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ZHANGXIN CHEN

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EXPANDED MIXED FINITE ELEMENT METHODS FOR QUASILINEAR SECOND ORDER ELLIPTIC PROBLEMS, II (*)

Zhangxin CHEN

Department of Mathematics, Box 156, Southern Methodist University, Dallas, Texas 75275-0156, USA
E-mail address zchen@golem math smu edu

Abstract — *A new mixed formulation recently proposed for linear problems is extended to quasilinear second-order elliptic problems This new formulation expands the standard mixed formulation in the sense that three variables are explicitly treated, i e , the scalar unknown, its gradient, and its flux (the coefficient times the gradient) Based on this formulation, mixed finite element approximations of the quasilinear problems are established Existence and uniqueness of the solution of the mixed formulation and its discretization are demonstrated Optimal order error estimates in L^p and H^{-s} are obtained for the mixed approximations A postprocessing method for improving the scalar variable is analyzed, and superconvergent estimates are derived Implementation techniques for solving the systems of algebraic equations are discussed Comparisons between the standard and expanded mixed formulations are given both theoretically and experimentally The mixed formulation proposed here is suitable for the case where the coefficient of differential equations is a small tensor and does not need to be inverted © Elsevier, Paris*

Résumé. — *Dans cet article, la formulation mixte précédemment proposée pour des problèmes linéaires est étendue à des problèmes quasi-linéaires elliptiques d'ordre deux On donne alors la méthode de résolution par éléments finis pour laquelle on dispose d'estimations d'erreurs en norme L^p et H^{-s} De plus, des résultats de superconvergence de la méthode utilisée pour la résolution sont montrés © Elsevier, Paris*

1. INTRODUCTION

This is the second paper of a series in which we develop and analyze expanded mixed formulations for the numerical solution of second-order elliptic problems. This new formulation expands the standard mixed formulation in the sense that three variables are explicitly treated; i.e., the scalar unknown, its gradient, and its flux (the coefficient times the gradient). It is suitable for the case where the coefficient of differential equations is a small tensor and does not need to be inverted. It applies directly to the flow equation with low permeability and to the transport equation with small dispersion in groundwater modeling and petroleum reservoir simulation.

In the first paper of the series [5], we analyzed the expanded mixed formulation for linear second-order elliptic problems. Optimal order and superconvergent error estimates for mixed approximations were obtained, and various implementation techniques for solving the system of algebraic equations were discussed.

In this paper, we consider the expanded mixed formulation for a general quasilinear second-order elliptic problem. The analysis for the nonlinear problem is completely different from that for the linear problem. First, existence and uniqueness of solution to the nonlinear expanded discretization need to be proven explicitly. This is accomplished through the Brouwer fixed point theorem. Second, the nonlinear error analysis heavily depends upon the established existence result and is much more difficult. Also, the post-processing scheme proposed here for the first time for nonlinear mixed methods is not a straightforward extension of their linear counterparts.

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This paper also gives a comparison between the standard mixed formulation and the expanded one. For certain nonlinear problems, we show that the expanded formulation is superior to the standard one in that the former leads to the derivation of optimal order error estimates, while the latter gives only suboptimal error estimates for the mixed method solution. This result is also justified through numerical results. In the previous papers [6, 7, 14], only the Raviart-Thomas spaces have been considered for nonlinear problems. Here we are able to consider all existing mixed finite element spaces [2, 3, 4, 8, 11, 15, 16, 17].

In the next section, we develop the expanded mixed formulation for a fairly general nonlinear second-order elliptic problem. It is proven that this formulation has a unique solution and is equivalent to the original differential problem. Then, in § 3 we show that all existing mixed finite elements apply to this formulation. In particular, it is demonstrated that the approximation formulation has a unique solution and gives optimal error estimates in L^p and H^{-s}. In § 4, we propose and analyze a postprocessing method for improving the scalar unknown and derive superconvergent estimates. In § 5, we extend the analysis to a nonlinear problem and discuss the difference between the usual mixed method and the standard one. Finally, in § 6 we briefly discuss implementation techniques for solving the system of algebraic equations arising from the expanded mixed method and present numerical examples to illustrate our theoretical results.

2. EXPANDED MIXED FORMULATION

Let Ω be a bounded domain in $\mathbb{R}^n$, $n = 2$ or 3, with the boundary $\partial\Omega$. We consider the quasilinear problem

$$Lu = -\nabla \cdot (a(u)\nabla u - b(u)) + c(u) = f \quad \text{in } \Omega, \tag{2.1a}$$

$$u = -g \qquad\qquad \text{on } \partial\Omega, \tag{2.1b}$$

where we assume that the coefficients $a : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$, $b : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}^n$, and $c : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ are twice continuously differentiable with bounded derivatives through second order; moreover, we assume that

$$(a(u)\mu, \mu) \geqslant a_0 \|\mu\|^2, \quad u \in \mathbb{R}, \quad \mu \in (L^2(\Omega))^n, a_0 > 0. \tag{2.1c}$$

($H^k(\Omega) = W^{k,2}(\Omega)$ is the Sobolev space of k differentiable functions in $L^2(\Omega)$ with the norm $\|\cdot\|_k$; we omit k when it is zero.) We also assume that for some $\varepsilon(0 < \varepsilon < 1)$ and each pair of functions $(f, g) \in H^{\varepsilon}(\Omega) \times H^{3/2+\varepsilon}(\partial\Omega)$ there exists a unique solution $u \in H^{2+\varepsilon}(\Omega)$ to (2.1).

Let

$$V = H(\mathrm{div}\,; \Omega) = \{v \in (L^2(\Omega))^n : \nabla \cdot v \in L^2(\Omega)\},$$

$$W = L^2(\Omega),$$

$$\Lambda = (L^2(\Omega))^n,$$

and let $(\cdot, \cdot)_S$ denote the $L^2(S)$ inner product (we omit S if $S = \Omega$). Then (2.1) is formulated in the following expanded mixed form for $(\sigma, \lambda, u) \in V \times \Lambda \times W$:

$$(a(u)\lambda, \mu) - (\sigma, \mu) + (b(u), \mu) = 0, \quad \forall \mu \in \Lambda, \tag{2.2a}$$

$$(\lambda, v) - (u, \nabla \cdot v) = (g, v \cdot \nu)_{\partial\Omega}, \quad \forall v \in V, \tag{2.2b}$$

$$(\nabla \cdot \sigma, w) + (c(u), w) = (f, w), \quad \forall w \in W, \tag{2.2c}$$

where ν is the outer unit normal to the domain Ω.

To analyze (2.2), let $U = W \times \Lambda$ with the usual product norm $\|\tau\|_U^2 = \|w\|^2 + \|\mu\|^2$, $\tau = (w, \mu) \in U$, and introduce the bilinear forms $\mathscr{A}(\,.\,,\,.\,) : U \times U \to \mathbb{R}$ and $\mathscr{B}(\,.\,,\,.\,) : U \times V \to \mathbb{R}$ by

$$\mathscr{A}(\chi, \tau) = (a(u)\,\lambda, \mu), \qquad \chi = (u, \lambda), \quad \tau = (w, \mu) \in U,$$
$$\mathscr{B}(\tau, v) = (w, \nabla \,.\, v) - (\mu, v) \quad \tau = (w, \mu) \in U, v \in V.$$

Then (2.2) can be written in the form for $(\chi, \sigma) \in U \times V$ such that

$$\mathscr{A}(\chi, \tau) + \mathscr{B}(\tau, \sigma) + \mathscr{C}(\chi, \tau) = \mathscr{F}(\tau) \quad \forall \tau \in U, \tag{2.3a}$$
$$\mathscr{B}(\chi, v) = -(g, v \,.\, \nu)_{\partial\Omega}, \qquad \forall v \in V, \tag{2.3b}$$

where

$$\mathscr{C}(\chi, \tau) = (b(u), \mu) + (c(u), w), \quad \tau = (w, \mu) \in U,$$
$$\mathscr{F}(\tau) = (f, w), \qquad \tau = (w, \mu) \in U.$$

Finally, we define

$$Z = \{\tau \in U : \mathscr{B}(\tau, v) = 0, \quad \forall v \in V\}.$$

The next result can be found in the first paper [5].

LEMMA 2.1: *Let $\tau = (w, \mu) \in U$. Then $\tau \in Z$ if and only if $w \in H_0^1(\Omega)$ and $\mu = -\nabla w$.*

THEOREM 2.2: *If $(\chi, \sigma) \in U \times V$ is the solution of (2.3) with $\chi = (u, \lambda)$, then $u \in H^1(\Omega)$ is the solution of (2.1) with $\lambda = -\nabla u$ and $u|_{\partial\Omega} = g$. Conversely, if $u \in H^1(\Omega)$ is the solution of (2.1) with $u|_{\partial\Omega} = g$, then (2.3) has the solution $(\chi, \sigma) \in U \times V$ with $\chi = (u, \lambda)$, $\lambda = -\nabla u$, and $\sigma = -(a(u)\nabla u - b(u))$.*

Proof. First, let $(\chi, \sigma) \in U \times V$ be the solution of (2.3) with $\chi = (u, \lambda)$. Without loss of generality, let $g = 0$ (otherwise, let $u_0 \in H^1(\Omega)$ such that $u_0|_{\partial\Omega} = g$ and consider $u - u_0$ [12]). Then (2.3b) with $g = 0$ implies that $\chi \in Z$ so that, by Lemma 2.1, $u \in H_0^1(\Omega)$ and $\lambda = -\nabla u$. Hence, for all $w \in H_0^1(\Omega)$ and $\mu = -\nabla w$, it follows from Lemma 2.1 that

$$\mathscr{A}(\chi, \tau) + \mathscr{C}(\chi, \tau) = \mathscr{F}(\tau), \quad \forall \tau = (w, \mu) \in Z;$$

i.e.,

$$(a(u)\nabla u, \nabla w) + (b(u), \nabla w) + (c(u), w) = (f, w), \quad \forall w \in H_0^1(\Omega).$$

Hence, u is a weak solution of (2.1); i.e., the solution of (2.1) [9].

