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AN ASYMPTOTIC STUDY OF A PLATE PROBLEM BY A REARRANGEMENT METHOD. APPLICATION TO THE MECHANICAL IMPEDANCE (*)

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Abstract — *We consider a vibrating plate Ω with a small inclusion ω_ε The motion of ω_ε is given and we note ε the ratio between the diameters of ω_ε and Ω We have already studied the behaviour of the solution when ε tends to zero by the matched asymptotic expansion method and we propose here to study this problem with a rearrangement method For a rigid movement of ω_ε, we apply the results to obtain some equivalent representations of the impedance terms © Elsevier, Paris*

Résumé — *On considère les vibrations de flexion d'une plaque mince Ω, donc une inclusion ω_ε a un mouvement donné En désignant par ε le rapport des diamètres de ω_ε et de Ω, nous avons déjà étudié le comportement de la solution de ce problème quand ε tend vers zéro, par des méthodes de raccordement de développements asymptotiques et nous proposons ici une nouvelle approche par une méthode de réarrangement Dans le cas où ω_ε est rigide, on applique les résultats obtenus à la détermination d'équivalents de termes d'impédance © Elsevier, Paris*

INTRODUCTION

We consider a Love-Kirchhoff plate subjected to vibration with given stresses in the presence of a small inclusion ω_ε, the movement of which is supposed to be given. The parameter ε is taken to denote the ratio of the diameter of the inclusion ω_ε to the plate diameter. An asymptotic description of the displacement solution is proposed for ε sufficiently small. Such a problem has already been considered by the authors in [1] and [2] where it was shown that inner and outer expansions problem can be sought with suitable matching requirements (*cf.* [4], [6], [9] and [11]). In those works, an equivalent impedance matrix for free boundary plate as well as for the rigid body motion of ω_ε were obtained. It is noted worthy to recall that the impedance matrix was used to compute the stresses applied to ω_ε by the plate (*cf.* [1]). These equivalent terms are rational in ln ε.

In the present paper, we propose to study this problem by another method: We use series expansions of solutions of limiting problems to write the displacement, and we describe the rearrangement procedure (*cf.* [3], [7], [8] and [9]) to obtain a suitable sequence of problems.

For brevity and convenience, we shall use the same notations employed in [2] and shall often refer to the results in that work. We start therefore by recalling the main notations and hypothesis.

Let Ω and ω be two bounded domains in $\mathbb{R}^2$, the contours of which being denoted by $\partial\Omega$ and $\partial\omega$ and are C^1. Note that ω is strictly included in Ω. Also, let O be a point in ω, choosen as the origin of the coordinates. The parameter ε is supposed to be positive and we set,

$$\omega_\varepsilon = \{x = (x_1, x_2) \in \mathbb{R}^2 \text{ with } \varepsilon^{-1} . x \in \omega\}$$

with,

$$\Omega(\varepsilon) = \Omega \setminus \bar{\omega}_\varepsilon .$$

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Let "a" be the usual bilinear form for plates, $\mathscr{D} = (\mathscr{D}_1, \mathscr{D}_2)$ and $\mathscr{D}^{\varepsilon}$ be the Dirichlet operators on $\partial\Omega$ and $\partial\omega_{\varepsilon}$ (displacement and normal derivative), and let $\mathscr{N} = (\mathscr{N}_1, \mathscr{N}_2)$ and $\mathscr{N}^{\varepsilon}$, be the Neumann operators on $\partial\Omega$ and $\partial\omega_{\varepsilon}$ (boundary forces and bending moments) (*cf.* [1]). The Green formula is given by,

$$a(u, U)_{\Omega(\varepsilon)} = (\Delta_x^2 u, U)_{\Omega(\varepsilon)} + (\mathscr{N} u, \mathscr{D} U)_{\partial\Omega} + (\mathscr{N}^{\varepsilon} u, \mathscr{D}^{\varepsilon} U)_{\partial\omega_{\varepsilon}} . \tag{1}$$

By the dilatation $\xi = x/\varepsilon$, we define $\mathscr{D}^{\omega}(\xi, \nabla_{\xi})$ and $\mathscr{N}^{\omega}(\xi, \nabla_{\xi})$respectively the Dirichlet and the Neumann operators on $\partial\omega$.

Let Λ be a real number. Hence, we have the general boundary value problem:

$$\Delta_x^2 u(\varepsilon, x) - \Lambda u(\varepsilon, x) = f(x) \quad \text{in } \Omega(\varepsilon) \tag{2}$$

$$\mathscr{B}(x, \nabla_x)\, u(\varepsilon, x) = g(x) = (g_1(x), g_2(x)) \quad \text{on } \partial\Omega \tag{3}$$

$$\mathscr{D}^{\varepsilon}(x, \nabla_x)\, u(\varepsilon, x) = h^{\varepsilon}(\varepsilon, x) = (h_1^{\varepsilon}(\varepsilon, x), h_2^{\varepsilon}(\varepsilon, x)) \quad \text{on } \partial\omega_{\varepsilon} \tag{4}$$

where $\mathscr{B} = (\mathscr{B}_1, \mathscr{B}_2)$ represents the operator corresponding with the arbitrary boundary conditions on $\partial\Omega$.

The boundary condition (4) can be expressed in terms of the fast coordinates ξ as given by,

$$\mathscr{D}^{\omega}(\xi, \nabla_{\xi})\, u(\varepsilon, \varepsilon\xi) = h^{\omega}(\xi) = (h_1^{\omega}(\xi), h_2^{\omega}(\xi)) \quad \text{on } \partial\omega . \tag{5}$$

Thus, we may write,

$$h_1^{\varepsilon}(\varepsilon, \varepsilon\xi) = \varepsilon h_1^{\omega}(\xi) \quad \text{and} \quad h_2^{\varepsilon}(\varepsilon, \varepsilon\xi) = h_2^{\omega}(\xi)$$

and suppose that f (resp. g and h^{ω}) are C^{∞} on $\bar{\Omega}$ (resp. $\partial\Omega$ and $\partial\omega$).

It is known that suitable variational spaces for these kinds of problems are Kondratiev weighted spaces (*cf.* [2], [5], [9]). Let ℓ be an integer, β an arbitrary real number (which is supposed in this study to belong to $]\ell + 1, \ell + 2[$), and let $\mathscr{P}$ be a subset of $\mathbb{R}^2$.

By $V_{\beta}^{\ell}(\mathscr{P})$, we denote the space of functions on $\mathscr{P}$ with the norm,

$$\| z ; V_{\beta}^{\ell}(\mathscr{P}) \| = \left(\sum_{k=0}^{\ell} \| x \to |x|^{\beta - \ell + k} \nabla_x^k z(x) ; L^2(\mathscr{P}) \|^2 \right)^{1/2} . \tag{6}$$

The norms in the suitable trace spaces are,

$$\| z_{\Omega}; V_{\beta}^{\ell - 1/2}(\partial\Omega) \| = \operatorname{Inf} \{ \| z ; V_{\beta}^{\ell}(\Omega(\varepsilon)) \| , \forall z \text{ with } z = z_{\Omega} \text{ on } \partial\Omega \}$$

$$\| z_{\omega}; V_{\beta}^{\ell - 1/2}(\partial\omega_{\varepsilon}) \| = \operatorname{Inf} \{ \| z ; V_{\beta}^{\ell}(\Omega(\varepsilon)) \| , \forall z \text{ with } z = z_{\omega} \text{ on } \partial\omega_{\varepsilon} \} .$$

The homogeneous Sobolev problem may then be defined as corresponding to the free vibrations of a plate Ω, with homogeneous boundary conditions on $\partial\Omega$ with O fixed. This problem takes the form,

$$\begin{aligned} \Delta_x^2 v - \Lambda v &= 0 \quad \text{in } \Omega \\ \mathscr{B}(x, \nabla_x)\, v &= 0 \quad \text{on } \partial\Omega \\ v(O) &= 0 \end{aligned} \tag{7}$$

Let us now introduce the space,

$$\mathscr{H} = \{ v \in H^2(\Omega) \text{ with } v(O) = 0 \text{ and } \mathscr{B}_j v = 0 \text{ on } \partial\Omega \text{ if } \sigma_j < 2 \ (j = 1, 2) \}$$

where σ_i denotes the order of the highest derivative in $\mathscr{B}_i$. We have then only kinematical boundary conditions on $\partial\Omega$. Then, there exists an eigenvalue sequence (Λ_n) for which the problem (7) has nonzero solutions in $\mathscr{H}$ (*cf.* [2] and [10]).

Note that Λ *is supposed not to be one of the eigenvalues* Λ_n

We introduce the fundamental solution of the biharmonic equation

$$\Phi(x) = \frac{|x|^2}{8\pi} \ln |x| = \frac{r^2}{8\pi} \ln r$$

and let $-\Phi^i$ be the x_i first derivative of Φ.

Within this framework, we can define two limiting problems.

The first is the outer problem which has the form,

$$\begin{aligned} \Delta_x^2 v - \Lambda v &= f^1 \quad \text{in } \Omega \\ \mathscr{B}(x, \nabla_x) v &= g^1 \quad \text{on } \partial\Omega \end{aligned} \tag{8}$$

for which the following results (*cf.* [2] Theorem 8) hold true:

If $\{f^1, g^1\}$ belongs to the space $R_\beta^\ell V(\Omega)$,

$$R_\beta^\ell V(\Omega) = V_\beta^\ell(\Omega) \times V_\beta^{\ell - \sigma_1 + 7/2}(\partial\Omega) \times V_\beta^{\ell - \sigma_2 + 7/2}(\partial\Omega) .$$

The homogeneous problem associated with (8), has two independent solutions η^1 and η^2 in the space $V_{\beta+1}^{\ell+4}(\Omega)$. They have the following representation,

$$\eta^j(x) = \Phi^j(x) - \Gamma_{j1} x_1 - \Gamma_{j2} x_2 + \tilde{\eta}^j(x) \tag{9}$$

where Γ_{jk} are constants depending on $\partial\Omega$ ($\Gamma_{12} = \Gamma_{21}$), and where $\tilde{\eta}^j$ belongs to $V_\beta^{\ell+4}(\Omega)$.

Then, problem (8) has a unique solution in $V_\beta^{\ell+4}(\Omega)$ if and only if f^1 and g^1 satisfy the compatibility equations,

$$(f^1, \eta^k)_\Omega + (g^1, \mathscr{T}\eta^k)_{\partial\Omega} = 0 \quad k = 1, 2 \tag{10}$$

where $\mathscr{T}$ is the dual operator of $\mathscr{B}$.

The second limiting problem is the inner problem. It can be writen in fast coordinates $\xi = x\varepsilon^{-1}$ as,

$$\begin{aligned} \Delta_\xi^2 w &= f^2 \quad \text{in } \mathbb{R}^2 \backslash \omega \\ \mathscr{D}^\omega(\xi, \nabla_\xi) w &= h^2 \quad \text{on } \partial\omega . \end{aligned} \tag{11}$$

Similarly, if $\{f^2, h^2\}$ belongs to $R_\beta^\ell V(\mathbb{R}^2 \backslash \omega)$,

$$R_\beta^\ell V(\mathbb{R}^2 \backslash \omega) = V_\beta^\ell(\mathbb{R}^2 \backslash \omega) \times V_\beta^{\ell + 7/2}(\partial\omega) \times V_\beta^{\ell + 5/2}(\partial\omega) .$$

We have the following properties (*cf.* [2], Theorem 11).

Let ζ^1 and ζ^2 be the solutions of the homogeneous problem (11) under the following form,

$$\zeta^1(\xi) = \Phi^j(\xi) - \gamma_{j1} \xi_1 - \gamma_{j2} \xi_2 + \tilde{\zeta}^j(\xi) \tag{12}$$

where γ_{jk} are some constants depending on $\partial\omega$, $(\gamma_{12} = \gamma_{21})$, and where $\tilde{\zeta}^j$ belongs to $V_{\beta+1}^{\ell+4}(\mathbb{R}^2 \backslash \omega)$.

Problem (11) admits a solution w in $V_\beta^{\ell+4}(\mathbb{R}^2 \backslash \omega)$ which is not unique and takes the form of an arbitrary linear combination of ζ^1 and ζ^2. Nevertheless, a unique solution can be obtained on using the orthogonality condition,

$$\int_{\partial\omega} \mathscr{D}^\omega \xi^k \,.\, \mathscr{N}^\omega w \, ds = 0 \quad k = 1, 2 \,. \tag{13}$$

In addition to the previous hypothesis, let r and θ be the polar coordinates in $Ox_1 x_2$ (ρ and θ in $O\xi_1 \xi_2$), and suppose that f, g et h^ω are polynomial in r and $\ln r$, the coefficients of which are smooth functions of θ on S^1.

