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INCREMENTAL UNKNOWNS METHOD AND COMPACT SCHEMES (*)

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Abstract — *In this article we establish a link between the Incremental Unknowns method (IU) and the compact scheme discretization techniques (CS) Thanks to the high order of accuracy of the CS, we introduce via an interpolation process high order Incremental Unknowns The combination of the IU methodology and of the interpolation compact schemes improves the usual properties of the data compression of the IUs and is presented as a new tool for the implementation of Nonlinear Galerkin Methods when finite differences are used for space discretization In addition we derive efficient high accurate solutions of elliptic problems, when the operators are discretized with compact schemes © Elsevier, Paris*

Résumé — *Nous proposons ici de coupler la méthode des Inconnues Incrémentales (II) avec les techniques de discrétisation de Schémas Compacts (SC) En exploitant la précision élevée des SC nous construisons des II d'ordre élevé à l'aide d'un processus d'interpolation Cette approche nous permet, non seulement d'améliorer les propriétés de compression de données des II, mais aussi de disposer de nouveaux outils pour l'implémentation de méthodes de type Galerkin non linéaire en différences finies Nous présentons également une méthode de résolution numérique très précise de problèmes elliptiques © Elsevier, Paris*

1. INTRODUCTION

The Nonlinear Galerkin Method (NGL) was introduced in the view of the long time approximation of dissipative evolution equations and of the simulation of the turbulence (see [15] and [16]). This method which is derived from the theory of inertial manifolds (see e.g. [18] and the references therein) is a means to modelize the interaction between the large and the small wavelengths by an exact or an approximate law. It was shown that a different and appropriate treatment of these two types of wavelengths, to which one can associate large structures (denoted by Y) and small structures (denoted by Z), is very efficient for long time integration. Several numerical simulations confirm the efficiency of the method (see e.g. [12]). One of the keys of the method lies in the decomposition of the solution into suitable structures and particularly one of the main technical argument is that

$$\|Z\| << \|Y\| ,$$

$$\left\|\frac{dZ}{dt}\right\| << \left\|\frac{dY}{dt}\right\| ,$$

where $\|\,.\,\|$ is some suitable Hilbert norm. This type of inequalities means that there exist two different scales in time and in space: one for the Y's and one for the Z's. Then, for instance, splitting up methods can be considered for solving such problems (see [14]).

In the spectral case one can express the approximated solution as a truncated sum like

$$U = \sum_{i=1}^{2n} \alpha_i w_i = \sum_{i=1}^{n} \alpha_i w_i + \sum_{i=n+1}^{2n} \alpha_i w_i = Y + Z ,$$

and the several scales appear naturally. Here the numbers α_i denote real or complex coefficients and $(\omega_i)_{i \in \mathbb{N}}$ is an *ad hoc* Hilbert basis. When finite differences are used for the space discretization, the above decomposition is not possible. For that purpose, R. Temam introduced in [17] the incremental unknowns method

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(IU) for the implementation of Nonlinear Galerkin Method when finite differences are used for the space discretization. Using several levels of grids, this method consists in generating several structures in distinct points of a grid. More precisely it consists in replacing the nodal values of the unknown function outside the coarse grid by proper increments to the values on the coarse grids. These increments are expected to be small and are considered as the small structures. The generation of these structures is realized with the use of a hierarchical preconditioner which can be related in certain cases (but not allways, see e.g. [9]) to those of Hierarchical bases in finite elements ([1, 20]). We dispose of various numerical results which illustrate the efficiency of the IU method. In [8] it was shown that the condition number of the underlying matrices associated to selfadjoint elliptic operators is considerably reduced and then algorithms like Conjugate Gradient are very efficient, [7]. In [3] and [4] the use of IU method gave efficient generalizations of the Marder and Weitzner scheme for solving nonlinear eigenvalue and bifurcation problems. In [5] the extension of the IU method to a shifted mesh of MAC type gave efficient hierarchical preconditioner for the Uzawa operator associated to a generalized Stokes problem.

Up to this time the Incremental Unknowns that have been used were essentially of first or second order: i.e. if h denotes the mesh size of the fine grid, the magnitude of the IU was $O(h)$ (for the first order IUs [9]) or $O(h^2)$ (for the second order IUs). Third order IUs were introduced in [6]. The incremental unknowns play here the role of the small structures described above for the implementation of the NLG method. Unfortunately they are not always "small" as in the spectral case, in certain practical cases. Indeed this situation arises typically when the number of points is not large enough and the discretized function has strong gradients. Consequently only fine meshes can be a garanty of the small size of the IUs and it is then natural to try to build higher order incremental unknowns in order to need less grid points for the implementation of Nonlinear Galerkin Method-like in finite differences. The construction of the IU, after a hierarchization, can be summarized into an interpolation step: the order of the IU is the same as of the order of the interpolation scheme used. The use of traditional high order interpolation scheme should complicate the implementation of the IU method because of the important number of points that must be used locally for the interpolation scheme.

The use of the compact schemes (CS) seems to be a solution to the above problems. These schemes were introduced for the high order discretization accuracy in finite differences. This accuracy is close to the spectral accuracy and one of the advantages of the CS is that they can be adapted to non periodic boundary conditions (see [13] and the references therein). One of the main applications of the compact schemes is the simulation of turbulence but also the solution of hyperbolic system (see e.g. [2]) and the calculations of shocks ([10]). Combined with the Incremental Unknowns methodology, the interpolation compact schemes become a suitable and powerful tool of data compression because of their simplicity and of their high order of accuracy. Indeed, if p^{th} order IUs are considered, less digits are needed to store them (on the i^{th} grid with mesh h_i : the unknowns are of order $O(h_i^p)$ instead of order $O(\|U\|)$, that of the solution on the usual coarse grid).

Our aim in this paper is to make a link between compact schemes and the IU method in the double context of hierarchical methods and Nonlinear Galerkin method. For that purpose in one hand we use compact scheme for defining high order incremental unknowns, and in the other hand we use the incremental unknowns for preconditioning the underlying matrices of high order discretization of elliptic problem. Developement of schemes of NLG-like Methods with high order IUs and their implementation will be discussed elsewhere.

This paper is organized as follows. In Section 2 we recall the construction of the IUs relating their order to the order of the interpolation scheme used. Then in Section 3 we present some general results concerning the compact schemes in the one dimensional case. After that in Section 4 we introduce high order IUs in dimension one, two and three with an emphazis for the fourth order case. Finally in Section 5 we present some numerical results which are related to the combination of the IU method and of the compact scheme. They concern two aspects of our approach. First of all we illustrate the efficiency of the data compression method and we compare the decay of magnitude of the structures according to the grid level to which they belong. After that we consider elliptic problems. We point out a saturation phenomenon of the hierarchical preconditioning of the matrices associated to the Dirichlet problem. When the contrary situation is considered, say when the compact schemes are used for the high accurate discretization of an elliptic operator, we use the second order IU for the preconditioning the martix and we obtain efficient high accurate solution of the Dirichlet problem. Finally, using a hierarchization process, we derive from the discretization of a given elliptic partial differential operator an associated interpolation scheme with which we build related Incremental Unknowns; they are then adapted to the boundary conditions and in that way we introduce second order and fourth order IUs associated to homogeneous Neumann boundary conditions.

2. INCREMENTAL UNKNOWNS AND INTERPOLATION

2.1. A general construction of the IUs

Along this section and the next two ones, we shall consider only IUs of homogeneous Dirichlet type i.e. IUs associated to functions that take a null value on the boundary of the domain.

For the sake of simplicity, let us consider two levels of discretization. We consider the regular grid of the domain Ω with the step $h = \frac{1}{2N}$; for our applications Ω will be $]0, 1[^n$, $n = 1, 2, 3$. At this point we distinguish the fine grid G_h which is associated to the mesh h, and the coarse grid G_H which is associated to the mesh $H = 2h = \frac{1}{N}$. The construction of incremental unknowns is decomposed in two steps.

2.1.1. *Hierarchization*

Let u be a regular function defined on Ω. We denote by U_i, $i = 1, \ldots, 2N-1$, (resp. $U_{i,j}$, $i, j = 1, \ldots, 2N-1$) the approximation of u at the grid point ($U_i \simeq u(i.h)$ in space dimension one, $U_{i,j} \simeq u(i.h, j.h)$ in space dimension two and $U_{i,j,k} \simeq u(i.h, j.h, k.h)$ in space dimension three).

2.1.1.a. *The one dimensional case*

The hierarchization consists in separating the nodal unknowns according to the grid to which they belong: we first consider the unknowns of G_H, denoted by Y, and then those of $G_h \backslash G_H$, denoted by U_f; of course each family is ordered in the standard way. Then the unknowns of the coarse grid are associated to points of even indices and those of the complementary grid, say $G_h \backslash G_H$, are associated to points of odd indices (see *fig. 1*).

. o × o × o × o × o .

Figure 1. — Space dimension 1, Ω =]0, 1[, ×: points in G_H, o: points in $G_h \backslash G_H$.

2.1.1.b. *The two dimensional case*

Here $\Omega =]0, 1[^2$ and as above, we consider the two-grid splitting. We denote here by $U_{i,j}$, $i, j = 1, \ldots, 2N-1$, the approximation of u at the grid point. The unknowns of the coarse grid G_H are $U_{2i,2j}$, the other are those of $G_h \backslash G_H$ (see *fig. 2*).

```
.  .  .  .  .  .  .  .  .
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.  o  ×  o  ×  o  ×  o  .
.  o  o  o  o  o  o  o  .
.  o  ×  o  ×  o  ×  o  .
.  o  o  o  o  o  o  o  .
.  o  ×  o  ×  o  ×  o  .
.  o  o  o  o  o  o  o  .
.  .  .  .  .  .  .  .  .
```

Figure 2. — Space dimension 2, Ω = ( ]0, 1[ )^2, ×: points in G_H, o: points in $G_h \backslash G_H$.

2.1.1.c. *The three dimensional case*

Here we consider $\Omega =]0, 1[^3$. As for lower dimension cases the coarse grid unknowns have all their coordinates even. We give a representation of the 3-D two level splitting in the following figure.

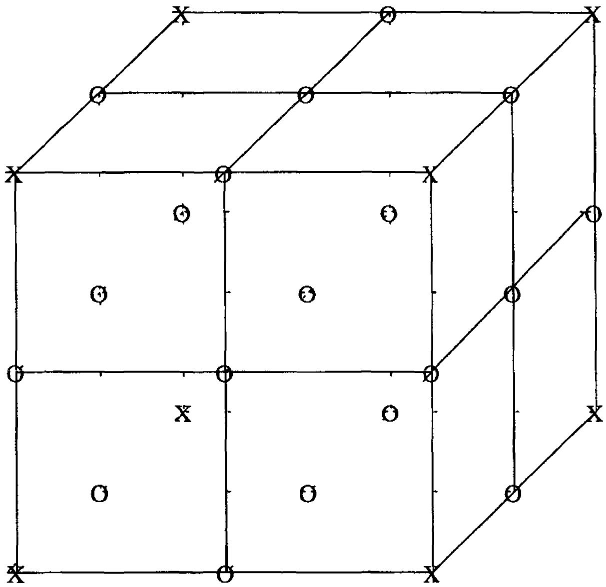


Figure 3. — Space dimension 3, $\Omega = (\,]0,1[\,)^3$, ×: points in G_H o points in $G_h \backslash G_H$

Notice that we can repeat recursively this hierarchization process using $l+1$, $l \geqslant 1$ level of discretization

2 1 2 *Change of variable*

Now, we introduce a change of variable operating only in $G_h \backslash G_H$ and which leaves the Y unknowns unchanged We can express it in the form

$$Z_f = U_f - R \,.\, Y\,, \tag{2 1}$$

where $R \; G_H \to G_h \backslash G_H$ is a p^{th} order interpolation operator

The numbers Z are the *Incremental Unknowns* According to Taylor's formula, their magnitude is expected to be $O(h^p)$ (for $p = 2$, it was shown in [7], for the dimensions one and two, and in [6], for the dimension three, using *a priori* estimates of energy type, that the IUs are indeed small as expected) As the second order case (see [7], [17]) this process can be repeated recursively with d levels of discretization Denoting by Z_i the successive Z-levels and by S the transfer matrix, we have

$$\begin{pmatrix} Y \\ U_{f1} \\ U_{f2} \\ \cdot \\ U_{fd} \end{pmatrix} = S \begin{pmatrix} Y \\ Z_1 \\ Z_2 \\ \\ Z_d \end{pmatrix},$$

with obvious notation

Notice that S has a lower triangular structure, this property is of course important for the implementation of the method

2.2. The second order case

We present here the second order Incremental Unknowns (IU2) in dimension one, two and three. These IUs appear as a particular case of those that will be defined in Section 4. We shall refer to these IUs along this paper for comparing them with the high order Incremental Unknowns.