Next, we assume that $u \in H_0^1(\Omega)$ is the solution of (2.1). Set $\chi = (u, \lambda)$ with $\lambda = -\nabla u$ and $\sigma = -(a(u)\nabla u - b(u))$. Then it follows from Lemma 2.1 that $\chi \in Z$, so (2.3b) with $g = 0$ holds. Thus, (2.3a) remains to be proved. For each $\tau \in U$ with $\tau = (w, \mu)$,

$$\begin{aligned}\mathscr{A}(\chi, \tau) + \mathscr{B}(\tau, \sigma) + \mathscr{C}(\chi, \tau) &= (a(u)\,\lambda, \mu) + (w, \nabla \,.\, \sigma) - (\mu, \sigma) + (b(u), \mu) + (c(u), w) \\ &= (w, -\nabla \,.\, (a(u)\nabla u - b(u)) + c(u)) \\ &= (f, w), \quad \forall w \in W,\end{aligned}$$

which implies (2.3a). □

3. MIXED FINITE ELEMENTS

To define a finite element method, we need a partition $\mathscr{E}_h$ of Ω into elements E, say, simplexes, rectangular parallelepipeds, and/or prisms, where only edges or faces on $\partial\Omega$ may be curved. In $\mathscr{E}_h$, it is also necessary that adjacent elements completely share their common edge or face; let $\partial\mathscr{E}_h$ denote the set of all interior edges $(n = 2)$ or faces $(n = 3)$ e of $\mathscr{E}_h$.

Since mixed finite element spaces are finite dimensional and defined locally on each element, for each $E \in \mathscr{E}_h$ let $V_h(E) \times W_h(E)$ denote one of the mixed finite element spaces introduced in [2, 3, 4, 8, 11, 15, 16, 17] for second-order elliptic problems. Then we define

$$\Lambda_h = \{\mu \in \Lambda : \mu|_E \in V_h(E) \text{ for each } E \in \mathscr{E}_h\},$$

$$V_h = \{v \in V : v|_E \in V_h(E) \text{ for each } E \in \mathscr{E}_h\},$$

$$W_h = \{w \in W : w|_E \in W_h(E) \text{ for each } E \in \mathscr{E}_h\}.$$

The expanded mixed finite element method for (2.1) is to find $(\sigma_h, \lambda_h, u_h) \in V_h \times \Lambda_h \times W_h$ such that

$$(3.1a) \qquad (a(u_h)\,\lambda_h, \mu) - (\sigma_h, \mu) + (b(u_h), \mu) = 0, \quad \forall \mu \in \Lambda_h,$$

$$(3.1b) \qquad (\lambda_h, v) - (u_h, \nabla . v) = (g, v . \nu)_{\partial\Omega}, \quad \forall v \in V_h,$$

$$(3.1c) \qquad (\nabla . \sigma_h, w) + (c(u_h), w) = (f, w), \quad \forall w \in W_h.$$

We shall establish existence, uniqueness, and convergence results for (3.1) in this section. For simplicity, we concentrate on the planar case; an extension to the space case is straightforward. We mention that while an extra unknown is introduced in (3.1), the computational cost for solving (3.1) is the same as that for solving the usual mixed method, as shown in § 6.

3.1. Existence

C and C_1 are generic constants below, where C_1 depends on $\|u\|_{2+\varepsilon}$, at most quadratically. Each of our mixed finite element spaces [2, 3, 4, 8, 11, 15, 16, 17] has the property that there are projection operators $\Pi_h : (H^1(\Omega))^n \to V_h$ and $P_h = L^2$-projection: $L^2(\Omega) \to W_h$ such that

$$(3.2a) \qquad \|v - \Pi_h v\| \leqslant C\|v\|_r h^r, \qquad 1 \leqslant r \leqslant k+1,$$

$$(3.2b) \qquad \|\nabla . (v - \Pi_h v)\| \leqslant C\|\nabla . v\|_r h^r, \quad 0 \leqslant r \leqslant k^*,$$

$$(3.2c) \qquad \|w - P_h w\|_{-s} \leqslant C\|w\|_r h^{r+s}, \qquad 0 \leqslant s, r \leqslant k^*,$$

and

$$(3.3a) \qquad (\nabla . (v - \Pi_h v), w) = 0, \quad \forall w \in W_h,$$

$$(3.3b) \qquad (\nabla . v, w - P_h w) = 0, \quad \forall v \in V_h,$$

where $k^* = k + 1$ for the Raviart-Thomas-Nedelec spaces [17, 15, 16] and the Brezzi-Douglas-Fortin-Marini spaces, $k^* = k$ for the Brezzi-Douglas-Marini spaces and Brezzi-Douglas-Durán-Fortin [4, 2], and the Chen-Douglas spaces include both cases. Also, let R_h be the L^2-projection onto Λ_h. Then we see that

$$(3.4) \qquad (\mu - R_h\mu, \tau) = 0, \quad \forall \mu \in \Lambda, \tau \in \Lambda_h,$$

and

$$\|\mu - R_h \mu\|_{-s} \leqslant C \|\mu\|_r h^{r+s}, \quad 0 \leqslant s, r \leqslant k+1 . \tag{3.5}$$

For the analysis below, we write

$$a(u_h) - a(u) = -\tilde{a}_u(u_h)(u - u_h) = -a_u(\dot{u})(u - u_h) + \tilde{a}_{uu}(u_h)(u - u_h)^2 , \tag{3.6}$$

where

$$\tilde{a}_u(u_h) = \int_0^1 a_u(u_h + t(u - u_h))\, dt ,$$

$$\tilde{a}_{uu}(u_h) = \int_0^1 (1-t)\, a_{uu}(u + t(u_h - u))\, dt ,$$

are bounded in $\bar{\Omega}$. Similarly, we write

$$b(u_h) - b(u) = -\tilde{b}_u(u_h)(u - u_h) = -b_u(u)(u - u_h) + \tilde{b}_{uu}(u_h)(u - u_h)^2 , \tag{3.7}$$

$$c(u_h) - c(u) = -\tilde{c}_u(u_h)(u - u_h) = -c_u(u)(u - u_h) + \tilde{c}_{uu}(u_h)(u - u_h)^2 , \tag{3.8}$$

where $\tilde{b}_u(u_h)$, $\tilde{b}_{uu}(u_h)$, $\tilde{c}_u(u_h)$ and $\tilde{c}_{uu}(u_h)$ are bounded functions in $\bar{\Omega}$. We now subtract (3.1) from (2.2) to obtain the error equations

$$(a(u)(\lambda - \lambda_h), \mu) - (\sigma - \sigma_h, \mu) + (b(u) - b(u_h), \mu) = ((a(u_h) - a(u))\lambda_h, \mu), \quad \forall \mu \in \Lambda_h , \tag{3.9a}$$

$$(\lambda - \lambda_h, v) - (u - u_h, \nabla . v) = 0, \quad \forall v \in V_h , \tag{3.9b}$$

$$(\nabla . (\sigma - \sigma_h), w) + (c(u) - c(u_h), w) = 0, \quad \forall w \in W_h . \tag{3.9c}$$

Substituting (3.6)-(3.8) into (3.9), we see that

$$(a(u)(\lambda - \lambda_h), \mu) - (\sigma - \sigma_h, \mu) + (\Gamma(u)(u - u_h), \mu)$$

$$= ((\tilde{a}_{uu}(u_h)\lambda + b^2_{uu}(u_h))(u - u_h)^2, \mu) + (\tilde{a}_u(u_h)(u - u_h)(\lambda - \lambda_h), \mu), \quad \forall \mu \in \Lambda_h , \tag{3.10a}$$

$$(\lambda - \lambda_h, v) - (u - u_h, \nabla . v) = 0, \quad \forall v \in V_h, \tag{3.10b}$$

$$(\nabla . (\sigma - \sigma_h), w) + (\gamma(u - u_h), w) = (\tilde{c}_{uu}(u_h)(u - u_h)^2, w), \quad \forall w \in W_h , \tag{3.10c}$$

where $\Gamma(u) = a_u(u)\lambda + b_u(u)$ and $\gamma(u) = c_u(u)$. Now let M: $H^2(\Omega) \to L^2(\Omega)$ be the linear operator

$$Mw = -\nabla . (a(u)\nabla w - \Gamma(u) w) + \gamma w ,$$

and let

$$\Phi : V_h \times \Lambda_h \times W_h \to V_h \times \Lambda_h \times W_h$$

be given by $\Phi((\tau, \eta, \rho)) = (x, y, z)$, where (x, y, z) is the solution of the system

$$(a(u)(\lambda - y), \mu) - (\sigma - x, \mu) + (\Gamma(u - z), \mu)$$

$$= ((\tilde{a}_{uu}(\rho)\lambda + \tilde{b}_{uu}(\rho))(u - \rho)^2, \mu) + (\tilde{a}_u(\rho)(u - \rho)(\lambda - \eta), \mu), \quad \forall \mu \in \Lambda_h, \tag{3.11a}$$

$$(\lambda - y, v) - (u - z, \nabla . v) = 0, \quad \forall v \in V_h, \tag{3.11b}$$

$$(\nabla . (\sigma - x), w) + (\gamma(u - z), w) = (\tilde{c}_{uu}(\rho)(u - \rho)^2, w), \quad \forall w \in W_h. \tag{3.11c}$$