In the first part of this paper, we shall study the spectral properties of biharmonic problems with right hand sides of the previous type, where we obtain asymptotic expansions for the outer and inner solutions of (8) and (11).

The second part is central for asymptotic analysis of our study. We give a formal representation of the solution u of (2), (3), (4); we use series expansions of solutions of suitable limiting problems and we describe the rearrangement method. We obtain a sequence of problems and we then explain how it is possible to solve them by iterative methods.

In the third part, an estimation of the remainder of the series expansion solution is made. Then it is justified *a posteriori* by an asymptotic method.

Finally, these results are applied to obtain equivalent expressions to impedance terms if the small inclusion is rigid. Then, we conclude by comparing this rearrangement method to matched asymptotic expansions method (*cf.* [2]).

1. ASYMPTOTIC DESCRIPTIONS OF OUTER AND INNER SOLUTIONS

1.1. Spectral properties

Let us consider the biharmonic equation,

$$\Delta_x^2 u = f \quad \text{in } \mathbb{R}^2 \backslash O \tag{1.1.1}$$

where f has the form,

$$f(x) = r^{\lambda - 3} F(\theta) \quad \lambda \in \mathbb{R}, \quad F \in C^\ell(S^1) \,. \tag{1.1.2}$$

We seek for a solution of (1.1.1) as,

$$u(x) = r^{\lambda + 1} U(\theta) \,. \tag{1.1.3}$$

It is easy to see that,

$$\Delta_x^2 u(x) = r^{\lambda - 3} P(\lambda, \partial_\theta) U(\theta) \tag{1.1.4}$$

where $P(\lambda, \partial_\theta)$ denote the pencil of operators (*cf.* [3, 9]),

$$P(\lambda, \partial_\theta) = \left[(\lambda - 1)^2 + \frac{\partial^2}{\partial\theta^2} \right] \left[(\lambda + 1)^2 + \frac{\partial^2}{\partial\theta^2} \right] . \tag{1.1.5}$$

Then, we have to solve in $C^\ell(S^1)$ the equation,

$$P(\lambda, \partial_\theta) U(\theta) = F(\theta) \,. \tag{1.1.6}$$

1.2. Spectral representation of the pencil *P*

If λ is a real number, then $P(\lambda, \partial_\theta)$ is selfadjoint, and the following spectral properties hold true. We shall consider four different cases.

1.2.1. *λ does not belong to $\mathbb{Z}$*

The homogeneous equation (1.1.6) has only the zero solution, and so the complete equation (1.1.6) has a unique solution in $C^{\ell+4}(S^1)$.

1.2.2. *λ belongs to $\mathbb{Z}$ and λ is different from* 1, − 1 *and* 0

Then, λ is an eigenvalue and its index is 1. The homogeneous equation (1.1.6) has four eigenfunctions,

$$V_1^0(\theta) = \cos(\lambda+1)\theta; \quad V_2^0(\theta) = \sin(\lambda+1)\theta; \quad V_3^0(\theta) = \cos(\lambda-1)\theta; \quad V_4^0(\theta) = \sin(\lambda-1)\theta$$

and there is no associated functions since the matrix

$$[(P'_\lambda(\lambda, \partial_\theta)\, V_k^0(\theta), V_j^0(\theta))_{L^2(S^1)}]_{k,j=1\ldots4} \tag{1.2.1}$$

is regular (*cf.* [3]).

1.2.3. $\lambda = 0$

0 is an eigenvalue and its index is 2. The eigenfunctions are,

$$V_1^0(\theta) = \cos\theta; \quad V_2^0(\theta) = \sin\theta$$

and the associated functions are solutions of,

$$P(0, \partial_\theta)\, V_i^1(\theta) = -P'_\lambda(0, \partial_\theta)\, V_i^0(\theta) \tag{1.2.2}$$

where,

$$P'_\lambda(\lambda, \partial_\theta) = 4\lambda\left[(\lambda^2-1) + \frac{\partial^2}{\partial\theta^2}\right].$$

The right hand side of (1.2.2) is not zero and we obtain an associated function of order 1,

$$V_i^1(\theta) = 0 \quad i = 1, 2.$$

The associated functions of order 2 are solutions of,

$$P(0, \partial_\theta)\, V_i^2(\theta) = -P'_\lambda(0, \partial_\theta)\, V_i^1(\theta) - \frac{1}{2!}P''_\lambda(0, \partial_\theta)\, V_i^0(\theta). \tag{1.2.3}$$

Since the matrix,

$$\left[\left(P'_\lambda(0, \partial_\theta)\, V_i^1(\theta) + \frac{1}{2!}P''_\lambda(0, \partial_\theta)\, V_i^0(\theta), V_k^0(\theta)\right)_{L^2(S^1)}\right]_{i,k=1,2} \tag{1.2.4}$$

is regular, there are no associated functions of order 2.

1.2.4. $\lambda = 1$ *or* -1

The index of this eigenvalue is 2 and the eigenfunctions are,

$$V_1^0(\theta) = \cos 2\theta\,; \quad V_2^0(\theta) = \sin 2\theta\,; \quad V_3^0(\theta) = 1\,.$$

The associated functions must satisfy,

$$P(\pm 1, \partial_\theta)\, V_i^1(\theta) = -\, P'_\lambda(\pm 1, \partial_\theta)\, V_i^0(\theta)\,. \tag{1.2.5}$$

It is easily seen that only the third eigenfunction has an associated function and that this function is equal to zero.

Setting,

$$\mathscr{F}_i(\theta) = -\, P'_\lambda(\pm 1, \partial_\theta)\, V_i^0(\theta) \quad i = 1, 2$$

$$\mathscr{F}_3(\theta) = -\, P'_\lambda(\pm 1, \partial_\theta)\, V_3^1(\theta) - \frac{1}{2} P''_\lambda(\pm 1, \partial_\theta)\, V_3^0(\theta)$$

the matrix,

$$[(\mathscr{F}_i(\theta), V_k^0(\theta))_{L^2(S^1)}]_{i,k=1,2,3} \tag{1.2.6}$$

is regular and then there is no associated function of higher order.

In this case, the geometric multiplicity of 1 and -1 is 3, the algebraic multiplicities are respectively 1, 1 and 2 and the total multiplicity is 4.

We see that the spectrum of P coincides with $\mathbb{Z}$ while each of its eigenvalues is of total multiplicity 4.

If λ is an eigenvalue and if V_i^0 is an eigenfunction, then,

$$r^{\lambda+1}\, V_i^0(\theta)$$

is the solution of homogeneous equation (1.1.1). If V_i^1 is an associated function, then,

$$r^{\lambda+1}(\ln r\, V_i^0(\theta) + V_i^1(\theta))$$

is the solution of homogeneous (1.1.1) in $\mathbb{R}^2 \backslash \{0\}$ (*cf.* [3, 9]).

So we meet again the fundamental solution Φ, its derivatives and $\Delta\Phi$.

The complete equation (1.1.6) has a solution in $C^{\ell+4}(S^1)$ if the right hand side F is orthogonal in $L^2(S^1)$ to eigenfunctions. The solution is define with an arbitrary eigenfunction.

1.3. *F* is a polynomial in ln *r*

We consider equation (1.1.1) with the right hand side,

$$f(x) = r^{\lambda-3}\, F(\theta, \ln r)$$

where F is a polynomial in $\ln r$, with coefficient in $C^\ell(S^1)$. We have the lemma,

LEMMA 1: *Let λ be a real number. If λ is an eigenvalue of $P(\lambda, \partial_\theta)$ with index 0, 1 or 2, then the equation,*

$$\Delta_x^2\, u(x) = r^{\lambda-3}\, F(\theta, \ln r) \quad sur \quad \mathbb{R}^2 \backslash O \tag{1.3.1}$$

where F is a polynomial in $\ln r$, *with coefficients in* $C^{\ell}(S^1)$, *has a solution in the form,*

$$u(x) = r^{\lambda+1}\, U(\theta, \ln r) \tag{1.3.2}$$

where U is a polynomial in $\ln r$, *with coefficients in* $C^{\ell+4}(S^1)$. *Moreover,*

$$\operatorname{Deg} U = \operatorname{Deg} F + s \tag{1.3.3}$$

Proof: Let $p = \operatorname{Deg} F$. We note,

$$F(\theta, \ln r) = \sum_{q=0}^{p} \frac{1}{q!} F_q(\theta)\,(\ln r)^q \tag{1.3.4}$$

and seek U in the form,

$$U(\theta, \ln r) = \sum_{k=0}^{p+s} \frac{1}{k!} U_k(\theta)\,(\ln r)^k . \tag{1.3.5}$$

By noting that,

$$\Delta_x^2 = r^{-4}\left[\left(r\frac{\partial}{\partial r} - 2\right)^2 + \frac{\partial^2}{\partial\theta^2}\right]\left[\left(r\frac{\partial}{\partial r}\right)^2 + \frac{\partial^2}{\partial\theta^2}\right]$$

and with u taken in the form (1.3.2), we must have,

$$\Delta_x^2 u = r^{-4} P(r\,\partial_r - 1, \partial_\theta)\,.\, r^{\lambda+1}\, U(\theta, \ln r)\,. \tag{1.3.6}$$

Moreover, since,

$$(r\,\partial_r)\,(r^{\lambda+1} U(\theta, \ln r)) = r^{\lambda+1}[(\lambda + 1 + \partial_t)\, U(\theta, t)]_{t=\ln r}$$

we find,

$$(r\,\partial_r - 1)^j (r^{\lambda+1} U(\theta, \ln r)) = r^{\lambda+1}[(\lambda + \partial_t)^j\, U(\theta, t)]_{t=\ln r}$$

and then,

$$\Delta_x^2 u = r^{\lambda-3}[P(\lambda + \partial_t, \partial_\theta)\, U(\theta, t)]_{t=\ln r}\,. \tag{1.3.7}$$

By (1.3.5) and by using the Taylor decomposition for P, we obtain,

$$\begin{aligned}\Delta_x^2 u &= r^{\lambda-3} \sum_{j=0}^{\operatorname{Deg} P} \sum_{k=0}^{p+s} \frac{1}{j!} \frac{1}{k!} P^{(j)}(\lambda, \partial_\theta) \, [\partial_t^j t^k U_k(\theta)]_{t=\ln r} \\ &= r^{\lambda-3} \sum_{j=0}^{\operatorname{Deg} P} \sum_{k=j}^{p+s} \frac{1}{j!} \frac{1}{(k-j)!} (\ln r)^{k-j} P^{(j)}(\lambda, \partial_\theta) \, U_k(\theta) \\ &= r^{\lambda-3} \sum_{q=0}^{p+s} \sum_{j=0}^{p+s-q} \frac{1}{j!} \frac{1}{q!} (\ln r)^q P^{(j)}(\lambda, \partial_\theta) \, U_{j+q}(\theta) \qquad (1.3.8)\end{aligned}$$

by setting $q = k - j$ and by shifting the sums.

Then we have to solve,

$$\forall q = 0 \ldots p + s, \quad F_q(\theta) = \sum_{j=0}^{p+s-q} \frac{1}{j!} P^{(j)}(\lambda, \partial_\theta) \, U_{j+q}(\theta) \,. \qquad (1.3.9)$$

Let us now consider the different resulting situations.

1.3.1. *Let* λ *not in* $\mathbb{Z}$ $(s = 0)$

Then, equation (1.3.9) for q equal to p, is,

$$P(\lambda, \partial_\theta) \, U_p(\theta) = F_p(\theta)$$

and by using results of § 1.2, there exists a unique solution U_p^0 in $C^{\ell+4}(S^1)$.

Then the equation for $q = p - 1$ is,

$$P(\lambda, \partial_\theta) \, U_{p-1}(\theta) = F_{p-1}(\theta) - P'_\lambda(\lambda, \partial_\theta) \, U_p^0(\theta)$$

and we have a unique solution U_{p-1}^0 in $C^{\ell+4}(S^1)$.

By iteration, we obtain the unique solution U with degree p.