Dimension one

Let U_j, $j = 0, \ldots, 2N-1$ be the nodals unknowns on G_h; we set

$$Z_{2j+1} = U_{2j+1} - \frac{1}{2}(U_{2j} + U_{2j+2}),$$
$$\text{for} \quad j = 0, \ldots, N-1 \tag{2.2}$$
$$(U_0 = U_{2N} = 0).$$

Dimension two

In dimension one the unknowns of the complementary grid $(G_h \backslash G_H)$ have the same geometric characteristics (see *fig. 1*). In dimension two, we distinguish in fact three kinds of points in $G_h \backslash G_H$: points of type f1, f2, and f3 (see *fig. 4*),

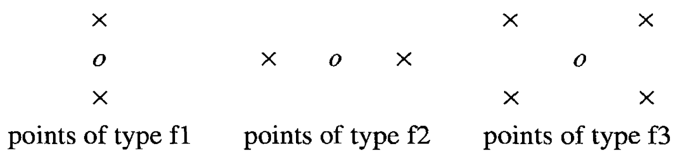


Figure 4. — The several type of points in $G_h \backslash G_H$

At this point we define the incremental unknowns which consist of the nodal values $Y_{2i,2j} = U_{2i,2j}$ at the coarse grid points $(2ih, 2jh)$, $i, j = 1, \ldots, N-1$ and of the following incremental quantities at the other points.

At the f1 points

$$Z_{2i,2j+1} = U_{2i,2j+1} - \frac{1}{2}(U_{2i,2j} + U_{2i,2j+2}). \tag{2.3}$$

At the f2 points

$$Z_{2i+1,2j} = U_{2i+1,2j} - \frac{1}{2}(U_{2i,2j} + U_{2i+2,2j}); \tag{2.4}$$

and at the f3 points

$$Z_{2i+1,2j+1} = U_{2i+1,2j+1} - \frac{1}{4}(U_{2i,2j} + U_{2i,2j+2} + U_{2i+2,2j} + U_{2i+2,2j+2}), \tag{2.5}$$

for i, $j = 0, \ldots, N-1$; $U_{\alpha,\beta} = 0$ if α or $\beta \in \{0, 2N\}$.

As for the dimensional one case, we recognize here the stencil of the definition of the IUs and the second order interpolation scheme.

Dimension three

As for the two dimensional case, we must distinguish several type of points of $G_h \backslash G_H$ according to their geometrical disposition. We obtain two sorts of complementary grid points (see [6]).

(i) Points on the coarse planes i.e. points of coordinates $(\alpha, \beta, 2k)$, α, $\beta = 1, ..., 2N-1$, $k = 1, ..., N-1$.
Coarse grid points: $U_{2i,2j,2k}$, i, j, $k = 1, ..., N-1$.
Points of type $f1$: $U_{2i,2j+1,2k}$, $j = 0, ..., N-1$, i, $k = 1, ..., N-1$.
Points of type $f2$: $U_{2i+1,2j,2k}$, $i = 0, ..., N-1$, j, $k = 1, ..., N-1$.
Points of type $f3$: $U_{2i+1,2j,2k}$, i, $j = 0, ..., N-1$, $k = 1, ..., N-1$.
(ii) Points on the complementary planes i.e. points of coordinates $(\alpha, \beta, 2k+1)$, α, $\beta = 1, ..., 2N-1$, $k = 0, ..., N-1$.
Points of type $f4$: $U_{2i,2j,2k+1}$, $i, j = 1, ..., N-1$, $k = 0, ..., N-1$.
Points of type $f5$: $U_{2i,2j+1,2k+1}$, $i = 1, ..., N-1$, j, $k = 0, ..., N-1$.
Points of type $f6$: $U_{2i,2j,2k+1}$, $j = 1, ..., N-1$, i, $k = 0, ..., N-1$.
Points of type $f7$: $U_{2i+1,2j+1,2k+1}$, i, j, $k = 0, ..., N-1$.
Now we define the 3-D second order IUs.
For the coarse plan points.

$$
\begin{cases}
\text{At the } f1 \text{ points :} \\
\quad Z_{2i,2j+1,2k} = U_{2i,2j+1,2k} - \frac{1}{2}(U_{2i,2j,2k} + U_{2i,2j+2,2k}). \\
\text{At the } f2 \text{ points :} \\
\quad Z_{2i+1,2j2k} = U_{2i+1,2j,2k} - \frac{1}{2}(U_{2i,2j,2k} + U_{2i,2,2j,2k}). \\
\text{At the } f3 \text{ points :} \\
\quad Z_{2i+1,2j+1,2k} = U_{2i+1,2j+1,2k} \\
\qquad - \frac{1}{4}(U_{2i,2j,2k} + U_{2i,2j+2,2k} + U_{2i+2,2j,2k} + U_{2i+2,2j+2,2k}),
\end{cases}
\tag{2.6}
$$

for i, $j = 0, ..., N-1$, $k = 1, ..., N-1$.
For the complementary plane points.

$$
\begin{cases}
\text{At the } f4 \text{ points} \\
\quad Z_{2i,2j,2k+1} = U_{2i,2j,2k+1} - \frac{1}{2}(U_{2i,2j,2k} + U_{2i,2j,2k+2}). \\
\text{At the } f5 \text{ points} \\
\quad Z_{2i,2j+1,2k+1} = U_{2i,2j+1,2k+1} \\
\qquad - \frac{1}{4}(U_{2i,2j,2k} + U_{2i,2j+2,2k} + U_{2i,2j,2k+2} + U_{2i,2j+2,2k+2}). \\
\text{At the } f6 \text{ points} \\
\quad Z_{2i+1,2j,2k+1} = U_{2i+1,2j,2k+1} \\
\qquad = \frac{1}{4}(U_{2i,2j,2k} + U_{2i+2,2j,2k} + U_{2i,2j,2k+2} + U_{2i+2,2j,2k+2}). \\
\text{At the } f7 \text{ points} \\
\quad Z_{2i+1,2j+1,2k+1} = U_{2i+1,2j+1,2k+1} \\
\qquad - \frac{1}{8}(U_{2i,2j,2k} + U_{2i+2,2j,2k} + U_{2i,2j,2k} + U_{2i+2,2j,2k} \\
\qquad + U_{2i,2j,2k+2} + U_{2i+2,2j,2k+2} + U_{2i,2j,2k+2} + U_{2i+2,2j,2k+2}).
\end{cases}
\tag{2.7}
$$

$(U_{\alpha,\beta,\gamma} = 0$ if α, β or $(\gamma \in \{0, 2N\})$.

2.3. The high order interpolation problem

As we have seen it above, the order of magnitude of the IUs is related to the order of the interpolation operator used to define them. The construction of high interpolation scheme by the usual techniques requires locally an important number of points. This should complicate the implementation of the IU method, particularly the transfer matrix will not have a simple structure, that is lower-triangular. The use of the compact schemes (see e.g. [13]) gives the possibility to conserve this particular form of the transfer matrix. In the sequel of the paper, we shall consider regular functions.

3. THE COMPACT SCHEMES

The compact schemes (CS) were originally introduced as a means to approach the spectral accuracy in the discretization of the partial differential operators when one uses finite differences (they mimmic the spectral global dependence). Thanks to their high accuracy they are a powerful tool for the simulation of turbulent flows. In the point of view of the implementation, the CS have a many advantages: firstly they involve just few points of discretization as compared to their accuracy which is obtained using a compact stencil and secondly, these schemes can be adapted to non periodic boundary conditions. Thirdly, their cost, regardless of the number of discretization points, involve only inversion of banded symmetric matrix. Finally they are well adapted to the parallel computing. One can summarize their definition as follows:

Let D the linear operator to be discretized; D can be a partial differential operator as well as an interpolation operator. The general form of the CS is

$$P \, . \, D \, . \, U = Q \, . \, U \, , \tag{3.8}$$

where U is the vector containing the approximations of a function at the points of a mesh. P and Q are two matrices; Q is a matrix of discretization of D (centered or no centered schemes) and P is a nonsingular matrix, easy to invert, generally P is symmetric and tri or pentadiagonal. With this techniques one can build schemes up to the tenth order accuracy. Notice that the traditional schemes correspond to the particular case: $P = Id$, where Id is the indentity matrix. Let us recall some particular compact schemes (see [13]).

3.1. Approximation of the derivatives

For the sake of simplicity we restrict ourselves to the one dimensional case.

Let f be a regular function. We denote by f_i and by f_i' the approximations of f and f' respectively, at the point of the meshing. The compact scheme associated to the approximation of f' has the general form:

$$\beta f_{i-2}' + \alpha f_{i-1}' + f_i' + \alpha f_{i+1}' + \beta f_{i+2}' = c\frac{f_{i+3} - f_{i-3}}{6h} + b\frac{f_{i+2} - f_{i-2}}{4h} + a\frac{f_{i+1} - f_{i-1}}{2h} \, . \tag{3.9}$$

The order of the scheme (3.9) is related to a linear system to be satisfied by the parameters a, b, c and α, β ; this system is deduced by application of the Taylor's expansion.

In the same way, we can define the compact scheme associated to the discretization of the second derivative:

$$\beta f_{i-2}'' + \alpha f_{i-1}'' + f_i'' + \alpha f_{i+1}'' + \beta f_{i+2}'' = c\frac{f_{i+3} - 2f_i + f_{i-3}}{9h^2} + b\frac{f_{i+2} - 2f_i + f_{i-2}}{4h^2} + a\frac{f_{i+1} - 2f_i + f_{i-1}}{h^2} \, . \tag{3.10}$$

Remark 1: If P is diagonal, say $\alpha = \beta = 0$, then only the second order approximation can be obtained (with appropriate values of a, b, c).

— If P is tridiagonal, say $\beta = 0$, then the scheme can have the fourth or the sixth accuracy: in taking $b = c = 0$ and appropriate values of α and a, we obtain the fourth order and in taking $c = 0$ and appropriate values of α, a and b we obtain the sixth order.

— The eighth and the tenth order of accuracy are obtained respectively letting $c = 0$ (and computing the other parameters with Taylor's formula) and determining all the parameters by Taylor's expansion.

In the practical cases we shall use only the tridiagonal form of the matrix P and consequently only compact schemes of order four or six.

Remark 2: If $\alpha = \frac{344}{1\,179}$ and $c = 0$, the scheme (3.10) coincides with a scheme given by Collatz [11] p. 538 (see [13]).

3.2. Boundary conditions

The previous formulas are defined for all points of Ω if the boundary conditions are periodic. In other cases, with which we are more concerned here, it is necessary to define in a different way the scheme for the points near the boundary in order to conserve the particular form of the matrix P. For example, the boundary scheme associated to the first derivative approximation with a $O(h^4)$ accuracy, say (3.9), is ([13])

$$f_1' + \alpha f_2' = \frac{af_1 + bf_2 + cf_3 + df_4}{h}. \tag{3.11}$$

In this case the matrix P is tridiagonal and the parameter α is the same as in the formula (3.9). Notice that this matrix can be easily factorized, and then inverted, by a classical L.U. Technique. Of course similar schemes can be used for the approximation of the second derivative near the boundary points. Formula as (3.11) will be called closure formula in the following of the paper.

3.3. Interpolation

Let us consider now the problem of the interpolation which is central for the definition of the IUs as we have seen it in the previous section. As for the differential operators, we can define a compact scheme for the mid-point interpolation ([13]) by the formula

$$\beta\hat{f}_{i-2} + \alpha\hat{f}_{i-1} + \hat{f}_i + \alpha\hat{f}_{i+1} + \beta\hat{f}_{i+2} = \frac{c}{2}\left(f_{i+\frac{5}{2}} + f_{i-\frac{5}{2}}\right) + \frac{b}{2}\left(f_{i+\frac{3}{2}} + f_{i-\frac{3}{2}}\right) + \frac{a}{2}\left(f_{i+\frac{1}{2}} + f_{i-\frac{1}{2}}\right), \tag{3.12}$$

where $\hat{f}_i$ are the interpolates of f at the points x_i.

