\text{as} \quad s_2 > s_1; \\ \alpha(L) - \alpha(0) &= 2\pi; \\ \alpha(s_2) - \alpha(s_1) &\leq 2\pi + |\Theta|, \quad &&\text{as} \quad s_2 - s_1 \leq L, \end{aligned} \tag{A.1}$$

where the third inequality is a consequence of the first two. Now let us define the sequence $s_i : i = 0, 1, \ldots$ as: $s_0 = 0$, and s_{i+1} is defined as the supremum of s on the set $s_i < s \leq L$ such that the variation of $\alpha(s)$ over $[s_i, s_{i+1}]$ is lower than $\pi - \beta$; where $\beta \in (0, \pi - |\Theta(\gamma)|)$. Let $\alpha_+ = \max_{s\in[s_i,s_{i+1}]} \alpha(s)$, $\alpha_- = \min_{s\in[s_i,s_{i+1}]} \alpha(s)$.

By the second of $(A1)$ it is

$$\alpha(s_i) - \alpha_- \geq -\Theta(\gamma),$$

then it must be

$$\alpha_+ = \alpha(s_{i+1}).$$

Then we have

$$\alpha(s_{i+1}) - \alpha(s_i) = \alpha_+ - \alpha_- + \alpha_- - \alpha(s_i) \geq \pi - \beta - |\Theta(\gamma)|,$$

and $\alpha(s_k) - \alpha(0) \geq k\,(\pi - |\Theta(\partial\Lambda)| - \beta)$. By the third $(A1)$, it cannot be $k > \dfrac{2\pi + |\Theta|}{\pi - |\Theta|}$, therefore we can divide the curve in at most $\left[\dfrac{2\pi + |\Theta(\partial\Lambda)|}{\pi - |\Theta(\partial\Lambda)| - \beta}\right] + 1$ pieces.

In other words, in order to have the variation of α on a piece of curve Γ not smaller than $\pi - \beta$, the turning angle must increase along Γ at least of the quantity $\pi - \beta - |\Theta(\gamma)|$. Then, being bounded by $2\pi + |\Theta(\gamma)|$ the maximum turning angle along a tract of γ, we have the previous bound.

Equation (4.1) follows in the limit $\beta \to 0$.

So we have divided the curve in a finite number of pieces and in any of this piece the total variation of the tangent is strictly lower then $\pi - \beta$ and therefore any of this piece may be represented as a function. Now let us consider a piece of curve, let say Γ_i. Γ_i may be represented by a function for which the variation of the tangent is lower than $\pi - \beta$, therefore, by choosing appropriately the coordinate system to represent the function as $y(x)$, we have that $|y'(x)| \leq \tan(\dfrac{\pi - \beta}{2})$, while the domain of the function, let say I, extend at most for the diameter diam(γ) of γ. Then in any of this piece we have

$$|\Gamma_i| = \int_I \sqrt{(1 + y'(x)^2)} \leq \mathrm{diam}(\gamma)\sqrt{1 + \tan^2\left(\frac{\pi - \beta}{2}\right)}. \tag{A.2}$$

Finally (4.2) may be obtained by summing on all the Γ_i and by choosing $\beta = \frac{\pi - |\Theta(\gamma)|}{2}$. Notice that this choice of β is, in general, not optimal.

The extension of the proof to a generic π–set may be obtained by a compactness (Ascoli–Arzela) argument: in fact the approximating polygons of a π–set, as Definition 4.3, are representable as union of a finite, and uniformly bounded, number of Lipschitz curves.

The fact that a $\pi-$set is a strictly non-trapping set, follows from the fact that the approximating $\pi-$polygons are strictly non-trapping sets [see Eqs. (3.10–3.11)], and we can pass to the limit in the proof by a compactness argument. □

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