1.3.2. *Let* λ *belong to* $\mathbb{Z}$ *and be different from* 1, -1 *and* 0 $(s = 1)$

From (1.3.9), we have for the terms of order $p + 1$ the following equation,

$$P(\lambda, \partial_\theta) \, U_{p+1}(\theta) = 0 \,.$$

The solution of which may be written in the form,

$$\sum_{n=1}^{4} c_{n,p+1} \, V_n^0(\theta)$$

where $c_{n,p+1}$ denote arbitrary constants (*cf.* § 1.2.2)). For the terms of order p, we have,

$$P(\lambda, \partial_\theta) \, U_p(\theta) + P'_\lambda(\lambda, \partial_\theta) \, U_{p+1}(\theta) = F_p(\theta)$$

i.e.

$$P(\lambda, \partial_\theta)\, U_p(\theta) = F_p(\theta) - \sum_{n=1}^{4} P'_\lambda(\lambda, \partial_\theta)\, c_{n,p+1}\, V_n^0(\theta)\,.$$

This problem has solutions if the right hand side satisfies to the compatibility conditions,

$$\left(F_p(\theta) - \sum_{n=1}^{4} P'_\lambda(\lambda, \partial_\theta)\, c_{n,p+1}\, V_n^0(\theta)\,, \quad V_m^0(\theta) \right)_{L^2(S^1)} = 0$$

for all m from 1 to 4. Since the matrix (1.2.1) is regular, we have four conditions and so it is possible to obtain the four constants $c_{n,p+1}$. Then we can find the solution $U_p(\theta)$ with an arbitrary eigenfunction, that is,

$$U_p(\theta) = U_p^0(\theta) + \sum_{n=1}^{4} c_{n,p}\, V_n^0(\theta)\,.$$

Likewise, for $p-1$ order terms, we can calculate $c_{n,p}$ in order to satisfy the corresponding compatibility equations; we obtain $U_{p-1}(\theta)$ with an arbitrary function. By repeating this procedure, the functions $U_k(\theta)$ are computed. Note that the $U_k(\theta)$ unless $U_0(\theta)$, are unique, but there are no compatibility equations for $U_0(\theta)$.

The solution U is a polynomial in $\ln r$ of degree $p+1$. We have,

$$U(\theta, \ln r) = U^0(\theta, \ln r) + \sum_{n=1}^{4} c_{n,0}\, V_n^0(\theta)\,.$$

1.3.3. *Let λ be equal to* $0\ (s = 2)$

$P'_\lambda(\lambda, \partial_\theta)$ is equal to zero. For the terms of order $p+2$ in (1.3.9) we find,

$$P(\lambda, \partial_\theta)\, U_{p+2}(\theta) = 0$$

where the solution has the form, $\sum_{n=1}^{2} c_{n,p+2}\, V_n^0(\theta)$. For order $p+1$, we have,

$$P(\lambda, \partial_\theta)\, U_{p+1}(\theta) = 0$$

and similarly the solution can be written in the form,

$$U_{p+1}(\theta) = \sum_{n=1}^{2} c_{n,p+1}\, V_n^0(\theta)\,.$$

For the terms of order p, we have,

$$P(\lambda, \partial_\theta)\, U_p(\theta) + \frac{1}{2!} P''_\lambda(\lambda, \partial_\theta)\, U_{p+2}(\theta) = F_p(\theta)\,.$$

The compatibility conditions give the constants $c_{n,p+2}$ because the matrix (1.2.5) is regular. So we obtain $U_p(\theta)$. The equation for the terms of order $p-1$ allow us to calculate the $c_{n,p+1}$ in order to satisfy the compatibility equations where we compute $U_{p-1}(\theta)$, and so on. Finally, we write the solution U as a polynomial in $\ln r$. Its degree is $p+2$ and the constants $c_{n,1}$ and $c_{n,0}$ are arbitrary.

1.3.4. *Let λ be equal to ± 1 $(s=2)$*

For order $p+2$ terms in (1.3.9), we have,

$$U_{p+2}(\theta) = \sum_{n=1}^{3} c_{n,p+2} V_n^0(\theta)$$

(*cf.* § 1.2.4). The order $p+1$ gives,

$$P(\pm 1, \partial_\theta)\, U_{p+1}(\theta) = -P'_\lambda(\pm 1, \partial_\theta)\, U_{p+2}(\theta)$$

and we have solutions only if $P'_\lambda(\pm 1, \partial_\theta)\, U_{p+2}(\theta)$ is orthogonal to the eigenfunctions.

So, the constants $c_{1,p+2}$ and $c_{2,p+2}$ are equal to zero. We have,

$$U_{p+1}(\theta) = \sum_{n=1}^{3} c_{n,p+1} V_n^0(\theta)\,.$$

The equation corresponding to order p is,

$$P(\pm 1, \partial_\theta)\, U_p(\theta) = F_p(\theta) - \sum_{n=1}^{2} c_{n,p+1} P'_\lambda(\pm 1, \partial_\theta)\, V_n^0(\theta) - \frac{1}{2!} P''_\lambda(\pm 1, \partial_\theta)\, c_{3,p+2} V_3^0(\theta)\,.$$

Likewise, the compatibility conditions give the constants because the matrix (1.2.6) is regular. Then we obtain U_p. By iteration, the solution U appears as a polynomial in $\ln r$. Its degree is $p+2$ and $c_{3,1}$ and $c_{n,0}$ are arbitrary. ■

1.4. Application to several examples

1. Let Ω be a domain, O a point inside and $\mathcal{V}$, the subspace of functions in $H^2(\Omega)$ which vanish in O. In $\mathcal{V}$, we seek for the solutions of

$$\Delta_x^2 v = 0 \quad in \quad \mathbb{R}^2 \backslash O$$

which can be written as a sum of $r^{k+1} U_k(\theta, \ln r)$.

F from lemma 1, is equal to zero.

If k is an integer, k is an eigenvalue of P and we can obtain a nonzero solution. Moreover, this solution belongs to $\mathcal{V}$ if k is positive and if U_0 does not depend on $\ln r$. U_0 is then a linear combination of the eigenfunctions $\cos\theta$ and $\sin\theta$.

Let us look for U_1 as a polynomial with degree 1 in $\ln r$,

$$U_{10}(\theta) + U_{11}(\theta) \ln r\,.$$

We then have,

$$P(1, \partial_\theta)\, U_{11}(\theta) = 0\,,$$

so that,

$$U_{11}(\theta) = c_{10} \cos 2\,\theta + c_{11} \sin 2\,\theta + c_{12}\,.$$

As,

$$P(1, \partial_\theta)\, U_{10}(\theta) + P'_\lambda(1, \partial_\theta)\, U_{11}(\theta) = 0$$

we obtain,

$$P(1, \partial_\theta)\, U_{10}(\theta) = 16(c_{10} \cos 2\,\theta + c_{11} \sin 2\,\theta)\,.$$

By writing the orthogonality of the right hand side with $\cos 2\,\theta$, $\sin 2\,\theta$ and 1, we see that c_{10} and c_{11} are equal to zero. So,

$$U_1(\theta, \ln r) = c_{12} \ln r + c_{00} \cos 2\,\theta + c_{01} \sin 2\,\theta + c_{02}\,.$$

We find again that $\cos 2\,\theta$ and $\sin 2\,\theta$ have no associated functions, and that the associated function to 1 is equal to zero. For k greater than 1, we show that $U_k(\theta, \ln r)$ is a linear combination of $\cos(k+1)\,\theta$, $\sin(k+1)\,\theta$, $\cos(k-1)\,\theta$ and $\sin(k-1)\,\theta$, without $\ln r$ terms, because they all disappear in compatibility equations (these eigenfunctions have no associated functions). We obtain the classical expression of biharmonical functions in $\mathscr{V}$,

$$c_1\, x_1 + c_2\, x_2 + c_{12}\, r^2 \ln r + r^2 (c_{00} \cos 2\,\theta + c_{01} \sin 2\,\theta + c_{02}) + \cdots$$

2. Let us study the solutions of $\Delta_x^2 v - \Delta v = 0$ in $\mathbb{R}^2 \backslash O$, $v \in \mathscr{V}$.

We look for a solution as a sum of $r^i\, V_i(\theta, \ln r)$ $(i \geqslant 1)$, where the V_i are polynomial in $\ln r$. The four first terms are biharmonic and so they are obtained as in example 1. Then, we have,

$$\Delta_x^2\, r^5\, V_5(\theta, \ln r) = \Delta r V_1(\theta, \ln r) = \Delta(c_1\, x_1 + c_2\, x_2)$$

and so we obtain,

$$r^5\, V_5(\theta, \ln r) = \frac{r^4}{192}\, \Delta(c_1\, x_1 + c_2\, x_2)\,.$$

Then, we have to solve,

$$\Delta_x^2\, r^6\, V_6(\theta, \ln r) = \Delta c_{12}\, r^2 \ln r$$

and so on...

3. Resolution of $\Delta_x^2\, u(\theta, \ln r) = r^{-3} \sin \theta$ in $\mathbb{R}^2 \backslash O$.

In this case, we have $\lambda = 0$ and $F(\theta, \ln r) = \sin \theta$. We look for solution $rU(\theta, \ln r)$ with,

$$U(\theta, \ln r) = U_0(\theta) + U_1(\theta) \ln r + \frac{U_2(\theta)}{2} (\ln r)^2\,.$$

Then, the functions $U_2(\theta)$ and $U_1(\theta)$ must belong to the kernel of $P(0, \partial_\theta)$ and so are written in the form,

$$U_i(\theta) = c_{i1} \cos \theta + c_{i2} \sin \theta ;$$

and U_0 is solution of,

$$P(0, \partial_\theta)\, U_0(\theta) + \frac{1}{2} P''_\lambda(0, \partial_\theta)\, U_2(\theta) = \sin \theta .$$

We have,

$$\frac{1}{2} P''_\lambda(0, \partial_\theta)\, U_2(\theta) = 2\left(\frac{\partial^2}{\partial\theta^2} - 1\right) U_2(\theta) = -\,4(c_{21} \cos \theta + c_{22} \sin \theta) .$$

The compatibility conditions can be written as,

$$\left(\sin \theta - \frac{1}{2} P''_\lambda(0, \partial_\theta)\, U_2(\theta), \sin \theta\right)_{L^2(S^1)} = 0$$

$$\left(\sin \theta - \frac{1}{2} P''_\lambda(0, \partial_\theta)\, U_2(\theta), \cos \theta\right)_{L^2(S^1)} = 0$$

and we obtain, $c_{21} = 0$ and $c_{22} = -\,1/4$. We find U_0 as an arbitrary linear combination of $\cos\theta$ and $\sin\theta$. Finally, the general solution is given by,

$$u(x) = rU(\theta, \ln r) = c_{01}\, r \cos \theta + c_{02}\, r \sin \theta + r \ln r(c_{11} \cos \theta + c_{12} \sin \theta) - \frac{r(\ln r)^2}{8} \sin \theta .$$

1.5. Representation of the outer solution

Let us consider the first limiting problem (8). We have properties (9) and (10). Moreover,

THEOREM 2: *If the function f^1 in* (8) *can be written as,*

$$f^1(x) = \sum_{j=1}^{L} r^{j-3} f^1_j(\theta, \ln r) + \tilde{f}^1_L(x) \tag{1.5.1}$$

where f^1_j is polynomial in $\ln r$, *f^1_j belongs to $C^\ell(S^1)$ and $\tilde{f}^1_L$ to $V^\ell_{\beta - L}(\Omega)$.*
Then the solution v belongs to $V^{\ell+4}_\beta(\Omega)$ and has the form,

$$v(x) = \sum_{j=1}^{L} r^{j+1} v_j(\theta, \ln r) + \tilde{v}_L(x) \tag{1.5.2}$$

where v_j is a polynomial in $\ln r$, *with coefficients in $C^{\ell+4}(S^1)$, and where $\tilde{v}_L$ belongs to $V^{\ell+4}_{\beta-L}(\Omega)$.*

Proof: v is the unique solution of the outer problem (8) if the right hand side satisfies the compatiblity equation (10).

So, the proof is given by lemma 1. Likewise, as in example 2, we successively obtain the v_j functions as solutions of equations (1.3.1). Moreover, as

$$\ell + 1 < \beta < \ell + 2 ,$$

r^{j-3}(resp. r^{j+1}) belongs to $V^{\ell}_{\beta-L}(\Omega)$ (resp. $V^{\ell+4}_{\beta-L}(\Omega)$) if and only if j is strictly greater than L (*cf.* [2], Theorem 8). ■

1.6. Representation of the inner solutions

Here we consider the second limiting problem (11), which has a unique solution w under the orthogonality condition (13). We have the theorem.