We assume that the restrictions of M and M^* (its adjoint) to $H^2(\Omega) \cap H_0^1(\Omega)$ have bounded inverses. This is satisfied if $c_u \geqslant 0$ [12]. Then, the existence and uniqueness of the solution to (3.11) is known [5] since (3.11) corresponds to the expanded mixed method for the linear operator M. Now we see that existence of a solution to (3.1) is equivalent to the problem that the map Φ has a fixed point. Consequently, the solvability of (3.1) follows from the Brouwer fixed point theorem if we can prove that Φ maps a ball of $V_h \times \Lambda_h \times W_h$ into itself. Toward that end, we need the following definition [10].

We say that Ω is $(s + 2, \theta)$-regular with respect to M if the Dirichlet problem

$$M^*\varphi = \psi \quad \text{in } \Omega, \tag{3.12a}$$

$$\varphi = 0 \quad \text{on } \partial\Omega \tag{3.12b}$$

is uniquely solvable for $\psi \in L^2(\Omega)$ and if

$$\|\varphi\|_{s+2,\theta} \leqslant C\|\psi\|_{s,\theta}. \tag{3.13}$$

LEMMA 3.1: *Assume that $2 \leqslant \theta < \infty$ and Ω is $(s + 2, \theta')$-regular with respect to M, where $\theta' = \theta/(\theta - 1)$ is the conjugate exponent of θ. Let $\xi \in L^2(\Omega)$, $\phi \in V$, $\zeta \in L^2(\Omega)$, and $r \in L^2(\Omega)$. If $\pi \in W_h$ satisfies the system*

$$(a(u)\xi, \mu) - (\phi, \mu) + (\Gamma\pi, \mu) = (\zeta, \mu), \quad \forall \mu \in \Lambda_h, \tag{3.14a}$$

$$(\xi, v) - (\pi, \nabla . v) = 0, \quad \forall v \in V_h, \tag{3.14b}$$

$$(\nabla . \phi, w) + (\gamma\pi, w) = (r, w), \quad \forall w \in W_h, \tag{3.14c}$$

then there is a constant $C = C(\theta, a, \Gamma, \gamma, \Omega)$ such that

$$\|\pi\|_{0,\theta} \leqslant C\{(\|\xi\| + \|\phi\|)h^{2/\theta} + h^{\min(1 + 2/\theta, k^*)}\|\nabla . \phi\| + \|\zeta\| + \|\gamma\|\}. \tag{3.15}$$

Moreover, if $\xi \in L^\theta(\Omega)$, $\phi \in W^{0,\theta}(\mathrm{div}\,; \Omega) = \{v \in L^\theta(\Omega)\,; \nabla . v \in L^\theta(\Omega)\}$, $\zeta \in L^\theta(\Omega)$, and $r \in L^\theta(\Omega)$, then for $0 \leqslant s \leqslant 2k^*$

$$\begin{aligned}\|\pi\|_{-s,\theta} \leqslant C\{&(\|\xi\|_{0,\theta} + \|\phi\|_{0,\theta})h^{\min(s+1,k+1)} + \|\zeta\|_{0,\theta}h^{\min(s+1,k^*)} \\ &+ (\|\nabla . \phi\|_{0,\theta} + \|r\|_{0,\theta})h^{\min(s+2,k^*)} + \|\zeta\|_{-s-1,\theta} + \|r\|_{-s-2,\theta}\}.\end{aligned} \tag{3.16}$$

Proof: We only prove (3.16); (3.15) can be shown more easily. Let $\psi \in W^{s,\theta'}(\Omega)$ and $\varphi \in W^{s+2,\theta'}(\Omega)$ be the solution of (3.12). Then, by (3.3), (3.14), and integration by parts, we see that

$$\begin{aligned}(\pi, \psi) &= (\pi, M^*\varphi) = (\pi, -\nabla \cdot (a(u)\nabla\varphi) - \Gamma\nabla\varphi + \gamma\varphi)\\ &= -(\xi, \Pi_h(a(u)\nabla\varphi)) - (\Gamma\pi, \nabla\varphi - R_h\nabla\varphi) - (\Gamma\pi, R_h\nabla\varphi) + (\gamma\pi, \varphi)\\ &= (\xi, a(u)\nabla\varphi - \Pi_h(a(u)\nabla\varphi)) - (a(u)\xi, \nabla\varphi - R_h\nabla\varphi) + (\phi, \nabla\varphi - R_h\nabla\varphi)\\ &\quad + (\nabla\cdot\phi, \varphi - P_h\varphi) + (\zeta, \nabla\varphi - R_h\nabla\varphi) - (\zeta, \nabla\varphi) + (r, R_h\varphi - \varphi) + (r, \varphi)\\ &\quad + (\Gamma\pi, \nabla\varphi - R_h\nabla\varphi) + (\gamma\pi, \varphi - P_h\varphi).\end{aligned} \tag{3.17}$$

Applying (3.2a), (3.2b) and (3.5), we observe that

$$|(\xi, a(u)\nabla\varphi - \Pi_h(a(u)\nabla\varphi))| \leq C\|\xi\|_{0,\theta}\|\varphi\|_{s+2,\theta'}h^{\min(s+1,k+1)},$$

$$|(a(u)\xi, \nabla\varphi - R_h\nabla\varphi)| \leq C\|\xi\|_{0,\theta}\|\varphi\|_{s+2,\theta'}h^{\min(s+1,k+1)},$$

$$|(\phi, \nabla\varphi - R_h\nabla\varphi)| \leq C\|\phi\|_{0,\theta}\|\varphi\|_{s+2,\theta'}h^{\min(s+1,k+1)},$$

$$|(\nabla\cdot\phi, \varphi - P_h\varphi)| \leq C\|\nabla\cdot\phi\|_{0,\theta}\|\varphi\|_{s+2,\theta'}h^{\min(s+2,k^*)},$$

$$|(\zeta, \nabla\varphi - R_h\nabla\varphi)| \leq C\|\zeta\|_{0,\theta}\|\varphi\|_{s+2,\theta'}h^{\min(s+1,k+1)},$$

$$|(\zeta, \nabla\varphi)| \leq \|\zeta\|_{-s-1,\theta}\|\varphi\|_{s+2,\theta'},$$

$$|(r, P_h\varphi - \varphi)| \leq C\|r\|_{0,\theta}\|\varphi\|_{s+2,\theta'}h^{\min(s+2,k^*)},$$

$$|(r, \varphi)| \leq \|r\|_{-s-2,\theta}\|\varphi\|_{s+2,\theta'},$$

$$|(\Gamma\pi, \nabla\varphi - R_h\nabla\varphi)| \leq C\|\pi\|_{0,\theta}\|\varphi\|_{s+2,\theta'}h^{\min(s+1,k+1)},$$

$$|(\gamma\pi, \varphi - P_h\varphi)| \leq C\|\pi\|_{0,\theta}\|\varphi\|_{s+2,\theta'}h^{\min(s+2,k^*)}.$$

Substitute these inequalities into (3.17) and use (3.13) to obtain

$$\begin{aligned}\|\pi\|_{-s,\theta} \leq C((&\|\xi\|_{0,\theta} + \|\phi\|_{0,\theta} + \|\zeta\|_{0,\theta})h^{\min(s+1,k+1)}\\ &+ \|\nabla\cdot\phi\|_{0,\theta}h^{\min(s+2,k^*)} + \|\zeta\|_{-s-1,\theta}\\ &+ \|r\|_{0,\theta}h^{\min(s+2,k^*)} + \|r\|_{-s-2,\theta} + \|\pi\|_{0,\theta}h^{\min(s+1,k^*)}).\end{aligned} \tag{3.18}$$