Of course, as above, we recover here the same difficulties with the points which are near the boundary. In order to conserve the accuracy of the schemes as (3.12), we propose to use a closure formula as (3.11).

For the sake of simplicity, we consider the case where P is tridiagonal, i.e. when $\beta = 0$. We let also $a = b = 0$. We can define $\hat{f}_1$ by

$$\hat{f}_1 + \alpha\hat{f}_2 = a' f_{\frac{1}{2}} + b' f_{\frac{3}{2}} + c' f_{\frac{5}{2}} + d' f_{\frac{7}{2}}. \tag{3.13}$$

Here again the symmetry of P is conserved.

4. HIGH ORDER IUs

The definition of the IUs built with the CS we give here is general and IUs of arbitrary order can be constructed but, higher is the order of the interpolation scheme fuller is the associated matrix P. Futhermore the complexity of the closure formulas increases with the order of the CS. For these reasons, we shall consider for our applications only the fourth or the sixth order IUs.

We study first the one dimensional case. The two and the three dimensional cases will be treated hereafter by adapting the techniques developped in the following subsection. We concentrate here on the case where the boundary condition are of homogeneous Dirichlet type; however other types of boundary conditions can be considered and we introduce in Section 5 high order IUs associated to homogeneous Neumann boundary conditions (the periodic boundary condition case is very easy to treat: there are no closure formulas).

4.1. The one dimensional case

As we stated it, modulo a hierarchization process, the construction of the IU is essentially realized with the use of an interpolation scheme. We consider the discretization of $\Omega = (0, 1)$ on two grids. We use (3.12) for the interpolation of the complementary grid points (which are in $G_h \backslash G_H$), and whose indices are odd, by points of the coarse grid G_H whose indices are even. We obtain

$$\beta \hat{f}_{2i-3} + \alpha \hat{f}_{2i-1} + \hat{f}_{2i+1} + \alpha \hat{f}_{2i+3} + \beta \hat{f}_{2i+5} = \frac{c}{2}(f_{2i-4} + f_{2i+6}) + \frac{b}{2}(f_{2i-2} + f_{2i+4}) + \frac{a}{2}(f_{2i} + f_{2i+2}) . \tag{4.14}$$

At this point, we propose two equivalent definitions of the IU

DEFINITION 1: *The Incremental Unknowns are the numbers:*

$$z_{2i+1} = f_{2i+1} - \hat{f}_{2i+1}, \quad i = 1, \dots, N-2 , \tag{4.15}$$

where $\hat{f}_{2i+1}$ is given by (4.14).

The values of Z associated to the points near the boundary are calculated with formulas of type (3.13).

DEFINITION 2: *Let P and Q be the matrices associated to an interpolation compact scheme. We define the Incremental Unknowns, with the notations of Section 2, as:*

$$Z_f = U_f - P^{-1} . Q . Y . \tag{4.16}$$

The interpolation operator is here $R = P^{-1} . Q$.

4.1.1. *Application: Fourth order 1-D IUs*

We apply here the techniques proposed in [13] using directly the Taylor's formula. We propose in Section 5 another method aimed at computing particular coefficients of the compact scheme.

We consider here a very simple compact interpolation scheme: we choose $\beta = 0$, $b = c = 0$. We have:

$$\alpha \hat{f}_{2i-1} + \hat{f}_{2i+1} + \alpha \hat{f}_{2i+3} = \frac{a}{2}(f_{2i} + f_{2i+2}) . \tag{4.17}$$

Let us first examine the inner points and let us compute α and a such as the scheme (4.17) is of $O(h^4)$ accuracy. The Taylor expansion gives:

$$f_{2i+3} + f_{2i-1} = 2\left(f_{2i+1} + \frac{(2h)^2}{2!}\frac{\partial^2 f}{\partial x^2}\Big|_{x=x_{2i+1}} + \frac{(2h)^2}{4!}\frac{\partial^4 f}{\partial x^4}\Big|_{x=x_{2i+1}} + O(h^6)\right),$$

$$f_{2i+2} + f_{2i} = 2\left(f_{2i+1} + \frac{h^2}{2!}\frac{\partial^2 f}{\partial x^2}\Big|_{x=x_{2i+1}} + \frac{h^4}{4!}\frac{\partial^4 f}{\partial x^4}\Big|_{x=x_{2i+1}} + O(h^6)\right).$$

Then the relations satisfied by the parameters are:

$$\begin{cases} 1 + 2\alpha = a & \text{(second order)}, \\ 8\alpha = a & \text{(fourth order)}. \end{cases} \tag{4.18}$$

We find $\alpha = \frac{1}{6}$ and $a = \frac{4}{3}$.

Remark 3: If we take $\alpha = 0$ and $a = 1$, we recover the interpolation scheme associated to second order IUs (see [17]).

Now, we must construct the closure formulas aimed at computing $\hat{f}_1$, and by a symmetry argument $\hat{f}_{2N-1}$, with the same $O(h^4)$ accuracy. From (3.13) we infer

$$\hat{f}_1 + \alpha \hat{f}_3 = a' f_0 + b' f_2 + c' f_4 + d' f_6 . \tag{4.19}$$

From the Taylor's formula we deduce the following sysem to be satisfied by these coefficients:

$$\begin{cases} 1 + \alpha & = a' + b' + c' + d' & \text{(first order)}, \\ 2\alpha & = -a' + b' + 3c' + 5d' & \text{(second order)}, \\ 4\alpha & = a' + b' + 9c' + 25d' & \text{(third order)}, \\ 8\alpha & = -a' + b' + 27c' + 125d' & \text{(fourth order)}, \end{cases} \tag{4.20}$$

whee of course the parameter α is the same as in (4.17), that is $\alpha = \frac{1}{6}$.

We find

$$a' = \frac{29}{96}, \quad b' = \frac{99}{96}, \quad c' = -\frac{21}{96} \quad \text{and} \quad d' = \frac{5}{96}.$$

Remark 4: Here the boundary conditions are of homogeneous Dirichlet type ($f_0 = f_{2N} = 0$) and a' is not used.

We can write this fourth order scheme in the following matricial form:

$$P \cdot \hat{f} = Q \cdot f,$$

where P is the $N \times N$ tridiagonal matrix,

$$P = \begin{pmatrix} 1 & \frac{1}{6} & 0 & . & . & 0 \\ \frac{1}{6} & 1 & \frac{1}{6} & 0 & . & . \\ 0 & \frac{1}{6} & 1 & \frac{1}{6} & . & . \\ . & . & . & . & . & 0 \\ . & . & 0 & \frac{1}{6} & 1 & \frac{1}{6} \\ 0 & . & . & 0 & \frac{1}{6} & 1 \end{pmatrix},$$

and where $\underline{Q}$ is the $N \times (N-1)$ matrix,

$$Q = \begin{pmatrix} \frac{99}{96} & -\frac{21}{96} & \frac{5}{96} & 0 & . & 0 \\ \frac{2}{3} & \frac{2}{3} & 0 & . & . & . \\ 0 & \frac{2}{3} & \frac{2}{3} & 0 & . & . \\ . & . & . & . & . & 0 \\ . & . & . & 0 & \frac{2}{3} & \frac{2}{3} \\ 0 & . & . & \frac{5}{96} & -\frac{21}{96} & \frac{99}{96} \end{pmatrix}.$$

The transfer matrix S is then

$$S = \begin{pmatrix} I_{N-1} & 0 \\ B_N & I_N \end{pmatrix}$$

where $B_N = P^{-1} \cdot Q$.

It is obvious that, as in the second order case, this process can be repeated recursively using d grid refinements. Furthermore, the matrix P can be either easily factorized in a L.U. form either easily inverted by a Conjugate Gradient Method thanks to its very good condition number ($\leqslant 2$ as one can check it applying the Gershgorin theorem).

4.2. The two dimensional case

We propose here a very simple extension to the two dimensional case of the interpolation schemes introduced for the dimension one. As for the second order IU (see Section 2), this extension is realized via a geometric classification of the complementary grid points in order to obtain a lower triangular structure of the transfer matrix S.

First of all we adapt the formula (4.14) in order to interpolate points of type $f1$ and $f2$ by the coarse grid points. We obtain the following schemes (for the inner points):

$$\begin{cases} \text{For the points of type } f1 \\ \beta\hat{f}_{2i,2j-3} + \alpha\hat{f}_{2i,2j-1} & \frac{c}{2}(f_{2i,2j-4} + f_{2i,2j+6}) \\ \qquad + \hat{f}_{2i,2j+1} = & + \frac{b}{2}(f_{2i,2j-2} + f_{2i,2j+4}) \\ + \alpha\hat{f}_{2i,2j+3} + \beta\hat{f}_{2i,2j+5} & + \frac{a}{2}(f_{2i,2j} + f_{2i,2j+2}) . \end{cases} \tag{4.21}$$

$$\begin{cases} \text{For the points of type } f2 \\ \beta\hat{f}_{2i-3,2j} + \alpha\hat{f}_{2i-1,2j} & \frac{c}{2}(f_{2i-4,2j} + f_{2i+6,2j}) \\ \qquad + \hat{f}_{2i+1,2j} = & + \frac{b}{2}(f_{2i-2,2j} + f_{2i+4,2j}) \\ + \alpha\hat{f}_{2i+3,2j} + \beta\hat{f}_{2i+5,2j} & + \frac{a}{2}(f_{2i,2j} + f_{2i+2,2j}) . \end{cases} \tag{4.22}$$

We do not use analogous schemes for the direct interpolation of points of type $f3$ by points of the coarse grid because, in this case, the corresponding matrix P will loss the lower triangular structure and, consequently, the implementation of the method will be heavy. Then, we propose to interpolate the points of type $f3$ by points of type $f1$ or $f2$. However the points of type $f3$ are computed indirectely, via the $f2$ ones, by points of the coarse grid and then the transfer matrix conserves its lower triangular form. We define the associated schemes as follows:

$$\begin{cases} \text{(a) Interpolation by points of type } f1 \\ \beta\bar{f}_{2i-3,2j+1} + \alpha\bar{f}_{2i-1,2j+1} + \bar{f}_{2i+1,2j+1} + \alpha\bar{f}_{2i+3,2j+1} + \beta\bar{f}_{2i+5,2j+1} = \frac{c}{2}(\hat{f}_{2i-4,2j+1} + \hat{f}_{2i+6,2j+1}) + \frac{b}{2}(\hat{f}_{2i-2,2j+1} + \hat{f}_{2i+4,2j+1}) + \frac{a}{2}(\hat{f}_{2i,2j+1} + \hat{f}_{2i+2,2j+1})\,; \\ \text{(b) Interpolation by points of type } f2 \\ \beta\bar{f}_{2i+1,2j-3} + \alpha\bar{f}_{2i+1,2j-1} + \bar{f}_{2i+1,2j+1} + \alpha\bar{f}_{2i+1,2j+3} + \beta\bar{f}_{2i+1,2j+5} = \frac{c}{2}(\hat{f}_{2i+1,2j-4} + \hat{f}_{2i+1,2j+6}) + \frac{b}{2}(\hat{f}_{2i+1,2j-2} + \hat{f}_{2i+1,2j+4}) + \frac{a}{2}(\hat{f}_{2i+1,2j} + \hat{f}_{2i+1,2j+2})\,. \end{cases} \tag{4.23}$$

At this point we can define the two dimensional IUs by

DEFINITION 3: *The 2-D Incremental Unknowns are the number Z given by the formulas:*

$$\begin{cases} Z_{2i,2j+1} = f_{2i,2j+1} - \hat{f}_{2i,2j+1}\,, \\ Z_{2i+1,2j} = f_{2i+1,2j} - \hat{f}_{2i+1,2j}\,, \\ Z_{2i+1,2j+1} = f_{2i+1,2j+1} - \bar{f}_{2i+1,2j+1}\,. \end{cases} \tag{4.24}$$

Here $\hat{f}_{2i,2j+1}$ and $\hat{f}_{2i+1,2j}$ are the interpolates given by (4.21) and (4.22) respectively. The numbers $\bar{f}_{2i+1,2j+1}$ are calculated either by $(4.23)_a$ either by $(4.23)_b$.

If the boundary conditions are not periodic, we close these systems with formula of type (4.19).