THEOREM 3: *If the function f^2 in (11) can be written as,*

$$f^2(\xi) = \sum_{j=0}^{L} \rho^{-j-3} f^2_j(\theta, \ln \rho) + \tilde{f}^2_L(\xi) \tag{1.6.1}$$

where f^2_j is polynomial in $\ln \rho$ ($\rho = r\varepsilon^{-1}$), f^2_j belong to $C^{\ell}(S^1)$ and $\tilde{f}^2_L$ to $V^{\ell}_{\beta+L+1}(\mathbb{R}^2\backslash\omega)$.
Then the solution w belonging to $V^{\ell+4}_{\beta}(\mathbb{R}^2\backslash\omega)$, has the form,

$$w(\xi) = \sum_{j=0}^{L} \rho^{1-j} w_j(\theta, \ln \rho) + \tilde{w}_L(\xi) \tag{1.6.2}$$

where w_j is a polynomial in $\ln \rho$, with coefficients in $C^{\ell+4}(S^1)$, and where $\tilde{w}_L$ belongs to $V^{\ell+4}_{\beta+L+1}(\mathbb{R}^2\backslash\omega)$.
The proof follows from lemma 1 and the properties stated in [2] (Theorem 11). Moreover, for β with,

$$\ell + 1 < \beta < \ell + 2\,,$$

we note that ρ^{-j-3} (resp. ρ^{1-j}) belongs to $V^{\ell+4}_{\beta+L+1}(\mathbb{R}^2\backslash w)$ (resp. $V^{\ell+4}_{\beta+L+1}(\mathbb{R}^2\backslash\omega)$) if and only if j is strictly greater than L.

2. FORMAL CONSTRUCTION OF AN ASYMPTOTIC EXPANSION

2.1. Preliminaries

We are giving an asymptotic expansion of the solution u of problem (2), (3), (4), using a rearrangement method. An asymptotic study of u was done in [1] and [2] by matching methods (*cf.* [4], [6]). By these methods, we could easily obtain all the first terms of the expansions, but it becames very cumbersome to obtain the following terms. The rearrangement method which was introduced and developed in [7] and [8] will give us a representation of u by using solutions of a sequence of outer and inner problems. All of them will have the same difficulties and we will solve them by iteration.

We look for u as,

$$u(\varepsilon, x) = \sum_{k=0}^{\infty} \varepsilon^k \{ v^k(x, \ln \varepsilon) + \varepsilon w^k(\varepsilon^{-1} x, \ln \varepsilon) \} \tag{2.1.1}$$

where the functions v^k and w^k belong respectively to $V^{\ell+4}_{\beta}(\Omega)$ and $V^{\ell+4}_{\beta}(\mathbb{R}^2\backslash\omega)$.

The solutions v^k and w^k will be rational fractions in $\ln \varepsilon$. Let us recall that all rational fraction R can be written uniquely in the form,

$$R = E + \frac{P}{Q}$$

where E, P and Q are polynomial functions and where $\deg P < \deg Q$. We call "denominator degree of R", the positive integer $\deg Q - \deg P$. So, the denominator degree of a sum of R_1 and R_2 is smaller or equal to the greatest denominator degree.

We show first, on a simple example, why it will be necessary to turn to the rearrangement of some discrepancies. Let us suppose that f is equal to zero. Then, we could think that v^k and w^k would be solutions of,

$$\Delta_x^2 v^k - \Lambda v^k = 0 \tag{2.1.2}$$

$$\Delta_\xi^2 w^k = \Lambda w^{k-4} \tag{2.1.3}$$

and so we should have,

$$\Delta_\xi^2 w^0 = 0 .$$

By using Theorem 3, we should have the decomposition,

$$w^0(\xi) = \rho w_0^0(\theta, \ln \rho) + w_1^0(\theta, \ln \rho) + \cdots$$

and w^0 would belong to $V_\beta^{\ell+4}(\mathbb{R}^2 \backslash \omega)$ but not to $V_\beta^\ell(\mathbb{R}^2 \backslash \omega)$.

For k equal to 4, the right hand side of (2.1.3) would not belong to the good space to apply Theorem 3. To avoid this difficulty, we have to proceed to the rearrangement of the problems.

2.2. Rearrangement

We propose to write a sequence of suitable problems for the functions v^k and w^k of (2.1.1).

Concerning w^k, we saw that some terms in ρ^j, with polynomial coefficients in $\ln \rho$, belong to $V_\beta^{\ell+4}(\mathbb{R}^2 \backslash \omega)$ but not to $V_\beta^\ell(\mathbb{R}^2 \backslash \omega)$. We have,

$$w^{k-4}(\xi, \ln \varepsilon) = \sum_{j=0}^{3} \rho^{1-j} w_j^{k-4}(\theta, \ln \rho, \ln \varepsilon) + \tilde{w}^{k-4}(\xi, \ln \varepsilon)$$

where $\tilde{w}^{k-4}$ belongs to $V_\beta^\ell(\mathbb{R}^2 \backslash \omega)$. Then, $\Lambda \tilde{w}^{k-4}$ is convenient for the inner problem.

The other terms are not suitable for the inner problem but they can be written in x-coordinates as,

$$\rho^{1-j} w_j^{k-4}(\theta, \ln \rho, \ln \varepsilon) = \varepsilon^{j-1} r^{1-j} w_j^{k-4}(\theta, \ln r - \ln \varepsilon, \ln \varepsilon)$$

and for j integer between 0 and 3, r^{1-j} belongs to $V_\beta^\ell(\Omega)$ and can be used in suitable right hand sides of the outer problem.

So, we will write suitable right hand sides of outer and inner problems. For the modified boundary conditions, it will be sufficient to introduce the good terms and to shift r into ρ or conversely.

The rearrangement method will consist in doing this procedure at each step of the asymptotic construction.

Let us note that $w_j^{k-4}(\theta, \ln \rho, \ln \varepsilon)$ is a polynomial in $\ln \rho$ and is rational in $\ln \varepsilon$.

So, $w_j^{k-4}(\theta, \ln r - \ln \varepsilon, \ln \varepsilon)$ is also a polynomial in $\ln \rho$ and is rational in $\ln \varepsilon$.

2.3. Construction of the problems

According to Section 2.2, we will write equations for $v^k(x, \ln \varepsilon)$ and $w^k(\xi, \ln \varepsilon)$. We formally replace u by the expansion (2.1.1) in the left hand side of equation (2),

$$\Delta_x^2 u(\varepsilon, x) - \Delta u(\varepsilon, x) = \sum_{k=0}^{\infty} [\varepsilon^k(\Delta_x^2 v^k(x, \ln \varepsilon) - \Delta v^k(x, \ln \varepsilon)) + \varepsilon^{k-3} \Delta_\xi^2 w^k(\xi, \ln \varepsilon) - \Delta \varepsilon^{k+1} w^k(\xi, \ln \varepsilon)] .$$

Due to remark 2.2, we can write,

$$\Delta_x^2 u(\varepsilon, x) - \Delta u(\varepsilon, x) = \sum_{k=0}^{\infty} \left\{ \left[\varepsilon^k \left(\Delta_x^2 v^k(x, \ln \varepsilon) - \Delta v^k(x, \ln \varepsilon) - \Delta \sum_{j=0}^{3} \varepsilon^{j-1} r^{1-j} w_j^k(\theta, \ln \rho, \ln \varepsilon) \right) \right] + [\varepsilon^{k-3} \Delta_\xi^2 w^k(\xi, \ln \varepsilon) - \Delta \varepsilon^{k+1} \tilde{w}_3^k(\xi, \ln \varepsilon)] \right\} .$$

We will try to solve,

$$\Delta_x^2 v^k(x, \ln \varepsilon) - \Delta v^k(x, \ln \varepsilon) = \Delta \sum_{i=0}^{3} r^{1-i} w_i^{k-i}(\theta, \ln r - \ln \varepsilon, \ln \varepsilon) + f(x) \delta_{0k}$$

and,

$$\Delta_\xi^2 w^k(\xi, \ln \varepsilon) = \Delta \tilde{w}_3^{k-4}(\xi, \ln \varepsilon)$$

where all the functions with negative index vanish. Concerning the boundary conditions (3) and (4), we have,

$$u(\varepsilon, x) = \sum_{k=0}^{\infty} \varepsilon^k [v^k(x, \ln \varepsilon) + \varepsilon(\rho w_0^k(\theta, \ln r - \ln \varepsilon, \ln \varepsilon) + \cdots + \rho^{-k} w_{k+1}^k(\theta, \ln r - \ln \varepsilon, \ln \varepsilon) + \tilde{w}_{k+1}^k(\xi, \ln \varepsilon))]$$

$$= \sum_{k=0}^{\infty} \varepsilon^k [v^k(x, \ln \varepsilon) + r w_0^k(\theta, \ln r - \ln \varepsilon, \ln \varepsilon) + \cdots + \varepsilon^{k+1} r^{-k} w_{k+1}^k(\theta, \ln r - \ln \varepsilon, \ln \varepsilon) + \varepsilon \tilde{w}_{k+1}^k(\varepsilon^{-1} x, \ln \varepsilon)] .$$

Then, by identifying ε^k terms, we obtain on $\partial\Omega$,

$$\mathscr{B}(x, \nabla_x) v^k(x, \ln \varepsilon) = g(x) \delta_{0k} - \mathscr{B}(r w_0^k(\theta, \ln r - \ln \varepsilon, \ln \varepsilon) + \cdots + r^{-k+1} w_k^0(\theta, \ln r - \ln \varepsilon, \ln \varepsilon)) .$$

Similarly,

$$u(\varepsilon, x) = \sum_{k=0}^{\infty} \varepsilon^k [(r^2 v_1^k(\theta, \ln r, \ln \varepsilon) + \cdots + r^{L+1} v_L^k(\theta, \ln r, \ln \varepsilon) + \tilde{v}_L^k (x, \ln \varepsilon)) + \varepsilon w^k(\xi, \ln \varepsilon)]$$

$$= \sum_{k=0}^{\infty} \varepsilon^{k+1} [(\varepsilon \rho^2 v_1^k(\theta, \ln \rho + \ln \varepsilon, \ln \varepsilon) + \cdots + \varepsilon^L \rho^{L+1} v_L^k(\theta, \ln \rho + \ln \varepsilon, \ln \varepsilon) + \tilde{v}_L^k(\varepsilon\xi, \ln \varepsilon)) + w^k(\xi, \ln \varepsilon)]$$

and we obtain the boundary condition (5) on $\partial\omega$,

$$\mathscr{D}^{\omega}(\xi, \nabla_{\xi}) w^k(\xi, \ln \varepsilon) = h^{\omega}(\xi) \delta_{0k} - \mathscr{D}^{\omega}(\rho^2 v_1^{k-1}(\theta, \ln \rho + \ln \varepsilon, \ln \varepsilon) + \cdots + \rho^{k+1} v_k^0(\theta, \ln \rho + \ln \varepsilon, \ln \varepsilon)) .$$

Finally, we obtain the following sequence of problems:

Let k be an integer, θ et r (ou ρ) the polar coordinates. The function v^k is solution of the outer problem $P_e(k)$,

$$\Delta_x^2 v^k(x, \ln \varepsilon) - \Lambda v^k(x, \ln \varepsilon) = f(x) \delta_{k0} + \Lambda \sum_{\iota=0}^{3} r^{1-\iota} w_{\iota}^{k-\iota}(\theta, \ln r - \ln \varepsilon, \ln \varepsilon) \quad \text{in } \Omega$$

$$\mathscr{B} v^k(x, \ln \varepsilon) = g(x) \delta_{k0} - \sum_{\iota=0}^{k} \mathscr{B}(r^{1-\iota} w_{\iota}^{k-\iota}(\theta, \ln r - \ln \varepsilon, \ln \varepsilon)) \quad \text{on } \partial\Omega \tag{2.3.1}$$

where all the functions with negative index vanish.

Moreover, the function w^k is solution of the inner problem $P_i(k)$,

$$\Delta_{\xi}^2 w^k(\xi, \ln \varepsilon) = \Lambda \tilde{w}_3^{k-4}(\xi, \ln \varepsilon) \quad \text{in } \mathbb{R}^2 \backslash \omega$$

$$\mathscr{D}^{\omega} w^k(\xi, \ln \varepsilon) = h^{\omega}(\xi) \delta_{k0} - \sum_{\iota=1}^{k} \mathscr{D}^{\omega}(\rho^{1+\iota} v_{\iota}^{k-\iota}(\theta, \ln \rho + \ln \varepsilon, \ln \varepsilon)) \quad \text{on } \partial\omega . \tag{2.3.2}$$

2.4. Method for solving (2.3.1) and (2.3.2)

We will describe an iterative procedure for solving these problems.

Let $F^{2,k}(\xi, \ln \varepsilon)$ and $H^k(\xi, \ln \varepsilon)$ (resp. $F^{1,k}(x, \ln \varepsilon)$ and $(G^k(x, \ln \varepsilon))$ be the right hand sides of (2.3.2) (resp. (2.3.1)). Let us suppose that $v^0, v^1, \ldots, v^{K-1}$ and $w^0, w^1, \ldots, w^{K-1}$ are known and admit the decompositions (1.5.2) et (1.6.2).

