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ON CONSERVATIVE AND ENTROPIC DISCRETE AXISYMMETRIC FOKKER-PLANCK OPERATORS (*)

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Abstract — *We study, in axisymmetric geometry, a discretization of the Fokker-Planck operator that preserves the physical properties which are decrease of the kinetic entropy and conservation of mass, momentum and energy and only those quantities*

For this purpose, we exhibit how the above properties are consequences, first, of the algebraic structure *of the Landau form of the Fokker-Planck operator and, secondly, of an* integration step *Then we show that, even in our particular geometry, it is easy to make discretizations preserving the* algebraic structure *Concerning the second point we provide an analysis inducing necessary and sufficient conditions on the discrete derivation operators Consequently, a discrete Fokker-Planck operator decreasing the kinetic entropy and conserving mass, momentum and energy is easy to build Yet, a discrete Fokker-Planck operator conserving only those quantities is not so easy to get and in particular it cannot involve vertex-independent finite difference operators We then build an actual implemented operator which we validate on physically realistic examples of plasma collisions © Elsevier, Paris*

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Résumé — *Nous étudions en géométrie axisymétrique, une discrétisation de l'opérateur de Fokker-Planck respectant les propriétés physiques importantes que sont la décroissance de l'entropie cinétique et la conservation de la masse, de l'impulsion, de l'énergie et exclusivement ces trois quantités*

Pour ce faire, nous montrons que ces propriétés sont la conséquence de la structure algébrique *de l'opérateur de Fokker-Planck écrit sous la forme de Landau d'une part, et d'autre part d'une* relation intégrale *Puis nous montrons que même en géométrie axisymétrique, il est simple de réaliser des discrétisations préservant la* structure algébrique *Concernant le second point, nous déduisons une condition nécessaire et suffisante sur les opérateurs de dérivation discrets pour préserver la* relation intégrale *En conséquence, il est simple de construire des opérateurs de Fokker-Planck discrets réalisant la décroissance de l'entropie cinétique et la conservation de la masse, de l'impulsion et de l'énergie En revanche, l'obtention d'un opérateur conservant exclusivement ces quantités et plus délicate En particulier, il ne peut se construire à l'aide d'opérateurs de dérivation discrets uniformément définis sur le maillage Enfin, nous construisons l'opérateur effectivement implémenté dans notre code que nous validons sur des exemples physiquement réalistes de collisions de plasmas © Elsevier, Paris*

1. INTRODUCTION

We present a discrete Fokker-Planck operator, in cylindrical coordinates $(v_\parallel, v_\perp)$ which, as does the continuous one in the following homogeneous in space Fokker-Planck equation

$$\begin{cases} \partial_t f = (\partial_t f)_{coll} = P(f(t,.), f(t,.)), \\ f_{|t=0} = f_0, \end{cases} \tag{1.1}$$

possesses important physical properties: decrease of the kinetic entropy, conservation of mass, momentum and energy and of only those quantities.

This study is carried out in two successive steps. First of all, following the idea developped in B. Lucquin-Desreux [18] and P. Degond & B. Lucquin-Desreux [12] for the whole 3D case, we observe that writing the

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Fokker-Planck operator in the Landau form involving logarithms, allows to derive the decrease of the kinetic entropy and a characterization of the collisional invariants by a system of first order differential equations Those properties are consequences of the mere *algebraic structure* of the operator, and they are then valid for both continuous and discrete Fokker-Planck operators as soon as the discretization does not break down this *algebraic structure* The second point, that we call *integration step,* consists in solving the system of differential equations obtained in the first step and gives for the continuous Fokker-Planck operator the conservation properties The main difficulty appears here, since at the discrete level the solutions derived are closely related to the particular choice of the discrete derivation operator used to approximate the gradient This means that in some cases, additional collisional invariants may appear, which do not have any physical meaning This situation, which was already present in [18] and [12] is far more drastic here, due to the context of axisymmetric geometry

We now present the starting point of our approach, in axisymmetric geometry, writing the Fokker-Planck operator in the Landau form with logarithms, i e

$$
\begin{aligned}
&P(f,f) = \mathrm{Div}\, p(f,f)\,, && (a)\\
&p(f,f)\,(v) = \int_{\mathscr{V}} f(V)\, f(V^1)\, \Phi(v - v^1)\,.\,(\mathrm{Grad\,Log} f(v) - \mathrm{Grad\,Log} f(v^1))\, d\sigma^1\, d\alpha^1\,, && (b)
\end{aligned}
\qquad (1\ 2)
$$

with $d\sigma^1 = v_\perp^1\, dv_\parallel^1\, dv_\perp^1$ In formula (1 2), $\mathscr{V} = \mathbb{R} \times \mathbb{R}_+^* \times (0, 2\pi)$, and $v = (V, \alpha) = (v_\parallel, v_\perp, \alpha)$ is a cylindrical system of coordinates (the notation $v_\parallel$ and $v_\perp$ will be precised later on), while Div and Grad denote the divergence and gradient operators The velocity distribution $f \equiv f(V)$ does not depend on α, yielding an operator $P(f,f)$ which also does not depend on α (we shall show this fact in Section 2) At last, $\Phi(w)$ is the tensor

$$
\Phi(w) = \frac{1}{|w|}\left[I - \frac{w \otimes w}{|w|^2}\right] \qquad (1\ 3)
$$

Since $\left[I - \frac{w \otimes w}{|w|^2}\right]$ is the projection operator onto the plane orthogonal to w, $\Phi(w)$ is semi-definite positive and its null set is

$$
\mathrm{Ker}\, \Phi(w) = w\mathbb{R} \qquad (1\ 4)
$$

These two purely algebraic properties of the tensor Φ, coupled with the fact that $(-\mathrm{Div})$ and (Grad) are adjoint operators, are precisely what we call the *algebraic structure* of the Fokker-Planck operator

Physically speaking, the equation (1 1) under consideration is a model for the evolution in time t of an α-independent velocity distribution $f(t, V)$ of a spatially uniformly distributed, fully ionised and hot plasma, made of one species of particles which is not submitted to any external force

Since, by use of a splitting in time algorithm, a numerical method for solving (1 1) also permits to simulate the evolution of a non spatially uniformly distributed plasma, the independence with respect to the position variable is actually not restrictive Yet, the α-independence is usually a consequence of some assumptions made on the spatial distribution of the plasma

One of these is when the spatial distribution is only varying in one fixed direction $\vec{r}$ Introducing then r as a coordinate in this direction, the Vlasov-Fokker-Planck equation describing the evolution of the plasma writes

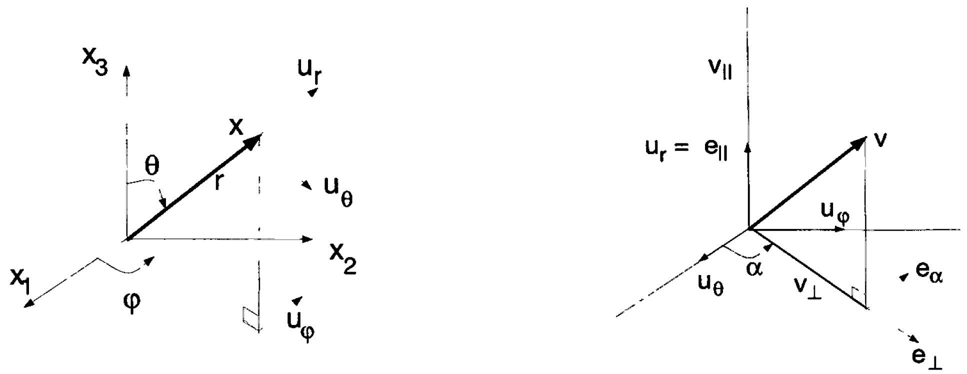
$$
\begin{cases}
(\partial_t f + v_\parallel\, \partial_r f)\,(t, r, V) = P(f(t, r, .\,), f(t, r, .\,))\,(V)\,,\\
f_{|t=0} = f_0
\end{cases}
\qquad (1\ 5)
$$

In this context, $v_\parallel = v\ \dfrac{\vec{r}}{|\vec{r}|}$

Another assumption yielding the α-independence is when the plasma is spatially isotropic. Then the distribution function only depends on the distance $r = |x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ between x and the origin, on the velocity component $v_\parallel$ parallel to $x \left(v_\parallel = v \cdot \frac{x}{r} \right)$, and on the modulus $v_\perp$ of the velocity projection onto the plane orthogonal to x. In order to give a clear meaning to the variables used there, we introduce the spherical coordinate system (r, θ, φ) for the position defined by

$$x_1 = r \sin \theta \cos \varphi, \quad x_2 = r \sin \theta \sin \varphi, \quad x_3 = r \cos \theta . \tag{1.6}$$

Denoting by (e_1, e_2, e_3) the basis associated with the coordinate system (x_1, x_2, x_3), the usual local basis associated with (r, θ, φ) is $(u_r, u_\theta, u_\varphi)$, defined by $u_r = x/|x| = \sin \theta \cos \varphi e_1 + \sin \theta \sin \varphi e_2 + \cos \theta e_3$, $u_\theta = \cos \theta \cos \varphi e_1 + \cos \theta \sin \varphi e_2 - \sin \theta e_3$ and $u_\varphi = - \sin \varphi e_1 + \cos \varphi e_2$ (see *fig.* 1.1.a); we denote by $(v_r, v_\theta, v_\varphi)$ the coordinates of the velocity v in this previously defined local basis. Introducing at last the cylindrical system (V, α), $V = (v_\parallel, v_\perp)$ from the coordinates $(v_r, v_\theta, v_\varphi)$ by the relation (see *fig.* 1.1.b)



a. Spherical coordinates for position **b. Cylindrical coordinates for velocity.**

Figure 1.1. — Change of coordinates.

$$v_r = v_\parallel , \quad v_\theta = v_\perp \cos \alpha, \quad v_\varphi = v_\perp \sin \alpha , \tag{1.7}$$

the Vlasov-Fokker-Planck equation for the distribution function $f \equiv f(t, r, v_\parallel, v_\perp)$ writes

$$\begin{cases} \left(\partial_t f + v_\parallel \, \partial_r f + \frac{v_\perp^2}{r} \partial_{v_\parallel} f - \frac{v_\parallel \, v_\perp}{r} \partial_{v_\perp} f \right) (t, r, V) = P(f(t, r, .\,), f(t, r, .\,)) \, (V) , \\ f_{|t=0} = f_0 . \end{cases} \tag{1.8}$$

For the derivation of the Fokker-Planck model, we refer to N. A. Krall & A. W. Trivelpiece [14] where the model is obtained from physical considerations. We also refer to A. V. Bobylev [7] and to P. Degond & B. Lucquin-Desreux [11] where the Fokker-Planck operator is obtained as the first term of an asymptotic expansion of the Boltzmann operator with screened Coulomb potential. See also A. A. Arsenev & O. E. Buryak [3] and L. Desvillettes [10] for a deduction of the Fokker-Planck operator from the Boltzmann one, but excluding the Coulomb case. From a theoretical viewpoint, A. A. Arsenev & N. V. Peskov [4] proved the existence of a solution to equation (1.1) for a short time.

The reader interested in older works concerning discretizations that do not destroy the decrease of entropy or the conservation properties is referred to J. C. Witney [22], I. F. Potapenko & V. A. Chuyanov [21], A. V. Bobylev, I. F. Potapento & V. A. Chuyanov [8], M. S. Pekker & V. N. Kudick [20] and Yu. A. Berezin, M. S. Pekker & V. N. Kudick [5].

Concerning actual numerical simulations of the Fokker-Planck equation, let us mention the pioneering work of W. M. Mac Donald, M. N. Rosenbluth & W. Chuck [19] who implemented a 1D code in the case of a distribution

f only depending on the velocity modulus $|v|$ Other simulations, making the same geometrical assumption as the one considered here, were done by S Jorna & L Wood [15] In the latter, the problem of conservation is not considered We also refer to O Larroche [16] who implemented a mass-conserving finite volume scheme An improvement of this method was realized by D Deck & G Samba [9] yielding the conservation of momentum and energy and using a correction method exposed in V V Aristov & F G Cheremisin [2] Last, we refer to M Lemou, C Buet, S Cordier & P Degond [17], for recent simulations of the 3D Fokker Planck equation, using the method described in [12] and [18] In this work, the cost induced by the 3D character of the problem is decreased using sub-mesh methods

The paper is organized in the following way In Section 2, we first analyze the whole continuous problem in the context of the axisymmetric geometry we show the decrease of the kinetic entropy and we characterize the collisional invariants In particular, we point out the crucial role played by the *algebraic structure* of the Fokker-Planck operator, which may be easily extended to the discrete case A class of discrete Fokker-Planck operators, involving finite differences, and preserving this *algebraic structure* is discussed in Section 3 Then necessary and sufficient conditions are given on the finite difference operators in order to preserve at the discrete level the solutions of the *integration step* We propose in Section 4 a discrete implemented operator that preserves all the expected quantities and only those ones Numerical results are finally given and compared with previous computations in Section 5

2. ALGEBRAIC STRUCTURE AND PHYSICAL PROPERTIES

In this Section, we show that the decrease of the kinetic entropy is a consequence of the mere *algebraic structure* of the Landau form of the Fokker-Planck operator Yet, the conservation properties are a consequence, in a first place, of this *algebraic structure* which yields a differential equation for the collisional invariants Then, in a second place, the *integration step* which consists in solving this equation, leads the conserved quantities which are mass, momentum and energy and only those ones

The velocity space is provided with a cylindrical coordinate system $(v_{\parallel}, v_{\perp}, \alpha)$ and is denoted by $\mathcal{V} = \Omega \times (0, 2\pi)$, $\Omega = \mathbb{R} \times \mathbb{R}_+^*$ The velocity variable is $v = (V, \alpha) \in \mathcal{V}$ with $V = (v_{\parallel}, v_{\perp}) \in \Omega$ and $\alpha \in (0, 2\pi)$ To each $v = (V, \alpha) \in \mathcal{V}$, is associated the classical orthonormal local basis $B_\alpha = (e_{\parallel}, e_{\perp}, e_\alpha)$, (see *fig* 1 1 b) and the coordinates of any vector A in B_α is denoted by $(A^{\parallel}, A^{\perp}, A^{\alpha})$ Let us adopt the following definitions

DEFINITION 2 1 *A real valued function* ψ $\Omega \times (0, 2\pi) \to \mathbb{R}$ *is called cylindrical if* $\psi(v) \equiv \psi(V)$ *does not depend on* α

DEFINITION 2 2 *A vector valued function* φ $\Omega \times (0, 2\pi) \to \mathbb{R}^3$ *is called cylindrical if its expression* $(\varphi(v))^{B_\alpha} = (\varphi^{\parallel}, \varphi^{\perp}, \varphi^{\alpha})$ *in the local basis* B_α *associated with any* $v = (V, \alpha) \in \mathcal{V}$, *satisfies*

$$\varphi^{\parallel} \text{ and } \varphi^{\perp} \text{ do not depend on } \alpha \text{ and } \varphi^{\alpha} = 0 \tag{2 1}$$

The Fokker Planck operator, which is considered as acting on cylindrical and positive functions writes

$$\begin{aligned} &P(f,f) = \operatorname{Div} p(f,f)\,, && (a) \\ &p(f,f)(v) = \int_{\mathcal{V}} f(V) f(V^1)\, \Phi(v - v^1) \quad (\operatorname{Grad} \operatorname{Log} f(v) - \operatorname{Grad}\operatorname{Log} f(v^1))\, d\sigma^1\, d\alpha^1 && (b) \end{aligned} \tag{2 2}$$

The gradient operator, acting on the cylindrical function $\operatorname{Log} f$, expresses in the local basis B_α associated with v :

$$(\operatorname{Grad} \operatorname{Log} f(v))^{B_\alpha} = \begin{pmatrix} \partial_{v_\parallel} \operatorname{Log} f(V) \\ \partial_{v_\perp} \operatorname{Log} f(V) \\ 0 \end{pmatrix}. \tag{2.3}$$

Of course, its expression can be given in any basis B_{α^1} associated with $v^1 = (V^1, \alpha^1)$ by

$$(\operatorname{Grad} \operatorname{Log} f(v))^{B_{\alpha^1}} = \begin{pmatrix} \partial_{v_\parallel} \operatorname{Log} f(V) \\ \partial_{v_\perp} \operatorname{Log} f(V) \cos(\alpha - \alpha^1) \\ \partial_{v_\perp} \operatorname{Log} f(V) \sin(\alpha - \alpha^1) \end{pmatrix}, \tag{2.4}$$

and then, computing the difference involved in (2.2), we get:

$$(\operatorname{Grad} \operatorname{Log} f(v))^{B_\alpha} - (\operatorname{Grad} \operatorname{Log} f(v^1))^{B_\alpha} = \begin{pmatrix} \partial_{v_\parallel} \operatorname{Log} f(V) - \partial_{v_\parallel} \operatorname{Log} f(V^1) \\ \partial_{v_\perp} \operatorname{Log} f(V) - \partial_{v_\perp} \operatorname{Log} f(V^1) \cos(\alpha^1 - \alpha) \\ - \partial_{v_\perp} \operatorname{Log} f(V^1) \sin(\alpha^1 - \alpha) \end{pmatrix}. \tag{2.5}$$

For different expressions of the tensor $\Phi(v - v^1)$, we refer to Annex A.