First, consider $s = 0$; for h sufficiently small, the $h\|\pi\|_{0,\theta}$ term on the right-hand side of (3.18) can be absorbed into the left-hand side, and the result (3.16) has been established for $s = 0$. Then, for $s > 0$, apply (3.18) again, the established result for $s = 0$, and the interpolation result [13]

$$\|r\|_{-2,\theta} \leq C\|r\|_{0,\theta}^{s/(s+2)}\|r\|_{-s-2,\theta}^{2/(s+2)} \leq C(h\|r\|_{0,\theta} + h^{-s/2}\|r\|_{-s-2,\theta}),$$

to obtain (3.16) since $k^* \leq k+1$ and $s \leq 2k^*$. $\square$

We now turn to existence of a solution to (3.1). For this we rewrite (3.11) by shifting (u, λ, σ) to $(P_h u, R_h \lambda, \Pi_h \sigma)$ and using (3.3a), (3.3b) and (3.4) as follows:

$$(a(u)(R_h \lambda - y), \mu) - (\Pi_h \sigma - x, \mu) + (\Gamma(P_h u - z), \mu) \tag{3.19a}$$

$$= ((\tilde{a}_{uu}(\rho) \lambda + \tilde{b}_{uu}(\rho))(u-\rho)^2, \mu) + (\tilde{a}_u(\rho)(u-\rho)(\lambda - \eta), \mu)$$

$$+ (a(u)(R_h \lambda - \lambda), \mu) + (\sigma - \Pi_h \sigma, \mu) + (\Gamma(P_h u - u), \mu), \quad \forall \mu \in \Lambda_h,$$

$$(R_h \lambda - y, v) - (P_h u - z, \nabla \cdot v) = 0, \quad \forall v \in V_h, \tag{3.19b}$$

$$(\nabla \cdot (\Pi_h \sigma - x), w) + (\gamma(P_h u - z), w) \tag{3.19c}$$

$$= (\tilde{c}_{uu}(\rho)(u-\rho)^2, w) + (\gamma(P_h u - u), w), \quad \forall w \in W_h.$$

Let $\mathscr{W}_h = W_h$ and $\mathscr{L}_h = \Lambda_h$ with the stronger norms $\|w\|_{\mathscr{W}_h} = \|w\|_{0,\theta}$ and $\|\mu\|_{\mathscr{L}_h} = \|\mu\|_{0,2+\varepsilon}$, respectively, where $\theta = (4 + 2\varepsilon)/\epsilon > 4$.

THEOREM 3.2: *For $\delta > 0$ sufficiently small (dependent of h), Φ maps a ball of radius δ of $V_h \times \mathscr{L}_h \times \mathscr{W}_h$ onto itself.*

Proof: Let

$$\|\Pi_h \sigma - \tau\|_V < \delta, \quad \|P_h u - \rho\|_{0,\theta} < \delta, \quad \|R_h \lambda - \eta\|_{0,2+\varepsilon} < \delta. \tag{3.20}$$

We now apply (3.15) to (3.19) with

$$\zeta = (\tilde{a}_{uu}(\rho)\lambda + \tilde{b}_{uu}(\rho))(u-\rho)^2 + \tilde{a}_u(\rho)(u-\rho)(\lambda-\eta) + a(u)(R_h \lambda - \lambda)$$

$$+ \sigma - \Pi_h \sigma + \Gamma(P_h u - u),$$

$$r = \tilde{c}_{uu}(\rho)(u-\rho)^2 + \gamma(P_h u - u).$$

First, note that, by (3.2a), (3.2b) and (3.5),

$$\|\zeta\| + \|r\| \leqslant C\{\|u-\rho\|_{0,4}^2 + \|u-\rho\|_{0,\theta}\|\lambda-\eta\|_{0,2+\varepsilon}$$

$$+ \|R_h \lambda - \lambda\| + \|\sigma - \Pi_h \sigma\| + \|P_h u - u\|\}$$

$$\leqslant C\{\|u - P_h u\|_{0,\theta}^2 + \|\rho - P_h u\|_{0,\theta}^2 + h\|u\|_2$$

$$+ (\|u - P_h u\|_{0,\theta} + \|P_h u - \rho\|_{0,\theta})(\|\lambda - R_h \lambda\|_{0,2+\varepsilon} + \|R_h \lambda - \eta\|_{0,2+\varepsilon})\},$$

so that, by (3.2), (3.5), (3.20), and the Sobolev embedding inequalities [1]

$$\|u\|_{2,2+\varepsilon} \leqslant C_\varepsilon \|u\|_{2+\varepsilon}, \quad \|u\|_{1,\theta} \leqslant C_\varepsilon \|u\|_{2+\varepsilon},$$

we see that

$$\|\zeta\| + \|r\| \leqslant C_1(h + \delta^2), \tag{3.21}$$

where $C_1 = C(\|u\|_{2+\varepsilon})$. If we take the last term on the left side of (3.19a) and (3.19c) over to the right side, the left side in (3.19) becomes the expanded mixed method for the differential operator $-\nabla\cdot(a(u)\nabla)$. It follows from [5] that

$$\|\Pi_h\sigma - x\|_V \leqslant C(\|P_h u - z\| + \|\zeta\| + \|r\|)\,, \tag{3.22a}$$

$$\|R_h\lambda - y\| \leqslant C(\|P_h u - z\| + \|\zeta\| + \|r\|)\,. \tag{3.22b}$$

Now, apply (3.15) to (3.19) to obtain

$$\|P_h u - z\|_{0,\theta} \leqslant C\{(\|R_h\lambda - y\| + \|\Pi_h\sigma - x\|)\,h^{2/\theta} + \|\nabla\cdot(\Pi_h\sigma - x)\|\,h^{\min(1+2/\theta,k^*)} + \|\zeta\| + \|r\|\}\,.$$

Consequently, it follows from (3.21) and (3.22) that

$$\|P_h u - z\|_{0,\theta} \leqslant C_1(h+\delta^2)\,, \tag{3.23}$$

for h sufficiently small. Exploit (3.21)-(3.23) again to see that

$$\|\Pi_h\sigma - x\|_V \leqslant C_1(h+\delta^2)\,, \tag{3.24a}$$

$$\|R_h\lambda - y\| \leqslant C_1(h+\delta^2)\,. \tag{3.24b}$$

Using the quasiregularity of T_h, we find that

$$\|R_h\lambda - y\|_{0,2+\varepsilon} \leqslant Ch^{-\varepsilon/(2+\varepsilon)}\|R_h\lambda - y\| \leqslant C_1 h^{-\varepsilon/(2+\varepsilon)}(h+\delta^2)\,. \tag{3.25}$$

Finally, let $h < (2C_1)^{-(4+2\varepsilon)/(2-\varepsilon)}$ and choose $\delta = 2C_1 h^{2/(2+\varepsilon)}$. Observe that, in order to have $C_1 h^{2/(2+\varepsilon)} \leqslant \delta/2$ ant $C_1 h^{-\varepsilon/(2+\varepsilon)}\delta^2 \leqslant \delta/2$, δ must belong to

$$[2C_1 h^{2/(2+\varepsilon)}, (2C_1)^{-1}h^{\varepsilon/(2+\varepsilon)}] \neq \Phi\,,$$

which is satisfied for h and δ as chosen. Now, by (3.23), (3.24a) and (3.25), for such chosen h and δ, we have

$$\|\Pi_h\sigma - x\|_V < \delta, \quad \|P_h u - z\|_{0,\theta} < \delta, \quad \|R_h\lambda - y\|_{0,2+\varepsilon} < \delta\,. \tag{3.26}$$

That is, Φ maps the ball of radius δ, centered at $(\Pi_h\sigma, R_h\lambda, P_h u)$ onto itself. □

3.2. L^2-error estimates

Assume momentarily that (3.1) has a unique solution which, at least for small h, will be established later. To obtain error estimates, we rewrite (3.9), by (3.6)-(3.8), as follows:

$$\begin{aligned}
&(a(u)(\lambda-\lambda_h),\mu) - (\sigma-\sigma_h,\mu) + ((\tilde{a}_u(u_h)\lambda_h + \tilde{b}_u(u_h))(u-u_h),\mu) = 0, && \forall\mu\in\Lambda_h\,,\\
&(\lambda-\lambda_h, v) - (u-u_h, \nabla\cdot v) = 0, && \forall v\in V_h\,,\\
&(\nabla\cdot(\sigma-\sigma_h), w) + (\tilde{c}_u(u_h)(u-u_h), w) = 0, && \forall w\in W_h\,.
\end{aligned}$$

Define

$$\alpha = \lambda - \lambda_h, \quad \beta = R_h \lambda - \lambda_h ,$$
$$d = \sigma - \sigma_h, \quad e = \Pi_h \sigma - \sigma_h ,$$
$$y = u - u_h, \quad z = P_h u - u_h .$$

We then have with $\tilde{\Gamma}_h = \tilde{a}_u(u_h)\,\lambda_h + \tilde{b}_u(u_h)$

$$(a(u)\,\alpha, \mu) - (d, \mu) + (\tilde{\Gamma}_h z, \mu) = (\tilde{\Gamma}_h(P_h u - u), \mu), \quad \forall \mu \in \Lambda_h , \tag{3.27a}$$
$$(\alpha, v) - (z, \nabla \cdot v) = 0, \quad \forall v \in V_h , \tag{3.27b}$$
$$(\nabla \cdot d, w) + (\tilde{c}_u(u_h)\, z, w) = (\tilde{c}_u(u_h)\,(P_h u - u), w), \quad \forall w \in W_h . \tag{3.27c}$$

Or, equivalently, as a result of (3.3b) and (3.4),

$$(a(u)\,\alpha, \mu) - (d, \mu) + (\tilde{\Gamma}_h z, \mu) = (\tilde{\Gamma}_h(P_h u - u), \mu), \quad \forall \mu \in \Lambda_h , \tag{3.28a}$$
$$(\beta, v) - (z, \nabla \cdot v) = 0, \quad \forall v \in V_h , \tag{3.28b}$$
$$(\nabla \cdot d, w) + (\tilde{c}_u(u_h)\, z, w) = (\tilde{c}_u(u_h)\,(P_h u - u), w), \quad \forall w \in W_h . \tag{3.28c}$$

Observe that (3.27) or (3.28) corresponds to the mixed method for the linear operator $N : H^2(\Omega) \to L^2(\Omega)$ given by $N_w = -\nabla \cdot (a(u)\,\nabla w - \tilde{\Gamma}_h w) + \tilde{c}_u(u_h)\, w$. As shown in [14], it follows from the results (3.26) in the proof of Theorem 3.2 that there is an h_0 such that the restriction of its adjoint N^* to $H^2(\Omega) \cap H_0^1(\Omega)$ has a bounded inverse for $h < h_0$. Now we prove the next result.