For k equal to K, the functions $\tilde{w}_3^{K-4}$, $v_{\iota}^{K-\iota}$ are well known and the right hand sides $F^{2,K}$ and H^K are determined. Due to (12) and Theorem 3, we have,

$$w^K = W^K + c_1^K \zeta^1 + c_2^K \zeta^2 \tag{2.4.1}$$

where W^K belongs to $V_{\beta+1}^{\ell+4}(\mathbb{R}^2 \backslash \omega)$.

As the functions ζ^ι have the form (12), we have,

$$\rho w_0^K = \rho \, W_0^K + \Xi^K$$

where Ξ^K is the following biharmonical function,

$$\Xi^K(\xi, \ln \varepsilon) = \sum_{\iota = 1, 2} c_\iota^K(\Phi^\iota(\xi) - \gamma_{\iota 1} \xi_1 - \gamma_{\iota 2} \xi_2) . \tag{2.4.2}$$

It then appears that the right hand side of (2.3.1) can be written in the form,

$$F^{1, K} = F^{0, K} + \Lambda r w_0^K \quad \text{and} \quad G^K = G^{0, K} - \mathscr{B}(r w_0^K)$$

where $F^{0, K}$ and $G^{0, K}$ are completely defined and where w_0^K depends on the factors c_1^K and c_2^K.

To obtain these two numbers, we will write the compatibility equations (10) for the outer problem.

We have,

$$(F^{0, K}(x, \ln \varepsilon) + \Lambda r w_0^K(\theta, \ln r - \ln \varepsilon, \ln \varepsilon), \eta^J(x))_\Omega$$
$$+ (G^{0, K}(x, \ln \varepsilon) - \mathscr{B}(r w_0^K(\theta, \ln r - \ln \varepsilon, \ln \varepsilon)), \mathscr{T}\eta^J(x))_{\partial\Omega} = 0$$

that is,

$$(F^{0, K}(x, \ln \varepsilon) + \Lambda r W_0^K(\theta, \ln r - \ln \varepsilon, \ln \varepsilon), \eta^J(x))_\Omega$$
$$+ (G^{0, K}(x, \ln \varepsilon) - \mathscr{B}(r W_0^K(\theta, \ln r - \ln \varepsilon, \ln \varepsilon)), \mathscr{T}\eta^J(x))_{\partial\Omega}$$
$$= - (\varepsilon \Lambda \Xi^K(\varepsilon^{-1} x, \ln \varepsilon), \eta^J(x))_\Omega + (\varepsilon \mathscr{B}(\Xi^K(\varepsilon^{-1} x, \ln \varepsilon)), \mathscr{T}\eta^J(x))_{\partial\Omega} . \tag{2.4.3}$$

Let us introduce the disk $d(O, \delta)$ and its contour C_δ. We define the form q by,

$$q(u, v) = (\mathscr{N}(u), \mathscr{D}(v))_{C_\delta} - (\mathscr{D}(u), \mathscr{N}(v))_{C_\delta} \tag{2.4.4}$$

q is bilinear and antisymmetric and has the following properties (*cf.* [1] and [2]),

$$q(x \to x_p, x \to x_q) = 0$$
$$q(\Phi^p, \Phi^q) = 0 \tag{2.4.5}$$
$$q(\Phi^p, x \to x_q) = \delta_{pq} .$$

By writing the Green formula on $\Omega_\delta = \Omega \backslash d(O, \delta)$, the right hand side of (2.4.3) becomes,

$$(\varepsilon \Xi^K(\varepsilon^{-1} x, \ln \varepsilon), \Delta_x^2 \eta^J(x) - \Lambda \eta^J(x))_{\Omega_\delta} + (\varepsilon \mathscr{T}(\Xi^K(\varepsilon^{-1} x, \ln \varepsilon)), \mathscr{B}\eta^J(x))_{\partial\Omega}$$
$$- (\varepsilon \mathscr{N}(\Xi^K(\varepsilon^{-1} x, \ln \varepsilon)), \mathscr{D}\eta^J(x))_{C_\delta} + (\varepsilon \mathscr{D}(\Xi^K(\varepsilon^{-1} x, \ln \varepsilon)), \mathscr{N}\eta^J(x))_{C_\delta}$$

and then,

$$- (\varepsilon \mathscr{N}(\Xi^K(\varepsilon^{-1} x, \ln \varepsilon)), \mathscr{D}\eta^J(x))_{C_\delta} + (\varepsilon \mathscr{D}(\Xi^K(\varepsilon^{-1} x, \ln \varepsilon)), \mathscr{N}\eta^J(x))_{C_\delta} = q(x \to - \varepsilon \Xi^K(\varepsilon^{-1} x, \ln \varepsilon), \eta^J)$$

because η^j is a solution of the homogeneous outer problem. Replacing Ξ^K by its expression (2.4.2) and η^j by its asymptotic expansion (9), we obtain (*cf.* [2]),

$$q(x \to -\varepsilon \Xi^K(\varepsilon^{-1} x, \ln \varepsilon), \eta^j) = \sum_{i=1,2} c_i^K\Big(\Gamma_{ij} + \frac{1}{4\pi} \ln \varepsilon\, \delta_{ij} - \gamma_{ij}\Big)$$

$$= T(\varepsilon) \begin{bmatrix} c_1^K \\ c_2^K \end{bmatrix}$$

where the matrix $T(\varepsilon)$ is defined by (*cf.* [2]),

$$T(\varepsilon) = \Big[\Gamma_{pq} + \frac{1}{4\pi} \ln \varepsilon\, \delta_{pq} - \gamma_{pq}\Big]_{p,q=1,2}. \tag{2.4.6}$$

This matrix is inversible for sufficiently small ε. Consequently, we can choose the constants c_i^K as,

$$\begin{bmatrix} c_1^K \\ c_2^K \end{bmatrix} = T(\varepsilon)^{-1}[(F^{0,K}(x, \ln \varepsilon) + \Lambda r W_0^K(\theta, \ln r - \ln \varepsilon, \ln \varepsilon), \eta^j(x))_\Omega$$

$$+ (G^{0,K}(x, \ln \varepsilon) - \mathscr{B}(r W_0^K(\theta, \ln r - \ln \varepsilon, \ln \varepsilon)), \mathscr{T}\eta^j(x))_{\partial\Omega}]_{j=1,2} \tag{2.4.7}$$

so that the outer problem has a unique solution in $V_\beta^{\ell+4}(\Omega)$.

Moreover, if $F^{1,K}$ and G^K are rational in $\ln \varepsilon$, as W_0^K, the c_i^K are also rational in $\ln \varepsilon$.

As the degree of the determinant of $T(\varepsilon)$ is 2, the product by $T(\varepsilon)^{-1}$ in (2.4.6) will increase the denominator degree to 2.

In this way, we have formally built step by step, the representation (2.1.1) of the solution u where v^k and w^k appear as rational functions in $\ln \varepsilon$. Their denominator degrees are less than $2k+2$.

Now, we must justify the validity of this solution u.

3. JUSTIFICATION OF THE METHOD. ESTIMATION OF THE REMAINDER

3.1. Approximated solution

We can write an approximated solution of problem (2), (3), (4) by the truncation of the expansion (2.1.1). Let N be an integer and,

$$u_N(\varepsilon, x) = \sum_{k=0}^{N} \varepsilon^k\{v^k(x, \ln \varepsilon) + \varepsilon w^k(\varepsilon^{-1} x, \ln \varepsilon)\}. \tag{3.1.1}$$

The functions v^k and w^k belong respectively to $V_\beta^{\ell+4}(\Omega)$ and $V_\beta^{\ell+4}(\mathbb{R}^2 \backslash \omega)$, and then to $V_\beta^{\ell+4}(\Omega(\varepsilon))$.

We have to estimate,

$$\|(\Delta_x^2 - \Lambda)(u_N - u), \mathscr{B}(u_N - u), \mathscr{D}^\varepsilon(u_N - u)\,;\, R_\beta^\ell V(\Omega(\varepsilon))\| \tag{3.1.2}$$

where,

$$R_\beta^\ell V(\Omega(\varepsilon)) = V_\beta^\ell(\Omega(\varepsilon)) \times V_\beta^{\ell-\sigma_1+7/2}(\partial\Omega) \times V_\beta^{\ell-\sigma_2+7/2}(\partial\Omega) \times V_\beta^{\ell+7/2}(\partial\omega_\varepsilon) \times V_\beta^{\ell+5/2}(\partial\omega_\varepsilon) \tag{3.1.3}$$

and then we will use the results of [2] (*cf.* Theorem 13) where the estimate of the norm of the operator inverse of $\{\Delta_x^2 - \Lambda, \mathscr{B}, \mathscr{D}^{\varepsilon}\} : V_{\beta}^{\ell+4}(\Omega(\varepsilon)) \to R_{\beta}^{\ell} V(\Omega(\varepsilon))$, was obtained.

3.2. Estimates of the remainders

We propose to evaluate the discrepancy introduced by using the approximate solution (3.1.1).

1. *Boundary condition on* $\partial\Omega$

For the equation (3),

$$\mathscr{B}(x, \nabla_x)\, u(\varepsilon, x) = g(x) \quad \text{on } \partial\Omega$$

we have,

$$\mathscr{B}u_N(\varepsilon, x) = \sum_{k=0}^{N} \varepsilon^k \{\mathscr{B}v^k(x, \ln\varepsilon) + \varepsilon\mathscr{B}w^k(\varepsilon^{-1}x, \ln\varepsilon)\}$$

and, from (2.3.1),

$$\mathscr{B}u_N(\varepsilon, x) = g(x) + \sum_{k=0}^{N} \varepsilon^{k+1}\, \mathscr{B}w^k(\varepsilon^{-1}x, \ln\varepsilon) - \sum_{k=0}^{N} \varepsilon^k \sum_{\iota=0}^{k} \mathscr{B}(r^{1-\iota} w_\iota^{k-\iota}(\theta, \ln r - \ln\varepsilon, \ln\varepsilon))$$

$$= g(x) + \sum_{k=0}^{N} \varepsilon^{k+1}\left(\mathscr{B}\left(w^k(\varepsilon^{-1}x, \ln\varepsilon) - \sum_{\iota=0}^{N-k} \rho^{1-\iota} w_\iota^k(\theta, \ln r - \ln\varepsilon, \ln\varepsilon)\right)\right).$$

Then by (1.6.2), we have,

$$\mathscr{B}u_N(\varepsilon, x) = g(x) + \sum_{k=0}^{N} \varepsilon^{k+1}(\mathscr{B}\tilde{w}_{N-k}^k(\varepsilon^{-1}x, \ln\varepsilon)) \tag{3.2.1}$$

where $\xi \to \tilde{w}_{N-k}^k(\xi, \ln\varepsilon)$ belongs to $V_{\beta+N-k+1}^{\ell+4}(\mathbb{R}^2 \backslash \omega)$.

So we are going to estimate $\mathscr{B}u_N(\varepsilon, x) - g(x)$ in $V_\beta^{\ell-\sigma_1+7/2}(\partial\Omega) \times V_\beta^{\ell-\sigma_2+7/2}(\partial\Omega)$ or equivalently, in $H^{\ell-\sigma_1+7/2}(\partial\Omega) \times H^{\ell-\sigma_2+7/2}(\partial\Omega)$ where σ_1 and σ_2 are derivation orders in $\mathscr{B}$.