First, the following property (see Annex B for its proof) shows the adequacy of the Fokker-Planck operator with respect to the notion of cylindrical functions.

PROPOSITION 2.3: *If f is a cylindrical function then $p(f, f)$ and $P(f, f)$ are cylindrical.*

From now on, we always suppose that f is a positive cylindrical function, yielding an operator $P(f, f)$ that does not depend on α.

The *algebraic structure* of the operator (essentially (Grad) and ($-$ Div) are mutually adjoint operators and $\Phi(v - v^1)$ is proportional to a projection tensor) yields the following key point from which the physical properties follow.

PROPOSITION 2.4: *For every real valued cylindrical function ψ we have*

$$\int_\Omega P(f, f)(V)\, \psi(V)\, d\sigma = -\pi \int_{\Omega^2} f(V) f(V^1)\, \Phi(v - v^1) \cdot (\operatorname{Grad} \operatorname{Log} f(v) - \operatorname{Grad} \operatorname{Log} f(v^1)) \cdot (\operatorname{Grad} \psi(v) - \operatorname{Grad} \psi(v^1))\, d\sigma\, d\sigma^1. \tag{2.6}$$

We recall briefly the proof of this proposition which is classical. We have, for every cylindrical function ψ, the following weak formulation of the Fokker-Planck operator

$$\int_{\mathscr{V}} P(f, f)\, \psi\, d\sigma\, d\alpha = - \int_{\mathscr{V}^2} f(V) f(V^1)\, \Phi(v - v^1) \cdot (\operatorname{Grad} \operatorname{Log} f(v) - \operatorname{Grad} \operatorname{Log} f(v^1)) \cdot \operatorname{Grad} \psi(v)\, d\sigma\, d\sigma^1\, d\alpha\, d\alpha^1, \tag{2.7}$$

i.e.

$$\int_\Omega P(f,f)\,\psi\,d\sigma = -2\pi\int_{\Omega^2} f(V)\,f(V^1)\,\Phi(v-v^1)\cdot$$

$$(\operatorname{Grad}\operatorname{Log} f(v) - \operatorname{Grad}\operatorname{Log} f(v^1))\cdot\operatorname{Grad}\psi(v)\,d\sigma\,d\sigma^1. \quad (2.8)$$

Exchanging then the role of $v=(V,\alpha)$ and $v^1=(V^1,\alpha^1)$ we obtain, since Φ is an even function

$$\int_\Omega P(f,f)\,\psi\,d\sigma = 2\pi\int_{\Omega^2} f(V)\,f(V^1)\,\Phi(v-v^1)$$

$$(\operatorname{Grad}\operatorname{Log} f(v) - \operatorname{Grad}\operatorname{Log} f(v^1))\cdot\operatorname{Grad}\psi(v^1)\,d\sigma\,d\sigma^1, \quad (2.9)$$

and formula (2.6) follows simply by summing (2.8) and (2.9). ■

As a first consequence of this Proposition, replacing ψ by $\operatorname{Log} f$ in (2.6) and since $\Phi(w)$ is semi-definite positive, we get:

$$\int_\Omega P(f,f)\,(V)\operatorname{Log} f(V)\,d\sigma \leqslant 0\,. \quad (2.10)$$

Consider then $f(t,V)$, solution of the homogeneous in space Fokker-Planck equation

$$\partial_t f = P(f(t,.),f(t,.)),\quad f_{|t=0}=f_0>0\,. \quad (2.11)$$

We have

$$\frac{d}{dt}\int_\Omega f\operatorname{Log} f\,d\sigma = \int_\Omega \partial_t f(\operatorname{Log} f+1)\,d\sigma = \int_\Omega P(f,f)\,(\operatorname{Log} f+1)\,d\sigma\,, \quad (2.12)$$

and since $\int_\Omega P(f,f)\,d\sigma = 0$ the inequality (2.10) implies the following result which is a part of the so called H-Theorem.

COROLLARY 2.5: *The kinetic entropy* $\left(2\pi\int_\Omega f\operatorname{Log} f\,d\sigma\right)$, *with* $f=f(t,V)$ *solution of the Fokker-Planck equation (2.11) decreases with time.*

We now attend to the conservation properties by introducing first the

DEFINITION 2.6: *A cylindrical real valued function* ψ *is called a collisional invariant if*

$$\forall f \text{ cylindrical},\ \int_\Omega P(f,f)\,(V)\,\psi(V)\,d\sigma = 0\,. \quad (2.13)$$

Let us denote by $\mathscr{C}$ *the space made of all collisional invariants; we call the set* $\exp(\mathscr{C})$ *the "thermodynamical equilibrium set".*

The last point of this definition results from the following characterization of $\mathscr{C}$:

THEOREM 2.7: *First, a cylindrical function* ψ *belongs to* $\mathscr{C}$ *if and only if there exist* $\lambda\in\mathbb{R}$ *and* $\kappa\in\mathbb{R}$ *such that, for all* $V\in\Omega$ *we have*

$$\partial_{v_\parallel}\psi(V) = \lambda v_\parallel + \kappa \quad \text{and} \quad \partial_{v_\perp}\psi(V) = \lambda v_\perp\,. \quad (2.14)$$

Secondly, $P(f,f) = 0$ *if and only if* $f \in \exp(\mathscr{C})$.

The proof of this Theorem is a consequence of the Proposition 2.4 which gives the following preliminary result.

LEMMA 2.8: *A cylindrical function* ψ *belongs to* $\mathscr{C}$ *if and only if*

$$\forall (V, V^1) \in \Omega^2 \text{ and } \forall (\alpha, \alpha^1) \in (0, 2\pi)^2, \quad \text{Grad}\, \psi(v) - \text{Grad}\, \psi(v^1) \in \text{Ker}\, \Phi(v - v^1), \tag{2.15}$$

with the notations $v = (V, \alpha)$, *and* $v^1 = (V^1, \alpha^1)$.

Proof: Applying Proposition 2.4, formula (2.15) obviously implies (2.13). On the other hand, choosing $f = \exp(\psi)$ in (2.13), we get

$$\int_\Omega P(\exp(\psi), \exp(\psi))\, \psi\, d\sigma = -\pi \int_{\Omega^2} \exp(\psi)(V) \exp(\psi)(V^1)\, \Phi(v - v^1) \cdot$$

$$(\text{Grad}\, \psi(v) - \text{Grad}\, \psi(v^1)) \cdot (\text{Grad}\, \psi(v) - \text{Grad}\, \psi(v^1))\, d\sigma\, d\sigma^1. \tag{2.16}$$

The expression (2.16) is zero if and only if (2.15) holds true, proving the Lemma. ■

Proof of Theorem 2.7: Once (2.14) is established, the second point of the Theorem is obvious. In fact, we first notice that if $P(f,f) = 0$ then $\int_\Omega P(f,f)\, \psi\, d\sigma = 0$ for any cylindrical function ψ and applying (2.6) and Lemma 2.8, we get that $f \in \exp(\mathscr{C})$. Now, if $f \in \exp(\mathscr{C})$ the characterization (2.14) of $\mathscr{C}$ shows that $\text{Log} f$ satisfies (2.15) yielding $P(f,f) = 0$.

Concerning the first point, let us notice that, since an element of $\mathit{Ker}\, \Phi(v - v^1)$ is proportional to $(v - v^1)$ we have, applying again Lemma 2.8, $\psi \in \mathscr{C}$ if and only if there exists a real number $\lambda(V, \alpha, V^1, \alpha^1)$ such that

$$\text{Grad}\, \psi(v) - \text{Grad}\, \psi(v^1) = \lambda(V, \alpha, V^1, \alpha^1)(v - v^1), \tag{2.17}$$

for every $(v, v^1) \in (\Omega \times (0, 2\pi))^2$, with $v \neq v^1$. The main point consists now in showing that λ is in fact independant of V, α, V^1 and α^1. Rewritting (2.17), we get

$$\begin{cases} \partial_{v_\parallel} \psi(V) - \partial_{v_\parallel} \psi(V^1) = \lambda(V, \alpha, V^1, \alpha^1)(v_\parallel - v^1_\parallel), \\ \partial_{v_\perp} \psi(V) - \partial_{v_\perp} \psi(V^1) \cos(\alpha^1 - \alpha) = \lambda(V, \alpha, V^1, \alpha^1)(v_\perp - v^1_\perp \cos(\alpha^1 - \alpha)), \\ \partial_{v_\perp} \psi(V^1) \sin(\alpha^1 - \alpha) = \lambda(V, \alpha, V^1, \alpha^1)\, v^1_\perp \sin(\alpha^1 - \alpha), \end{cases} \tag{2.18}$$

for every $(V, V^1) \in \Omega^2$ and $(\alpha, \alpha^1) \in (0, 2\pi)^2$, $(V, \alpha) \neq (V^1, \alpha^1)$. The first equation of (2.18) gives $\lambda(V, \alpha, V^1, \alpha^1) = \lambda(V, V^1)$ does not depend on α and α^1 as soon as $v_\parallel \neq v^1_\parallel$. If $v_\parallel = v^1_\parallel$, and $\alpha \neq \alpha^1 + k\pi$, $k \in \mathbb{Z}$, the third equation leads to the same conclusion. At last, in the case when $v_\parallel = v^1_\parallel$, $\alpha = \alpha^1 + k\pi$ with $v \neq v^1$ (i.e. k odd or $(v_\perp \neq v^1_\perp)$ the second equation enables us to conclude.

Fixing then α and α^1, $\alpha \neq \alpha^1 + k\pi$, equation (2.18) becomes

$$\begin{cases} \partial_{v_\parallel} \psi(V) - \partial_{v_\parallel} \psi(V^1) = \lambda(V, V^1)(v_\parallel - v^1_\parallel), & (a) \\ \partial_{v_\perp} \psi(V) = \lambda(V, V^1)\, v_\perp, & (b) \\ \partial_{v_\perp} \psi(V^1) = \lambda(V, V^1)\, v^1_\perp, & (c) \end{cases} \tag{2.19}$$

for every $(V, V^1) \in \Omega^2$. As on Ω, $v_\perp > 0$ (2.19.b) yields $\lambda(V, V^1) = \lambda(V)$, and (2.19.c) $\lambda(V, V^1) = \lambda(V^1)$. Using those facts in (2.19.a) we get

$$\partial_{v_\parallel} \psi(V) - \partial_{v_\parallel} \psi(V^1) = \lambda(V)(v_\parallel - v_\parallel^1) = \lambda(V^1)(v_\parallel - v_\parallel^1). \tag{2.20}$$

Then,

$$\lambda(V) = \lambda(V^1), \tag{2.21}$$

for all $(V, V^1) \in \Omega^2$ such that $v_\parallel \neq v_\parallel^1$. At last if $v_\parallel = v_\parallel^1$, we have for some $\delta \in \mathbb{R}^*$ $\lambda((v_\parallel, v_\perp)) = \lambda((v_\parallel + \delta, v_\perp)) = \lambda((v_\parallel, v_\perp^1))$ yielding $\lambda(V) = \lambda$ is constant on Ω, giving

$$\begin{cases} \partial_{v_\parallel} \psi(V) = \partial_{v_\parallel} \psi(V^1) = \lambda(v_\parallel - v_\parallel^1), & \forall (V, V^1) \in \Omega^2, \\ \partial_{v_\perp} \psi(V) = \lambda v_\perp, & \forall V \in \Omega. \end{cases} \tag{2.22}$$

Now, fixing V^1 and setting $\kappa = \partial_{v_\parallel} \psi(V^1) - \lambda v_\parallel^1$, we get (3.9). Since the reverse is obvious, the Theorem follows. ∎

Remark 2.9: Notice that (2.10) and Theorem 2.7 are consequences, via algebraic manipulations of the mere *algebraic structure* of the Fokker-Planck operator. Hence, they remain valid for any operator (continuous or discrete) having the same *algebraic structure,* i.e. writting

$$\begin{cases} Q(f,f) = -D^* \cdot q(f,f) \\ q(f,f)(v) = \displaystyle\int_0^{2\pi} \mathscr{L}_I(f(V) f(\,.\,)\, \Phi(v - (\,.\,, \alpha^1)) \cdot (D \operatorname{Log} f(v) - D \operatorname{Log} f(\,.\,, \alpha^1)))\, d\alpha^1, \end{cases} \tag{2.23}$$

where $f(V)$ is defined on a set $I \subset \Omega$ (which can be discrete) and where $v = (V, \alpha)$ for any α. The operator D acts on real valued cylindrical functions ψ and gives a cylindrical vector valued function $D\psi$. For any $v = (V, \alpha)$, $V \in I$, its expression in the basis B^α is given by

$$(D\psi(v))^{B_\alpha} = \begin{pmatrix} \partial^\parallel \psi(V) \\ \partial^\perp \psi(V) \\ 0 \end{pmatrix}, \tag{2.24}$$

where $\partial^\parallel$ and $\partial^\perp$ are two linear operators on real valued cylindrical functions. $\mathscr{L}_I$ is a linear form having a behaviour comparable with the one of an integral operator, and in particular satisfying

$$(\psi \geq 0 \Rightarrow \mathscr{L}_I(\psi) \geq 0) \quad \text{and} \quad ([\psi \geq 0 \text{ and } \mathscr{L}_I(\psi) = 0] \Rightarrow [\psi = 0]). \tag{2.25}$$

Last $(D^* \cdot\,)$ is the adjoint operator of D defined by

$$\mathscr{L}_I(D^* \cdot \varphi\psi) = \mathscr{L}_I(\varphi \cdot D\psi), \tag{2.26}$$

for any real valued cylindrical function ψ and any vector valued cylindrical function φ.

As a consequence of the fact that (2.10) remains valid in this framework, if the operator D satisfies the additional condition $\partial^{\parallel} 1$ is a constant and $\partial^{\perp} 1$ is zero, the kinetic entropy $(2\pi\, \mathscr{L}_I(f \operatorname{Log} f))$ of f solution of

$$\partial_t f = Q(f(t,.), f(t,.)), \quad f_{|t=0} = f_0 > 0, \tag{2.27}$$

decreases with time. ■

Remark 2.10: The framework evoked in Remark 2.9 can be improved a bit without altering the validity of (2.10) and Theorem 2.7. We may consider in (2.23) that $f(V)$ is defined on $\mathcal{J} \supset I$, which may contain points V such that $v_\perp = 0$. Then D is an operator acting on real valued cylindrical functions defined on $\mathcal{J}$ and leading vector valued cylindrical functions defined on I. In this context, (2.26) has to be replaced by

$$\mathscr{L}_{\mathcal{J}}(D^* \cdot \varphi\psi) = \mathscr{L}_I(\varphi \cdot D\psi), \tag{2.28}$$

where $\mathscr{L}_{\mathcal{J}}$ satisfies the same properties (2.25) than $\mathscr{L}_I$. There, if the additional conditions $\partial^{\parallel} 1$ is a constant and $\partial^{\perp} 1$ is zero are satisfied, the entropy $(2\pi\, \mathscr{L}_{\mathcal{J}}(f \operatorname{Log} f))$ of f decreases. We shall need this type of context later on in Sections 3 and 4, especially when working in a bounded velocity domain, in view of practical computations. ■

The whole *algebraic structure* of the Fokker-Planck operator has been exploited when writing (2.14), achieving then the first step of the study. It remains now to solve the system (2.14), composed of two ordinary first order differential equations. This step, that we call *integration step,* is no more related to the operator itself. In the continuous case, the integration of system (2.14) stands to reason, and gives the following characterization of the collisional invariants:

PROPOSITION 2.11: *The collisional invariant space $\mathscr{C}$ is given by*

$$\mathscr{C} = \operatorname{Span}\{1, v_\parallel, v_\parallel^2 + v_\perp^2\}. \tag{2.29}$$

Then, since

$$\frac{d}{dt}\int_\Omega f\psi\, d\sigma = \int_\Omega P(f,f)\, \psi\, d\sigma, \tag{2.30}$$

for f solution of the Fokker-Planck equation (2.11), $\left(\int_\Omega f\psi\, d\sigma\right)$ is conserved if and only if ψ belongs to $\mathscr{C}$. Hence, Proposition 2.11 gives the

COROLLARY 2.12: *Let f be a solution of (2.11); then the mass $\left(2\pi\int_\Omega f\, d\sigma\right)$, the momentum $\left(2\pi\int_\Omega f v_\parallel\, d\sigma\right)$ and the energy $\left(2\pi\int_\Omega f(v_\parallel^2 + v_\perp^2)\, d\sigma\right)$ (and their linear combinations) are the only linear integral quantities conserved with time.*

Another direct consequence of Proposition 2.11 is the second part of the H-Theorem:

COROLLARY 2.13: *The stationary solutions of the Fokker-Planck equation, i.e. the functions f such that $P(f,f) = 0$, are the Maxwellian functions defined by $f \in \exp(\mathscr{C})$; moreover, these ones are the only functions realising the minimum of the kinetic entropy.*

Remark 2.14: The geometrical assumption under consideration enables us to disregard the α-dependent quantities. However, since in the basis B_α the velocity expresses $(v_\parallel, v_\perp \cos(\alpha), v_\perp \sin(\alpha))$, the α-integration yields that the components of the momentum perpendicular to $v_\parallel$ $\left(\int_0^{2\pi}\int_\Omega f v_\perp \cos(\alpha)\, d\sigma\, d\alpha\right.$ and $\left.\int_0^{2\pi}\int_\Omega f v_\perp \sin(\alpha)\, d\sigma\, d\alpha\right)$ are indentically zero and thus naturally conserved. ■

3. DISCRETE FOKKER-PLANCK OPERATORS

We now turn to the discrete case using an approximation of finite difference type. In this context, we reproduce the same approach based on two successive steps. Concerning the first one, as suggested in Remarks 2.9 and 2.10, it is actually easy to build a discrete Fokker-Planck operator having an *algebraic structure* similar to the continuous one and consequently decreasing the kinetic entropy and conserving mass, momentum and energy; we state here rapidly these properties.