Notice that we recover here the second order IUs (see Section 2) by taking $\alpha = \beta = b = c = 0$ and $a = 1$.

Remark 5: This method can be easily extended to the 3-*D* case as we shall see it in later.

4.2.1. *Application: Fourth order 2-D IUs*

We assume that the boundary conditions are of type homogeneous Dirichlet. Taking the same parameters as in the dimension one case we propose the following method:

Step 1: Computation of the IUs of type $f1$.

$$\begin{cases} \text{Points near the boundary} \\ \hat{f}_{2i,1} + \frac{1}{6}\hat{f}_{2i,3} = \frac{99}{96}f_{2i,2} - \frac{21}{96}f_{2i,4} + \frac{5}{96}f_{2i,6}\,, \quad \text{for } i = 1, \dots, N-1 \\ \text{Inner points} \\ \frac{1}{6}\hat{f}_{2i,2j-1} + \hat{f}_{2i,2j+1} + \frac{1}{6}\hat{f}_{2i,2j+3} = \frac{2}{3}(f_{2i,2j} + f_{2i,2j+2})\,. \\ \text{for } i = 1, \dots, N-1\,; \quad j = 2, \dots, N-2\,, \\ Z_{2i,2j+1} = f_{2i,2j+1} - \hat{f}_{2i,2j+1}\,. \end{cases} \tag{4.25}$$

Step 2: Computation of the IUs of type f2.

$$\begin{cases} \text{Points near the boundary} \\ \hat{f}_{1,2j} + \frac{1}{6}\hat{f}_{3,2j} = \frac{99}{96}f_{2,2j} - \frac{21}{96}f_{4,2j} + \frac{5}{96}f_{6,2j}, \quad \text{for } j = 1, \dots, N-1 . \\ \text{Inner points} \\ \frac{1}{6}\hat{f}_{2i-1,2j} + \hat{f}_{2i+1,2j} + \frac{1}{6}\hat{f}_{2i+3,2j} = \frac{2}{3}(f_{2i,2j} + f_{2i+2,2j}) , \\ \text{for } i = 2, \dots, N-2 ; \quad j = 1, \dots, N-1 , \\ Z_{2i+1,2j} = f_{2i+1,2j} - \hat{f}_{2i+1,2j} . \end{cases} \tag{4.26}$$

Step 3: Computation of the IUs of type f3 by points of type f2.

$$\begin{cases} \text{Points near the boundary} \\ \bar{f}_{2i+1,1} + \frac{1}{6}\bar{f}_{2i+1,3} = \frac{99}{96}\hat{f}_{2i+1,2} - \frac{21}{96}\hat{f}_{2i+1,4} + \frac{5}{96}\hat{f}_{2i+1,6}, \quad \text{for } i = 1, \dots, N-1 . \\ \text{Inner points} \\ \frac{1}{6}\bar{f}_{2i+1,2j-1} + \bar{f}_{2i+1,2j+1} + \frac{1}{6}\bar{f}_{2i+1,2j+3} = \frac{2}{3}(\hat{f}_{2i+1,2j} + \hat{f}_{2i+1,2j+2}) , \\ \text{for } i = 1, \dots, N-1 ; \quad j = 2, \dots, N-2 , \\ Z_{2i+1,2j+1} = f_{2i+1,2j+1} - \bar{f}_{2i+1,2j+1} . \end{cases} \tag{4.27}$$

Remark 6: Step 1 and Step 2 are independent and can be computed in parallel and Step 3 can be realized using the result of Step 2.

From the previous formulas, we deduce the structure of the transfer matrix S:

$$S = \begin{pmatrix} I_{(N-1)\times(N-1)} & 0 & 0 & 0 \\ S_1 & I_{N\times(N-1)} & 0 & 0 \\ S_2 & 0 & I_{N\times(N-1)} & 0 \\ S_3 & 0 & 0 & I_{N\times N} \end{pmatrix} \text{where} \begin{cases} S_1 = P_1^{-1} \cdot Q_1 \\ S_2 = P_2^{-1} \cdot Q_2 \\ S_3 = P_3^{-1} \cdot Q_3 \cdot P_2^{-1} \cdot Q_2 \end{cases}$$

with obvious notations.

4.3. The three dimensional case

We define the 3-D high order IUs with the use of an extension of the interpolation schemes introduced for the two dimension case. As in Section 2, we distinguish coarse and complementary planes. We propose the following procedure which is valid for inner points. As usual the boundary points are treated with the use of closure formulas.

Step 1: High order interpolation on the coarse planes

$$
\left\{
\begin{array}{l}
\text{(a) For the points of type } f1 \\
\begin{aligned}
\beta\hat{f}_{2i,2j-3,2k} + \alpha\hat{f}_{2i,2j-1,2k} & \quad \tfrac{c}{2}\left(f_{2i,2j-4,2k} + f_{2i,2j+6,2k}\right) \\
+\hat{f}_{2i,2j+1,2k} & = +\tfrac{b}{2}\left(f_{2i,2j-2,2k} + f_{2i,2j+4,2k}\right) \\
+\alpha\hat{f}_{2i,2j+3,2k} + \beta\hat{f}_{2i,2j+5,2k} & \quad +\tfrac{a}{2}\left(f_{2i,2j,2k} + f_{2i,2j+2,2k}\right).
\end{aligned} \\
\text{(b) For the points of type } f2 \\
\begin{aligned}
\beta\hat{f}_{2i-3,2j,2k} + \alpha\hat{f}_{2i-1,2j,2k} & \quad \tfrac{c}{2}\left(f_{2i-4,2j,2k} + f_{2i+6,2j,2k}\right) \\
+\hat{f}_{2i+1,2j,2k} & = +\tfrac{b}{2}\left(f_{2i-2,2j,2k} + f_{2i+4,2j,2k}\right) \\
+\alpha\hat{f}_{2i+3,2j,2k} + \beta\hat{f}_{2i+5,2j,2k} & \quad +\tfrac{a}{2}\left(f_{2i,2j,2k} + f_{2i+2,2j,2k}\right).
\end{aligned} \\
\text{(c) For the points of type } f3 \\
\begin{aligned}
\beta\bar{f}_{2i+1,2j-3,2k} + \alpha\bar{f}_{2i+1,2j-1,2k} & \quad \tfrac{c}{2}\left(\hat{f}_{2i+1,2j-4,2k} + \hat{f}_{2i+1,2j+6,2k}\right) \\
+\bar{f}_{2i+1,2j+1,2k} & = +\tfrac{b}{2}\left(\hat{f}_{2i+1,2j-2,2k} + \hat{f}_{2i+1,2j+4,2k}\right) \\
+\alpha\bar{f}_{2i+1,2j+3,2k} + \beta\bar{f}_{2i+1,2j+5,2k} & \quad +\tfrac{a}{2}\left(\hat{f}_{2i+1,2j,2k} + \hat{f}_{2i+1,2j+2,2k}\right).
\end{aligned}
\end{array}
\right. \tag{4.28}
$$

Step 2: High order interpolation on the complementary planes

$$
\left\{
\begin{array}{l}
\text{(d) For the points of type } f4 \\
\begin{aligned}
\beta\hat{f}_{2i,2j,2k-3} + \alpha\hat{f}_{2i,2j,2k-1} & \quad \tfrac{c}{2}\left(f_{2i,2j,2k-4} + f_{2i,2j,2k+6}\right) \\
+\hat{f}_{2i,2j,2k+1} & = +\tfrac{b}{2}\left(f_{2i,2j,2k-2} + f_{2i,2j,2k+4}\right) \\
+\alpha\hat{f}_{2i,2j,2k+3} + \beta\hat{f}_{2i,2j,2k+5} & \quad +\tfrac{a}{2}\left(f_{2i,2j,2k} + f_{2i,2j,2k+2}\right).
\end{aligned} \\
\text{(e) For the points of type } f5 \\
\begin{aligned}
\beta\bar{f}_{2i,2j+1,2k-3} + \alpha\bar{f}_{2i,2j+1,2k-1} & \quad \tfrac{c}{2}\left(\hat{f}_{2i,2j+1,2k-4} + \hat{f}_{2i,2j+1,2k+6}\right) \\
+\bar{f}_{2i,2j+1,2k+1} & = +\tfrac{b}{2}\left(\hat{f}_{2i,2j+1,2k-2} + \hat{f}_{2i,2j+1,2k+4}\right) \\
+\alpha\bar{f}_{2i,2j+1,2k+3} + \beta\bar{f}_{2i,2j+1,2k+5} & \quad +\tfrac{a}{2}\left(\hat{f}_{2i,2j+1,2k} + \hat{f}_{2i,2j+1,2k+2}\right).
\end{aligned} \\
\text{(f) For the points of type } f6 \\
\begin{aligned}
\beta\bar{f}_{2i+1,2j,2k-3} + \alpha\bar{f}_{2i+1,2j,2k-1} & \quad \tfrac{c}{2}\left(\hat{f}_{2i+1,2j,2k-4} + \hat{f}_{2i+1,2j,2k+6}\right) \\
+\bar{f}_{2i+1,2j,2k+1} & = +\tfrac{b}{2}\left(\hat{f}_{2i+1,2j,2k-2} + \hat{f}_{2i+1,2j,2k+4}\right) \\
+\alpha\bar{f}_{2i+1,2j,2k+3} + \beta\bar{f}_{2i+1,2j,2k+5} & \quad +\tfrac{a}{2}\left(\hat{f}_{2i+1,2j,2k} + \hat{f}_{2i+1,2j,2k+2}\right).
\end{aligned} \\
\text{(g) For the points of type } f7 \\
\begin{aligned}
\beta\bar{\bar{f}}_{2i+1,2j+1,2k-3} + \alpha\bar{\bar{f}}_{2i+1,2j+1,2k-1} & \quad \tfrac{c}{2}\left(\hat{f}_{2i+1,2j+1,2k-4} + \hat{f}_{2i+1,2j+1,2k+6}\right) \\
+\bar{\bar{f}}_{2i+1,2j+1,2k+1} & = +\tfrac{b}{2}\left(\hat{f}_{2i+1,2j+1,2k-2} + \hat{f}_{2i+1,2j+1,2k+4}\right) \\
+\alpha\bar{\bar{f}}_{2i+1,2j+1,2k+3} + \beta\bar{\bar{f}}_{2i+1,2j+1,2k+5} & \quad +\tfrac{a}{2}\left(\hat{f}_{2i+1,2j+1,2k} + \hat{f}_{2i+1,2j+1,2k+2}\right).
\end{aligned}
\end{array}
\right. \tag{4.29}
$$

We can now give the following definition:

DEFINITION 4: *The 3-D Incremental Unknowns are the numbers Z given by the formula:*

$$\begin{cases} Z_{2i,2j+1,2k} & = f_{2i,2j+1,2k} - \hat{f}_{2i,2j+1,2k}, \\ Z_{2i+1,2j,2k} & = f_{2i+1,2j,2k} - \hat{f}_{2i+1,2j,2k}, \\ Z_{2i+1,2j+1,2k} & = f_{2i+1,2j+1,2k} - \bar{f}_{2i+1,2j+1,2k}, \\ Z_{2i,2j,2k+1} & = f_{2i,2j,2k+1} - \hat{f}_{2i,2j,2k+1}, \\ Z_{2i,2j+1,2k+1} & = f_{2i,2j+1,2k+1} - \bar{f}_{2i,2j+1,2k+1}, \\ Z_{2i+1,2j,2k+1} & = f_{2i+1,2j,2k+1} - \bar{f}_{2i+1,2j,2k+1}, \\ Z_{2i+1,2j+1,2k+1} & = f_{2i+1,2j+1,2k+1} - \bar{\bar{f}}_{2i+1,2j+1,2k+1}. \end{cases} \tag{4.30}$$

Notice that, as for the lower dimension cases we recover here the second order IUs (see Section 2) in taking $\alpha = \beta = b = c = 0$ and $a = 1$. Moreover the transfer matrix S is here indeed lower triangular.

The fourth order 3-D IUs are easily defined taking

$$\alpha = \frac{1}{6}, \quad \beta = 0,$$

and

$$a = \frac{4}{3}, \quad b = 0, \quad c = 0,$$

exactly as in the lower dimension cases.