If z_Ω is the trace of z on $\partial\Omega$, we known (*cf.* [2], Lemma 1) that we have for any real γ,

$$\| z_\Omega ; H^{\ell-1/2}(\partial\Omega) \| \leqslant c \| z ; V_\gamma^\ell(\Omega(\varepsilon)) \| .$$

We obtain,

$$\| \mathscr{B}z_\Omega ; H^{\ell-\sigma_1+7/2}(\partial\Omega) \times H^{\ell-\sigma_2+7/2}(\partial\Omega) \| \leqslant c \| z ; V_\gamma^{\ell+4}(\Omega(\varepsilon)) \|$$

and then,

$$Q_1 = \| x \rightarrow \mathscr{B} u_N(\varepsilon, x) - g(x) \, ; H^{\ell - \sigma_1 + 7/2}(\partial\Omega) \times H^{\ell - \sigma_2 + 7/2}(\partial\Omega) \|$$

$$\leqslant c \sum_{k=0}^{N} \varepsilon^{k+1} \| x \rightarrow \tilde{w}^k_{N-k}(\varepsilon^{-1} x, \ln \varepsilon) \, ; V^{\ell+4}_{\gamma}(\Omega(\varepsilon)) \| \, .$$

Now,

$$\| x \rightarrow \tilde{w}^k_{N-k}(\varepsilon^{-1} x, \ln \varepsilon) \, ; V^{\ell+4}_{\gamma}(\Omega(\varepsilon)) \| = \varepsilon^{\gamma - (\ell+4)+1} \| \xi \rightarrow \tilde{w}^k_{N-k}(\xi, \ln \varepsilon) \, ; V^{\ell+4}_{\gamma}(\{\xi/\varepsilon\xi \in \Omega(\varepsilon)\}) \|$$

$\tilde{w}^k_{N-k}$ belonging to $V^{\ell+4}_{\beta+N-k+1}(\mathbb{R}^2 \backslash \omega)$, by choosing $\gamma = \beta + N - k + 1$, we have,

$$Q_1 \leqslant c \sum_{k=0}^{N} \varepsilon^{k+1} \varepsilon^{\beta+N-k+1-\ell-3} \varepsilon^{-\delta}$$

where δ is an arbitrary positive number and where ε^{δ} follows from the rational dependence on $\ln \varepsilon$.

Finally,

$$Q_1 \leqslant c\varepsilon^{\beta-\ell-1} \varepsilon^{N-\delta} \, . \tag{3.2.2}$$

2. *Boundary condition on* $\partial\omega_\varepsilon$

In fast coordinates, we have the Dirichlet condition (5),

$$\mathscr{D}^{\omega}(\xi, \nabla_{\xi}) \, u(\varepsilon, \varepsilon\xi) = h^{\omega}(\xi) \quad \text{on } \partial\omega \, .$$

We have,

$$\mathscr{D}^{\omega} u_N(\varepsilon, \varepsilon\xi) = \sum_{k=0}^{N} \varepsilon^k \{ \mathscr{D}^{\omega} v^k(\varepsilon\xi, \ln \varepsilon) + \varepsilon \mathscr{D}^{\omega} w^k(\xi, \ln \varepsilon) \}$$

and due to (2.3.2),

$$\mathscr{D}^{\omega} u_N(\varepsilon, \varepsilon\xi) = h^{\omega}(\varepsilon, \xi) + \sum_{k=0}^{N} \varepsilon^k \mathscr{D}^{\omega} v^k(\varepsilon\xi, \ln \varepsilon) - \varepsilon \sum_{k=1}^{N} \left(\sum_{\iota=1}^{k} \mathscr{D}^{\omega}(\rho^{1+\iota} v^{k-\iota}_{\iota}(\theta, \ln \rho + \ln \varepsilon, \ln \varepsilon)) \right)$$

and as in (3.2.1), we obtain,

$$\mathscr{D}^{\omega} u_N(\varepsilon, \varepsilon\xi) = h^{\omega}(\xi) + \sum_{k=0}^{N} \varepsilon^k (\mathscr{D}^{\omega} \tilde{v}^{\,k}_{N-k}(\varepsilon\xi, \ln \varepsilon)) \tag{3.2.3}$$

where $x \to \tilde{v}^{\,k}_{N-k}(x, \ln \varepsilon)$ (*cf.* (1.5.2)), belongs to $V^{\ell+4}_{\beta-N+k}(\Omega)$, and where we have set the convention,

$$\tilde{v}^{\,N}_{0} = v^N .$$

We are going to estimate the norm Q_2 of $\mathscr{D}^{\omega} u_N(\varepsilon, \varepsilon\xi) - h^{\omega}(\xi) - h^{\omega}(\xi)$ in $V^{\ell+7/2}_{\beta}(\partial\omega) \times V^{\ell+5/2}_{\beta}(\partial\omega)$.

Now, if z_{ω} is the trace of z on $\partial\omega$, we have for any γ (*cf.* [2], Lemma 1),

$$\| z_{\omega} ; H^{\ell - 1/2}(\partial\omega) \| \leqslant c\varepsilon^{\ell - 1 - \gamma} \| z ; V^{\ell}_{\gamma}(\Omega(\varepsilon)) \|$$

then,

$$\| \mathscr{D}^{\omega} z_{\omega} ; H^{\ell + 7/2}(\partial\omega) \times H^{\ell + 5/2}(\partial\omega) \| \leqslant c\varepsilon^{\ell + 2 - \gamma} \| z ; V^{\ell+4}_{\gamma}(\Omega(\varepsilon)) \| .$$

So we have,

$$Q_2 \leqslant c \sum_{k=0}^{N} \varepsilon^{k + \ell + 2 - \gamma} \| x \to \tilde{v}^{\,k}_{N-k}(x, \ln \varepsilon) ; V^{\ell+4}_{\gamma}(\Omega(\varepsilon)) \|$$

with

$$\| x \to \tilde{v}^{\,k}_{N-k}(x, \ln \varepsilon) ; V^{\ell+4}_{\gamma}(\Omega(\varepsilon)) \| \leqslant \| x \to \tilde{v}^{\,k}_{N-k}(x, \ln \varepsilon) ; V^{\ell+4}_{\gamma}(\Omega) \| .$$

As $\tilde{v}^{\,k}_{N-k}$ belongs to $V^{\ell+4}_{\beta-N+k}(\Omega)$, we choose $\gamma = \beta - N + k$ and we obtain,

$$Q_2 \leqslant c \sum_{k=0}^{N} \varepsilon^{k + \ell + 2 - (\beta - N + k)} \varepsilon^{-\delta}$$

and therefore,

$$Q_2 \leqslant c\varepsilon^{-\beta + \ell + 2} \varepsilon^{N - \delta} . \tag{3.2.4}$$

3. *Estimate of the remainder in the equation (2)*

This equation is,

$$\Delta^2_x u(\varepsilon, x) - \Lambda u(\varepsilon, x) = f(x) \quad \text{in } \Omega(\varepsilon) .$$

By using the approximated solution u_N, we have,

$$(\Delta_x^2 - \Lambda)\, u_N(\varepsilon, x) = \sum_{k=0}^{N} \varepsilon^k (\Delta_x^2 - \Lambda)\,(v^k(x, \ln \varepsilon) + \varepsilon w^k(\varepsilon^{-1} x, \ln \varepsilon))$$

$$= f(x) + \sum_{k=0}^{N} \Lambda \varepsilon^k \left[\sum_{\iota=1}^{3} r^{1-\iota} w_\iota^{k-\iota}(\theta, \ln r - \ln \varepsilon, \ln \varepsilon) + \varepsilon^{-3} \tilde{w}_3^{k-4}(\xi, \ln \varepsilon) - \varepsilon w^k(\varepsilon^{-1} x, \ln \varepsilon) \right]$$

$$= f(x) + \Lambda \sum_{k=0}^{N} \left(\sum_{\iota=1}^{3} \varepsilon^{k+1-\iota} \rho^{1-\iota} w_\iota^{k-\iota}(\theta, \ln \rho, \ln \varepsilon) \right)$$

$$+ \Lambda \left(\sum_{s=0}^{N-4} \varepsilon^{s+1} (\tilde{w}_3^s(\xi, \ln \varepsilon) - w^s(\xi, \ln \varepsilon)) - \sum_{s=N-3}^{N} \varepsilon^{s+1} w^s(\xi, \ln \varepsilon) \right)$$

$$= f(x) + \Lambda \sum_{s=0}^{N-4} \varepsilon^{s+1} \left(\sum_{\iota=1}^{3} \rho^{1-\iota} w_\iota^s(\theta, \ln \rho, \ln \varepsilon) + \tilde{w}_3^s(\xi, \ln \varepsilon) - w^s(\xi, \ln \varepsilon) \right)$$

$$+ \Lambda \sum_{s=N-3}^{N} \varepsilon^{s+1} \left(\sum_{\iota=1}^{N-s} \rho^{1-\iota} w_\iota^s(\theta, \ln \rho, \ln \varepsilon) - w^s(\xi, \ln \varepsilon) \right)$$

by shifting the sums. From (1.6.2),

$$(\Delta_x^2 - \Lambda)\, u_N(\varepsilon, x) = f(x) - \Lambda \sum_{s=N-3}^{N} \varepsilon^{s+1} \tilde{w}_{N-s}^s(\xi, \ln \varepsilon)$$

and we propose to evaluate,

$$Q_3 = \| (\Delta_x^2 - \Lambda)\, u_N - f \, ; V_\beta^\ell(\Omega(\varepsilon)) \|$$

that is,

$$Q_3 = \left\| x \to \Lambda \sum_{s=N-3}^{N} \varepsilon^{s+1} \tilde{w}_{N-s}^s(\varepsilon^{-1} x, \ln \varepsilon) \, ; V_\beta^\ell(\Omega(\varepsilon)) \right\|$$

$$\leq c \sum_{s=N-3}^{N} \varepsilon^{s+1} \varepsilon^{\beta - \ell + 1} \| \xi \to \tilde{w}_{N-s}^s(\xi, \ln \varepsilon) \, ; V_{\beta+4}^{\ell+4}(\{\xi : \varepsilon\xi \in \Omega(\varepsilon)\}) \|$$

by using the imbedding of $V^{\ell+4}_{\beta+4}$ into V^{ℓ}_{β}. We have,

$$Q_3 \leqslant c \sum_{s=N-3}^{N} \varepsilon^{s+2+\beta-\ell} \| \xi \to \tilde{w}^s_{N-s}(\xi, \ln \varepsilon) \, ; V^{\ell+4}_{\beta+N-s+1}(\{\xi : \varepsilon\xi \in \Omega(\varepsilon)\}) \|$$
$$\leqslant c\varepsilon^{N-1+\beta-\ell} \| \xi \to \tilde{w}^s_{N-s}(\xi, \ln \varepsilon) \, ; V^{\ell+4}_{\beta}(\mathbb{R}^2 \backslash \omega) \|$$
$$\leqslant c\varepsilon^{\beta-(\ell+1)} \varepsilon^{N-\delta} . \tag{3.2.5}$$

Finally, due to (3.2.2), (3.2.4) and (3.2.5), we have obtained an over-estimation of the norm (3.1.2),

$$\| (\Delta^2_x - \Lambda)(u_N - u), \mathscr{B}(u_N - u), \mathscr{D}^{\varepsilon}(u_N - u) \, ; R^{\ell}_{\beta} V(\Omega(\varepsilon)) \| \leqslant c(\varepsilon^{\beta-(\ell+1)} + \varepsilon^{(\ell+2)-\beta}) \, \varepsilon^{N-\delta} . \tag{3.2.6}$$

Let us recall that if U is the solution of problem (2), (3), (4), with right hand sides F, G and H, then we have the estimate (*cf.* [2], Theorem 13),

$$\| U \, ; V^{\ell+4}_{\beta}(\Omega(\varepsilon)) \| \leqslant c\varepsilon^{-|\beta-\ell-2|} \, . \, |\ln \varepsilon|^{-1} \| \{F, G, H\} \, ; R^{\ell}_{\beta} V(\Omega(\varepsilon)) \|$$

so we obtain,

$$\| u_N - u \, ; V^{\ell+4}_{\beta}(\Omega(\varepsilon)) \| \leqslant c(\varepsilon^{2\beta-2\ell-3} + 1) \, . \, |\ln \varepsilon|^{-1} \varepsilon^{N-\delta}$$

and finally, for any positive δ, we have the estimate,

$$\| u_N - u \, ; V^{\ell+4}_{\beta}(\Omega(\varepsilon)) \| \leqslant c(\varepsilon^{2\beta-2\ell-3} + 1) \, \varepsilon^{N-2\delta} . \tag{3.2.7}$$

As β belongs to $]\ell+1, \ell+2[$, we can write a rougher over-estimation,

$$\| u_N - u \, ; V^{\ell+4}_{\beta}(\Omega(\varepsilon)) \| \leqslant c\varepsilon^{N-1} . \tag{3.2.8}$$