The tricky point consists in showing that mass, momentum and energy are the only conserved quantities. This fact is a consequence of the *integration step,* i.e. the characterization of the solutions of the discrete analogue of system (2.14). Unfortunately, we shall see that if we consider finite difference operators which are uniformly defined on the mesh, it is not possible to preserve the form of the solution of (2.14): unexpected additional solutions appear giving rise to additional conserved quantities. This fact is specific to the axisymmetric geometry.

We now precise the mesh of the velocity domain and the general finite difference operators under consideration.

Finite differences on a regular mesh

Let Z be a regular mesh defined by

$$Z = \Delta v_\parallel\, \mathbb{Z} \times \Delta v_\perp\, \mathbb{Z}^+, \quad \Delta v_\parallel\, \Delta v_\perp \neq 0\,, \tag{3.1}$$

and let I and $\mathcal{J}$ be two submeshes satisfying $I \subset \mathcal{J} \subset Z$, $I \subset \Omega$ and $\mathcal{J} \subset \overline{\Omega}$. Since $\Omega = \mathbb{R} \times \mathbb{R}_+^*$, the submesh I cannot contain vertices such that $v_\perp = 0$; in the opposit, nothing excludes those vertices from $\mathcal{J}$. The reason for introducing two submeshes is the following: since the local basis B_α is not defined in $v_\perp = 0$, the vector valued functions cannot be defined along this axis so they are defined on I. Yet, there is no reason for a real valued function not to be defined in $v_\perp = 0$. Hence those last are defined on $\mathcal{J}$.

Let $\partial^\parallel$ and $\partial^\perp$ be two finite difference operators acting on real valued cylindrical functions and defined for any $V \in I$, by:

$$\begin{cases} \partial^\perp \psi(V) = \displaystyle\sum_{i \in M}\sum_{j \in J(i)} a_{ij}\, \psi(V + \alpha_j\, \Delta v_\parallel + \beta_i\, \Delta v_\perp)\,, \\ \partial^\parallel \psi(V) = \displaystyle\sum_{i \in M'}\sum_{j \in J'(i)} a'_{ij}\, \psi(V + \alpha'_j\, \Delta v_\perp + \beta'_i\, \Delta v_\parallel)\,. \end{cases} \tag{3.2}$$

In (3.2), M, M', $J(i)$ and $J'(i)$ are finite sets and the coefficients a_{ij} and a'_{ij} are non zero. The shifts α_j, α'_j, β_i and β'_i belong to $\mathbb{Z}$ and $j \to \alpha_j$, $j \to \alpha'_j$, $i \to \beta_i$ and $i \to \beta'_i$ are one to one such that $(V + \alpha_j\, \Delta v_\parallel + \beta_i\, \Delta v_\perp)$ and $(V + \alpha'_j \Delta v_\perp + \beta'_i \Delta v_\parallel)$ belong to $\mathcal{J}$. The definitions of sets, coefficients and shifts may depend on the vertex V.

Then denoting by $v = (V, \alpha)$ for any α, the discrete gradient operator, expressed in the basis B_α, is defined by

$$(D\psi(v))^{B_\alpha} = \begin{pmatrix} \partial^{\parallel}\psi(V) \\ \partial^{\perp}\psi(V) \\ 0 \end{pmatrix}, \tag{3.3}$$

and the discrete integrations by

$$\mathscr{L}_{\mathscr{I}}(\psi) = \sum_{V \in \mathscr{I}} \psi(V)\,\rho^1(V), \quad \mathscr{L}_I(\varphi) = \sum_{V \in I} \varphi(V)\,\rho^2(V), \tag{3.4}$$

for ψ defined on $\mathscr{I}$ and φ on I, where ρ^1 and ρ^2 are two non-vanishing approximations of $d\sigma$ on $\mathscr{I}$ and I respectively. We approximate the divergence operator by $(-D^* \cdot)$ where $(D^* \cdot)$ is the adjoint operator of D, i.e. satisfying

$$\mathscr{L}_{\mathscr{I}}(D^* \cdot \varphi\psi) = \mathscr{L}_I(\varphi \, . \, D\psi), \tag{3.5}$$

for all real valued cylindrical function ψ, and all vector valued cylindrical function φ. Notice that (3.5) makes sense and gives rise to an operator $(D^* .)$ without any singularity (unlike the continuous divergence operator which is not defined along the axis $(v_\perp = 0)$ since ρ^1 and ρ^2 are non-vanishing. Last, we define the discrete Fokker-Planck operator by

$$\begin{cases} Q(f,f) = -D^* \cdot q(f,f), \\ q(f,f)(v) = \displaystyle\int_0^{2\pi} \mathscr{L}_I(f(V) f(\,.\,)\,\Phi(v - (\,.\,, \alpha^1)) \cdot (D\,\mathrm{Log} f(v) - D\,\mathrm{Log} f(\,.\,, \alpha^1))\, d\alpha^1,) \end{cases} \tag{3.6}$$

where $\mathscr{L}_I$ acts componant by componant. As noticed in Remarks 2.9 and 2.10 and since the *algebric structure* is preserved, the proofs of (2.10) and Theorem 2.7 remain valid with $\partial_{v_\parallel}$ and $\partial_{v_\perp}$ respectively replaced by $\partial^{\parallel}$ and $\partial^{\perp}$. Hence we have:

PROPOSITION 3.1: *For all real valued cylindrical functions f we have*

$$\mathscr{L}_{\mathscr{I}}(Q(f,f)\,\mathrm{Log} f) \leqslant 0\,. \tag{3.7}$$

As a consequence of this, we easily obtain:

COROLLARY 3.2: *If $\partial^{\parallel} 1$ is a constant and $\partial^{\perp} 1$ is zero, then the discrete kinetic entropy $(2\pi \mathscr{L}_{\mathscr{I}}(f\,\mathrm{Log} f))$ decreases with time.*

By analogy with the continuous case, a real valued cylindrical function ψ defined on $\mathscr{I}$ is called a collisional invariant if, for every cylindrical function f defined on $\mathscr{I}$, we have

$$\mathscr{L}_{\mathscr{I}}(Q(f,f)\,\psi) = 0\,. \tag{3.8}$$

Now, since the characterization of such ψ is closely related to the particular choice of the discrete operator D, we shall denote by $\mathscr{C}(Q)$ the collisional invariant set associated with the discrete operator Q defined by (3.6). The equivalent of Theorem 2.7 is given by

THEOREM 3 3 *A real valued cylindrical functin* ψ *belongs to* $\mathscr{C}(Q)$ *if and only if there exist* $\lambda \in \mathbb{R}$ *and* $\kappa \in \mathbb{R}$ *such that, for all* $V \in I$, *we have*

$$\partial^{\parallel} \psi(V) = \lambda v_{\parallel} + \kappa \quad \text{and} \quad \partial^{\perp}\psi(V) = \lambda v_{\perp} \tag{3 9}$$

Secondly, $Q(f,f) = 0$ *if and only if* $f \in \exp(\mathscr{C}(Q))$

We now need to characterize the solutions of (3 9) Recall that, in order to have the right conservation properties and the right thermodynamical equilibrium set, we need $\mathscr{C}(Q) = \text{Span}\{1, v_{\parallel}, v_{\parallel}^2 + v_{\perp}^2\}$ Concerning this, as a direct consequence of Theorem 3 3 we have the

COROLLARY 3 4 *The space* $\text{Span}\{1, v_{\parallel}, v_{\parallel}^2 + v_{\perp}^2\} \subset \mathscr{C}(Q)$ *if and only if there exist* λ_1, λ_2, λ_3, κ_1, κ_2 *and* κ_3 *such that*

$$\begin{cases} \partial^{\parallel} 1 = \lambda_1 v_{\parallel} + \kappa_1, & \partial^{\perp} 1 = \lambda_1 v_{\perp}, \\ \partial^{\parallel} v_{\parallel} = \lambda_2 v_{\parallel} + \kappa_2, & \partial^{\perp} v_{\parallel} = \lambda_2 v_{\perp}, \\ \partial^{\parallel}(v_{\parallel}^2 + v_{\perp}^2) = \lambda_3 v_{\parallel} + \kappa_3, & \partial^{\perp}(v_{\parallel}^2 + v_{\perp}^2) = \lambda_3 v_{\perp} \end{cases} \tag{3 10}$$

Moreover, $\mathscr{C}(Q) \subset \textit{Span}\{1, v_{\parallel}, v_{\parallel}^2 + v_{\perp}^2\}$ *if and only if for all* $\lambda \in \mathbb{R}$ *and* $\kappa \in \mathbb{R}$,

$$\begin{cases} \partial^{\parallel} \psi = \lambda v_{\parallel} + \kappa \\ \partial^{\perp}\psi = \lambda v_{\perp} \end{cases} \Rightarrow \psi \in \text{Span}\{1, v_{\parallel}, v_{\parallel}^2 + v_{\perp}^2\} \tag{3 11}$$

Remark 3 5 A fundamental difference with the whole 3D case studied in P Degond & B Lucquin-Desreux [12] appears clearly at this level, simply by looking at the last equation in formula (3 10) the right hand side does not contain a constant term In particular, if $\partial^{\perp}$ is a first order approximation of $\partial_{v_{\perp}}$, this relation is not satisfied, and so the energy is not conserved ■

Example Let $\partial^{\parallel}$ be a first order finite difference approximation of $\partial_{v_{\parallel}}$ and $\partial^{\perp}$ a second order approximation of $\partial_{v_{\perp}}$ This choice of course satisfies condition (3 10) with $\lambda_1 = \kappa_1 = 0$, $\lambda_2 = 0$, $\kappa_2 = 1$, $\lambda_3 = 2$ and $\kappa_3 = \Delta v_{\parallel}$ Hence the resulting discrete Fokker-Planck operator decreases entropy and its collisional invariant space satisfies $\text{Span}\{1, v_{\parallel}, v_{\parallel}^2 + v_{\perp}^2\} \subset \mathscr{C}(Q)$

Now, with this second order approximation of $\partial_{v_{\perp}}$, condition (3 11) is not satisfied (see [12] for details) and the collisional invariant space associated with the resulting discrete Fokker-Planck operator is bigger as it has to be

We shall now see that condition (3 11) is not so easy to satisfy and, in particular, it is always violated by a vertex independent operator $\partial^{\perp}$

Vertex-independent operators

If the definition of $\partial^{\parallel}$ is the same for all vertices of I, then for every fixed $V \in I$ and every $\gamma = (\gamma_{\parallel}, \gamma_{\perp}) \in \mathbb{R}^2$ such that $V + \gamma = (v_{\parallel} + \gamma_{\parallel}, v_{\perp} + \gamma_{\perp}) \in I$ we have

$$\partial^{\parallel}(\psi(V+\gamma)) = (\partial^{\parallel}\psi)(V+\gamma), \tag{3 12}$$

for all real valued cylindrical functions ψ Hence (3 10) yields the following

COROLLARY 3.6: *If the definition of* $\partial^{\|}$ *is the same for all vertices of I, the space* $\operatorname{Span}\{1, v_{\|}, v_{\|}^2 + v_{\perp}^2\} \subset \mathscr{C}(Q)$ *if and only if we have (3.10) with*

$$\lambda_1 = 0, \quad \lambda_2 = \kappa_1 = 0, \quad \lambda_3 = 2\,\kappa_2 \text{ and } \partial^{\|} v_{\perp} = 0\,. \tag{3.13}$$

Proof: First, if (3.12) holds true and if $\psi \equiv \psi(v_{\perp})$ does not depend on $v_{\|}$, then $\partial^{\|}\psi$ does not depend on $v_{\|}$. Indeed, let us set $g(v_{\|}, v_{\perp}) := \partial^{\|}\psi$. For $\gamma = (\gamma_{\|}, 0)$ we have on the one hand $\partial^{\|}(\psi(V+\gamma)) = \partial^{\|}(\psi(V)) = g(v_{\|}, v_{\perp})$ and on the other hand, according to (3.12), $\partial^{\|}(\psi(V+\gamma)) = (\partial^{\|}\psi)(V+\gamma) = g(v_{\|} + \gamma_{\|}, v_{\perp})$, yielding $g(v_{\|}, v_{\perp}) = g(v_{\perp})$.

Exactly in the same way we may prove that, if $\psi \equiv \psi(v_{\|})$ then, $\partial^{\|}\psi \equiv g(v_{\|})$.

In order to get $\lambda_1 = 0$, we just write $\partial^{\|}(1(V+\gamma)) = (\partial^{\|}1)(V+\gamma) = \lambda_1(v_{\|} + \gamma_{\|}) + \kappa_1$, $(1(V+\gamma)$ means function 1 taken in $((V+\gamma))$, but also $\partial^{\|}(1(V+\gamma)) = \partial^{\|}1 = \lambda_1 v_{\|} + \kappa_1$, leading to the conclusion.

Now, $\partial^{\|}(v_{\|}(V+\gamma)) = (\partial^{\|}v_{\|})(V+\gamma) = \lambda_2(v_{\|} + \gamma_{\|}) + \kappa_2$. On the other hand, $\partial^{\|}(v_{\|}(V+\gamma)) = \partial^{\|}(v_{\|} + \gamma_{\|}) = \lambda_2 v_{\|} + \kappa_2 + \kappa_1\gamma_{\|}$. Then, $\lambda_2 = \kappa_1$.

In order to get the two other relations and $\kappa_1 = 0$, consider $g(v_{\perp}) := \partial^{\|} v_{\perp}$ and write $\partial^{\|}((v_{\|}^2 + v_{\perp}^2)(V+\gamma)) = \lambda_3(v_{\|} + \gamma_{\|}) + \kappa_3$. As a direct computation gives

$$\begin{aligned}\partial^{\|}((v_{\|}^2 + v_{\perp}^2)(V+\gamma)) &= \partial^{\|}(v_{\|}^2 + v_{\perp}^2 + 2\,v_{\|}\,\gamma_{\|} + 2\,v_{\perp}\gamma_{\perp} + \gamma_{\|}^2 + \gamma_{\perp}^2)\\ &= \lambda_3\,v_{\|} + \kappa_3 + 2\,\lambda_2\,v_{\|}\,\gamma_{\|} + 2\,\kappa_2\,\gamma_{\|} + 2\,g(v_{\perp})\,\gamma_{\perp} + \kappa_1\,\gamma_{\|}^2 + \kappa_1\,\gamma_{\perp}^2,\end{aligned}$$

we deduce $(\lambda_3 - 2\,\kappa_2)\,\gamma_{\|} + 2\,\kappa_1\,v_{\|}\,\gamma_{\|} + \kappa_1\gamma_{\|}^2 + \kappa_1\gamma_{\perp}^2 + 2\,g(v_{\perp})\,\gamma_{\perp} = 0$ for every $v_{\|}$, $\gamma_{\|}$ and $\gamma_{\perp}$ such that V and $V+\gamma \in I$. Then, $g(v_{\perp}) = 0$, $\kappa_1 = 0$ and $\lambda_3 = 2\,\kappa_2$, prooving (3.13). ■

In the same spirit, we also have

COROLLARY 3.7: *If* $\partial^{\perp}$ *is a vertex-independent operator, then* $\operatorname{Span}\{1, v_{\|}, v_{\|}^2 + v_{\perp}^2\} \subset \mathscr{C}(Q)$ *if and only if we have (3.10) with*

$$\lambda_1 = 0, \quad \lambda_2 = 0 \text{ and } \lambda_3 = 2\,\partial^{\perp} v_{\perp}. \tag{3.14}$$

Hence, applying the two last Corollaries, we get that if $\partial^{\|}$ and $\partial^{\perp}$ are vertex-independent operators, the conditions in order to have $\operatorname{Span}\{1, v_{\|}, v_{\|}^2 + v_{\perp}^2\} \subset \mathscr{C}(Q)$ (see (3.10)) are

$$\begin{cases}\partial^{\|}1 = \partial^{\perp}1 = 0\\ \partial^{\|}v_{\|} = \kappa_2, \quad \partial^{\|}v_{\perp} = 0\\ \partial^{\|}(v_{\|}^2 + v_{\perp}^2) = 2\,\kappa_2\,v_{\|} + \kappa_3, \quad \partial^{\perp}(v_{\|}^2 + v_{\perp}^2) = \lambda_3\,v_{\perp}\,,\\ \partial^{\perp}v_{\perp} = \lambda_3/2\,.\end{cases} \tag{3.15}$$

Hopeless Theorem

Unfortunately we have the

THEOREM 3.8: *Among the operators* $\partial^{\perp}$ *satisfying* $(\partial^{\perp}(v_{\perp}^2) \not\equiv 0)$ *and* $(\psi \equiv \psi(v_{\|}) \Rightarrow \partial^{\perp}\psi \equiv 0)$, *there exists no vertex-independent operator such that* $\mathscr{C}(Q) = \operatorname{Span}\{1, v_{\|}, v_{\|}^2 + v_{\perp}^2\}$.