THEOREM 3.3: *Assume that Ω is (2,2)-regular with respect to M. Then for h sufficiently small*

$$\|u - u_h\| \leqslant C_1(\|u\|_r h^r + \|u\|_{r_1 + \delta_{1k^*}} h^{r_1}), \tag{3.29a}$$
$$2 \leqslant r \leqslant k+2,\ 1 \leqslant r_1 \leqslant k^* ,$$

$$\|\lambda - \lambda_h\| + \|\sigma - \sigma_h\| \leqslant C_1(\|u\|_{r+1} h^r + \|u\|_{r_1} h^{r_1} + \|\nabla \cdot \sigma\|_{r_1} h^{r_1 + \min(2, k^*)}), \tag{3.29b}$$
$$1 \leqslant r \leqslant k+1,\ 0 \leqslant r_1 \leqslant k^* ,$$

$$\|\nabla \cdot (\sigma - \sigma_h)\| \leqslant C_1(\|u\|_{r+1} h^r + \|u\|_{r_1} h^{r_1} + \|\nabla \cdot \sigma\|_{r_1} h^{r_1}), \tag{3.29c}$$
$$1 \leqslant r \leqslant k+1,\ 0 \leqslant r_1 \leqslant k^* .$$

Proof: Using (3.26) with $\delta = 2\, C_1\, h^{2/(2+\varepsilon)}$, the embedding relation $H^{1+\varepsilon}(\Omega) \subset W^{\varepsilon/2, \infty}(\Omega)$, and the quasiregularity of T_h, we see that

$$\begin{aligned} \|\lambda_h\|_{0,\infty} &\leqslant \|\beta\|_{0,\infty} + \|P_h \lambda\|_{0,\infty} \\ &\leqslant Ch^{-2/(2+\varepsilon)} \|\beta\|_{0, 2+\varepsilon} + \|\lambda - P_h \lambda\|_{0,\infty} + \|\lambda\|_{0,\infty} \\ &\leqslant C_1(\|u\|_{2+\varepsilon}), \end{aligned} \tag{3.30}$$

so that $\|\tilde{\Gamma}_h\|_{0,\infty}$ is bounded by C_1. Now, apply (3.16) to (3.27) to obtain

$$\begin{aligned}\|z\| \leqslant C\{ & (\|\alpha\| + \|d\| + \|\tilde{\Gamma}_h(P_h u - u)\|) h \\ & + (\|\nabla \cdot d\| + \|\tilde{c}_u(u_h)(P_h u - u)\|) h^{\min(2, k^*)} \\ & + \|\tilde{\Gamma}_h(P_h u - u)\|_{-1} + \|\tilde{c}_u(u_h)(P_h u - u)\|_{-2} \} .\end{aligned} \tag{3.31}$$

Furthermore, by (3.2c) and (3.30), we see that

$$\|\tilde{\Gamma}_h(P_h u - u)\| h + \|\tilde{\Gamma}_h(P_h u - u)\|_{-1} \leqslant C_1 \|u\|_{r_1} h^{r_1 + 1}, \qquad 0 \leqslant r_1 \leqslant k^* , \tag{3.32}$$

$$\|\tilde{c}_h(u_h)(P_h u - u)\| h + \|\tilde{c}_h(u_h)(P_h u - u)\|_{-2} \leqslant C_1 \|u\|_{r_1} h^{r_1 + 1} \quad 0 \leqslant r_1 \leqslant k^* . \tag{3.33}$$

It remains to estimate α, d, and $\|\nabla \cdot d\|$. As in the proof of Theorem 3.2, it follows from (3.28) [5] that

$$\|\beta\| + \|e\|_V \leqslant C_1(\|\Pi_h \sigma - \sigma\| + \|\lambda - R_h \lambda\| + \|y\|) ,$$

so that, by (3.2a) and (3.5),

$$\|\beta\| + \|e\|_V \leqslant C_1(\|u\|_{r+1} h^r + \|y\|), \quad 1 \leqslant r \leqslant k + 1 . \tag{3.34}$$

Now, apply (3.2a), (3.2b), (3.5) and (3.34) to obtain

$$\|\alpha\| \leqslant C_1(\|u\|_{r+1} h^r + \|y\|), \qquad 1 \leqslant r \leqslant k + 1 , \tag{3.35a}$$

$$\|d\| \leqslant C_1(\|u\|_{r+1} h^r + \|y\|), \qquad 1 \leqslant r \leqslant k + 1 , \tag{3.35b}$$

$$\|\nabla \cdot d\| \leqslant C_1(\|\nabla \cdot \sigma\|_{r_1} h^{r_1} + \|u\|_r h^r + \|y\|), \; 0 \leqslant r_1 \leqslant k^*, 1 \leqslant r \leqslant k + 1 . \tag{3.35c}$$

Substitute (3.35a)-(3.35c) into (3.31) and use (3.2c), (3.32) and (3.33) to obtain

$$\|z\| \leqslant C_1(\|u\|_{r+1} h^{r+1} + \|u\|_{r_1} h^{r_1 + 1} + \|\nabla \cdot \sigma\|_{r_1} h^{r_1 + \min(2, k^*)}), \quad 1 \leqslant r \leqslant k + 1, 0 \leqslant r_1 \leqslant k^* , \tag{3.36}$$

for h sufficiently small. Now, combine (3.2c), (3.35) and (3.36) to yield the desired result (3.29). □

We remark that the L^2-error estimates in Theorem 3.3 are optimal both in rate (for any h) and in regularity. Also, as a result of (3.36), we have

$$\|P_h u - u_h\| \leqslant C_1 \|u\|_r h^{k^* + 1}, \quad r = \max(k^* + 1, 3) , \tag{3.37}$$

which is a superconvergence result and is needed in the analysis of the later postprocessing method. Note that in the case where $k^* = k + 1$ we have the superconvergence order $O(h^{k+2})$, and in the case where $k^* = k$ we have $O(h^{k+1})$. For the linear case where a does not depend on the solution u and $b = c \equiv 0$ in (2.1), we have shown a superconvergence result, which is of order $O(h^{k+2})$ for both cases [5]. We have a superconvergence only of order $O(h^{k+1})$ for the latter case for the present nonlinear problem because the coefficient a depends on u and b and c are not zero. The same remark applies to the postprocessing method proposed in § 4.

3.3. Uniqueness

We now demonstrate the uniqueness of the solution to (3.1). Let $(u^i, \lambda^i, \sigma^i) \in W_h \times \Lambda_h \times V_h$ be solutions of (3.1), $i = 1, 2$. Note that it follows from Theorem 3.3 that these two solutions satisfy the error bounds in (3.29) provided they satisfy (3.26). Then the quasi regularity of T_h and the error bounds imply that λ^i is bounded by $\|u\|_{2+\varepsilon}$, $i = 1, 2$. Let $\bar{u} = u^1 - u^2$, $\bar{\lambda} = \lambda^1 - \lambda^2$, and $\bar{\sigma} = \sigma^1 - \sigma^2$. Then, by (3.1), we see that

$$(3.38a) \qquad (a(u^1)\bar{\lambda}, \mu) - (\bar{\sigma}, \mu) + (\tilde{a}_u(u^2)\lambda^2 + \tilde{b}_u(u_2))\bar{u}, \mu = 0, \quad \forall \mu \in \Lambda_h,$$

$$(3.38b) \qquad (\bar{\lambda}, v) - (\bar{u}, \nabla . v) = 0, \quad \forall v \in V_h,$$

$$(3.38c) \qquad (\nabla . \bar{\sigma}, w) + (\bar{c}_u(u^2)\bar{u}, w) = 0, \quad \forall w \in W_h.$$

Then, as in the linear case [5], we have

$$(3.39) \qquad \|\bar{\lambda}\| + \|\bar{\sigma}\|_V \leqslant C_1 \|\bar{u}\|.$$

Also, we rewrite (3.38) in the form

$$\begin{aligned}
&(a(u)\bar{\lambda}, \mu) - (\bar{\sigma}, \mu) + ((\tilde{a}_u(u^2)\lambda^2 + \tilde{b}_u(u_2))\bar{u}, \mu) = (\tilde{a}_u(u^1)\bar{\lambda}(u - u^1), \mu), && \forall \mu \in \Lambda_h, \\
&(\bar{\lambda}, v) - (\bar{u}, \nabla . v) = 0, && \forall v \in V_h, \\
&(\nabla . \bar{\sigma}, w) + (\bar{c}_u(u^2)\bar{u}, w) = 0, && \forall w \in W_h.
\end{aligned}$$

Then, apply (3.15) to this system to see that

$$\|\bar{u}\| \leqslant C_1(\|\bar{\lambda}\| + \|\bar{\sigma}\|_V)h,$$

which, together with (3.39), implies that

$$\|\bar{u}\| \leqslant C_1 h \|\bar{u}\|.$$

Thus, $u^1 = u^2$ for h small enough. So, (3.39) yields that $\lambda^1 = \lambda^2$ and $\sigma^1 = \sigma^2$. Hence, the uniqueness is shown.