4. APPLICATION TO THE IMPEDANCE MATRIX

4.1. Impedance matrix

We are in the framework of the introduction. $\Omega(\varepsilon)$ is a plate, the boundary $\partial\Omega$ is free and ω_ε has a rigid motion. So we have the following problem (2), (3), (4),

$$\begin{aligned} \Delta^2_x u(\varepsilon, x) - \Lambda u(\varepsilon, x) &= 0 && \text{in } \Omega(\varepsilon) \\ \mathscr{N}(x, \nabla_x) \, u(\varepsilon, x) &= 0 && \text{on } \partial\Omega \\ \mathscr{D}^{\varepsilon}(x, \nabla_x) \, u(\varepsilon, x) &= (h_1(\varepsilon, x), h_2(\varepsilon, x)) && \text{on } \partial\omega_\varepsilon \end{aligned} \tag{4.1.1}$$

with,

$$\begin{aligned} h_1(\varepsilon, x) &= U_0 - \Theta_2 x_1 + \Theta_1 x_2 \\ h_2(\varepsilon, x) &= \partial_n(-\Theta_2 x_1 + \Theta_1 x_2) \end{aligned} \tag{4.1.2}$$

where U_0 denotes a translation perpendicular to the plate and Θ_1 and Θ_2, the rotations around Ox_1 and Ox_2 If U_0 is non zero, the Dirichlet condition on $\partial\omega$ is not of the same kind as that in (5) We will have to adapt it To have more clear notations, we set,

$$\alpha_0 = U_0, \quad \alpha_1 = -\Theta_2 \quad \text{and} \quad \alpha_2 = \Theta_1$$

and we denote by u^α the solution of the problem We calculate the reduction elements of the stresses applied by $\Omega(\varepsilon)$ on ω_ε,

$$R_0 = -(\mathcal{N}^\varepsilon u^\alpha, \mathcal{D}^\varepsilon 1)_{\partial\omega_\varepsilon} = q(x \to 1, u^\alpha) = \sum_{k=0}^{2} J_{0k}(\varepsilon)\,\alpha_k$$

$$-M_2 = -(\mathcal{N}^\varepsilon u^\alpha, \mathcal{D}^\varepsilon x_1)_{\partial\omega_\varepsilon} = q(x \to x_1, u^\alpha) = \sum_{k=0}^{2} J_{1k}(\varepsilon)\,\alpha_k \tag{4 1 3}$$

$$M_1 = -(\mathcal{N}^\varepsilon u^\alpha, \mathcal{D}^\varepsilon x_2)_{\partial\omega_\varepsilon} = q(x \to x_2, u^\alpha) = \sum_{k=0}^{2} J_{2k}(\varepsilon)\,\alpha_k$$

where q is the bilinear form (2 4 4) The terms $J_{ik}(\varepsilon)$ are impedance terms (in an other order (*cf* [1] and [2]) and we propose to apply the previous results to study their behaviours when ε goes to zero

4.2. Asymptotic expansion of the impedance terms

We propose to write an approximation of the impedance terms by using a truncation of the solution in (4 1 3) We set,

$$u^\alpha(\varepsilon, x) = \sum_{k=0}^{N} \varepsilon^k\{v^k(x, \ln\varepsilon) + \varepsilon w^k(\varepsilon^{-1}x, \ln\varepsilon)\} + \tilde{u}^\alpha(\varepsilon, x) \tag{4 2 1}$$

we then have

PROPOSITION 5 *Let N be an integer Then the impedance terms defined by (4 1 3) admit asymptotic expansions in the following forms,*

$$J_{pq}(\varepsilon) = \sum_{k=0}^{N} \varepsilon^k J^k_{pq}(\ln\varepsilon) + O(\varepsilon^{N+\kappa}) \quad \text{for} \quad p = 1, 2 \tag{4 2 2}$$

$$J_{0q}(\varepsilon) = \sum_{k=0}^{N-1} \varepsilon^k J^k_{0q}(\ln\varepsilon) + O(\varepsilon^{N-1+\kappa}) \tag{4 2 3}$$

where κ is arbitrary in $]0, \frac{1}{2}[$, and where the $J^k_{pq}(\ln\varepsilon)$ $(p = 0, 1, 2)$ are rational fractions in $\ln\varepsilon$

Proof: 1. We suppose that α_0 is equal to zero.

1.1. Let us prove (4.2.2).

We replace the solution u^α by (4.2.1) in $q(x \to x_\iota, u^\alpha)$. We have the estimate,

$$|q(x \to x_\iota, \tilde{u}^\alpha)| = |(\mathscr{D}^\varepsilon x_\iota, \mathscr{N} \tilde{u}^\alpha(\varepsilon, x))_{\partial\omega_\varepsilon}|$$

$$\leqslant \|x \to (0, 0, \mathscr{D}^\varepsilon x_\iota) ; R^\ell_{2\ell - \beta + 4} V(\Omega(\varepsilon))\| \cdot \|\tilde{u}^\alpha ; V^{\ell + 4}_\beta(\Omega(\varepsilon))\|$$

(*cf.* [2], § 7), and we have also (*cf.* [2], Lemma 1),

$$\|(0, 0, \mathscr{D}^\varepsilon x_\iota) ; R^\ell_{2\ell - \beta + 4} V(\Omega(\varepsilon))\| \leqslant \|x_\iota ; V^{\ell + 4}_{2\ell - \beta + 4}(0 \leqslant |x| \leqslant c\varepsilon)\| \leqslant c\varepsilon^{\ell - \beta + 2} .$$

So, due to (3.2.7), it remains

$$|q(x \to x_\iota, \tilde{u}^\alpha)| \leqslant c(\varepsilon^{\ell - \beta + 2} + \varepsilon^{-\ell - 1 + \beta})\, \varepsilon^{N - 2\delta} \tag{4.2.4}$$

where δ is an arbitrary positive number. As β is arbitrary in $]\ell + 1, \ell + 2[$, we have,

$$|q(x \to x_\iota, \tilde{u}^\alpha)| \leqslant c\varepsilon^{N + \kappa} \tag{4.2.5}$$

where κ is arbitrary in $]0, \frac{1}{2}[$.

Moreover, the quantity,

$$q(x \to x_\iota, x \to \varepsilon w^k(\varepsilon^{-1} x, \ln \varepsilon))$$

is rational in $\ln \varepsilon$: Indeed, this term can be written as,

$$-(x \to \mathscr{D}^\varepsilon x_\iota, x \to \mathscr{N}^\varepsilon \varepsilon w^k(\varepsilon^{-1} x, \ln \varepsilon))_{\partial\omega_\varepsilon}$$

and in fast coordinates,

$$-(\xi \to \mathscr{D}^\varepsilon \xi_\iota, \xi \to \mathscr{N}^\omega w^k(\xi, \ln \varepsilon))_{\partial\omega}$$

and we know that $w^k(\xi, \ln \varepsilon)$ is rational in $\ln \varepsilon$ (*cf.* § 2.4).

Now we have to estimate,

$$q(x \to x_\iota, x \to v^k(x, \ln \varepsilon)) .$$

From (1.5.2), we can write,

$$v^k(x, \ln \varepsilon) = \sum_{j=1}^{S} r^{j + 1} v^k_j(\theta, \ln r, \ln \varepsilon) + \tilde{v}^k_S(x, \ln \varepsilon)$$

where v^k_j and $\tilde{v}^k_S$, belonging to $V^{\ell + 4}_{\beta - S}(\Omega)$, are rational in $\ln \varepsilon$. We have,

$$\begin{aligned} &q(x \to x_\iota, x \to r^{j + 1} v^k_j(\theta, \ln r, \ln \varepsilon)) \\ &= -\varepsilon^j(\xi \to \mathscr{D}^\omega \xi_\iota, \xi \to \mathscr{N}^\omega \rho^{j + 1} v^k_j(\theta, \ln \rho - \ln \varepsilon, \ln \varepsilon))_{\partial\omega} \\ &= \varepsilon^j Q^k_j(\ln \varepsilon) \end{aligned}$$

where Q^k_j is rational in $\ln \varepsilon$.

Finally, as for (4.2.4), noting $B(C\varepsilon)$ the ball $B(O, \varepsilon)$, we have

$$|q(x \to x_i, x \to \tilde{v}_S^k(x, \ln \varepsilon))|$$

$$\leq \|x \to (0, 0, \mathscr{D}^\varepsilon x_i) ; R_{2\ell-\beta+4}^{\ell} V(\Omega(\varepsilon))\| \cdot \|\tilde{v}_S^k ; V_\beta^{\ell+4}(B(C\varepsilon)\backslash\omega_\varepsilon)\|$$

$$\leq c\varepsilon^{\ell+2-\beta} \varepsilon^S \|\tilde{v}_S^k ; V_{\beta-S}^{\ell+4}(B(C\varepsilon)\backslash\omega_\varepsilon)\|$$

because r is equivalent to ε in $B(C\varepsilon)\backslash\omega_\varepsilon$.

Finally,

$$|q(x \to x_i, x \to \tilde{v}_S^k(x, \ln \varepsilon))| \leq c\varepsilon^{\ell+2-\beta} \varepsilon^S \tag{4.2.6}$$

and we have obtained,

$$q(x \to x_i, x \to v^k(x, \ln \varepsilon)) = \sum_{j=1}^{S} \varepsilon^j Q_j^k(\ln \varepsilon) + o(\varepsilon^S) . \tag{4.2.7}$$

So we have an asymptotic expansion of the impedance terms in the form (4.2.2).

1.2. The proof of (4.2.3) is similar. We have to estimate,

$$q(x \to 1, u^\alpha) .$$

The only difference comes from,

$$\|(0, 0, \mathscr{D}^\varepsilon 1) ; R_{2\ell-\beta+4}^{\ell} V(\Omega(\varepsilon))\| \leq \|x_i ; V_{2\ell-\beta+4}^{\ell+4}(0 \leq |x| \leq c\varepsilon)\| \leq c\varepsilon^{\ell-\beta+1}$$

we obtain,

$$|q(x \to 1, \tilde{u}^\alpha)| \leq c\varepsilon^{N-1+\kappa} \tag{4.2.8}$$

where κ is arbitrary in $]0, \frac{1}{2}[$, and,

$$|q(x \to 1, x \to \tilde{v}_S^k(x, \ln \varepsilon))| \leq c\varepsilon^{\ell+1-\beta} \varepsilon^S . \tag{4.2.9}$$

We have only the order $N-1$ for a truncation of the solution at order N. So the proof of (4.2.3) and of proposition 5 is finished in the case α_0 equal to zero.

2. If α_0 is not zero, we study (4.1.1) with the right hand side $h_1(\varepsilon, x) = \varepsilon\alpha_0$, $h_2 = 0$.

By linearity, we obtain the results with expansions at order $N+1$. We will see a concrete example in 4.4. ■

4.3. Leading part of the impedance terms. Case of a rigid rotation of ω_ε ($\alpha_0 = 0$)

We propose to use the previous results to find the leading part of impedance terms. We already know these approximates by using the matching methods (*cf.* [1] et [2]).

We first suppose that the rigid motion of ω_ε is a rotation ($\alpha_0 = 0$). We then have,

$$h_1^\omega(\xi) = (\alpha_1 \xi_1 + \alpha_2 \xi_2) \quad \text{and} \quad h_2^\omega(\xi) = \partial_n(\alpha_1 \xi_1 + \alpha_2 \xi_2)$$

in (5). We write the expansion (2.1.1) of the solution u^{α},

$$u^{\alpha}(\varepsilon, x) = \sum_{k=0}^{\infty} \varepsilon^{k}\{v^{k}(x, \ln \varepsilon) + \varepsilon w^{k}(\varepsilon^{-1} x, \ln \varepsilon)\} \tag{4.3.1}$$

where v^k and w^k are the solutions of (2.3.1) and (2.3.2). Then we calculate the impedance matrix by using a truncation at order 0 or 1.

Let us look at the first terms of (4.3.1). w^0 is solution of $P_i(0)$ (*cf.* (2.3.2)),

$$\Delta_{\xi}^{2} w^{0}(\xi, \ln \varepsilon) = 0 \quad \text{in } \mathbb{R}^{2} \backslash \omega$$

$$w^{0}(\xi, \ln \varepsilon) = (\alpha_1 \xi_1 + \alpha_2 \xi_2) \quad \text{on } \partial\omega$$

and we will have all the solutions by adding an arbitrary linear combination of the functions ζ^{j} (*cf.* (12)),

$$w^{0} = W^{0} + c_1^{0} \zeta^{1} + c_2^{0} \zeta^{2}$$

with,

$$W^{0}(\xi) = \alpha_1 \xi_1 + \alpha_2 \xi_2 .$$

We deduce the first term of (1.6.2) associated to w^0,

$$w_0^{0}(\theta, \ln \rho) = \alpha_1 \cos \theta + \alpha_2 \sin \theta + \sum_{j=1,2} c_j^{0}(\rho^{-1} \Phi^{j}(\xi) - \gamma_{j1} \cos \theta - \gamma_{j2} \sin \theta) .$$

Then, we can write the outer problem $P_e(0)$ whose solution is v^0 (*cf.* (2.3.1)),

$$\Delta_x^{2} v^{0}(x, \ln \varepsilon) - \Lambda v^{0}(x, \ln \varepsilon) = \Lambda(\alpha_1 x_1 + \alpha_2 x_2) + \sum_{j=1,2} c_j^{0}(\varepsilon \Phi^{j}(\varepsilon^{-1} x) - \gamma_{j1} x_1 - \gamma_{j2} x_2) \quad \text{in } \Omega$$

$$\mathcal{N} v^{0}(x, \ln \varepsilon) = - \mathcal{N}\left(\sum_{j=1,2} c_j^{0}(\varepsilon \Phi^{j}(\varepsilon^{-1} x))\right) \quad \text{on } \partial\Omega$$

because $\mathcal{N}(x_i)$ is equal to zero.