Proof: Under condition (3.14) of Corollary 3.7, we shall build a function $\psi : \mathscr{I} \to \mathbb{R}$, not belonging to $\operatorname{Span}\{1, v_{\|}, v_{\|}^2 + v_{\perp}^2\}$, such that $D\psi \equiv 0$ and thus belonging to $\mathscr{C}(Q)$.

Let us recall that the vertex-independent operator $\partial^{\perp}$ is defined by

$$\forall V \in I, \quad \partial^{\perp}\psi(V) = \sum_{i \in M} \sum_{j \in J(i)} a_{ij}\, \psi(V + \alpha_j \Delta v_{\parallel} + \beta_i \Delta v_{\perp}), \tag{3.16}$$

where the sets M, $J(i)$, the coefficients a_{ij} and the shifts α_j, β_i are independent of vertex V itself.

First, under the assumption $\partial^{\perp}(\psi(v_{\parallel})) \equiv 0$, we have $\sum_{i \in M} \sum_{j \in J(i)} a_{ij}\, \alpha_j = 0$ and $\sum_{i \in M} \sum_{j \in J(i)} a_{ij}\, \alpha_j^2 = 0$. Secondly, $\partial^{\perp}(v_{\perp}^2) \not\equiv 0$ implies $\lambda_3 \neq 0$ in formula (3.10).

Hence, in view of (3.10) and (3.14), the conditions in order to have $\mathrm{Span}\{1, v_{\parallel}, v_{\parallel}^2 + v_{\perp}^2\} \subset \mathscr{C}(Q)$ are

$$\begin{cases} \sum_{i \in M} \sum_{j \in J(i)} a_{ij} = 0, & (a) \\ 2\,\Delta v_{\perp} \sum_{i \in M} \sum_{j \in J(i)} a_{ij}\, \beta_i = \lambda_3 \neq 0, & (b) \\ \sum_{i \in M} \sum_{j \in J(i)} a_{ij}\, \beta_i^2 = 0. & (c) \end{cases} \tag{3.17}$$

We now give the construction of the expected function ψ which, as we shall see, does not depend on $v_{\parallel}$. First, notice that a function $\psi(v_{\parallel}, v_{\perp}) \equiv \tilde{\psi}(v_{\perp})$ such that $\partial^{\perp}\psi \equiv 0$ then satisfies:

$$\sum_{i \in M} \left(\sum_{j \in J(i)} a_{ij} \right) \tilde{\psi}(v_{\perp} + \beta_i \Delta v_{\perp}) = 0\,. \tag{3.18}$$

Then, we see that the indices $i \in M$ such that $\sum_{j \in J(i)} a_{ij} \neq 0$ and the other ones do not play the same role. Hence let $M_0 = \left\{ i \in M, \sum_{j \in J(i)} a_{ij} = 0 \right\}$ and $M_c = M - M_0$. We have $\mathrm{Card}\,(M_c) \geq 2$. Indeed, because of (3.17.b), $\mathrm{Card}\,(M_c) \neq 0$. Now, suppose $\mathrm{Card}\,(M_c) = 1$. Denoting by i_0 its single element, we have $\sum_{j \in J(i)} a_{i_0 j} \neq 0$ and $\sum_{i \neq i_0} \sum_{j \in J(i)} a_{ij} = 0$. The second of those relations and (3.17.a) give $\sum_{j \in J(i)} a_{i_0 j} = 0$ contradicting the first. Then $\mathrm{Card}\,(M_c) \geq 2$.

Now, for $v_{\perp}$ such that $V = (v_{\parallel}, v_{\perp}) \in I$, we denote $P_c(v_{\perp}) = \{v_{\perp}^1 = v_{\perp} + \beta_i \Delta v_{\perp}, i \in M_c\}$. Consider $v_{\perp}^0 = \min\{v_{\perp}, V \in I\}$, and set $\tilde{\psi}(\min\{v_{\perp}^1, v_{\perp}^1 \in P_c(v_{\perp}^0)\}) = 1$ and $\tilde{\psi}(v_{\perp}) = 0$ if $v_{\perp} < \max\{v_{\perp}^1, v_{\perp}^1 \in P_c(v_{\perp}^0)\}$. The value of $\tilde{\psi}$ in $\max\{v_{\perp}^1, v_{\perp}^1 \in P_c(v_{\perp}^0)\}$ is given by the relation (3.18) which has a solution since $\mathrm{Card}\,(M_c) \geq 2$. We can build $\tilde{\psi}(v_{\perp})$ for every $v_{\perp} \leq \max\{v_{\perp}^1, v_{\perp}^1 \in P_c(v_{\perp}^2), V^2 \in I\}$ using the same relation. Fixing at last $\tilde{\psi}(v_{\perp})$ to 0 every $v_{\perp} > \max\{v_{\perp}^1, v_{\perp}^1 \in P_c(v_{\perp}^2), V^2 \in I\}$, we define the function ψ by setting:

$$\forall V = (v_{\parallel}, v_{\perp}) \in \mathscr{I}, \quad \psi(V) = \tilde{\psi}(v_{\perp})\,. \tag{3.19}$$

By construction, ψ satisfies $D\psi \equiv 0$ and then belongs to $\mathscr{C}(Q)$. As we shall see soon, (3.17.c) implies

$$\{v_{\perp}, (\min\{v_{\perp}^1, v_{\perp}^1 \in P_c(v_{\perp}^0)\} < v_{\perp} < \max\{v_{\perp}^1, v_{\perp}^1 \in P_c(v_{\perp}^0)\})\} \neq \Phi\,, \tag{3.20}$$

yielding that ψ is not constant. Since a function not depending on $v_{\parallel}$ and belonging to $\mathrm{Span}\{1, v_{\parallel}, v_{\parallel}^2 + v_{\perp}^2\}$ is a constant function, $\psi \notin \mathrm{Span}\{1, v_{\parallel}, v_{\parallel}^2 + v_{\perp}^2\}$, and the Theorem is proved.

Let us show (3.20). For this purpose, assume the contrary, i.e. assume that there is no $v_\perp$ such that $\min\{v_\perp^1, v_\perp^1 \in P_c(v_\perp^0)\} < v_\perp < \max\{v_\perp^1, v_\perp^1 \in P_c(v_\perp^0)\}$. Then $\operatorname{Card} M_c = 2$, i.e. $M_c = \{1, 2\}$, $\beta_1 = \beta \in \mathbb{Z}$ and $\beta_2 = \beta + 1$. Equation (3.17.a) gives $\sum_{j \in J(1)} a_{1j} = - \sum_{j \in J(2)} a_{2j}$. Equation (3.17.b) yields $2\,\Delta v_\perp \sum_{j \in J(2)} a_{2j} = \lambda_3$ and at last equation (3.17.c) leads to

$$\lambda_3(2\beta + 1) = 0\,. \tag{3.21}$$

Then we may conclude that either $\lambda_3 = 0$ or $\beta = -1/2 \notin \mathbb{Z}$ both contradicting the assumptions. Hence (3.20) is true. ∎

Remark 3.9: Approximating the partial differentiation $\partial_{v_\perp}$ by a second order vertex-dependent operator enables us to build discrete Fokker-Planck operators satisfying $\operatorname{Span}\{1, v_\|, v_\|^2 + v_\perp^2\} = \mathscr{C}(Q)$. Nevertheless, the choice of D sets the operator $(D^* \cdot\,)$ by formula (3.5), and we did not manege to build a vertex-dependent operator D leading an operator $(-D^* \cdot\,)$ consistent with the divergence. ∎

4. ACTUAL IMPLEMENTED OPERATOR

In view of numerical experiments, we have to build a discrete axisymmetric Fokker-Planck operator Q_{imp} on a bounded velocity domain. Moreover, in order to have the right conservation properties and the right thermodynamical equilibrium set, the collisional invariant space $\mathscr{C}(Q_{imp})$ has to be $\operatorname{Span}\{1, v_\|, v_\|^2 + v_\perp^2\}$. Then considering the situation explained in Theorem 3.8 and Remark 3.9, we use the following discrete operator

$$Q_{imp} = (1 - \varepsilon)\, Q + \varepsilon Q_0\,, \tag{4.1}$$

for a small parameter $0 < \varepsilon << 1$. In this expression, the operator Q involves constant coefficient finite difference operators; it is built from the continuous Fokker-Planck operator defined on a bounded velocity domain. Since the complementary set of $\operatorname{Span}\{1, v_\|, v_\|^2 + v_\perp^2\}$ in $\mathscr{C}(Q)$ is not empty, its thermodynamical equilibrium set is polluted. In order to remove this pollution, we perturb it with an operator Q_0 involving non constant coefficient operators and satisfying $\mathscr{C}(Q_0) = \operatorname{Span}\{1, v_\|, v_\|^2 + v_\perp^2\}$.

Construction of Q via a finite element method in a bounded velocity domain

Let $\mathscr{V}_b \subset \mathscr{V}$ be a bounded cylinder $\mathscr{V}_b = \Omega_b \times (0, 2\pi)$, $\Omega_b = \{V, v_\|^{Min} \leq v_\| \leq v_\|^{Max}, 0 < v_\perp \leq v_\perp^{Max}\}$. We introduce on Ω_b the regular mesh $\mathscr{I}$ (see *fig.* 4.1) with $\mathscr{I} = \mathscr{Z} \cap \overline{\Omega}_b$, where $\mathscr{Z} = \Delta v_\| \,\mathbb{Z} \times \Delta v_\perp \mathbb{Z}^+$ ($v_\|^{Min}/\Delta v_\|$, $v_\|^{Max}/\Delta v_\|$ and $v_\perp^{Max}/\Delta v_\perp$ are supposed to be integers). The construction of the operator Q is made in three successive steps, following the process described in [12] for the whole 3D case. The first step consists in using artificial boundary conditions, of Robin type, so as to preserve the weak formulation (2.7) of the Fokker-Planck operator. The continuous initial-boundary value problem we deal with writes

$$\begin{cases} \partial_t f = P(f,f)\,(V) \text{ for } v \in \mathscr{V}_b, \quad t > 0\,, \\ f_{|t=0} = f_0\,. \end{cases} \tag{4.2}$$

with

$$\begin{cases} P(f,f) = \operatorname{Div} p(f,f)\,, \\ p(f,f)\,(v) = \displaystyle\int_{\mathscr{V}_b} f(V) f(V^1)\, \Phi(v - v^1) \cdot (\operatorname{Grad} \operatorname{Log} f(v) - \operatorname{Grad} \operatorname{Log} f(v^1))\, d\sigma^1\, d\alpha^1, \end{cases} \tag{4.3}$$

and the boundary conditions are given by

$$p(f,f)\,(v)\ \ n(v) = 0, \quad \text{for } v \in \Gamma, \tag{4 4}$$

where Γ denotes the boundary of $\mathscr{V}_b$, and $n(v)$ its outer normal vector This choice of boundary conditions allows to keep the *algebraic structure* of the operator inducing then the decrease of the kinetic entropy and conservation of mass, momentum and energy

Moreover, the weak formulation of (4 3), writing for all $\psi \equiv \psi(V)$ regular enough,

$$\int_{\mathscr{V}_b} (P(f,f)\,\psi)\,(V)\,d\sigma\,d\alpha = -\int_{\mathscr{V}_b} (p(f,f)\ \ \text{Grad}\,\psi)\,(V)\,d\sigma\,d\alpha\,, \tag{4 5}$$

suggests a finite element discretization, of Q-1 type, the cells being rectangular this is the second step The third one consists then in choosing "good" quadrature formulae in the finite element formulation, so as to recover a finite difference scheme for the internal nodes of the mesh In parallel, it produces boundary conditions for the boundary nodes

The main advantage of this approach lies in the fact that the boundary conditions are naturally taken into account while they would be less easy to treat via a direct finite difference approximation Let us point out that in this process, the finite element formulation is just a tool to construct the right boundary conditions for the finite difference scheme

Since the main difference with the whole 3D case concerns the third step, we only detail this point in the proof of the next proposition, which gives the final expression of Q

PROPOSITION 4 1 *The discrete Fokker-Planck operator Q, built from P defined by (4 3), via a Q 1 finite element discretization and quadrature formulae, is defined by*

$$\begin{cases} Q(f,f) = -D^*\ \ q(f,f) \\ q(f,f)\,(v) = \displaystyle\int_0^{2\pi} \mathscr{L}_I(f(V)\,f(\ .\ )\,\Phi(v-(\ .\ ,\alpha^1))\ \ (D\,\mathrm{Log} f(v) - D\,\mathrm{Log} f(\ .\ ,\alpha^1)))\,d\alpha^1 \end{cases} \tag{4 6}$$

where v stands for (V,α) for any α, and with I given by $I = \{V \in \mathscr{I},\ v_\parallel \neq v_\parallel^{Max},\ v_\perp \neq 0,\ v_\perp \neq v_\perp^{Max}\}$, and D by

$$(D\psi)^{B_\alpha} = \begin{pmatrix} \partial^\parallel \psi(V) = \dfrac{1}{\Delta v_\parallel}\,(\psi(V+\Delta v_\parallel) - \psi(V)) \\ \partial^\perp \psi(V) = \dfrac{1}{2\,\Delta v_\perp}\,(\psi(V+\Delta v_\perp) - \psi(V-\Delta v_\perp)) \\ 0 \end{pmatrix} \tag{4 7}$$

The two linear forms $\mathscr{L}_{\mathscr{I}}$ and $\mathscr{L}_I$ are defined by

$$\mathscr{L}_{\mathscr{I}}(\psi) = \sum_{V \in \mathscr{I}} \psi(V)\,\rho^1(V), \quad \mathscr{L}_I(\varphi) = \sum_{V \in I} \varphi(V)\,\rho^2(V)\,, \tag{4 8}$$

where the approximated measures ρ^1 and ρ^2 are the volumes of the cells of two different grids (see (4 21), (4 18), (4 23) and (4 19) for precise definitions)

Finally, $(D^*\ \ )$ is the operator defined by

$$\forall V \in \mathscr{I}, \quad D^*\ \ \varphi(V) = \frac{1}{\rho^1(V)}\,[R_{\mathscr{I}}\tilde{D}\ \ E_I(\rho^2\,\varphi)]\,(V) \tag{4 9}$$

where $(\tilde{D}\cdot\ )$ *is the operator defined on the whole mesh* $\Delta v_\parallel\ \mathbb{Z}\times\Delta v_\perp\ \mathbb{Z}$ *by*

$$\tilde{D}\cdot\varphi(V) = -\frac{1}{\Delta v_\parallel}(\varphi^\parallel(V)-\varphi^\parallel(V-\Delta v_\parallel)) - \frac{1}{2\,\Delta v_\perp}(\varphi^\perp(V+\Delta v_\perp)-\varphi^\perp(V-\Delta v_\perp))\ ; \tag{4.10}$$

in expression (4.9) E_I *is a prolongation operator which for any function* φ *defined on I, associates the function* $E_I\,\varphi$ *defined on* $\Delta v_\parallel\ \mathbb{Z}\times\Delta v_\perp\ \mathbb{Z}$ *by*

$$[E_I\,\varphi]\,(V) = \varphi(V)\ \text{if}\ V\in I,\quad \text{and } 0 \text{ otherwise}, \tag{4.11}$$

while $R_{\mathcal{J}}$ *is a restriction operator acting on every function* ψ *defined on* $\Delta v_\parallel\ \mathbb{Z}\times\Delta v_\perp\ \mathbb{Z}$ *in the following way:*

$$R_{\mathcal{J}}\,\psi(V) = \psi(V),\quad \text{for}\ V\in\mathcal{J}\ . \tag{4.12}$$

Then, this operator $(D^*\cdot\ )$ *is the adjoint operator of D in the sense of identity (3.5), i.e. we have*

$$\mathcal{L}_{\mathcal{J}}(D^*\cdot\varphi\psi) = \mathcal{L}_{\mathcal{J}}(\varphi\cdot D\psi)\ , \tag{4.13}$$

for any φ *defined on I and any* ψ *defined on* $\mathcal{J}$.