5. NUMERICAL RESULTS

We present here some numerical results which are related to the combination of the incremental unknowns method with the compact scheme. First of all we concentrate on the data compression and we compare the decay of magnitude of the structures according to the grid level to which they belong. Here we use high order hierarchical preconditioner. After that we consider elliptic problems. We point out a saturation phenomenon of the hierarchical preconditioning of the matrices associated to the Dirichlet problem. At this point we consider the contrary situation: the compact schemes are used for a high accurate discretization of the underlying operators, in this case $-\Delta$, and the second order IUs are used for preconditioning. Particularly we recover comparable results of speed of convergence with the second accurate case (see [7]) when we use (bi)gradient methods. The results we give here correspond to homogeneous boundary conditions. However, concerning the data compression aspect, we recover same results when periodic boundary conditions are considered. All the numerical solutions of the Dirichlet problems were realized on a CRAY YMP at the Université Paris XI, Orsay, France.

Notations

— We introduce now the following notation which will be used along this section: we shall say that a grid has a $C_{k,l}$ configuration if it is obtained with l dyadic refinements of a grid composed of k points in each direction of the domain. The fine grid is thus composed of $2^l(k+1) - 1$ points in each direction.

— We denote by IUp the incremental unknowns of order p.

5.1. Compression of the data

The numerical results we present here illustrate the improvement of the data compression in using high order IUs instead of second order ones. For that purpose we compare the decay of the magnitude of the incremental unknowns according to the grid level to wich they belong, in the fourth and the second order case. We consider functions with numerous oscillations and with strong gradients. We observe that less discretization points are required with high order IU than second order ones to obtain a given order of magnitude of the smallest structures.

5 1 1 *The dimension one*

In (*fig 5*) we consider the discretization of the oscillating function $f(x) = \sin(19\pi x)$ The associated vector is written in both the IU2 and the IU4 base We see that when the number M of discretization is large enough (here when $M \geqslant 31$ which is, with this grid configuration, the smallest possible M larger than the number of oscillations of f), the decay of the IU4 is much more accentued that one of the IU2 Indeed in the finest grid we can observe that there is about a factor 1 000 between the magnitude of this two types of IUs

In (*fig 6*) the function considered has strong gradients near the points $x = \frac{1}{12}, \frac{5}{12}, \frac{3}{4}$ As in the above illustration starting from a certain grid level, the decay of the magnitude of the IU4 is greather than that of the IU2 We find about a factor 100

5 1 2 *The dimension two*

We realize here the same kind of test but in dimension two The function we consider is $f(x,y) = \sin(72 \,.\, x \,.\, y \,.\, (1-x) \,.\, (1-y))$ We can see in (*fig 7*), as in dimension one, that the decay of the magnitude of the IU4 is greather than that of the IU2

5 1 3 *The dimension three*

We consider here a $C_{3\ 4}$ grid i e the fine grid is composed of 63^3 points and the function discretized is $f(x,y,z) = \sin(\pi x) \,.\, \sin(2\pi y) \,.\, \sin(3\pi z)$ As in the lower dimension cases, we can observe in (*fig 8*) that the decay of the magnitude of the IU4 is greather than that of the IU2

5.2. Solution of the Dirichlet problem

We consider the classical Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega =]0,1[^n \quad n = 2, 3\,, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{5 31}$$

We arrange the unknowns in the hierarchical order and we introduce the Incremental Unknowns via the transfer matrix S (here both the order of the hierarchical preconditioner and that of the discretization matrix of the Laplacian are not noticed) The discrete problem to solve is then

$$AS \,.\, \hat{X} = F\,, \tag{5 32}$$

where A is the discretization matrix of $-\Delta$ written in the hierarchical basis, using a centered second order difference scheme After the symmetrization of (5 32) by multiplying, on the left, each term by ${}^tS$, we obtain the symmetric system

$${}^tSAS \,.\, \hat{X} = {}^tSF = \hat{F} \tag{5 33}$$

For a second order discretization (5 33) is symmetric again and can be solved by a conjugate gradient method

In space dimension two when S is the second order hierarchical transfer matrix, it was shown by M Chen and R Temam in [8] that the condition number of ${}^tSAS$, $K({}^tSAS)$, is $C_1 \,.\, (l+1)^2 = C_1 \,.\, \left(\mathrm{Log}_2\left(\frac{1}{h}\right)\right)^2$ which is much smaller than $K(A) = \frac{C_2}{h^2}$ where C_1 and C_2 are positive numbers independent of the mesh size Consequently, the conjugate gradient method is very efficient and well adapted to this problem

In space dimension three it was shown in [6] that, for a second order hierarchical preconditioning, the condition number of ${}^tSAS$ is $O(h^{-1}(\ln(h))^4)$ instead of $O(h^{-2})$ when the usual nodal unknowns are used, h being the fine grid mesh size

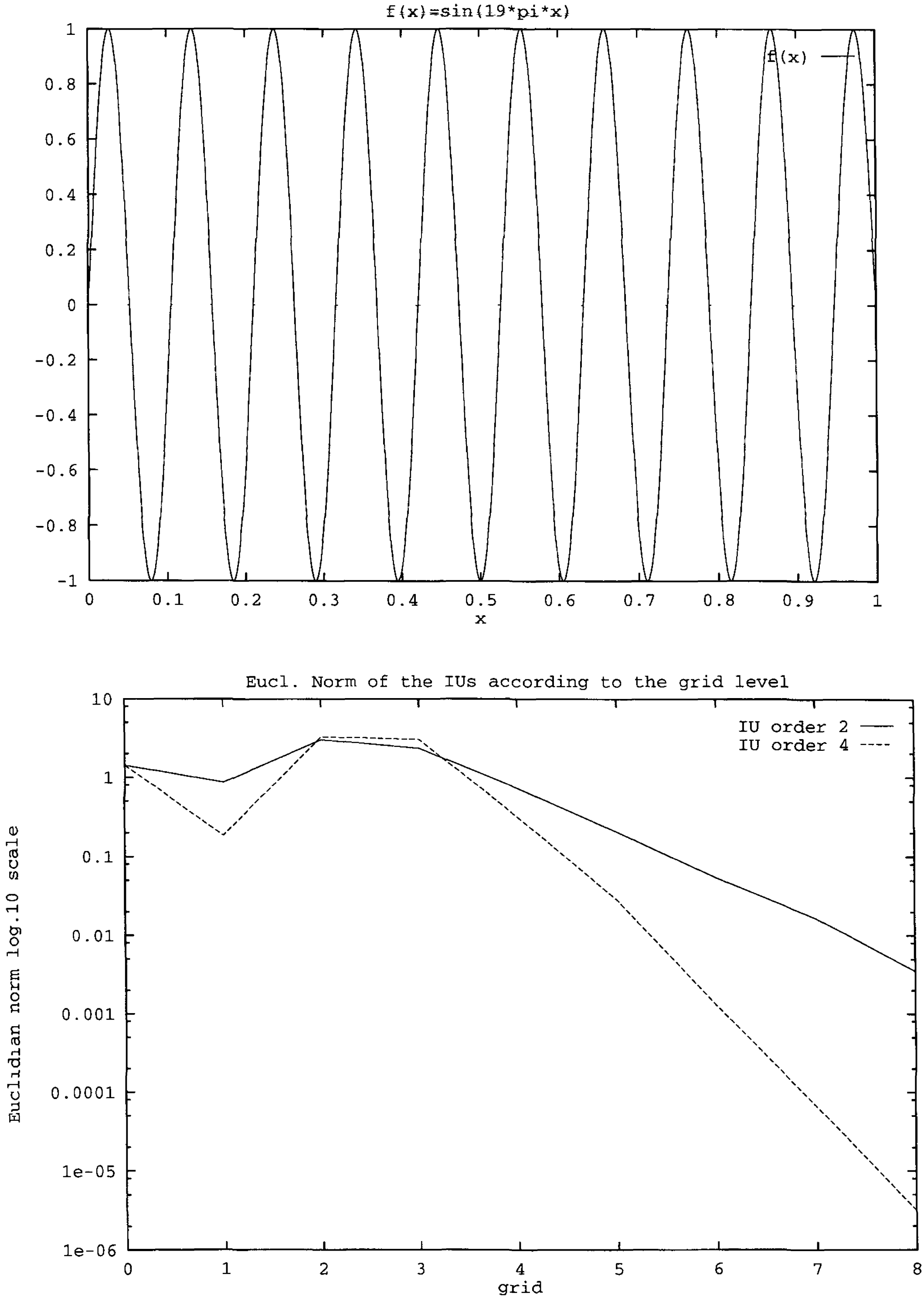


Figure 5. — Data compression in space dimension one. (a) The function $f(x) = \sin(19\pi x)$. (b) Decay of the structures according to the grid level to which they belong. Comparison between the second and the fourth order IUs. The grid is of type $C_{3,8}$.

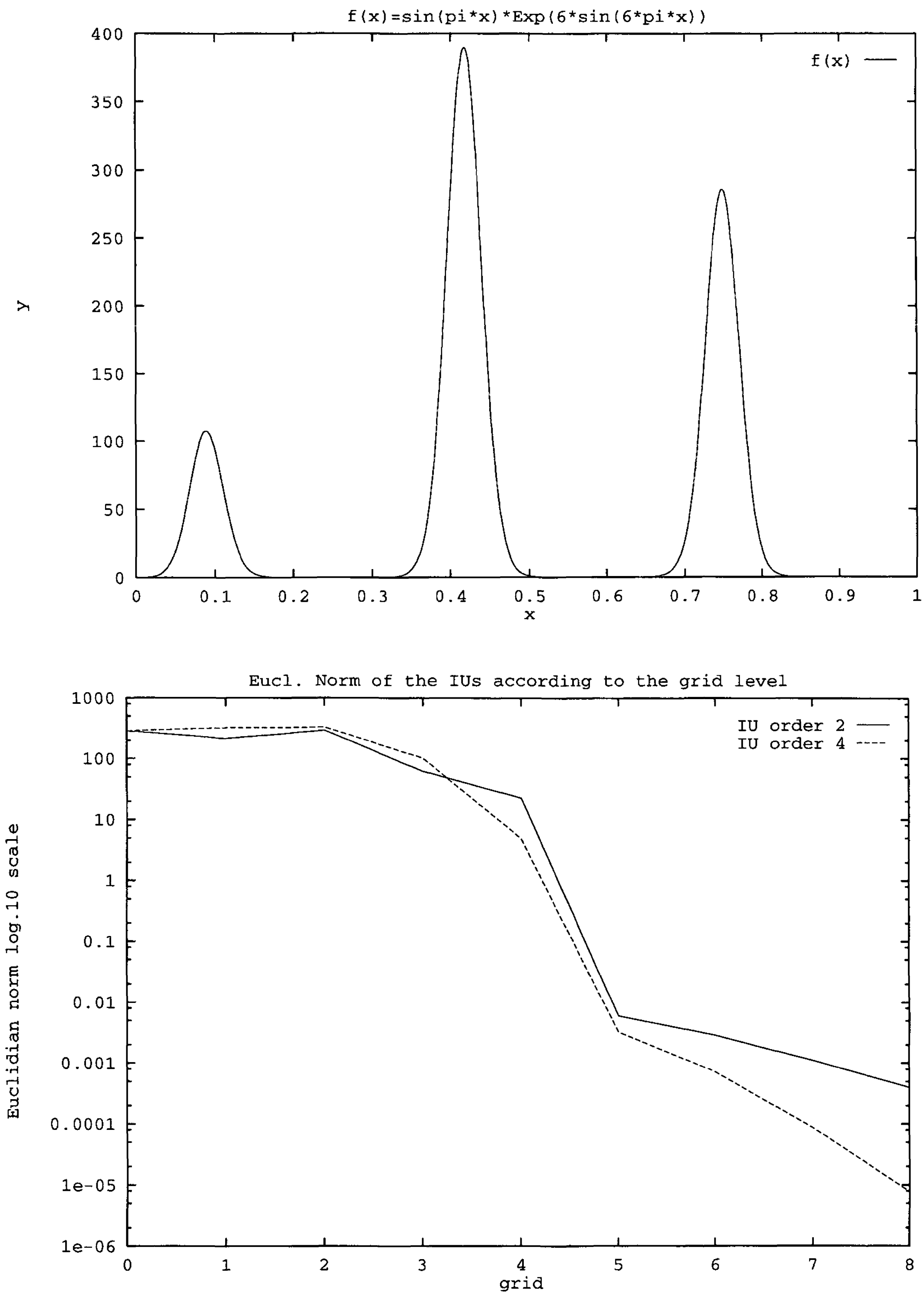


Figure 6. — Data compression in space dimension one. (a) The function $f(x) = \sin(\pi x)\, e^{6\sin(6\pi x)}$. (b) Decay of the structures according to the grid level to which they belong. Comparison between the second and the fourth order IUs. The grid is of type $C_{3,8}$.