3.4. $H^{-s}(\Omega)$-error estimates

Apply (3.16) to (3.27) with $\theta = 2$ to see that

$$\begin{aligned}
\|z\|_{-s} \leqslant C_1\{ &(\|\alpha\| + \|d\|)h^{\min(s+1, k+1)} + \|P_h u - u\| h^{\min(s+1, k^*)} \\
(3.40) \qquad &+ \|\nabla . d\| h^{\min(s+2, k^*)} + \|P_h u - u\|_{-s-1}\}.
\end{aligned}$$

Then it follows from (3.2c) and (3.29) that

$$\|u - u_h\|_{-s} \leqslant \|u - P_h u\|_{-s} + \|z\|_{-s}$$

$$
(3.41) \qquad \leqslant C_1 \begin{cases} \|u\|_r h^{r+s} + \|u\|_{r_1} h^{r_1+s}, \\ \qquad 0 \leqslant s \leqslant k^* - 2, 2 \leqslant r \leqslant k+1, 2 \leqslant r_1 \leqslant k^*, \\ \|u\|_{r+1} h^{r+k^*} + \|u\|_{r_1+1} h^{r_1+k^*-1}, \\ \qquad s = k^* - 1, 1 \leqslant r \leqslant k+1, 1 \leqslant r_1 \leqslant k^*, \\ \|u\|_{r+1} h^{r+k^*} + \|u\|_{r_1+2} h^{r_1+k^*}, \\ \qquad s = k^*, 1 \leqslant r \leqslant k+1, 0 \leqslant r_1 \leqslant k^*. \end{cases}
$$

Now, let $\varphi \in H^s(\Omega)$. By (3.3a) and (3.27), we have

$$
\begin{aligned}
(d, \varphi) &= (d, \varphi - R_h \varphi) + (d, R_h \varphi) \\
&= (d, \varphi - R_h \varphi) + (a(u)\,\alpha, R_h \varphi) + (\tilde{\Gamma}_h z, R_h \varphi) - (\tilde{\Gamma}_h(P_h u - u), R_h \varphi) \\
&= (d, \varphi - R_h \varphi) - (a(u)\,\alpha, \varphi - R_h \varphi) + (\alpha, a\varphi - \Pi_h(a\varphi)) \\
&\quad + (z, \nabla . (a(u)\,\varphi)) - (\tilde{\Gamma}_h z, \varphi - R_h \varphi) + (\tilde{\Gamma}_h z, \varphi) \\
&\quad + (\tilde{\Gamma}_h(P_h u - u), \varphi - R_h \varphi) - (\tilde{\Gamma}_h(P_h u - u), \varphi),
\end{aligned}
$$

so that

$$
\begin{aligned}
|(d, \varphi)| \leqslant C_1\{(&\|d\| + \|\alpha\| + \|z\| + \|P_h u - u\|\, h^{\min(s, k+1)}) \\
&+ \|z\|_{-s+1} + \|P_h u - u\|_{-s}\}\, \|\varphi\|_s .
\end{aligned}
$$

This inequality, together with (3.29b), (3.2c) and (3.40), implies that

$$
(3.42) \qquad \|\sigma - \sigma_h\|_{-s} \leqslant C_1 \begin{cases} \|u\|_{r+1} h^{r+s} + \|u\|_{r_1+1} h^{r_1+s}, \\ \qquad 0 \leqslant s \leqslant k^* - 1, 1 \leqslant r \leqslant k+1, 1 \leqslant r_1 \leqslant k^*, \\ \|u\|_{r+1} h^{r+k^*} + \|u\|_{r_1+2} h^{r_1+k^*}, \\ \qquad s \leqslant k^*, 1 \leqslant r \leqslant k+1, 0 \leqslant r_1 \leqslant k^*. \end{cases}
$$

The same result holds for $\lambda - \lambda_h$ by means of a similar argument. Finally, using (3.2c) and (3.27c), we see that, for $\varphi \in H^s(\Omega)$,

$$
\begin{aligned}
(\nabla . d, \varphi) &= (\nabla . d, \varphi - P_h \varphi) + (\nabla . d, P_h \varphi) \\
&= (\nabla . d, \varphi - P_h \varphi) - (\tilde{c}_u(u_h)\, z, \varphi) + (\tilde{c}_u(u_h)\, z, \varphi - P_h \varphi) \\
&\quad + (\tilde{c}_u(u_h)\,(P_h u - u), \varphi) + (\tilde{c}_u(u_h)\,(P_h u - u), P_h \varphi - \varphi).
\end{aligned}
$$

Consequently, we have

$$
(3.43) \qquad \begin{aligned}
\|\nabla . (\sigma - \sigma_h)\|_{-s} &\leqslant C\{(\|\nabla . d\| + \|z\| + \|P_h u - u\|)\, h^{\min(s, k^*)} \\
&\qquad + \|z\|_{-s} + \|P_h u - u\|_{-s}\} \\
&\leqslant C_1 \|u\|_{r+2} h^{r+s}, 0 \leqslant s, r \leqslant k^*.
\end{aligned}
$$

The results in (3.41)-(3.43) can be summarized in the following theorem.

THEOREM 3.4: *Let Ω be $(s+2,2)$-regular with respect to M. Then for h sufficiently small the results in* (3.41)-(3.43) *hold.*

3.5. L^p-error estimates

The next theorem can be easily shown from (3.2c), (3.3b), the triangle inequality, and the quasiregularity of T_h.

THEOREM 3.5: *There exists a constant C_1 independent of h such that*

$$\|u-u_h\|_{0,p} \leqslant C_1(\|u\|_{r+1}h^r + \|u\|_{r_1,p}h^{r_1} + \|\nabla\cdot\sigma\|_{r_1}h^{r_1+\min(1,k^*-1)}),$$
$$1 \leqslant r \leqslant k+1, \quad 0 \leqslant r_1 \leqslant k^*, \quad 2 \leqslant p \leqslant \infty .$$

4. POSTPROCESSING AND SUPERCONVERGENCE

In this section, we consider a postprocessing scheme, which leads to a new, more accurate approximation to the solution than u_h. The present scheme is an extension to the nonlinear case of the postprocessing procedure considered in [5] for the expanded mixed method for the linear problem. A similar approach for the usual linear method is given in [18]. Let

$$W_h^* = \{w \in W : W|_E \in R(E) \text{ for each } E \in \mathscr{E}_h\},$$

where $R(E) = P_{k^*}(E)$ if $E \in \mathscr{E}_h$ is a triangle and $R(E) = P_{k^*}(E) \otimes P_{k^*}(E)$ if $E \in \mathscr{E}_h$ is a rectangle. Then the postprocessing scheme is given for $u_h^* \in W_h^*$ as the solution of the system

$$(u_h^*, 1)_E = (u_h, 1)_E, E \in \mathscr{E}_h, \tag{4.1a}$$

$$(a(u_h^*)\nabla u_h^* - b(u_h^*), \nabla v)_E + (c(u_h), v)_E = (f, v)_E - \langle \sigma_h \cdot \nu_E, v\rangle_{\partial E}, \quad \forall v \in R(E), E \in \mathscr{E}_h, \tag{4.1b}$$

where (u_h, σ_h) is the solution of (3.1) and ν_E is the outer unit normal to E.

To see that there exists at least one solution u_h^* to (4.1), let us consider the map S: $W_h^* \to W_h^*$ defined by

$$(Sy, 1)_E = (u_h, 1)_E, E \in \mathscr{E}_h, \tag{4.2a}$$

$$(a(y)\nabla(Sy) - b(y), \nabla v)_E + (c(u_h), v)_E = (f, v)_E - \langle \sigma_h \cdot \nu_E, v\rangle_{\partial E}, \quad \forall v \in R(E), E \in \mathscr{E}_h, \tag{4.2b}$$

for $y \in W_h^*$. Note that, by (3.1c),

$$(c(u_h), v)_E = (f, v)_E - \langle \sigma_h \cdot \nu_E, v\rangle_{\partial E}, \quad \forall v \in P_0(E),$$

so that the linear equations (4.2) define S uniquely. Now, choose $v = Sy$ in (4.2b) to see that the range of S is contained in a ball. Since S is clearly continuous, the Brower fixed point theorem implies that (3.4) has a solution, as illustrated in Theorem 3.2. The argument in § 3.3 can also be used to show uniqueness of the solution for h sufficiently small.