Proof: As mentioned above, the way to build the operator $(D^*\cdot\ )$ is precisely motivated by the preservation, at the discrete level, of the weak formulation (4.5). Thus, (4.13) is a direct consequence of this construction that we now explain, only detailing the differences with the whole 3D case.

The starting point is the discretization, by use of Q-1 finite elements, of the weak formulation (4.5) which writes, after simplification by the factor 2π,

$$\int_{\Omega_b}(P(f,f)\,\psi)\,(V)\,d\sigma = -\int_{\Omega_b}(p(f,f)\cdot\mathrm{Grad}\,\psi)\,(V)\,d\sigma\ . \tag{4.14}$$

The domain $\overline{\Omega}_b$ is first partitioned in cells C_V, for $V\in\mathcal{J}$, defined by

$$C_V = \{(v_\parallel^1, v_\perp^1),\ v_\parallel\leqslant v_\parallel^1\leqslant v_\parallel+\Delta v_\parallel,\ v_\perp\leqslant v_\perp^1+\Delta v_\perp\}\cap\overline{\Omega}_b\ , \tag{4.15}$$

(see *fig.* 4.1 for a visualization of cells C_V and other forthcoming notations). The finite element space we use is generated by the basis $(\xi_V)_{V\in\mathcal{J}}$, ξ_V being a continuous function defined on $\overline{\Omega}_b$, whose restriction to each cell C_W is a polynomial of degree 1 in each variable and which satisfies for every $W\in\mathcal{J}$, $\xi_V(W)=1$ if $V=W$ and 0 otherwise. Because of the importance of the role played by "Log f" in the *algebraic structure* of the Fokker-Planck operator, the approximation $\tilde{f}$ of f is chosen such that Log $\tilde{f}$ belongs to the finite element space. Since, from now on, we only work with this approximated function, we simplify the notation and replace $\tilde{f}$ by f. We then have

$$\mathrm{Log}\,f = \sum_{V\in\mathcal{J}}\mathrm{Log}\,f(V)\,\xi_V\ , \tag{4.16}$$

and the discrete weak formulation writes

$$\int_{\Omega_b}(P(f,f)\,\xi_{V^0})\,(V)\,d\sigma = -\int_{\Omega_b}(p(f,f)\cdot\mathrm{Grad}\,\xi_{V^0})\,(V)\,d\sigma\ , \tag{4.17}$$

for any $V^0\in\mathcal{J}$.

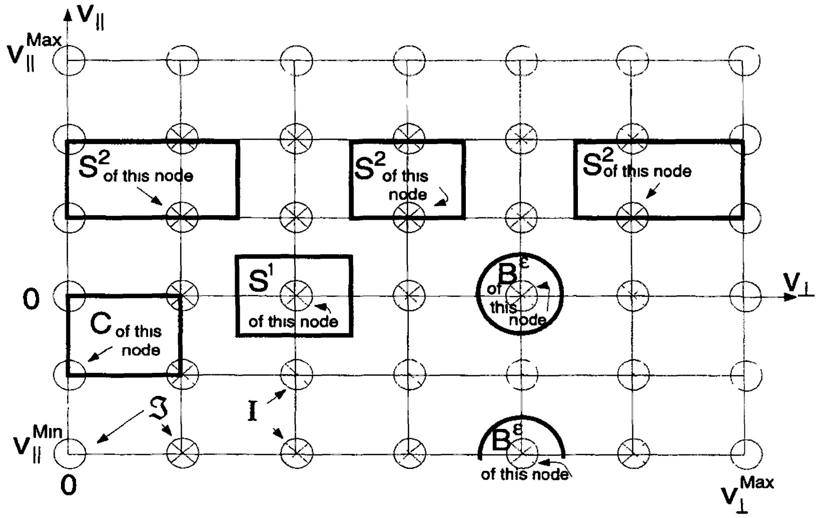


Figure 4.1. — A vizualization of $\mathcal{I}$, I, C_V, S_V^1, S_V^2 and B_V^ε.

We now have to choose quadrature formulae in order to compute the integrals appearing above. Since the discrete operators $Q(f,f)$ and $q(f,f)$, that respectively approximate the continuous operators $P(f,f)$ and $p(f,f)$, are not *a priori* defined on the same set of nodes, the two sides of equality (4.17) are computed via two different quadrature formulae, defined on two different grids. More precisely, let us consider, for $V \in \mathcal{I}$, the cell S_V^1 defined by

$$S_V^1 = \left\{ V^1, \left(v_\| - \frac{\Delta v_\|}{2} \right) < v_\|^1 \leqslant \left(v_\| + \frac{\Delta v_\|}{2} \right), \left(v_\perp - \frac{\Delta v_\perp}{2} \right) < v_\perp^1 \leqslant \left(v_\perp + \frac{\Delta v_\perp}{2} \right) \right\} \cap \Omega_b . \tag{4.18}$$

We also set, for any $V \in I$,

$$\begin{aligned} S_V^2 &= \left\{ V^1, v_\| < v_\|^1 \leqslant (v_\| + \Delta v_\|), \left(v_\perp - \frac{\Delta v_\perp}{2} \right) < v_\perp^1 \leqslant \left(v_\perp - \frac{\Delta v_\perp}{2} \right) \right\} \text{ if } v_\perp \neq \Delta v_\perp, \quad v_\perp \neq v_\perp^{Max} - \Delta v_\perp, \\ &= \left\{ V^1, v_\| < v_\|^1 \leqslant (v_\| + \Delta v_\|), 0 < v_\perp^1 \leqslant \tfrac{3}{2} \Delta v_\perp \right\} \text{ if } v_\perp = \Delta v_\perp, \\ &= \left\{ V^1, v_\| < v_\|^1 \leqslant (v_\| + \Delta v_\|), \left(v_\perp^{Max} - \tfrac{3}{2} \Delta v_\perp \right) < v_\perp^1 \leqslant v_\perp^{Max} \right\} \text{ if } v_\perp = v_\perp^{Max} - \Delta v_\perp . \end{aligned} \tag{4.19}$$

These cells are represented on *figure* 4.1; notice that they both recover the whole domain Ω_b.

In order to evaluate the left hand side of expression (4.17), we use a centered quadrature formula in each elementary cell S_V^1, $V \in \mathcal{J}$. This formula writes

$$\int_{S_V^1} \psi(V^1)\, d\sigma^1 \sim \rho^1(V)\, \psi(V)\,, \tag{4.20}$$

where:

$$\rho^1(V) = |S_V^1| = \int_{S_V^1} d\sigma^1\,. \tag{4.21}$$

The right hand side of (4.17) is computed via a decentered quadrature formula defined on each cell S_V^2, $V \in I$, by

$$\int_{S_V^2} \varphi(V^1)\, d\sigma^1 \sim \rho^2(V)\, \varphi(V)\,, \tag{4.22}$$

where:

$$\rho^2(V) = |S_V^2| = \int_{S_V^2} d\sigma^1\,. \tag{4.23}$$

Using the definition of the basis functions ξ_{V^0}, the equality (4.17) then writes, after approximation

$$\rho^1(V^0)\, Q(f,f)\,(V^0) = -\sum_{V \in I} \rho^2(V)\,(q(f,f)\cdot(\operatorname{Grad} \xi_{V^0})_{|S_V^2})\,(V)\,, \tag{4.24}$$

for all $V^0 \in \mathcal{J}$. Let us precise the right hand side of this last equality. First, we notice that the function $\operatorname{Grad} \xi_{V^0}$ is not continuous on S_V^2. We can however give a sense to this expression, adopting the following convention: if φ is not continuous in a point V, we set

$$\varphi(V) = \lim_{\varepsilon \to 0} \frac{1}{|B_V^\varepsilon|} \int_{B_V^\varepsilon} \varphi(V^1)\, d\sigma^1\,, \tag{4.25}$$

with $B_V^\varepsilon = \{V^1,\ |V - V^1| < \varepsilon\} \cap S_V^2$, and $|B_V^\varepsilon| = \int_{B_V^\varepsilon} d\sigma^1$.

Secondly, the computation of $q(f,f)$ that approximates $p(f,f)$ is carried out using in each elementary cell the second quadrature formula (4.22), with the convention (4.25) for the function $\operatorname{Log} f$. Now using the expression of ξ_{V^0} we get, after some easy computations (we set $v = (V, \alpha)$ for any α)

$$q(f,f)\,(v) = \int_0^{2\pi} \mathcal{L}_I(f(V) f(\,.\,)\, \Phi(v - (\,.\,, \alpha^1)) \cdot (D \operatorname{Log} f(v) - D \operatorname{Log} f(\,.\,, \alpha^1)))\, d\alpha^1, \tag{4.26}$$

with $\mathcal{L}_I$ defined by (4.8).

By analogy with the continuous case, we set for every $V^0 \in \mathcal{J}$,

$$(D^* \cdot \varphi)\,(V^0) = \frac{1}{\rho^1(V^0)} \sum_{V \in I} \rho^2(V)\,(\varphi \cdot (\operatorname{Grad} \xi_{V^0})_{|S_V^2})\,(V)\,, \tag{4.27}$$

so that (4.24) simply writes

$$Q(f,f)\,(V^0) = -\,(D^* \cdot q(f,f))\,(V^0)\,, \tag{4.28}$$

which, coupled to (4.26), gives the scheme (4.6).

Now, using once more the explicit expression of ξ_{V^0}, the computation of the operator $(D^* \cdot\ )$ defined by equation (4.27) is straitforward, and we get

$$D^* \cdot \varphi = \frac{1}{\rho^1} R_{\mathcal{J}} \tilde{D} \cdot E_I(\rho^2 \varphi)\,, \tag{4.29}$$

where $(\tilde{D}\,.\ )$ is given by (4.10). Let us notice that this operator $(\tilde{D}\,.\ )$ is the formal adjoint of the operator D whose definition would have been extended to the whole mesh $\Delta v_\parallel\,\mathbb{Z} \times \Delta v_\perp\,\mathbb{Z}$.

Finally, by construction, we obviously have the discrete weak formulation

$$\sum_{V \in \mathcal{J}} (D^* \cdot \varphi\psi)\,(V)\,\rho^1(V) = \sum_{V \in I} (\varphi\, D\psi)\,(V)\,\rho^2(V)\,, \tag{4.30}$$

which, using the definition (4.8) of $\mathcal{L}_{\mathcal{J}}$ and $\mathcal{L}_I$, gives exactly:

$$\mathcal{L}_{\mathcal{J}}(D^* \cdot \varphi\psi) = \mathcal{L}_I(\varphi\,.\,D\psi)\,. \tag{4.31}$$

■

Remark 4.2: The operator $(D^* \cdot\ )$ is consistent with the divergence operator. Moreover, since $\rho^1(V)$ is a non vanishing approximation of $v_\perp$, the singularity of the divergence along the axis $v_\perp = 0$ is removed. ■

The operator $\partial^\parallel$ being a first order operator, and $\partial^\perp$ a second order operator, the operator Q decreases the entropy and conserve mass, momentum and energy. The collisional invariant space is polluted since it writes $\mathscr{C}(Q) = \mathrm{Span}\,\{1, v_\parallel, v_\parallel^2 + v_\perp^2, \mathcal{O}\}$ with $\mathcal{O}(V) = 1$ if $v_\parallel = 2\,k\,\Delta v_\perp$, $k \in \mathbb{N}$, and 0 otherwise. Hence we perturb it by a second operator.

The operator Q_0

We use exactly the same sets $\mathcal{J}$ and I, the same linear forms $\mathcal{L}_{\mathcal{J}}$ and $\mathcal{L}_I$, and the same operators as for Q except for points of I where $v_\perp = 2\,k\,\Delta v_\perp$, $k \in \mathbb{N}^*$, where $\partial^\perp$ is the following second order finite difference opeator

$$\partial^\perp \psi(V) = \frac{1}{2\,\Delta v_\perp}\,(3\,\psi(V) - 4\,\psi(V - \Delta v_\perp) + \psi(V - 2\,\Delta v_\perp))\,. \tag{4.32}$$

Proceeding as well, condition (3.10) is satisfied, and a straightforward computation shows that (3.11) is also satisfied, leading to $\mathscr{C}(Q_0) = \mathrm{Span}\,\{1, v_\parallel, v_\parallel^2 + v_\perp^2\}$, so that we finally have $\mathscr{C}(Q_{imp}) = \mathrm{Span}\,\{1, v_\parallel, v_\parallel^2 + v_\perp^2\}$.

We easily obtain the following expression for the adjoint operator,

$$D_0^* \cdot \varphi(V) = \frac{1}{\rho^1(V)}\,[R_{\mathcal{J}} \tilde{D}_0 \cdot E_I(\rho^2 \varphi)]\,(V)\,, \tag{4.33}$$

with

$$\tilde{D}_0 \cdot \varphi(V) = -\frac{1}{\Delta v_{\|}}(\varphi^{\|}(V) - \varphi^{\|}(V - \Delta v_{\|}))$$
$$+\frac{1}{2\,\Delta v_{\perp}}(\varphi^{\perp}(V + 2\,\Delta v_{\perp}) - \varphi^{\perp}(V + \Delta v_{\perp}) + 3\,\varphi^{\perp}(V) + \varphi^{\perp}(V - \Delta v_{\perp})) \quad \text{if } v_{\perp} = 2\,k\,\Delta v_{\perp},$$
$$= -\frac{1}{\Delta v_{\|}}(\varphi^{\|}(V) - \varphi^{\|}(V - \Delta v_{\|})) - \frac{1}{2\,\Delta v_{\perp}}(4\,\varphi^{\perp}(V + \Delta v_{\perp})) \quad \text{if } v_{\perp} = (2\,k+1)\,\Delta v_{\perp}. \tag{4.34}$$

The α^1 -integration

Now, in order to achieve the discretization, we have to do the computation of $q(f, f)$ given by formula (4.6). This computation relies on an α^1-integration which cannot be done analytically. We refer to Annex C for its numerical computation (involving elliptic integrals and finite differences) and for the actual implemented expression of (4.6).

The time discretization

At last, we implement an explicit time discretization. Then, the velocity distribution f is approximated by $(f^n(V))_{n\in\mathbb{N}}$, $f^n(V) \sim f\left(\sum_{i=0}^{n-1} \Delta t^i, V\right)$ solution of

$$\begin{cases} f^{n+1}(V) = f^n(V) + \Delta t^n\, Q_{imp}(f^n, f^n)(V), & V \in \mathcal{J}, n \in \mathbb{N}, \\ f^0(V) = (f_0)(V), & V \in \mathcal{J}. \end{cases} \tag{4.35}$$

Easily, we have that the conservation properties are satisfied for the solution of (4.35).

On another hand, in order for the solution to be positive, Δt^n has to be such that

$$\Delta t^n < \inf_{\{V \in \mathcal{J},\, Q_{imp}(f^n, f^n)(V) < 0\}} \left(\frac{-f^n(V)}{Q_{imp}(f^n, f^n)(V)}\right), \tag{4.36}$$

and for the decrease of entropy, $\Delta t^n < \tau^n$ where τ^n realizes the minimum of the entropy in the direction $Q_{imp}(f^n, f^n)$, i.e.

$$\mathcal{L}_{\mathcal{J}}(f^n + \tau^n\, Q_{imp}(f^n, f^n)\,\mathrm{Log}\,(f^n + \tau^n\, Q_{imp}(f^n, f^n))) =$$
$$\min_{\tau\in\mathbb{R}^+} \mathcal{L}_{\mathcal{J}}(f^n + \tau Q_{imp}(f^n, f^n)\,\mathrm{Log}\,(f^n + \tau Q_{imp}(f^n, f^n))). \tag{4.37}$$

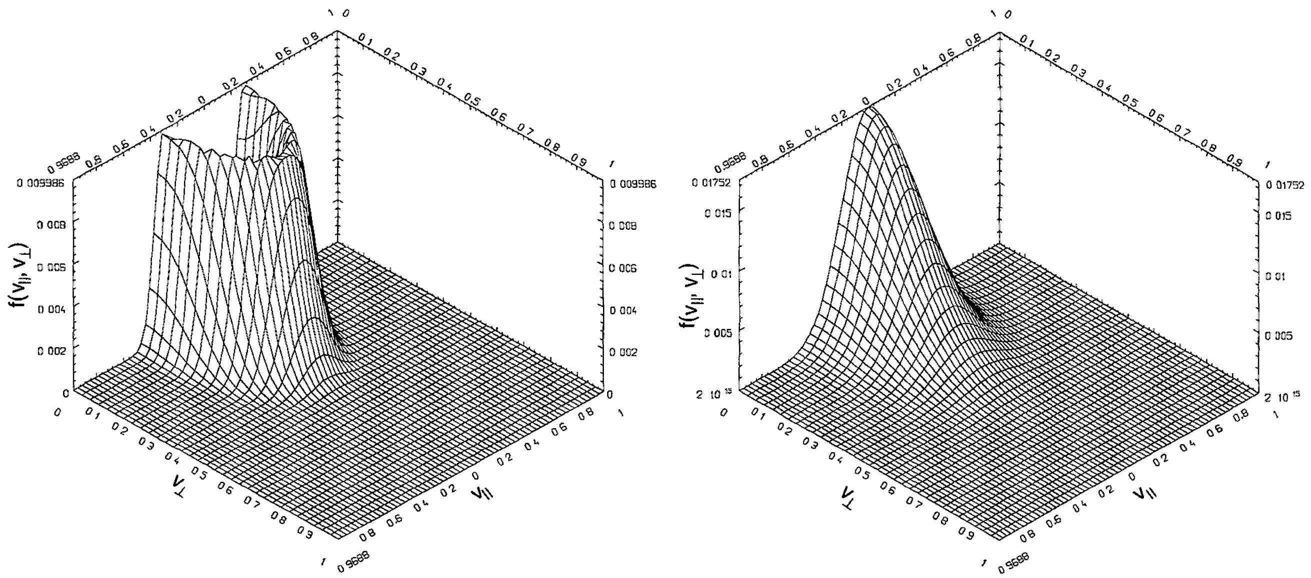
Since $\mathcal{L}_{\mathcal{J}}(f \log f)$ is a convex function of f, τ^n exits and is unique.