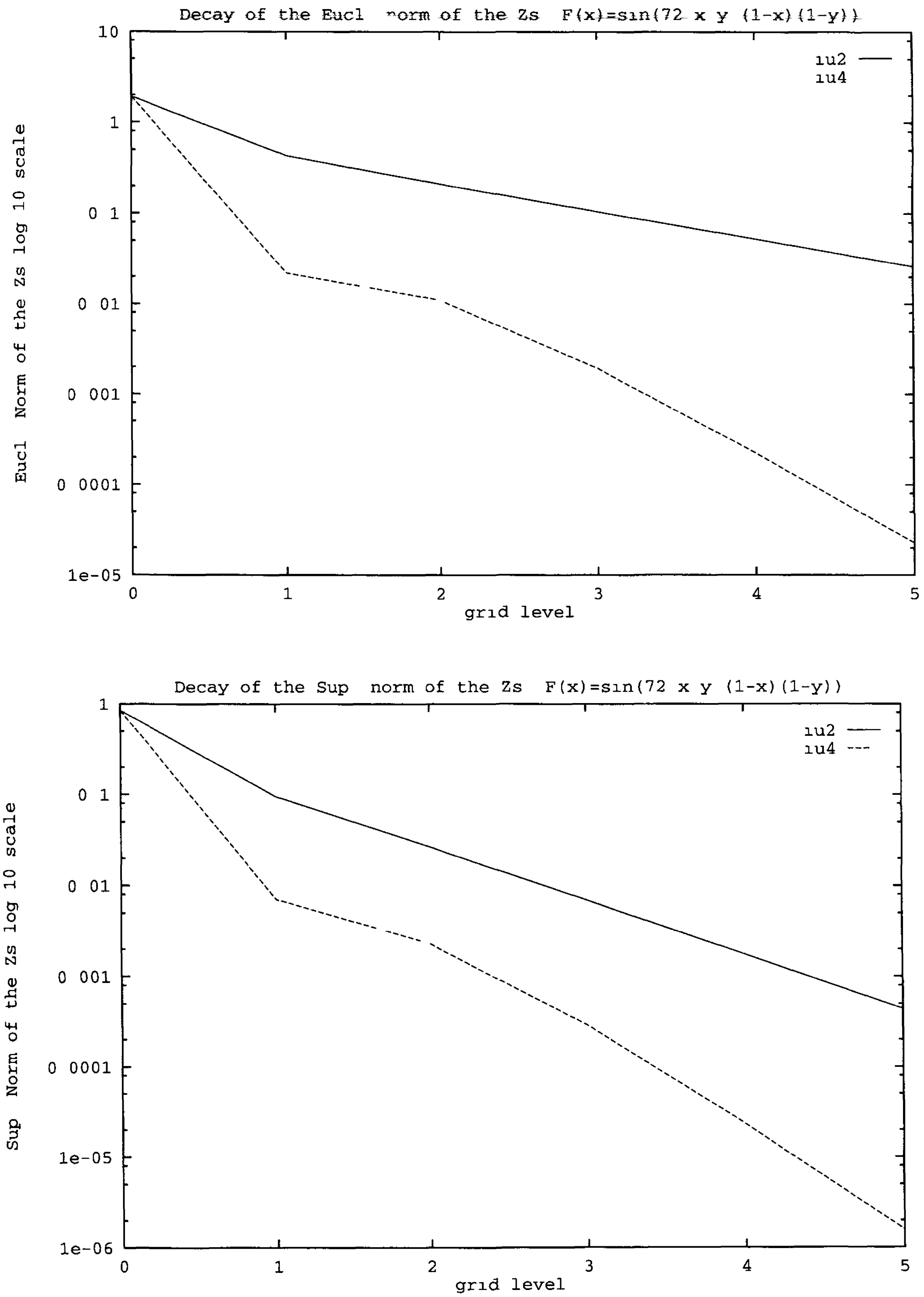


Figure 7. — Data compression in space dimension two, $f(x,y) = \sin(72x.y.(1-x)(1-y))$ Decay of the structures according to the grid level to which they belong. (a) Euclidian norm. (b) Sup norm. The grid is of type $C_{3\,5}$.

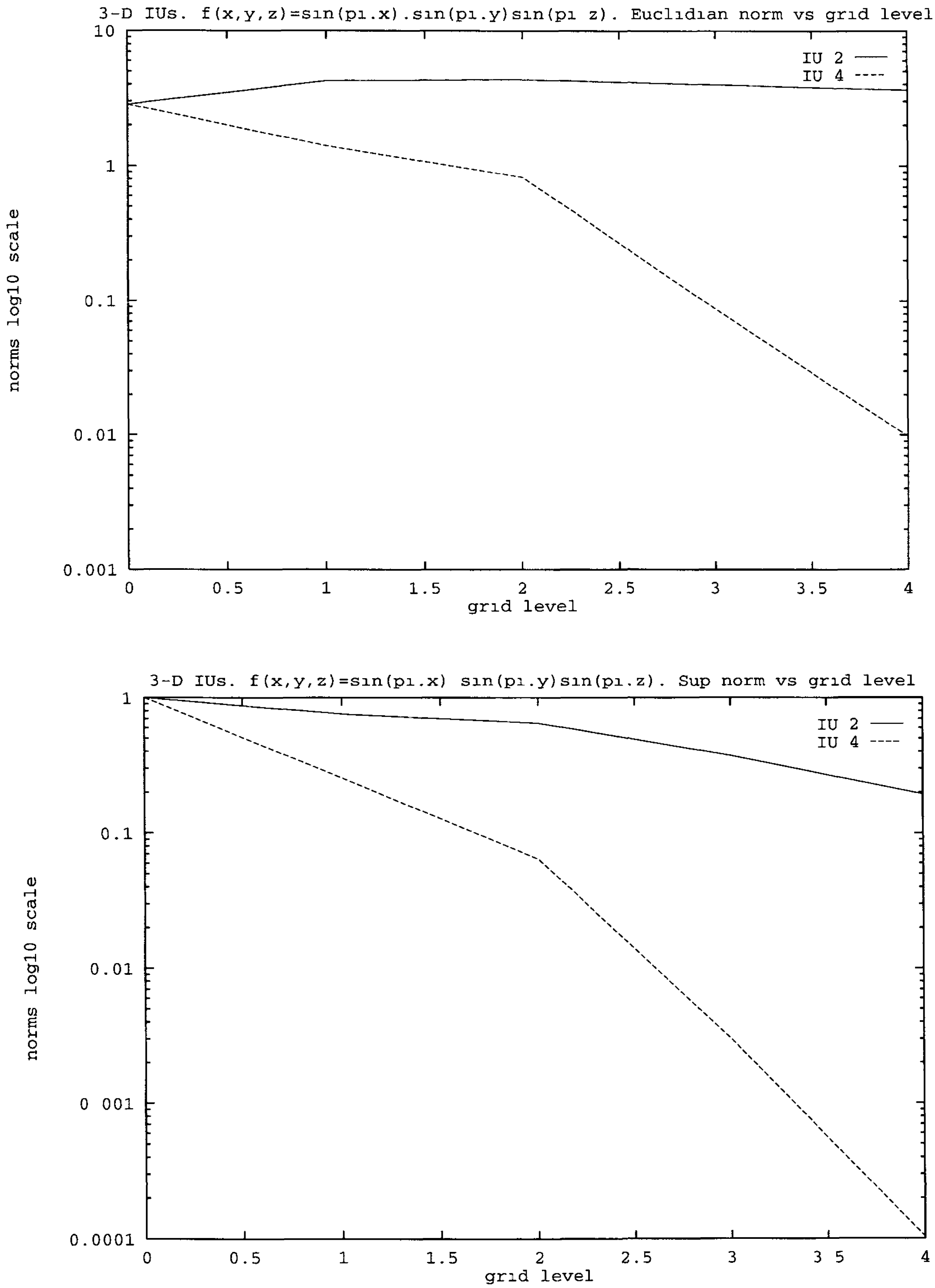


Figure 8. — Data compression in space dimension three, $f(x, y, z) = \sin(\pi x) . \sin(2\pi y) . \sin(3\pi z)$ Decay of the structures according to the grid level to which they belong. (a) Euclidian norm. (b) Sup norm. The grid is of type $C_{3\,4}$.

When S is a high order hierarchical transfer matrix, we observe, as in [6] for the dimension three case with the IU3, a saturation phenomenon: the condition number of ${}^tSAS$ has the same asymptotic behaviour as in the second order case, as it is illustrated in the following section.

5.2.1. *High order hierarchical preconditioning*

We consider here the Poisson problem with a *second order discretization,* that is the usual centered three points scheme in dimension one, and the five points scheme in dimension two.

We compute here the condition number of the matrix ${}^tS_iAS_i$, $i = 1, 2$; S_1 is the transfer matrix associated to second order IUs, S_2 that of the fourth order IUs. The results are summarized in the following tables.

5.2.1.a. *The one dimensional case*

The following results are obtained for $C_{3,l}$ grids.

Table 1. — Condition number of the matrix.

l	Nodal basis	Second order I.U.	Fourth order I.U.
$l = 1$ (N = 7)	25.27	6.82	13.07
$l = 2$ (N = 15)	103.08	13.65	21.74
$l = 3$ (N = 31)	412.72	21.31	39.80
$l = 4$ (N = 63)	1 647.53	54.62	78.56
$l = 5$ (N = 127)	6 589.85	109.25	156.91
$l = 6$ (N = 255)	26 359.10	218.50	313.80
$l = 7$ (N = 511)	105 436.13	437.02	627.59
$l = 8$ (N = 1 023)	421 744.27	874.04	1 255.18

Table 2. — Asymptotic behaviour of $C(A)$.

l	Nodal basis: $C(A)/h^2$	$C({}^tS_1AS_1)/h$	$C({}^tS_2AS_2)/h$
$l = 1$ (N = 7)	0.39490	0.8535	1.63416
$l = 2$ (N = 15)	0.40268	0.8535	1.35922
$l = 3$ (N = 31)	0.40304	0.8535	1.24388
$l = 4$ (N = 63)	0.40223	0.8535	1.22751
$l = 5$ (N = 127)	0.40221	0.8535	1.22590
$l = 6$ (N = 255)	0.40220	0.8535	1.22577
$l = 7$ (N = 511)	0.40220	0.8535	1.22576
$l = 8$ (N = 1 023)	0.40220	0.8535	1.22576

As we can see $C({}^tS_2AS_2)/h$ has the save asymptotic behaviour as $C({}^tS_1AS_1)/h$. Hence, the high order preconditioning does not improve the results of the second order preconditioning: this is a saturation phenomenon. The condition number of ${}^tS_1AS_1$ is $C_1 . h$ and we conjecture that $C({}^tS_2AS_2)$ is $C_2 . h$ with $C_2 > C_1$. There is no advantage to solve the Dirichlet problem with a high order hierarchical preconditioner.

5.2.2.b. *The two dimensional case*

As above we compute here the condition number of the matrix ${}^tS_iAS_i$, $i = 1, 2$ using the same notations.

Table 3. — Condition number of the matrix.

l	Nodal basis	Second order I.U.	Fourth order I.U.
$l = 1$ (N = 7)	25.21	7.22	19.04
$l = 2$ (N = 15)	101.46	10.19	42.67
$l = 3$ (N = 31)	410.37	14.58	66.58
$l = 4$ (N = 63)	1 652.47	20.35	92.34
$l = 5$ (N = 127)	6 585.48	27.47	120.61

Table 4. — Asymptotic behaviour of $C(A)$.

l	Nodal basis: $C(A)/h^2$	$C({}^tS_1AS_1)/(l+1)^2$	$C({}^tS_2AS_2)/(l+1)^2$
$l = 1$ (N = 7)	0.39390	1.8135	4.9113
$l = 2$ (N = 15)	0.3963	1.1329	4.7419
$l = 3$ (N = 31)	0.4007	0.9113	4.1613
$l = 4$ (N = 63)	0.4034	0.8142	3.6936
$l = 5$ (N = 127)	0.40195	0.7630	3.3503

The conclusions are the same as for the dimension one case. As it was shown in [8] the condition number of ${}^tS_1AS_1$ is $C_1 . (l+1)^2$. We conjecture that $C({}^tS_2AS_2)$ is $C_2 . (l+1)^2$ with $C_2 > C_1$. This saturation phenomenon shows that there is no advantage to solve the 2D-Dirichlet problem with IU4. Futhermore each iteration is much more costly because of the implicit nature of the IU4 (additional linear systems must be solved).