To carry out an error analysis for (4.1), we also need a family $\{U_h\}_{0<h<1}$ of continuous spaces in $\bar{\Omega}$, which are piecewise polynomials over $\mathscr{E}_h$, such that

$$\inf\{\|v-\xi\| + h\|\nabla(v-\xi)\| + h^2\|v-\xi\|_{1,6} : \xi \in U_h\} \leqslant C\|v\|_s h^s, \tag{4.3}$$

if $2 \leqslant s \leqslant k^*+1$. Finally, let P_E denote the L^2-projection onto $P_0(E)$. Because of the finite dimensionality of each U_h, the infimum in (4.3) is achieved. Let $\tilde{u}_h \in V_h$ be such that $\|u-\tilde{u}_h\| + h\|\nabla(u-\tilde{u}_h)\| + h^2\|u-\tilde{u}_h\|_{1,6}$ is minimal. Then it follows from (4.3) that

$$\|\nabla\tilde{u}_h\|_{0,6} \leqslant C\|u\|_{1,6} \leqslant C\|u\|_{2+\varepsilon}. \tag{4.4}$$

THEOREM 4.1: *Let $u \in H^{2+\varepsilon}(\Omega) \cap H^{k+2}(\Omega)$ be the solution of* (2.1) *and u_h^* be the solution of* (4.1). *Then*

$$\|u-u_h^*\| \leqslant C_1\|u\|_r h^{k^*+1}, \quad r = \max(k^*+1, 3). \tag{4.5}$$

Proof: By (2.1) and the relation $\sigma = -(a(u)\nabla u - b(u))$, we see that

$$(a(u)\nabla u - b(u), \nabla v)_E + (c(u), v)_E = (f, v)_E - \langle \sigma \cdot \nu_E, v\rangle_{\partial E}, \quad \forall v \in R(E). \tag{4.6}$$

Consequently, subtract (4.1) from (4.6) to yield the error equation

$$\begin{aligned}(a(u)\nabla u - a(u_h^*)\nabla u_h^*, \nabla v)_E - (b(u) - b(u_h^*), v)_E + (c(u) - c(u_h^*), v)_E \\ = \langle(\sigma - \sigma_h)\cdot\nu_E, v\rangle_{\partial E}, \quad \forall v \in R(E).\end{aligned}$$

This inequality, together with (2.1c), implies that

$$\begin{aligned}
&a_0\|\nabla(\tilde{u}_h - u_h^*)\|_E^2 \\
&= a_0\|\nabla(I-P_E)(\tilde{u}_h - u_h^*)\|_E^2 \\
&\leqslant (a(u_h^*)\nabla(I-P_E)(\tilde{u}_h - u_h^*), \nabla(I-P_E)(\tilde{u}_h - u_h^*))_E \\
&= (a(u)\nabla(\tilde{u}_h - u), \nabla(\tilde{u}_h - u_h^*))_E + ([a(u_h^*) - a(u)]\nabla\tilde{u}_h, \nabla(\tilde{u}_h - u_h^*))_E \\
&\quad + (b(u) - b(u_h^*), \nabla(\tilde{u}_h - u_h^*))_E - (c(u) - c(u_h), (I-P_E)(\tilde{u}_h - u_h^*))_E \\
&\quad - \langle(\sigma - \sigma_h)\cdot\nu_E, (I-P_E)(\tilde{u}_h - u_h^*)\rangle_{\partial E} \qquad (4.7) \\
&\leqslant C\|\nabla(\tilde{u}_h - u_h^*)\|_E\,\|\nabla(\tilde{u}_h - u_h^*)\|_E \\
&\quad + \|a(u_h^*) - a(u)\|_{0,3,E}\,\|\nabla\tilde{u}_h\|_{0,6,E}\,\|\nabla(\tilde{u}_h - u_h^*)\| \\
&\quad + \|b(u) - b(u_h^*)\|_E\,\|\nabla(\tilde{u}_h - u_h^*)\|_E + \|c(u) - c(u_h)\|_E\,\|(I-P_E)(\tilde{u}_h - u_h^*)\|_E \\
&\quad + \left(h_E\int_{\partial E}|(\sigma_h - \sigma)\cdot\nu_E|^2\,ds\right)^{1/2}\left(h_E^{-1}\int_{\partial E}|(I-P_E)(\tilde{u}_h - u_h^*)|^2\,ds\right)^{1/2}.
\end{aligned}$$

Note that a scaling argument implies that

$$\|(I-P_E)(\tilde{u}_h - u_h^*)\|_E \leqslant Ch_E\|\nabla(I-P_E)(\tilde{u}_h - u_h^*)\|_E. \tag{4.8}$$

Exploit (4.4), (4.7) and (4.8) to obtain

$$\begin{aligned}\|\nabla(\tilde{u}_h - u_h^*)\|_E \leqslant C_1\{ & \|\nabla(\tilde{u}_h - u)\|_E + \|a(u_h^*) - a(u)\|_{0,3,E} \\ & + \|b(u) - b(u_h^*)\|_E + h_E\|c(u) - c(u_h)\|_E \\ & + \left(h_E\int_{\partial E} |(\sigma_h - \sigma)\cdot\nu_E|^2\, ds\right)^{1/2}\right\}.\end{aligned} \tag{4.9}$$

Now, using the interpolation result

$$\|\phi\|_{0,3,E} \leqslant C\|\phi\|_E^{1/2}\ \|\nabla\phi\|_E^{1/2},$$

it follows from (4.8), (4.9), and the assumption on the coefficients a, b and c that

$$\begin{aligned}\|\tilde{u}_h - u_h^*\|_E \leqslant C_1\, h_E\{ & \|\nabla(\tilde{u}_h - u)\|_E + \|u - u_h^*\|_E + h_E\|u - u_h\|_E \\ & + \left(h_E\int_{\partial E} |(\sigma_h - \sigma)\cdot\nu_E|^2\, ds\right)^{1/2}\right\} + \|P_E(\tilde{u}_h - u_h^*)\|_E.\end{aligned} \tag{4.10}$$

Since P_E is bounded, it follows by (4.1a) that

$$\|P_E(\tilde{u}_h - u_h^*)\|_E \leqslant \|\tilde{u}_h - u\|_E + \|P_h u - u_h\|_E,$$

which, together with (4.10), yields that

$$\begin{aligned}\|\tilde{u}_h - u_h^*\|_E \leqslant C_1\, h_E\{ & \|\nabla(\tilde{u}_h - u)\|_E + \|u - u_h^*\|_E + h_E\|u - u_h\|_E \\ & + \left(h_E\int_{\partial E} |(\sigma_h - \sigma)\cdot\nu_E|^2\, ds\right)^{1/2}\right\} \\ & + \|\tilde{u}_h - u\|_E + \|P_h u - u_h\|_E.\end{aligned}$$

Sum this expression over all $E \in \mathscr{E}_h$ to obtain

$$\begin{aligned}\|\tilde{u}_h - u_h^*\| & \\ \leqslant C_1\Bigg\{ h\Bigg(& \|\nabla(\tilde{u}_h - u)\| + h\|u - u_h\| + \left(\sum_{E\in\mathscr{E}_h} h_E\int_{\partial E} |(\sigma - \Pi_h\sigma)\cdot\nu_E|^2\, ds\right)^{1/2} \\ & + \left(\sum_{E\in\mathscr{E}_h} h_E\int_{\partial E} |(\Pi_h\sigma - \sigma_h)\cdot\nu_E|^2\, ds\right)^{1/2}\Bigg) + \|\tilde{u}_h - u\| + \|P_h u - u_h\|\Bigg\} \\ \leqslant C_1\Bigg\{ h\Bigg(& \|\nabla(\tilde{u}_h - u)\| + h\|u - u_h\| + \left(\sum_{E\in\mathscr{E}_h} h_E\int_{\partial E} |(\sigma - \Pi_h\sigma)\cdot\nu_E|^2\, ds\right)^{1/2} \\ & + \|\sigma - \Pi_h\sigma\| + \|\sigma - \sigma_h\|\Bigg) + \|\tilde{u}_h - u\| + \|P_h u - u_h\|\Bigg\},\end{aligned}$$

for h sufficiently small. Finally, apply (3.29), (3.37), (4.3), and the approximation property of Π_h to obtain the desired result (4.5). □

5. EXTENSION TO A NONLINEAR PROBLEM

In this section, we extend the previous analysis to the nonlinear problem

$$-\nabla . A(x, \nabla u) = f(x) \quad \text{in } \Omega , \tag{5.1a}$$
$$u = -g \quad \text{on } \partial\Omega , \tag{5.1b}$$

and point out a difference between the usual mixed method and the expanded mixed method. We assume that A: $\bar{\Omega} \times \mathbb{R}^n \to \mathbb{R}^n$ is twice continuously differentiable with bounded derivatives through second order and that (5.1) is strictly elliptic at λ in the sense that there is a constant $a_0 > 0$ such that

$$\xi^T DA(x, \lambda)\, \xi \geq a_0 \| \xi \|^2_{\mathbb{R}^n}, \quad \xi \in \mathbb{R}^n, \quad (x, \lambda) \in \bar{\Omega} \times \mathbb{R}^n , \tag{5.2}$$

where $DA(x, \lambda) = (\partial A_i / \partial \lambda_j)$ is the $n \times n$ Jacobian matrix. The variable x is omitted in the notation below.