5. TESTS

Test 1

First we simulate the dimensionless equation

$$f^{n+1} = f^n + \Delta t^n \frac{1}{4\,\pi} Q_{imp}(f^n, f^n), \tag{5.1}$$



a. **Initial data.** b. $f(v_{\|}, v_{\perp})$ **at** t **= 89,17**

Figure 5.1. —f^0 **and** f **after relaxation**

with the spherically distributed initial data

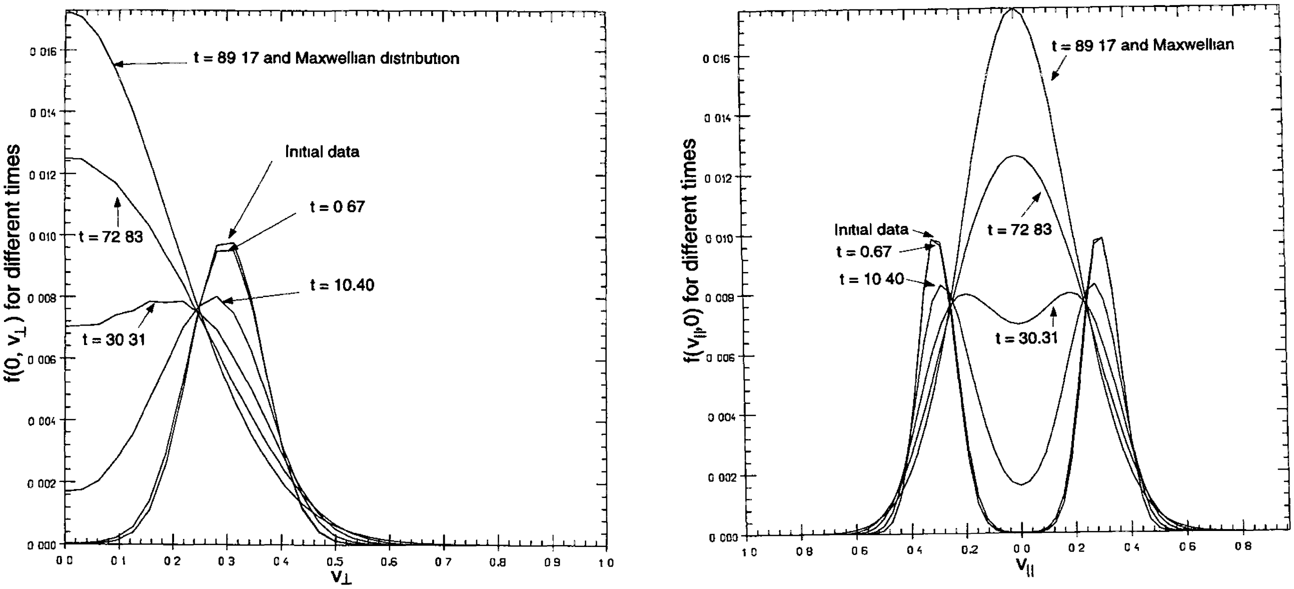
$$f^0(V) = 0.01 \exp\{-10[(|v| - 0.3)/0.3]^2\}, \tag{5.2}$$

the velocity modulus $|v|$ being $\sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{v_{\|}^2 + v_{\perp}^2}$, on the domaine $(-1, 1) \times (0, 1)$ discretized with a 65×33 regular mesh. Function f^0 is drawn on *figure* 5.2.a. By the way, we also give the distribution function after relaxation on *figure* 5.2.b. W. M. Mac Donald, M. N. Rosenbluth & W. Chuck [19] simulated the considered problem using the 1D character induced by the spherical symmetry assumption with a 1D explicit in time and finite difference scheme. Their results are shown on *figure* 5.3. As on the straighlines $(v_{\|} = 0)$ and $(v_{\perp} = 0)$ the velocity modulus $|v|$ equals $v_{\perp}$ and $v_{\|}$ respectively, in order to compare our results with their ones, we give on *figure* 5.2 the functions $f(0, v_{\perp})$ and $f(v_{\|}, 0)$ for the same times as they did. Despite the spherical symmetry is not a natural configuration for our code, the results of *figures* 5.2 and 5.3 are correlated with a good degree of accuracy.

This test exhibit the good behaviour of our method, which in addition to the decrease of entropy and the conservation properties leading to the relaxation to the right Maxwellian distribution, generates no numerical drift one the Maxwellian state if reached (as happens on *figure* 5.3).

Test 2

The second test consists in simulating a collision of two plasmas constituted of the same species of particles whose charge and mass numbers are $Z = 11$ and $A = 27$. For a complete and spatially non homogeneous simulation of this problem with a fluid code, we refer to R. L. Berger *et al.* [5]. A kinetic and spatially homogeneous simulation of this problem was done by O. Larroche [16], who implemented a mass-conserving finite volume and implicit in time scheme for solving the Fokker-Planck equation. We shall compare our results with their ones.



a. $f(v_{\parallel} = 0, v_{\perp})$ b. $f(v_{\parallel}, v_{\perp} = 0)$.

Figure 5.2. — $f(v_{\parallel} = 0, v_{\perp})$ and $f(v_{\parallel}, 0)$ for different times

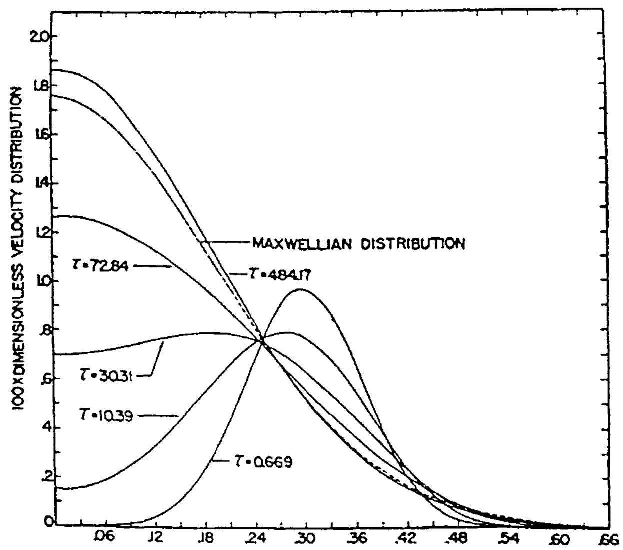
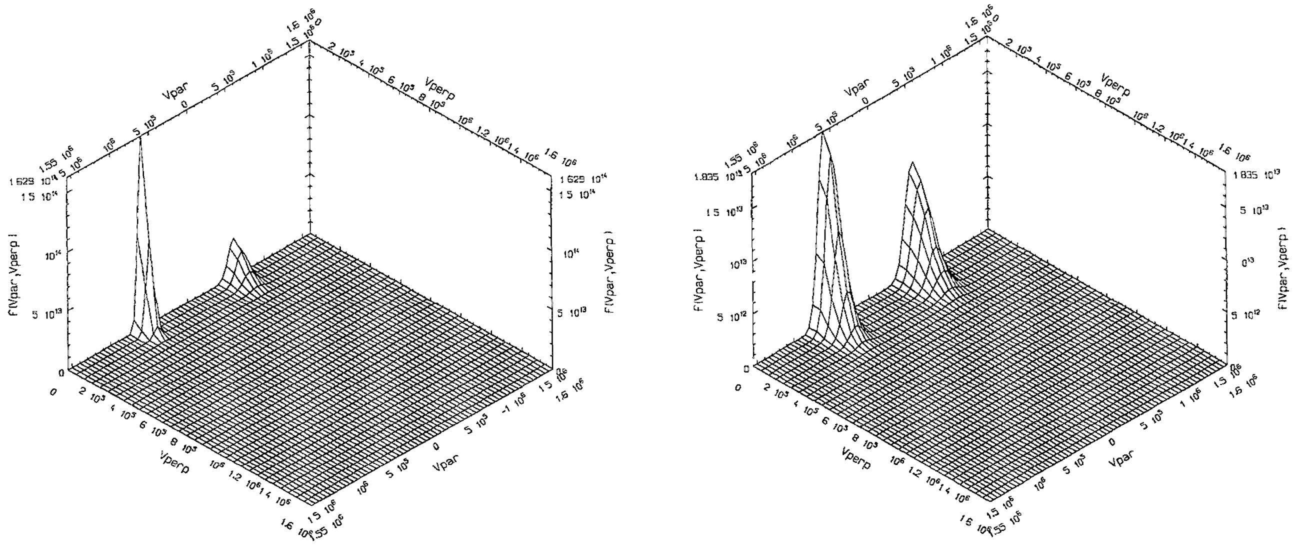


Figure 5.3. — Simulation of W. M. Mac Donald, M. N. Rosenbluth & W. Chuck.

The initial distribution is composed of two Maxwellian beams with density $n_1 = n_2 = 2 \times 10^{29}\ \mathrm{m}^{-3}$, velocity $v_1 = -v_2 = 6 \times 10^5\ \mathrm{ms}^{-1}$ and temperature $T_1 = 5.8 \times 10^6\ K(= 0.5\ \mathrm{kev})$ and $T_2 = 17.4 \times 10^6\ K\ (= 1.5\ \mathrm{kev})$. The equation to simulate is

$$f^{n+1} = f^n + \Delta t^n \frac{Z^4 e^4 \operatorname{Log}(\Lambda)}{\varepsilon_0 A^2 m_p^2} Q_{imp}(f^n, f^n) \tag{5.3}$$



a. Initial data. **b.** $t = 25\,83\,\tau_c$.

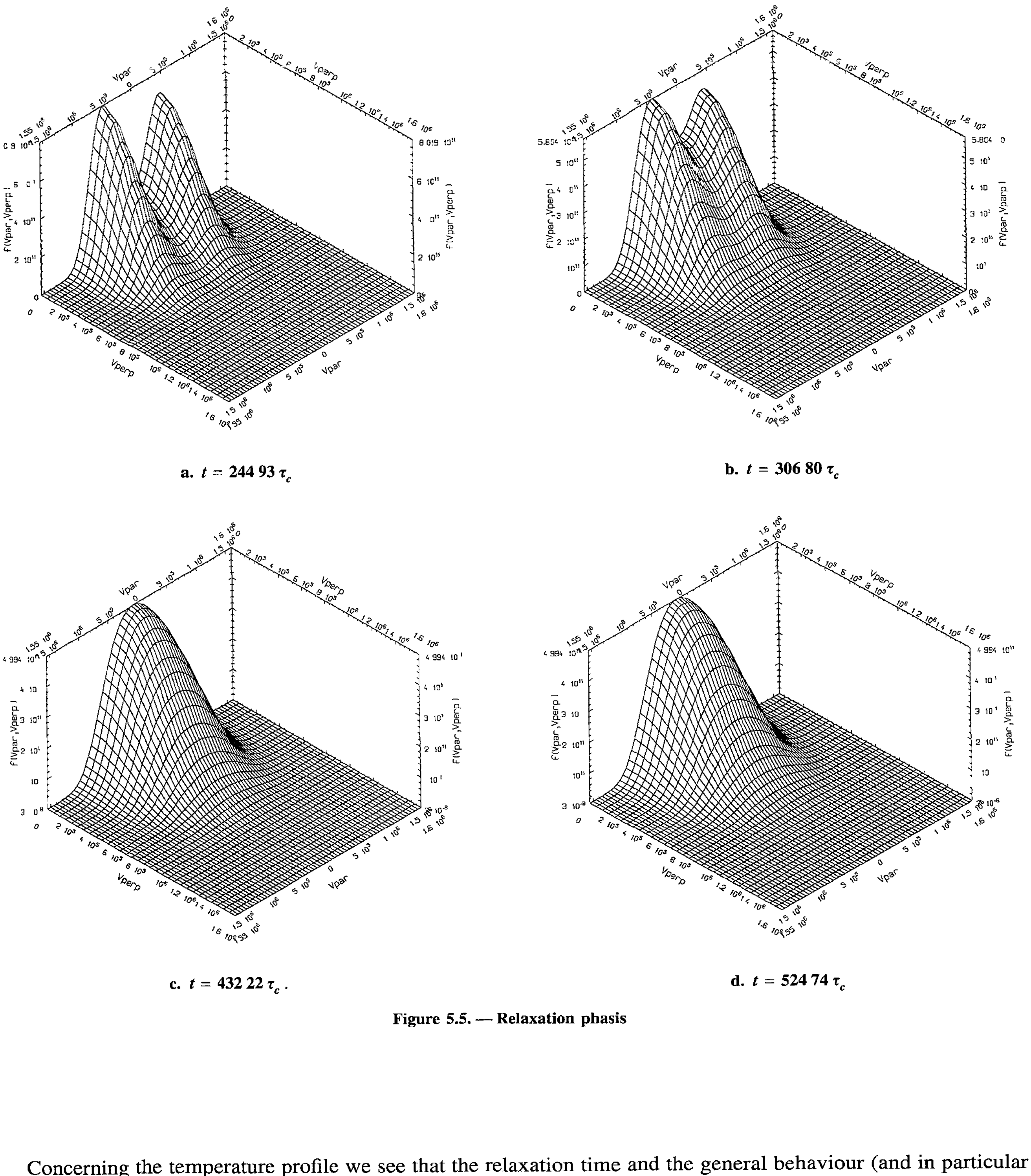
Figure 5.4. — Heating of the beams

where Log (Λ) is the Coulomb Logarithm with $\Lambda = \dfrac{3\,\varepsilon_0}{Ze^3}\left(\dfrac{k(T_1+T_2)}{2(n_1+n_2)}\right)^{3/2}$, ε_0 the dielectric permittivity of vacuum, e the elementary charge, m_p the proton mass and k the Boltzmann constant. In the results to come, we use as time unit the ion-ion collision time τ_c of a plasma constituted of the same species with density $n_0 = n_1 + n_2$ and $T_0 = 1/2(T_1 + T_2)$, $\tau_c = \dfrac{\varepsilon_0 (kT_0)^{3/2} A^{1/2} m_p^{1/2}}{n_0 Z^4 e^4 \operatorname{Log}(\Lambda)}$. First of all, we see on *figure* 5.4 that the two beams heat on each other. Then their relative velocity tends to zero (see *fig.* 5.5) until the complete relaxation shown on *figure* 5.6.