5.2.2. *High accuracy efficient solution of the Dirichlet problem*

In this section we consider the discretization of the Laplacian operator with a fourth order accurate compact scheme. The implicit expression of the matrix to be inverted not allows the use of any classical preconditioner which exploits the structure of the matrix.

If $\Omega =]0, 1[^n$, the discret problem to be solved can be expressed as

$$\left(\sum_{i=1}^{n} P_i^{-1} Q_i\right) . X = F, \tag{5.34}$$

where $P_i^{-1} Q_i$ is the compact scheme associated to the discretization of the operator $-\frac{\partial^2}{\partial x_i^2}$. Notice that expressed under the form (5.34), the system is not symmetric. Consequently we shall use a bigradient method e.g. Bicgstab [19] for the solutionof (5.34). We thus solve the equivalent system

$${}^tS\left(\sum_{i=1}^{n} P_i^{-1} Q_i\right) . S . \hat{X} = F, \tag{5.35}$$

where $X = S . \hat{X}$, S being the transfer matrix.

The speed of convergence obtained is obviously comparable to that observed in the second order discretization case but here the accuracy is higher (we recall that we can obtain up to the tenth order accuracy using a pentadiagonal form of the matrices P_i). We give in the following some results related to the solution of Dirichlet problem with a $O(h^4)$ accuracy. As noticed in the previous subsection there is a saturation phenomenon of the hierarchical preconditioning and consequently it suffices to take a second order hierarchical preconditioner for solving the problem, that is to use the transfer matrix associated to second order IUs.

For illustrating our purpose we present numerical results in both dimension two *(fig. 9, 10)* and dimension three *(fig. 11)*. In each case we take $F = 0$: there is no loss of generality in taking a null source term. We have chosen as initial data the functions:

$$\hat{X}^0(x, y) = \sin(16 \,.\, x \,.\, y \,.\, (1 - x) \,.\, (1 - y)) \text{ in dimension two and}$$

$$\hat{X}^0(x, y, z) = \sin(16 \,.\, x \,.\, y \,.\, z \,.\, (1 - x) \,.\, (1 - y) \,.\, (1 - z))\, e^{x + y + z} \text{ in dimension three.}$$

Remark 7: The (compact) discretization scheme of the laplacian we have implemented here is built by using the (compact) discretizations schemes of $-\frac{\partial^2}{\partial x_i^2}$. Truly multi-dimensional compact schemes for differential operators, e.g., those introduced by Collatz in [11] p. 542, can be used also. However these schemes cannot be derived by a simple composition of the corresponding 1-D formulas and their implementation is then more heavy.

5.3. Relation between differentiation matrices and incremental unknowns

5.3.1. *Discrete differentiation and interpolation*

We point out here a close relation between the differentiation schemes and the interpolation schemes. This relation is established via a hierarchization process. For the sake of simplicity we examine only the one dimensional case and, as usual, the following results can be easily generalized for higher space dimensions by using the extension technique presented in Section 4.

Let us consider the approximation of the second derivative by the usual second order scheme. Letting $f = -\frac{\partial^2 u}{\partial x^2}$ we have:

$$2 \,.\, U_i - U_{i-1} - U_{i+1} = h^2 F_i \,.$$

Here h denotes the spatial mesh size.

At this point we consider a splitting of the unknowns U_i according to the parity of the indice i: this is in fact the hierarchization step. For the odd indices which are associated to the complementary grid and where belong the IU components, the above relation is written as

$$2 \,.\, U_{2i+1} = U_{2i} + U_{2i+2} + h^2 F_{2i+1} \,.$$

Hence,

$$U_{2i+1} = \frac{1}{2}(U_{2i} + U_{2i+2}) + O(h^2)\,,$$

and we recognize on the right hand side the second order interpolation scheme used for defining the second order IUs.

More generally the approximation of $f = \frac{\partial^{2p} u}{\partial x^{2p}}$, by a classical finite difference scheme is (after a renormalization):

$$\sum_{k=-p}^{p} a_k U_{i+k} = h^{2p} F_i$$

with $a_k = a_{-k}$ and $a_k \neq 0\ \forall k$. As above, we split the unknowns according to the parity of their indices. We obtain:

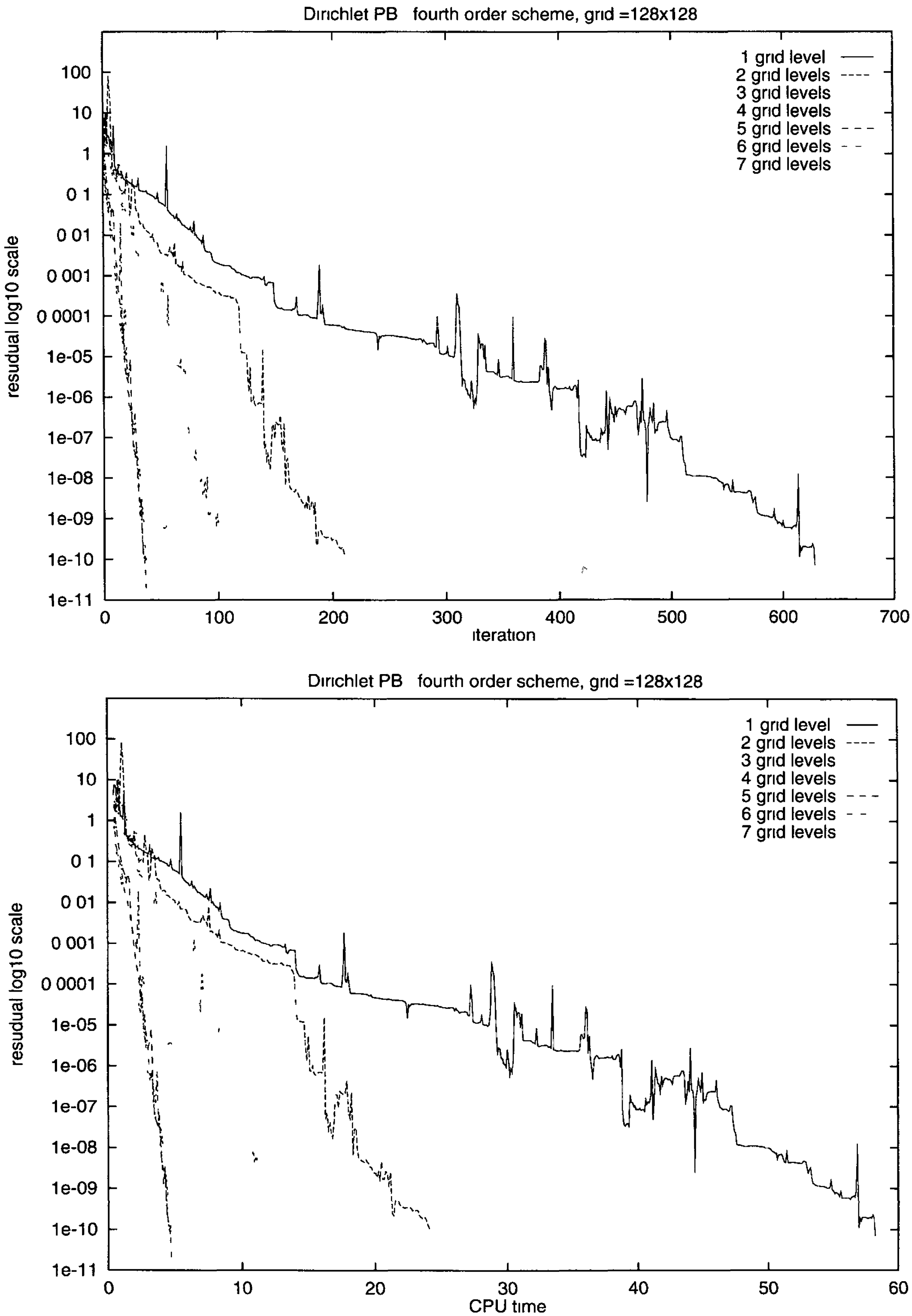
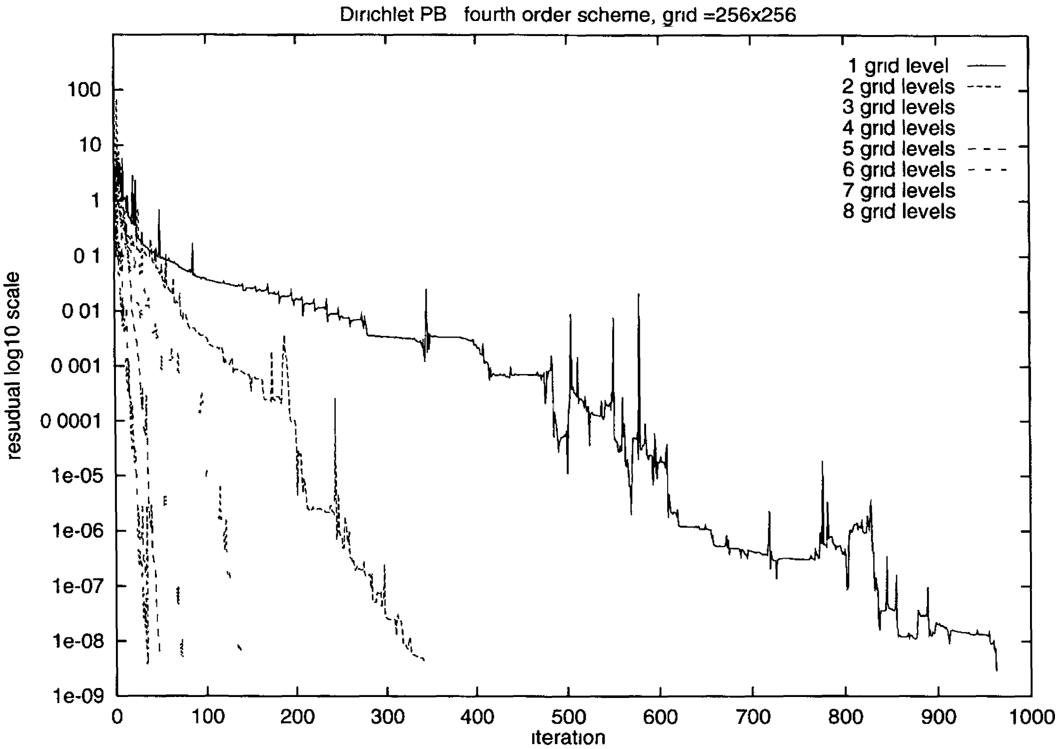


Figure 9. — Solution of the $O(h^4)$ accuracy bidimensional Dirichlet problem using a second order hierarchical preconditioner. (a) Residual vs iterations. (b) Residual vs CPU Time. The finest grid is composed of 128 points in each direction.