Using the previous notation, the expanded mixed form for (5.1) is formulated as follows:
Find $(\sigma, \lambda, u) \in V \times \Lambda \times W$ such that

$$(A(\lambda), \mu) + (\sigma, \mu) = 0, \quad \forall \mu \in \Lambda , \tag{5.3a}$$
$$(\lambda, v) + (u, \nabla . v) = (g, v . \nu)_{\partial\Omega}, \quad \forall v \in V , \tag{5.3b}$$
$$(\nabla . \sigma, w) = (f, w), \quad \forall w \in W .$$

As in Theorem 2.2, it can be shown that (5.3) has a unique solution and is equivalent to (5.1) through the relations

$$\lambda = \nabla u \quad \text{and} \quad \sigma = -A(\nabla u) .$$

The expanded mixed solution of (5.1) is $(\sigma_h, \lambda_h, u_h) \in V_h \times \Lambda_h \times W_h$ satisfying

$$(A(\lambda_h), \mu) + (\sigma_h, \mu) = 0, \quad \forall \mu \in \Lambda_h , \tag{5.4a}$$
$$(\lambda_h, v) + (u_h, \nabla . v) = 0, \quad \forall v \in V_h , \tag{5.4b}$$
$$(\nabla . \sigma_h, w) = (f, w), \quad \forall w \in W_h . \tag{5.4c}$$

Also, using the arguments in § 3, it can be seen that (5.4) has a unique solution for $h > 0$ sufficiently small and produces optimal error estimates in L^p and H^{-s}. In particular, we state the L^2 - error estimates as follows:

$$\| u - u_h \| \leq C_1 \begin{cases} \| u \|_r h^r, & 2 \leq r \leq k^*, k \geq 2 , \\ \| u \|_2 h, & k = 1, \text{in the case of } k^* = k , \\ \| u \|_2 h^{k+1}, & k = 0, 1, \text{in the case of } k^* = k + 1 , \end{cases} \tag{5.5a}$$

$$\| \lambda - \lambda_h \| \leq C_1 \| u \|_{r+1} h^r, \quad 1 \leq r \leq k + 1 , \tag{5.5b}$$
$$\| \sigma - \sigma_h \| \leq C_1 \| u \|_{r+1} h^r, \quad 1 \leq r \leq k + 1 , \tag{5.5c}$$
$$\| \nabla . (\sigma - \sigma_h) \| \leq C_1 \| \nabla . \sigma \|_r h^r, \quad 0 \leq r \leq k^* , \tag{5.5d}$$

$$\|u_h - P_h u\| \leqslant C_1 \begin{cases} \|u\|_{k+2} h^{k+2}, & k \geqslant 2, \\ \|u\|_3 h^2, & k = 1, \text{ in the case of } k^* = k, \\ \|u\|_3 h^{k+2}, & k = 0, 1, \text{ in the case of } k^* = k+1. \end{cases} \tag{5.5e}$$

The postprocessing scheme can be easily defined here; using (5.5e), analogous superconvergence results can be obtained. In the present case, we are able to obtain the superconvergence result (5.5e), which is of order $O(h^{k+2})$ in both cases where $k^* = k$ and $k^* = k + 1$, since the coefficient A depends on λ instead of u. The vector variable has the error estimate of higher order, as shown in (5.5b).

We point out that attempts at using the usual mixed method based on the Brezzi-Douglas-Marini mixed finite elements ($n = 2$) [4] and the Brezzi-Douglas-Durán-Fortin mixed finite elements ($n = 3$) [2] (or some of the Chen-Douglas mixed finite elements [8]) for (5.1) are not entirely successful, as shown in [7], because the error equations couple the scalar variable u and the flux variable σ. Consequently, the errors of the scalar influence those of the flux. Hence, the error estimates for the flux variable are not optimal since these mixed spaces use higher-order polynomials for this variable than for the scalar. However, the expanded mixed method decouples the flux error equations from the scalar equations; as a consequence, optimal error estimates can be obtained for both the flux and scalar variables, as shown in (5.5).

6. IMPLEMENTATION AND NUMERICAL RESULTS

In this section we present numerical results for the model problem

$$-\nabla \cdot (a(u) \nabla u) = f \quad \text{in } \Omega, \tag{6.1a}$$

$$u = -g \quad \text{on } \partial\Omega. \tag{6.1b}$$

Before this, we need to consider implementation techniques for solving for the corresponding mixed method solution $(\sigma_h, \lambda_h, u_h) \in V_h \times \Lambda_h \times W_h$, where

$$(a(u_h) \lambda_h, \mu) - (\sigma_h, \mu) = 0, \quad \forall \mu \in \Lambda_h, \tag{6.2a}$$

$$(\lambda_h, v) - (u_h, \nabla \cdot v) = (g, v \cdot \nu)_{\partial\Omega}, \quad \forall v \in V_h, \tag{6.2b}$$

$$(\nabla \cdot \sigma_h, w) = (f, w), \quad \forall w \in W_h. \tag{6.2c}$$

A linearized version of (6.2) is constructed as follows. Starting from any $(\sigma_h^0, \lambda_h^0, u_h^0) \in V_h \times \Lambda_h \times W_h$, we construct the sequence $(\sigma_h^m, \lambda_h^m, u_h^m) \in V_h \times \Lambda_h \times W_h$, by solving

$$(a(u_h^{m-1}) \lambda_h^m, \mu) - (\sigma_h^m, \mu) = 0, \quad \forall \mu \in \Lambda_h, \tag{6.3a}$$

$$(\lambda_h^m, v) - (u_h^m, \nabla \cdot v) = (g, v \cdot \nu)_{\partial\Omega}, \quad \forall v \in V_h, \tag{6.3b}$$

$$(\nabla \cdot \sigma_h^m, w) = (f, w), \quad \forall w \in W_h. \tag{6.3c}$$

The ideas in [6] can be used to show that the sequence $\{(\sigma_h^m, \lambda_h^m, u_h^m)\}$ converges to $(\sigma_h, \lambda_h, u_h)$. Consequently, since (6.3) is linear for each m, the implementation techniques discussed in [5] for the linear expanded mixed method (e.g., alternating-direction iterative methods, hybridization methods, and preconditioned iterative methods) can be applied here.

We now present two two-dimensional problems on the unit square with the Dirichlet boundary condition (5.1b) or (6.1b). In the first example, the coefficient $a(u)$ in (6.1a) is taken to be of the form $a(u) = u$. The true solution is

$$u(x, y) = x^2 + y^2 + \sin(x) \cos(y),$$

with f and g defined accordingly by (6.1). The expanded mixed formulation is discretized by means of the lowest-order Brezzi-Douglas-Marini space [4] on rectangles as in (6.4). Namely, we solve a cell-centered finite difference system for the scalar u over a uniform rectangular decomposition of Ω. In Table 1 we show the errors and convergence rates. Note that the orders of convergence in L^2 and L^∞ are two in all cases. So, in fact, we have a superconvergent result for the scalar u.

Table 1. — Convergence rates for the scalar in example one.

$1/h$	L^∞-error($\times 10^2$)	L^∞-order	L^∞-error($\times 10^2$)	L^∞-order
5	1.550	—	1.470	—
10	0.470	1.73	0.380	1.95
20	0.120	1.97	0.091	2.06
40	0.029	2.05	0.022	2.05

In the second example, the coefficient $A(\nabla u)$ in (5.1a) is defined by

$$A(v) = (v_1, 3\, v_2/2 - \sin(2\, v_2)/4), \quad v = (v_1, v_2),$$

$g \equiv 0$ in (5.1b), and f in (5.1a) is given by

$$f(x, y) = 2(y - y^2) + (x - x^2)(3 - \cos(2(x - x^2)(1 - 2y))).$$

Problem (5.1) has a unique solution [12] for such chosen functions. The Brezzi-Douglas-Marini space [4] of lowest order on a uniform triangular decomposition of Ω is exploited this time. Tables 2 and 3 show the errors and convergence rates for the scalar and the flux variable, respectively. The convergence rate for the scalar is $O(h)$, while it is $O(h^2)$ for the flux. The numerical results in Tables 1, 2 and 3 confirm the theoretical results from the previous sections.

Table 2. — Convergence rates for the scalar in example two.

$1/h$	L^∞-error($\times 10^2$)	L^∞-order	L^2-error($\times 10^2$)	L^∞-order
5	3.57	—	2.50	—
10	1.89	0.91	1.20	0.99
20	0.99	0.93	0.63	1.02
40	0.52	0.98	0.30	1.09

Table 3. — Convergence rates for the flux in example two.

$1/h$	L^∞-error($\times 10^2$)	L^∞-order	L^2-error($\times 10^2$)	L^∞-order
5	1.870	—	1.540	—
10	0.540	1.79	0.430	1.84
20	0.140	1.94	0.110	1.97
40	0.032	2.12	0.027	2.03

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