In order to compare these results with those obtained by O. Larroche [16], we give the profiles of the mass, velocity, and temperatures which are defined by

$$
\begin{aligned}
&\rho = \mathcal{L}_{\mathcal{G}}(f)\,,\\
&u_\parallel = 1/\rho\ \mathcal{L}_{\mathcal{G}}(f v_\parallel)\,,\\
&T = A m_p/(3\,k\rho)\ \mathcal{L}_{\mathcal{G}}(f((v_\parallel - u_\parallel)^2 + v_\perp^2))\,,\\
&T_\parallel = A m_p/(k\rho)\ \mathcal{L}_{\mathcal{G}}(f(v_\parallel - u_\parallel)^2)\,,\\
&T_\perp = A m_p/(2\,k\rho)\ \mathcal{L}_{\mathcal{G}}(f v_\perp^2)\,.
\end{aligned}
\tag{5.4}
$$

These profiles, which are given on *figures* 5.7 and 5.8, show that the conservation properties are satisfied with a very good degree of accuracy.



a. $t = 244\ 93\ \tau_c$

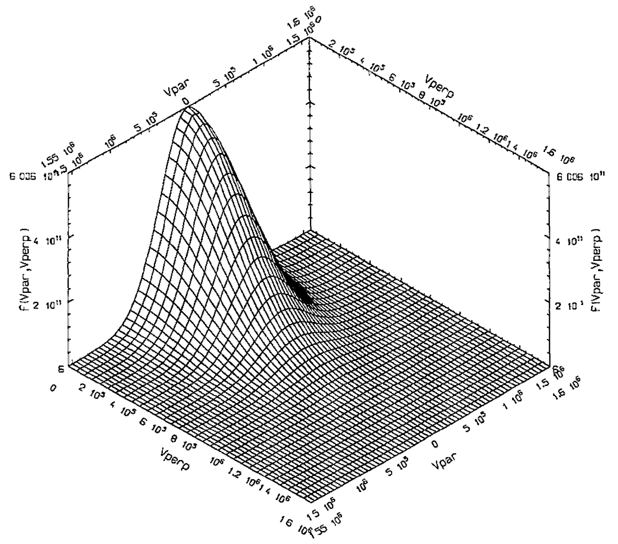
b. $t = 306\ 80\ \tau_c$

c. $t = 432\ 22\ \tau_c$.

d. $t = 524\ 74\ \tau_c$

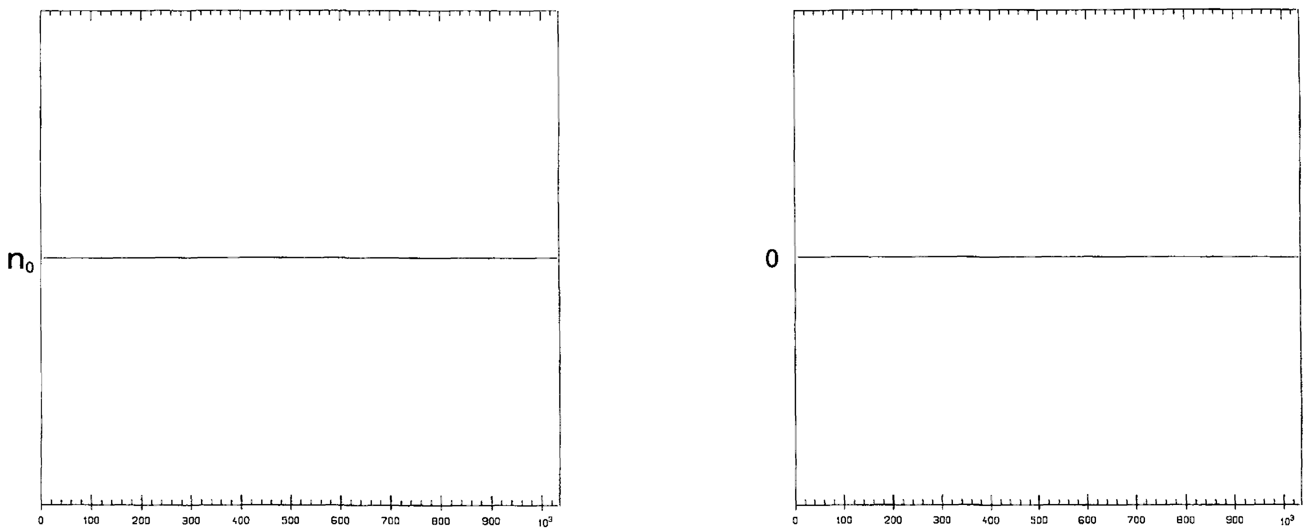
Figure 5.5. — Relaxation phasis

Concerning the temperature profile we see that the relaxation time and the general behaviour (and in particular the angles β and γ) are quite similar to the one obtained in [16]. We point out that in [16] the presence of electrons gives rise to a drift of the temperature in the end of the computation which is not the case here since there is only one species of ion. In spite of the improvement of D. Deck & G. Samba [9] to the method of O. Larroche [16],



$t = 1\,446.76\,\tau_c$.

Figure 5.6. — Maxwellian distribution after relaxation



a. Mass. **b. Velocity.**

Figure 5.7. — Evolution of the mass and the velocity

the resulting scheme does not ensure decrease of the kinetic entropy in every case. Moreover, the correction raises difficulties when the method is applied to multi-species plasmas. Our approach seems to be better adapted to this problem.

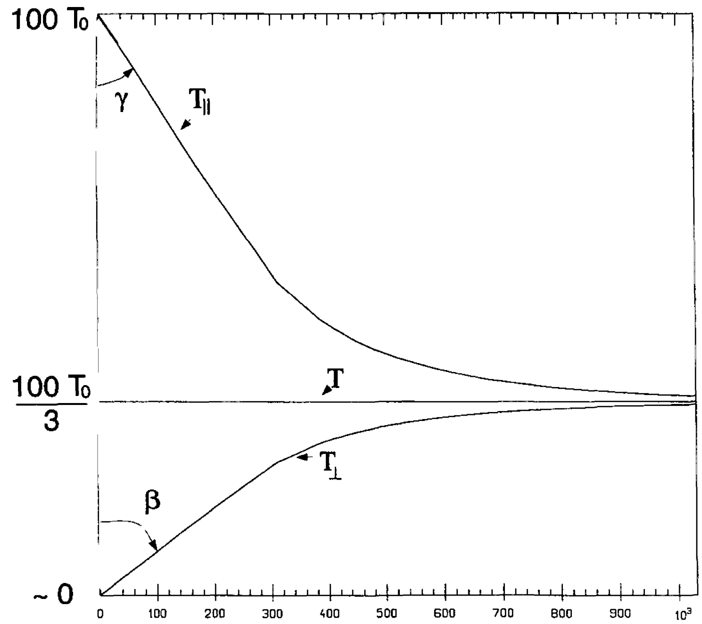


Figure 5.8. — Evolution of the Temperatures in T_0 unit.

ANNEXES

A. Derivatives of $|v - v^1|$ and computation of $\Phi(v - v^1)$

We have

$$|v - v^1|^2 = (v_\parallel - v_\parallel^1)^2 + (v_\perp - v_\perp^1 \cos(\alpha^1 - \alpha))^2 + v_\perp^{1^2} \sin^2(\alpha^1 - \alpha) \tag{A.1}$$

$$= (v_\parallel - v_\parallel^1)^2 + (v_\perp^1 - v_\perp \cos(\alpha - \alpha^1))^2 + v_\perp^2 \sin^2(\alpha - \alpha^1)$$

$$= (v_\parallel - v_\parallel^1)^2 + v_\perp^2 + v_\perp^{1^2} - 2\, v_\perp v_\perp^1 \cos(\alpha^1 - \alpha)\,.$$

The partial derivatives of $|v - v^1|$ are then

$$\begin{aligned}
\frac{\partial}{\partial v_\parallel}(|v - v^1|) &= \frac{v_\parallel - v_\parallel^1}{|v - v^1|}, & \frac{\partial}{\partial v_\parallel^1}(|v - v^1|) &= -\frac{v_\parallel - v_\parallel^1}{|v - v^1|},\\
\frac{\partial}{\partial v_\perp}(|v - v^1|) &= \frac{v_\perp - v_\perp^1 \cos(\alpha^1 - \alpha)}{|v - v^1|}, & \frac{\partial}{\partial v_\perp^1}(|v - v^1|) &= \frac{v_\perp^1 - v_\perp \cos(\alpha^1 - \alpha)}{|v - v^1|},\\
\frac{\partial}{\partial \alpha}(|v - v^1|) &= -\frac{v_\perp v_\perp^1 \sin(\alpha^1 - \alpha)}{|v - v^1|}, & \frac{\partial}{\partial \alpha^1}(|v - v^1|) &= \frac{v_\perp - v_\perp^1 \sin(\alpha^1 - \alpha)}{|v - v^1|},
\end{aligned} \tag{A.2}$$

while the derivatives of $|v - v^1|^{-1}$ are given by:

$$\begin{aligned}
\frac{\partial}{\partial v_\parallel}\left(\frac{1}{|v-v^1|}\right) &= \frac{v_\parallel - v^1_\parallel}{|v-v^1|^3}, & \frac{\partial}{\partial v^1_\parallel}\left(\frac{1}{|v-v^1|}\right) &= \frac{v_\parallel - v^1_\parallel}{|v-v^1|^3},\\
\frac{\partial}{\partial v_\perp}\left(\frac{1}{|v-v^1|}\right) &= -\frac{v_\perp - v^1_\perp \cos(\alpha^1-\alpha)}{|v-v^1|^3}, & \frac{\partial}{\partial v^1_\perp}\left(\frac{1}{|v-v^1|}\right) &= -\frac{v^1_\perp - v_\perp \cos(\alpha^1-\alpha)}{|v-v^1|^3},\\
\frac{\partial}{\partial \alpha}\left(\frac{1}{|v-v^1|}\right) &= \frac{v_\perp v^1_\perp \sin(\alpha^1-\alpha)}{|v-v^1|^3}, & \frac{\partial}{\partial \alpha^1}\left(\frac{1}{|v-v^1|}\right) &= -\frac{v_\perp v^1_\perp \sin(\alpha^1-\alpha)}{|v-v^1|^3}.
\end{aligned} \tag{A.3}$$

Then the second partial derivatives of $|v - v^1|$ are:

$$\frac{\partial^2}{\partial v_\parallel^2}(|v-v^1|) = \frac{1}{|v-v^1|} - \frac{(v_\parallel - v^1_\parallel)^2}{|v-v^1|^3},$$

$$\frac{\partial^2}{\partial v_\perp^2}(|v-v^1|) = \frac{1}{|v-v^1|} - \frac{(v_\perp - v^1_\perp \cos(\alpha^1-\alpha))^2}{|v-v^1|^3},$$

$$\frac{\partial^2}{\partial v_\perp\,\partial v^1_\perp}(|v-v^1|) = -\frac{\cos(\alpha^1-\alpha)}{|v-v^1|} - \frac{(v_\perp - v^1_\perp \cos(\alpha^1-\alpha))\,(v^1_\perp - v_\perp \cos(\alpha^1-\alpha))}{|v-v^1|^3},$$

$$\frac{\partial^2}{\partial v_\parallel\,\partial v_\perp}(|v-v^1|) = -\frac{(v_\parallel - v^1_\parallel)\,(v_\perp - v^1_\perp \cos(\alpha^1-\alpha))}{|v-v^1|^3},$$

$$\frac{\partial^2}{\partial v_\parallel\,\partial v^1_\perp}(|v-v^1|) = -\frac{(v_\parallel - v^1_\parallel)\,(v^1_\perp - v_\perp \cos(\alpha^1-\alpha))}{|v-v^1|^3}, \tag{A.4}$$

$$\frac{\partial^2}{\partial \alpha^1\,\partial \alpha}(|v-v^1|) = -\frac{v_\perp v^1_\perp \cos(\alpha^1-\alpha)}{|v-v^1|} + \frac{v^2_\perp v^{1^2}_\perp \sin^2(\alpha^1-\alpha)}{|v-v^1|^3},$$

$$\frac{\partial^2}{\partial v_\parallel\,\partial \alpha^1}(|v-v^1|) = -\frac{(v_\parallel - v^1_\parallel)\,(v_\perp v^1_\perp \sin(\alpha^1-\alpha))}{|v-v^1|^3},$$

$$\frac{\partial^2}{\partial v_\perp\,\partial \alpha^1}(|v-v^1|) = \frac{v^1_\perp \sin(\alpha^1-\alpha)}{|v-v^1|} - \frac{v_\perp v^1_\perp \sin(\alpha^1-\alpha)\,(v_\perp - v^1_\perp \cos(\alpha^1-\alpha))}{|v-v^1|^3}$$

$$\frac{\partial^2}{\partial v^1_\perp\,\partial \alpha^1}(|v-v^1|) = \frac{v_\perp \sin(\alpha^1-\alpha)}{|v-v^1|} - \frac{v_\perp v^1_\perp \sin(\alpha^1-\alpha)\,(v^1_\perp - v_\perp \cos(\alpha^1-\alpha))}{|v-v^1|^3}.$$

Since the tensor $\Phi(v - v^1)$, expressed in the local basis B_α, is given by:

$$\Phi^{B_\alpha}(v - v^1) =$$

$$\begin{pmatrix} \frac{1}{|v-v^1|} - \frac{(v_\parallel - v_\parallel^1)^2}{|v-v^1|^3} & -\frac{(v_\parallel - v_\parallel^1)(v_\perp - v_\perp^1\cos(\alpha^1-\alpha))}{|v-v^1|^3} & \frac{(v_\parallel - v_\parallel^1)(v_\perp^1\sin(\alpha^1-\alpha))}{|v-v^1|^3} \\ -\frac{(v_\parallel - v_\parallel^1)(v_\perp - v_\perp^1\cos(\alpha^1-\alpha))}{|v-v^1|^3} & \frac{1}{|v-v^1|} - \frac{(v_\perp - v_\perp^1\cos(\alpha^1-\alpha))^2}{|v-v^1|^3} & \frac{(v_\perp - v_\perp^1\cos(\alpha^1-\alpha))(v_\perp^1\sin(\alpha^1-\alpha))}{|v-v^1|^3} \\ \frac{(v_\parallel - v_\parallel^1)(v_\perp^1\sin(\alpha^1-\alpha))}{|v-v^1|^3} & \frac{(v_\perp - v_\perp^1\cos(\alpha^1-\alpha))(v_\perp^1\sin(\alpha^1-\alpha))}{|v-v^1|^3} & \frac{1}{|v-v^1|} - \frac{v_\perp^{1^2}\sin^2(\alpha^1-\alpha)}{|v-v^1|^3} \end{pmatrix}, \quad (A.5)$$

using (A.4), it may also be given in terms of the derivatives of $|v - v^1|$:

$$\Phi^{B_\alpha}(v - v^1) =$$

$$\begin{pmatrix} \frac{\partial^2}{\partial v_\parallel^2}(|v-v^1|) & \frac{\partial^2}{\partial v_\parallel\,\partial v_\perp}(|v-v^1|) & -\frac{1}{v_\perp}\frac{\partial^2}{\partial v_\parallel\,\partial\alpha^1}(|v-v^1|) \\ \frac{\partial^2}{\partial v_\parallel\,\partial v_\perp}(|v-v^1|) & \frac{\partial^2}{\partial v_\perp^2}(|v-v^1|) & -\frac{1}{v_\perp}\frac{\partial^2}{\partial v_\perp\,\partial\alpha^1}(|v-v^1|) - \frac{\partial}{\partial\alpha^1}(|v-v^1|) \\ -\frac{1}{v_\perp}\frac{\partial^2}{\partial v_\parallel\,\partial\alpha^1}(|v-v^1|) & -\frac{1}{v_\perp}\frac{\partial^2}{\partial v_\perp\,\partial\alpha^1}(|v-v^1|) - \frac{\partial}{\partial\alpha^1}(|v-v^1|) & \frac{1}{|v-v^1|} - \frac{v_\perp^{1^2}\sin^2(\alpha^1-\alpha)}{|v-v^1|^3} \end{pmatrix}. \quad (A.6)$$

B. Proof of Proposition 2.3

Since the divergence of any vector valued cylindrical function $(\varphi(v))^{B_\alpha} = (\varphi^\parallel(V), (\varphi^\perp(V), 0)$ writes $\mathrm{D}iv\,\varphi = \frac{1}{v_\perp}(\partial_{v_\parallel}(v_\perp\varphi^\parallel) + \partial_{v_\perp}(v_\perp\varphi^\perp))$, it is a real valued cylindrical function; so it suffices to prove the result for $p(f,f)$.