We have a unique solution provided the right hand side satisfies the compatibility equations (10). We obtain the constants c_j^0 by (2.4.7), that is here,

$$\begin{bmatrix} c_1^{0} \\ c_2^{0} \end{bmatrix} = T(\varepsilon)^{-1}[(\Lambda(\alpha_1 x_1 + \alpha_2 x_2), \eta^{j}(x))_{\Omega}]_{j=1,2} .$$

From the properties of the functions η^j (*cf.* [2], Lemma 9), it remains,

$$\begin{bmatrix} c_1^{0} \\ c_2^{0} \end{bmatrix} = - T(\varepsilon)^{-1} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} . \tag{4.3.2}$$

The unique solution v^0 of $P_\varepsilon(0)$, belonging to $V_\beta^{\ell+4}(\Omega)$, is,

$$v^0(x, \ln \varepsilon) = -(\alpha_1 x_1 + \alpha_2 x_2) - \sum_{j=1,2} c_j^0(\varepsilon \Phi^j(\varepsilon^{-1} x) - \gamma_{j1} x_1 - \gamma_{j2} x_2 + \eta^j(x))$$

$$= \sum_{j=1,2} c_j^0 \tilde{\eta}^j(x)$$

by (9) and (4.3.1).

We may obtain an approximate of u^α by keeping only one term of (4.3.1), i.e.

$$u_0(\varepsilon, x) = v^0(x, \ln \varepsilon) + \varepsilon w^0(\varepsilon^{-1} x, \ln \varepsilon)$$

$$= \sum_{j=1,2} c_j^0(\eta^j(x) + \varepsilon \tilde{\zeta}^j(\varepsilon^{-1} x)) \tag{4.3.3}$$

where (12) was used.

We will find an approximated value of the impedance terms by replacing (4.3.3) in (4.1.3). By using the properties of the form q (*cf.* (2.4.4) and [2]) and (9), we write,

$$q(x \to x_i, u_0) = q\left(x \to x_i, \sum_{j=1,2} c_j^0 \Phi^j\right) = -\delta_{ij} c_j^0$$

and then by (4.3.2), we have the equivalent of the impedance matrix $[J_{pq}]$ $(p, q = 1, 2)$, equal to the inverse matrix of $T(\varepsilon)$. Finally,

$$J_{pq} - T_{pq}^{-1}(\varepsilon) + O(\varepsilon^\kappa). \tag{4.3.4}$$

This result is the same as in [1] and [2], that we have obtained by matching expansion methods. The leading term, $T_{pq}^{-1}(\varepsilon)$, is a rational fraction in $\ln \varepsilon$.

To obtain the leading parts of J_{01} and J_{02}, we have to calculate,

$$q(x \to 1, r^2 v_1^0 + \varepsilon w^0).$$

The $\tilde{\eta}^j$ functions have the representation (*cf.* [1]),

$$\tilde{\eta}^j(x) = \Gamma_{j0} \Phi(x) + \Gamma_{j3} \frac{x_1^2}{\sqrt{2}} + \Gamma_{j4} x_1 x_2 + \Gamma_{j5} \frac{x_2^2}{\sqrt{2}} + \tilde{\tilde{\eta}}^j(x) \tag{4.3.5}$$

where $\tilde{\tilde{\eta}}^j$ belongs to $V_{\beta-1}^{\ell+4}(\Omega)$.

By using the classical properties of the q form (*cf.* [1]), we obtain,

$$q(x \to 1, r^2 v_1^0 + \varepsilon w^0) = q\left(x \to 1, \sum_{j=1,2} c_j^0 \Gamma_{j0} \Phi(x)\right)$$

$$= [\Gamma_{10} \Gamma_{20}] \, T(\varepsilon)^{-1} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

and it follows that,

$$J_{0q} = \Gamma_{10} T_{1q}^{-1}(\varepsilon) + \Gamma_{20} T_{2q}^{-1}(\varepsilon) + O(\varepsilon^{\kappa}) . \tag{4.3.6}$$

This equivalent representation is of course the same rational fraction in $\ln \varepsilon$ as in the previous studies (*cf.* [1] et [2]).

4.4. Leading parts of impedance terms. Case of a rigid translation of ω_ε ($\alpha_1 = \alpha_2 = 0$)

Let the rigid motion of ω_ε be a translation defined by α_0. We have,

$$h_1^{\omega}(\xi) = \alpha_0 \quad \text{and} \quad h_2^{\omega}(\xi) = 0 .$$

And we have to study εu^{α}. We seek a solution of the form,

$$\varepsilon u^{\alpha}(\varepsilon, x) = \sum_{k=0}^{\infty} \varepsilon^k \{ v^k(x, \ln \varepsilon) + \varepsilon w^k(\varepsilon^{-1} x, \ln \varepsilon) \} \tag{4.4.1}$$

Then w^0 is solution of $P_i(0)$ (*cf.* (2.3.2)),

$$\Delta_\xi^2 w^0(\xi, \ln \varepsilon) = 0 \quad \text{in } \mathbb{R}^2 \backslash \omega$$

$$w^0(\xi, \ln \varepsilon) = \alpha_0 \quad \text{on } \partial\omega$$

so we have,

$$W^0(\xi, \ln \varepsilon) = \alpha_0 .$$

The first term W_0^0 of (1.6.2) associated to W^0 is equal to zero. Due to (2.4.5), the constants c_j^0 vanish and we obtain,

$$w^0(\xi, \ln \varepsilon) = w_1^0(\xi, \ln \varepsilon) = \alpha_0 .$$

The outer problem $P_e(0)$ is homogeneous and the only suitable solution is zero. The inner problem $P_i(1)$ (*cf.* (2.3.2)) is also homogeneous and the solution is,

$$w^1 = c_1^1 \zeta^1 + c_2^1 \zeta^2 .$$

The function W^1 associated to w^1 in (2.4.1) is then equal to zero. We can write $P_e(1)$ which has the form,

$$\Delta_x^2 v^1(x, \ln \varepsilon) - \Lambda v^1(x, \ln \varepsilon) = \Lambda \alpha_0 + \Lambda \sum_{j=1,2} c_j^1 (\varepsilon \Phi^j(\varepsilon^{-1} x) - \gamma_{j1} x_1 - \gamma_{j2} x_2) \quad \text{in } \Omega$$

$$\mathcal{N} v^1(x, \ln \varepsilon) = - \mathcal{N}\left(\sum_{j=1,2} c_j^1 (\varepsilon \Phi^j(\varepsilon^{-1} x)) \right) \quad \text{on } \partial\Omega .$$

The compatibility conditions show that,

$$\begin{bmatrix} c_1^1 \\ c_2^1 \end{bmatrix} = T(\varepsilon)^{-1}[q(x \to \alpha_0, \eta^j)]$$

$$= -\alpha_0\, T(\varepsilon)^{-1} \begin{bmatrix} \Gamma_{10} \\ \Gamma_{20} \end{bmatrix}$$

where the constants Γ_{j0} are defined in (4.3.5).

Let η^0 the solution in $H^2(\Omega)$ of the problem,

$$\Delta_x^2 \eta^0 - \Delta\eta^0 = 1 \quad \text{in } \Omega$$

$$\mathcal{N}\eta^0 = 0 \quad \text{on } \partial\Omega$$

so that η^0 has the representation (*cf.* [1] and [2]),

$$\eta^0(x) = \Gamma_{10}\, x_1 + \Gamma_{20}\, x_2 + \Gamma_{00}\, \Phi(x) + \Gamma_{03} \frac{x_1^2}{\sqrt{2}} + \Gamma_{04}\, x_1\, x_2 + \Gamma_{05} \frac{x_2^2}{\sqrt{2}} + \tilde{\tilde{\eta}}^0(x) \tag{4.4.2}$$

where $\tilde{\tilde{\eta}}^0$ belongs to $V_{\beta-1}^{\ell+4}(\Omega)$.

We obtain a unique solution of $P_\varepsilon(1)$ in $V_\beta^{\ell+4}(\Omega)$,

$$v^1(x, \ln\varepsilon) = \alpha_0\left(\Gamma_{00}\, \Phi(x) + \Gamma_{03} \frac{x_1^2}{\sqrt{2}} + \Gamma_{04}\, x_1\, x_2 + \Gamma_{05} \frac{x_2^2}{\sqrt{2}} + \tilde{\tilde{\eta}}^0(x)\right)$$
$$+ \sum_{j=1,2} c_j^1(\eta^j(x) - \Gamma_{j1}\, x_1 - \Gamma_{j2}\, x_2)$$

Then we calculate the approximates of the impedance terms by (4.3.5),

$$q(x \to x_i, u_1) = \varepsilon^{-1} q(x \to x_i\,, r^2 v_1^0 + \varepsilon w^0 + \varepsilon r^2 v_1^1 + \varepsilon^2) + O(\varepsilon^\kappa)$$
$$= \varepsilon^{-1} q\left(x \to x_i\,, \varepsilon\alpha_0\left(1 + \Gamma_{00}\, \Phi(x) + \Gamma_{03} \frac{x_1^2}{\sqrt{2}} + \Gamma_{04}\, x_1\, x_2 + \Gamma_{05} \frac{x_2^2}{\sqrt{2}}\right)\right)$$
$$+ \sum_{j=1,2} c_j^1\left(\Gamma_{j0}\, \Phi(x) + \Gamma_{j3} \frac{x_1^2}{\sqrt{2}} + \Gamma_{j4}\, x_1\, x_2 + \Gamma_{j5} \frac{x_2^2}{\sqrt{2}} + \varepsilon\zeta^j(\varepsilon^{-1} x)\right) + O(\varepsilon^\kappa)$$

and, by the properties of the q form, we have,

$$q(x \to x_i, u_1) = q(x \to x_i, c_1^1\, \Phi^1(x) + c_2^1\, \Phi^2(x)) + O(\varepsilon^\kappa)$$
$$= -c_i^1 + O(\varepsilon^\kappa)$$

and we obtain the equivalent relation,

$$J_{p0} = T_{p1}^{-1}(\varepsilon)\, \Gamma_{10} + T_{p2}^{-1}(\varepsilon)\, \Gamma_{20} + O(\varepsilon^\kappa) \quad (p = 1, 2)\,. \tag{4.4.3}$$

The matrix $T(\varepsilon)$ is symmetric and we find the same expression as in (4.3.6) and this is in accordance with the symmetry of the impedance matrix.

It remains to estimate J_{00}. We have,

$$q(x \to 1, u_2) = q\left(x \to 1, \alpha_0 \Gamma_{00} \Phi(x) + \sum_{j=1,2} c_j^1 \Gamma_{j0} \Phi(x)\right) + O(\varepsilon^\kappa)$$

$$= -\left(\alpha_0 \Gamma_{00} + \sum_{j=1,2} c_j^1 \Gamma_{j0}\right) + O(\varepsilon^\kappa)$$

which gives us the last equivalent representation,

$$J_{00} = -\Gamma_{00} + [\Gamma_{10}\,\Gamma_{20}]\; T(\varepsilon)^{-1} \begin{bmatrix} \Gamma_{10} \\ \Gamma_{20} \end{bmatrix} + O(\varepsilon^\kappa)\,. \tag{4.4.5}$$

This rational term in $\ln \varepsilon$, is the same as that we have obtained in [1] by the matching methods.

CONCLUSION

With this rearrangement method, we have found a new representation of the solution "u" of problem (2), (3), (4) and then we have justified the approximations of impedance terms.

First, we note that the determination of equivalents in the case of a rigid motion of the inclusion, is only an example. This method can be used with more general displacement conditions.

Moreover, the rational form of the coefficients in expressions like (4.2.2) or (4.2.3), has been proved. It was not the case with the matching methods.

However, the main advantage of rearrangement lies in the structure of the two methods. In the matched asymptotic expansion method, we calculate the first terms of outer and inner solutions; then we match, and then we write again the solutions; We match again and so on. We have more and more difficulties to obtain the expressions of the solutions because of the increasing number of terms. In the rearrangement method, all the problems we use to obtain the expansion of "u", are of the same kind. The difficulty of their resolutions does not increase as the iteration dictates.

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