According to expression (A.6), a straightforward computation gives

$$\begin{aligned} p^\alpha(f,f)(V) = \int_0^{2\pi}\int_\Omega f(V)f(V^1)\Bigg(\Bigg(-\frac{1}{v_\perp}\frac{\partial^2}{\partial v_\parallel\,\partial\alpha^1}|v-v^1|\Bigg)(\partial^\parallel \operatorname{Log} f(V) - \partial^\parallel \operatorname{Log} f(V^1)) \\ -\frac{1}{v_\perp}\Bigg(\Bigg(\frac{\partial^2}{\partial v_\perp\,\partial\alpha^1} - \frac{\partial}{\partial\alpha^1}\Bigg)|v-v^1|\Bigg)(\partial^\perp \operatorname{Log} f(V)) \\ +\frac{1}{v_\perp}\Bigg(-\frac{v_\perp\sin(\alpha^1-\alpha)}{|v-v^1|} + \frac{v_\perp v_\perp^1\sin(\alpha^1-\alpha)(v_\perp^1 - v_\perp\cos(\alpha^1-\alpha))}{|v-v^1|^3}\Bigg)(\partial^\perp \operatorname{Log} f(V^1))\Bigg)\,d\sigma^1\,d\alpha^1. \end{aligned} \quad (B.1)$$

The last term of the integrand can then be simplified by use of (A.4) which yields to the final expression

$$\begin{aligned} p^\alpha(f,f)(V) &= \int_0^{2\pi} -\frac{1}{v_\perp}\frac{\partial}{\partial\alpha^1}\int_\Omega f(V)f(V^1)\Bigg(\Bigg(\frac{\partial}{\partial v_\parallel}|v-v^1|\Bigg)(\partial^\parallel \operatorname{Log} f(V) - \partial^\parallel \operatorname{Log} f(V^1)) \\ &\quad + \Bigg(\Bigg(\frac{\partial}{\partial v_\perp} - 1\Bigg)|v-v^1|\Bigg)(\partial^\perp \operatorname{Log} f(V)) + \Bigg(\frac{\partial}{\partial v_\perp^1}|v-v^1|\Bigg)(\partial^\perp \operatorname{Log} f(V^1))\Bigg)\,d\sigma^1\,d\alpha^1, \\ &= 0. \end{aligned} \quad (B.2)$$

Secondly, in (2.5) and (A.5), α and α^1 only appear through sinusoidal functions of $(\alpha^1-\alpha)$. Then the integration with respect to α^1 over a whole period occuring in (2.2) remove the α-dependence. The conditions (2.1) are both satisfied and the proposition is thus proved. ■

C. The α^1-integration

This Annex is devoted to the computation of $q(f,f)$ (see (4.6)). This computation relies on the knowlege of

$$\pi(v,v^1)=\int_{(0,2\pi)}\{\Phi(v-v^1).(D\,\mathrm{Log}f(v)-D\,\mathrm{Log}f(v^1))\}\,d\alpha^1, \tag{C.1}$$

for every $(V,V^1)\in I^2$. Denoting by $\pi^{B_\alpha}(v,v^1)=(\pi^{\parallel}(V,V^1),\pi^{\perp}(V,V^1),\pi^{\alpha}(V,V^1))$, the expression of $\pi(v,v^1)$ in the basis B_α, we recall that

$$\pi^{\alpha}(V,V^1)=0,\quad \forall(V,V^1)\in I^2, \tag{C.2}$$

and that $\pi^{\parallel}(V,V^1)$ and $\pi^{\perp}(V,V^1)$ do not depend on α. Then using the expressions of Φ^{B_α} (see (A.5), (A.6)), we get

$$\begin{aligned}
\pi^{\parallel}(V,V^1)=\int_0^{2\pi}\Bigg(&\left\{\frac{\partial^2}{\partial v_{\parallel}^2}(|v-v^1|)\right\}(\partial^{\parallel}\,\mathrm{Log}f(V)-\partial^{\parallel}\,\mathrm{Log}f(V^1))\\
&+\left\{\frac{\partial^2}{\partial v_{\parallel}\,\partial v_{\perp}}(|v-v^1|)\right\}(\partial^{\perp}\mathrm{Log}f(V))\\
&+\left\{\frac{(v_{\parallel}-v_{\parallel}^1)(v_{\perp}-v_{\perp}^1\cos(\alpha^1-\alpha))}{|v-v^1|^3}\right\}(\partial^{\perp}\mathrm{Log}f(V^1)\cos(\alpha^1-\alpha))\\
&-\left\{\frac{(v_{\parallel}-v_{\parallel}^1)(v_{\perp}^1\sin(\alpha^1-\alpha))}{|v-v^1|^3}\right\}(\partial^{\perp}\mathrm{Log}f(V^1)\sin(\alpha^1-\alpha))\Bigg)\,d\alpha^1,\\
=&\left\{\int_0^{2\pi}\frac{\partial^2}{\partial v_{\parallel}^2}(|v-v^1|)\,d\alpha^1\right\}(\partial^{\parallel}\,\mathrm{Log}f(V)-\partial^{\parallel}\,\mathrm{Log}f(V^1))\\
&+\left\{\int_0^{2\pi}\frac{\partial^2}{\partial v_{\parallel}\,\partial v_{\perp}}(|v-v^1|)\,d\alpha^1\right\}(\partial^{\perp}\mathrm{Log}f(V)-\partial^{\perp}\mathrm{Log}f(V^1))\\
&-\left\{\int_0^{2\pi}\frac{(v_{\parallel}-v_{\parallel}^1)[(v_{\perp}-v_{\perp}^1\cos(\alpha^1-\alpha))+(v_{\perp}^1-v_{\perp}\cos(\alpha^1-\alpha))]}{|v-v^1|^3}\,d\alpha^1\right\}(\partial^{\perp}\mathrm{Log}f(V^1)),\\
=&\left\{\int_0^{2\pi}\frac{\partial^2}{\partial v_{\parallel}^2}(|v-v^1|)\,d\alpha^1\right\}(\partial^{\parallel}\,\mathrm{Log}f(V)-\partial^{\parallel}\,\mathrm{Log}f(V^1))\\
&+\left\{\int_0^{2\pi}\frac{\partial^2}{\partial v_{\parallel}\,\partial v_{\perp}}(|v-v^1|)\,d\alpha^1\right\}(\partial^{\perp}\mathrm{Log}f(V)-\partial^{\perp}\mathrm{Log}f(V^1))\\
&+\left\{\int_0^{2\pi}\left(\frac{\partial^2}{\partial v_{\parallel}\,\partial v_{\perp}}+\frac{\partial^2}{\partial v_{\parallel}\,\partial v_{\perp}^1}\right)(|v-v^1|)\,d\alpha^1\right\}(\partial^{\perp}\mathrm{Log}f(V^1)).
\end{aligned}\tag{C.3}$$

A similar computation gives the second component

$$\pi^{\perp}(V, V^1) = \left\{ \int_0^{2\pi} \frac{\partial^2}{\partial v_{\parallel}\, \partial v_{\perp}} (|v - v^1|)\, d\alpha^1 \right\} (\partial^{\parallel} \operatorname{Log} f(V) - \partial^{\parallel} \operatorname{Log} f(V^1)) + \left\{ \int_0^{2\pi} \frac{\partial^2}{\partial v_{\perp}^2} (|v - v^1|)\, d\alpha^1 \right\} (\partial^{\perp} \operatorname{Log} f(V) - \partial^{\perp} \operatorname{Log} f(V^1)) + \left\{ \int_0^{2\pi} \left(\frac{\partial^2}{\partial v_{\perp}^2} + \frac{\partial^2}{\partial v_{\perp}\, \partial v_{\perp}^1} \right) (|v - v^1|)\, d\alpha^1 \right\} (\partial^{\perp} \operatorname{Log} f(V^1)) . \tag{C.4}$$

Hence setting

$$\begin{aligned}
\mathcal{U}^1(V, V^1) &= \int_0^{2\pi} \frac{\partial^2}{\partial v_{\parallel}^2} (|v - v^1|)\, d\alpha^1 , \\
\mathcal{U}^2(V, V^1) &= \int_0^{2\pi} \frac{\partial^2}{\partial v_{\parallel}\, \partial v_{\perp}} (|v - v^1|)\, d\alpha^1, \qquad \mathcal{U}(V, V^1) = \begin{pmatrix} \mathcal{U}^1(V, V^1) & \mathcal{U}^2(V, V^1) \\ \mathcal{U}^2(V, V^1) & \mathcal{U}^4(V, V^1) \end{pmatrix}, \\
\mathcal{U}^3(V, V^1) &= \int_0^{2\pi} \left(\frac{\partial^2}{\partial v_{\parallel}\, \partial v_{\perp}} + \frac{\partial^2}{\partial v_{\parallel}\, \partial v_{\perp}^1} \right) (|v - v^1|)\, d\alpha^1, \\
\mathcal{U}^4(V, V^1) &= \int_0^{2\pi} \frac{\partial^2}{\partial v_{\perp}^2} (|v - v^1|)\, d\alpha^1, \qquad \mathcal{W}(V, V^1) = \begin{pmatrix} \mathcal{U}^3(V, V^1) \\ \mathcal{U}^5(V, V^1) \end{pmatrix}, \\
\mathcal{U}^5(V, V^1) &= \int_0^{2\pi} \left(\frac{\partial^2}{\partial v_{\perp}^2} + \frac{\partial^2}{\partial v_{\perp}\, \partial v_{\perp}^1} \right) (|v - v^1|)\, d\alpha^1 ,
\end{aligned} \tag{C.5}$$

$\pi(v, v^1)$ expresses

$$\pi^{B_\alpha}(v, v^1) = \begin{pmatrix} \mathcal{U}(V, V^1) \begin{pmatrix} \partial^{\parallel} \operatorname{Log} f(V) - \partial^{\parallel} \operatorname{Log} f(V^1) \\ \partial^{\perp} \operatorname{Log} f(V) - \partial^{\perp} \operatorname{Log} f(V^1) \end{pmatrix} + \partial^{\perp} \operatorname{Log} f(V^1)\, \mathcal{W}(V, V^1) \\ 0 \end{pmatrix} . \tag{C.6}$$

Since the computation of the coefficients $\mathcal{U}^i$ is not easy, or even not possible (for instance if $V^1 = V$ the integrands are not integrable functions), we invert the integration and the derivation operators. Therefore, we define

$$\begin{aligned}
U^1(V, V^1) &= \frac{\partial^2}{\partial v_{\parallel}^2} \left(\int_0^{2\pi} |v - v^1|\, d\alpha^1 \right), \\
U^2(V, V^1) &= \frac{\partial^2}{\partial v_{\parallel}\, \partial v_{\perp}} \left(\int_0^{2\pi} |v - v^1|\, d\alpha^1 \right), \qquad U(V, V^1) = \begin{pmatrix} U^1(V, V^1) & U^2(V, V^1) \\ U^2(V, V^1) & U^4(V, V^1) \end{pmatrix}, \\
U^3(V, V^1) &= \left(\frac{\partial^2}{\partial v_{\parallel}\, \partial v_{\perp}} + \frac{\partial^2}{\partial v_{\parallel}\, \partial v_{\perp}^1} \right) \left(\int_0^{2\pi} |v - v^1|\, d\alpha^1 \right), \\
U^4(V, V^1) &= \frac{\partial^2}{\partial v_{\perp}^2} \left(\int_0^{2\pi} |v - v^1|\, d\alpha^1 \right), \qquad W(V, V^1) = \begin{pmatrix} U^3(V, V^1) \\ U^5(V, V^1) \end{pmatrix} . \\
U^5(V, V^1) &= \left(\frac{\partial^2}{\partial v_{\perp}^2} + \frac{\partial^2}{\partial v_{\perp}\, \partial v_{\perp}^1} \right) \left(\int_0^{2\pi} |v - v^1|\, d\alpha^1 \right),
\end{aligned} \tag{C.7}$$

Then, the operator $q(f,f)$ is replaced by

$$\mathscr{L}_I\left(f(V)\, f(V^1) \left(U(V,V^1) \begin{pmatrix} \partial^{\parallel} \operatorname{Log} f(V) - \partial^{\parallel} \operatorname{Log} f(V^1) \\ \partial^{\perp} \operatorname{Log} f(V) - \partial^{\perp} \operatorname{Log} f(V^1) \\ 0 \end{pmatrix} + \partial^{\perp} \operatorname{Log} f(V^1)\, W(V,V^1) \right) \right), \tag{C.8}$$

the derivation operators involved in (C.7) being replaced by finite difference operators. And, in order to access to an approximated value of

$$\int_0^{2\pi} |v - v^1| \, d\alpha^1 , \tag{C.9}$$

we express it in terms of elliptic integrals. Setting

$$a = (v_{\parallel} - v_{\parallel}^1)^2 + v_{\perp}^2 + v_{\perp}^{1^2} \quad \text{and} \quad b = 2\, v_{\perp}\, v_{\perp}^1 , \tag{C.10}$$

and using expression (A.1) of annexe A, we have

$$\int_0^{2\pi} |v - v^1| \, d\alpha^1 = 2 \int_{(0,\pi)} \sqrt{a - b \cos \alpha^1} \, d\alpha^1 . \tag{C.11}$$

As $a \geqslant b \geqslant 0$, applying formula n° 2.576, page 156 of I. S. Gradshteyn & I. M. Ryzhik [13], we deduce that

$$\begin{aligned} \int_0^{2\pi} |v - v^1| \, d\alpha^1 &= \left[4 \sqrt{a+b}\, \mathbf{E}\left(\arcsin \sqrt{\frac{(a+b)(1-\cos\alpha^1)}{2(a - b\cos\alpha^1)}}, \sqrt{\frac{2b}{a+b}} \right) \right]_{\alpha^1 = 0}^{\alpha^1 = \pi} \\ &= 4\sqrt{a+b} \left(\mathbf{E}\left(\frac{\pi}{2}, \sqrt{\frac{2b}{a+b}} \right) - \mathbf{E}\left(0, \sqrt{\frac{2b}{a+b}} \right) \right) \\ &= 4\sqrt{a+b} \left(\mathbf{E}\left(\frac{\pi}{2}, \sqrt{\frac{2b}{a+b}} \right) \right), \end{aligned} \tag{C.12}$$

where **E** is the second kind elliptic integral. In order to compute **E** we apply the method described in M. Abrahamowitz & A. I. Stegun [1], chapter 17.6, page 598.

REFERENCES

[1] M. ABRAHAMOWITZ & I. A. STEGUN. Handbook of mathematematical functions. *Dover Publications, INC, New York.*

[2] V. V. ARISTOV & F. G. CHEREMISIN, 1980, The conservative splitting method for solving Boltzmann's equation. *U.S.R.R. Comput. Maths. Math. Phys.,* Vol. 20, No. 1, p. 208-225.

[3] A. A. ARSENEV & O. E. BURYAK, 1991, On the connection between a solution of the Boltzmann equation and a solution of the Landau-Fokker-Planck equation. *Math. USSR Sbornik,* Vol. 69, No. 2, p. 465-478.

[4] A. A. ARSENEV & N. V. PESKOV, 1978, On the existence of a generalized solution of Landau's equation. *U.S.S.R. Comput. Maths Math. Phys.,* Vol. 17, p. 241-246.

[5] Yu. A. BEREZIN, M. S. PEKKER & V. N. KUDICK, 1987, Conservative Finite-Difference Schemes for the Fokker-Planck Equation Not Violating the Low of an Increasing Entropy. *Jour. of comp. phys.,* Vol. 69, p. 163-174.

[6] R. L. BERGER, J. R. ALBRITTON, C. J. RANDALL, E. A. WILLIAMS, W. L. KRUER, A. B. LANGDON & C. J. HANNA, 1990, Stopping and thermalization of interpenetrating plasma streams. *Phys. Fluids B,* Vol. 3, No. 1.

[7] A. V. BOBYLEV, 1981, Expansion of the Boltzmann collision integral in a Laudau series. *Sov. Phys. Dolk.,* Vol. 20, No. 11, p. 740-742.

[8] A. V. BOBYLEV, I. F. POTAPENKO & V. A. CHUYANOV, Kinetic equations of the Landau type as a model of the Boltzmann equation and completely conservative difference schemes. *U.S.R.R. Comput. Maths. Math. Phys.,* Vol. 20, No. 4, p. 190-201.

[9] D. DECK & G. SAMBA, 1994, Le code Procions. *Note CEA No. N 2780, CEA/CEL-V,* F-94195 Villeneuve St. Georges Cedex.

[10] L. DESVILLETTES, 1992, On asymptotics of the Boltzmann equation when the collisions become grazing. *Trans. Th. and Stat. Phys.,* Vol. 21, No. 3, p. 259-276.

[11] P. DEGOND & B. LUCQUIN-DESREUX, 1992, The Fokker-Plank assymptotics of the Boltzmann operator in the Coulomb case. *Math. Mod. and Meth. in Appl. Sc.,* Vol. 2, No. 2, p. 167-182.

[12] P. DEGOND & B. LUCQUIN-DESREUX, 1994, An entropy scheme for the Fokker-Planck collision operator of plasma kinetic theory. *Numer. Math.,* Vol. 68, p. 239-262.

[13] I. S. GRADSHTEYN & I. M. RYZHIK, Table of integrals, series and products. *Academic press.*

[14] N. A. KRALL & A. W. TRIVELPIECE, 1973, Principles of plasma physics. *Mc Graw Hill book company.*

[15] S. JORNA & L. WOOD, 1987, Fokker-Planck calculations on relaxation of anisotropic velocity distributions in plasmas. *Phys. rev. A,* Vol. 36, No. 1.

[16] O. LARROCHE, 1993, Kinetic simulation of a plasma collision experiment. *Phys. Fluids B,* Vol. 5, No. 8.

[17] M. LEMOU, C. BUET, S. CORDIER & P. DEGOND, A numerical, conservative and entropic scheme for the Fokker-Planck equation. *In preparation.*

[18] B. LUCQUIN-DESREUX, 1992, Discrétisation de l'opérateur de Fokker-Planck dans le cas homogène. *C. R. Acad. Sci.,* Paris, t. 314, p. 407-411.

[19] W. M. MAC DONALD, M. N. ROSENBLUTH & W. CHUCK, 1957, Relaxation of a system of particles with Coulomb interactions. *Phys. Rev.,* Vol. 107, No. 2.

[20] M. S. PEKKER & V. N. KUDICK, 1984, Conservative Difference Schemes for the Fokker-Planck Equation. *U.S.R.R. Comput. Maths. Math. Phys.,* Vol. 24, No. 3, p. 206-210.

[21] I. F. POTAPENKO & V. A. CHUYANOV, 1979, A completely conservative difference scheme for the two-dimensional Landau equation. *U.S.R.R. Comput. Maths. Math. Phys.,* Vol. 20, No. 2, p. 249-253.

[22] J. C. WITNEY, 1970, Finite Difference Methods for the Fokker-Planck Equation. *J. Comp. Phys.,* Vol. 6, p. 483-509.