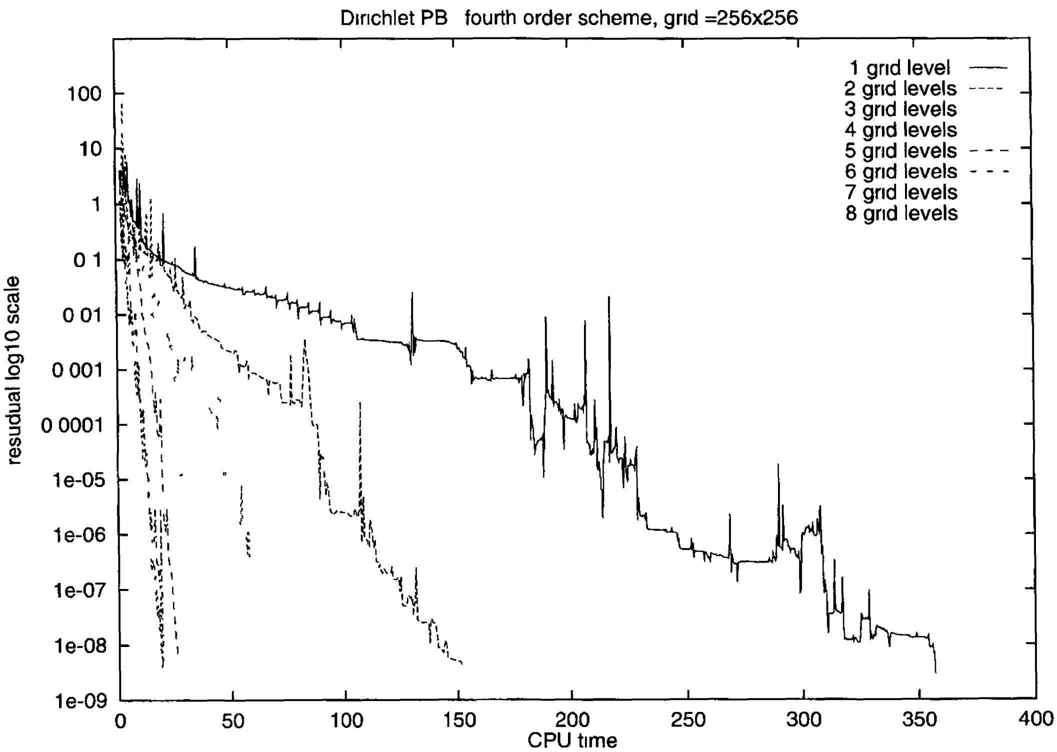
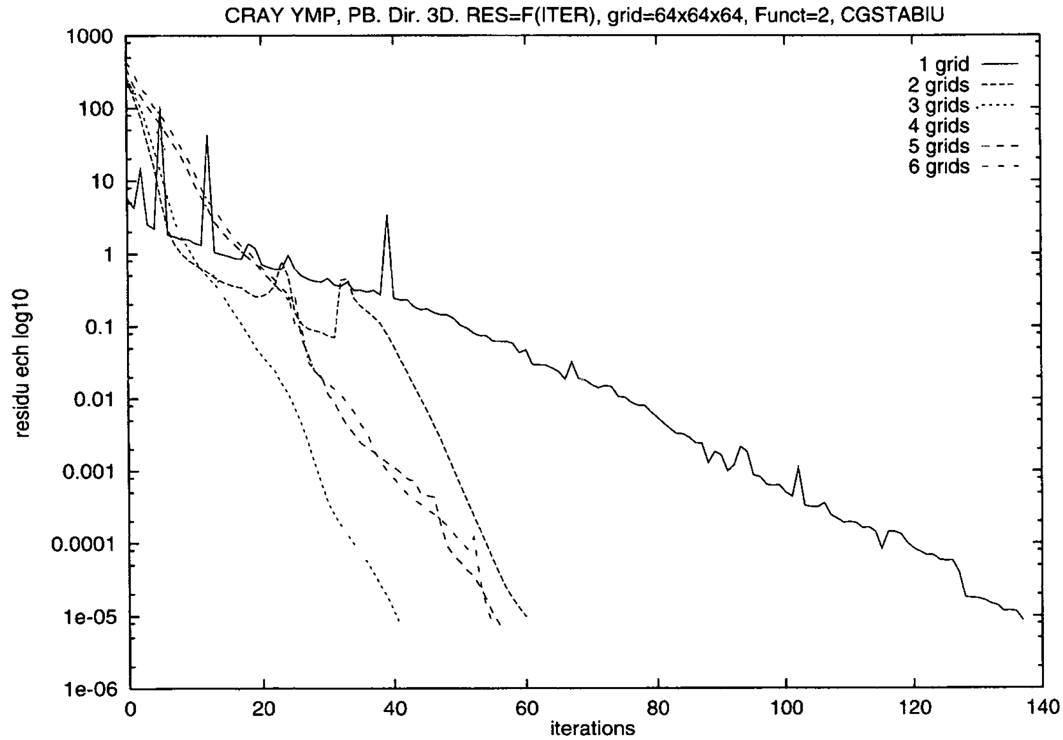


Figure 10. — Solution of the $O(h^4)$ accuracy bidimensional Dirichlet problem using a second order hierarchical preconditioner. (a) Residual vs iterations. (b) Residual vs CPU Time. The finest grid is composed of 256 points in each direction.



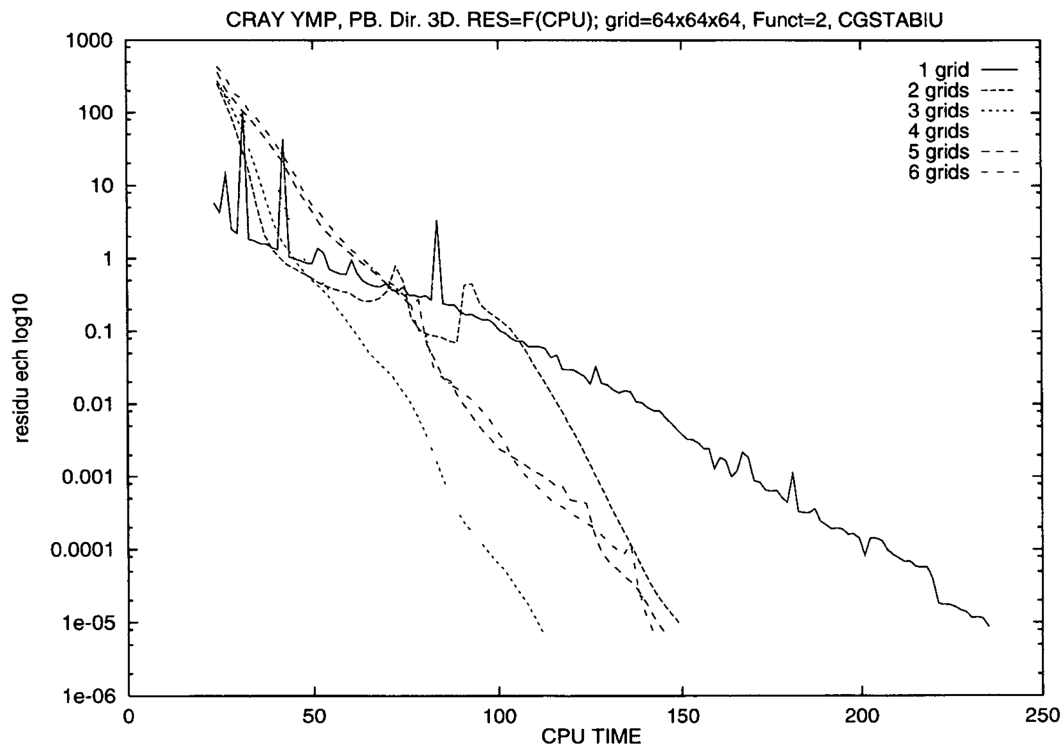


Figure 11. — Solution of the $O(h^4)$ accuracy tridimensional Dirichlet problem using a second order hierarchical preconditioner. (a) Residual vs iterations. (b) Residual vs CPU Time. The finest grid is composed of 64 points in each direction. The initial data is $\hat{X}^0 = \sin(16 . x . y . z . (1-x) . (1-y) . (1-z)) e^{x+y+z}$.

• if $p = 2q$, we have

$$\sum_{k=-q}^{q} a_{2k} U_{2i+1+2k} + \sum_{k=-q}^{q-1} a_{2k+1} U_{2i+2+2k} = h^{2p} F_i ,$$

thus

$$\sum_{k=-q}^{q} a_{2k} U_{2i+1+2k} = - \sum_{k=-q}^{q-1} a_{2k+1} U_{2i+2+2k} + O(h^{2p}) .$$

We set $b_k = \frac{ak}{a_0}$ and the above relation becomes

$$U_{2i+1} + \sum_{k=-q, k \neq 0}^{q} b_{2k} U_{2i+1+2k} = - \sum_{k=-q}^{q-1} b_{2k+1} U_{2i+2+2k} + O(h^{2p}) .$$

We recognize here the stencil of the $2p^{th}$ order interpolation compact scheme.

• if $p = 2q + 1$ we obtain comparable expressions and the conclusion is the same.

5.3.2. *Calculation of the coefficients of compact interpolation schemes*

The above relations are valid only for the inner points (i.e. points of indice i such that $i \geqslant p$ and $N - i \geqslant p$) or if the boundary conditions are periodic. For the inner points precisely we can derive a very simple method for computing the coefficients a_k (the coefficients of the boundary points will be calculated with a closure formula).

Let A be the matrix associated to the operator $-\frac{\partial^2}{\partial x^2}$. Then, for the inner points, the coefficients a_k which are associated to $(-1)^p \frac{\partial^{2p}}{\partial x^{2p}}$ are those of the matrix A^p, the p^{th} power of A.

5.3.3. *Incremental Unknowns and boundary conditions*

Up to this time, the Incremental Unknowns were essentially associated to periodic or homogeneous Dirichlet boundary conditions. As we have seen it above, we can relate closely the second order IUs to the discretization of some elliptic operators. Using the same approach, we show that the associated IU can heritate the underlying boundary conditions. In that way we can defined for each boundary condition suitable incremental unknowns. Indeed, in one hand, for the construction of the IU2 of Dirichlet type (see also [7]) we can start from the discretization scheme of the Laplacian.

We let $F_i = -\frac{\partial^2 u}{\partial x^2}|_{x=ih}$ and $h = \frac{1}{2N}$. We now consider the linear system which corresponds to the discretization of the Dirichlet problem:

$$2 . U_i - U_{i-1} - U_{i+1} = h^2 F_i \quad \text{for} \quad i = 1, ..., 2N-1 ,$$

$$U_0 = U_{2N} = 0 .$$

We consider a splitting of the unknowns U_i according to the parity of the indices: this is in fact the hierarchization step. For the odd indices which are associated to complementary grids and where belong the IU, the above relation is written as

$$2 . U_{2i+1} = U_{2i} + U_{2i+2} + h^2 F_{2i+1} ,$$

hence

$$U_{2i+1} = \frac{1}{2}(U_{2i} + U_{2i+2}) + O(h^2),$$

and we recognize on the right hand side the second order interpolation scheme used for defining the second order IUs.

In the other hand, we easily build the IU2 associated to periodic boundary conditions considering the related discretization matrix of the Laplacian:

$$B = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & . & . & -1 \\ -1 & 2 & -1 & 0 & . & . \\ 0 & -1 & 2 & -1 & & . \\ . & . & . & . & . & 0 \\ . & . & 0 & -1 & 2 & -1 \\ -1 & . & . & 0 & -1 & 2 \end{pmatrix},$$

using a similar hierarchization process, we find

$$Z_{2i+1} = U_{2i+1} - \frac{1}{2}(U_{2i} + U_{2i+2}); \quad \text{for } i = 0, N-1,$$

$$U_0 = U_{2N}.$$

Note that high order (periodic) IUs are very easily constructed by deriving the coefficients of the interpolation scheme of the powers of B.

Using the relation between operators and IUs, we shall define Incremental unknowns adapted to other boundary conditions.

Homogeneous Neumann IUs

a) The second order IUs case

We consider the (singular) discretization matrix of the Laplacian associated to homogeneous Neumann boundary conditions:

$$C = \frac{1}{h^2}\begin{pmatrix} 1 & -1 & 0 & . & . & . \\ -1 & 2 & -1 & 0 & . & . \\ 0 & -1 & 2 & -1 & . & . \\ . & . & . & . & . & 0 \\ . & . & 0 & -1 & 2 & -1 \\ . & . & . & 0 & -1 & 1 \end{pmatrix}.$$

We let $F_i = -\frac{\partial^2 u}{\partial x^2}\big|_{x=ih}$ and we consider the linear system $C.U = F$. We have:

$$U_1 - U_2 = h^2 F_1,$$

$$2.U_i - U_{i-1} - U_{i+1} = h^2 F_i \quad \text{for } i = 2, \ldots, 2N-2,$$

$$U_{2N-1} - U_{2N-2} = h^2 F_{2N-1}.$$

We now apply the hierarchization procedure and we find the corresponding Incremental Unknowns which are defined by:

DEFINITION 5: *The IU2 attached to homogeneus Neumann boundary conditions are the numbers Z:*

$$\begin{aligned} Z_1 &= U_1 - U_2 , \\ Z_{2i+1} &= U_{2i+1} - \frac{1}{2}(U_{2i} + U_{2i+2}) , \quad \textit{for } i = 2, \dots, N-2 , \\ Z_{2N-1} &= U_{2N-1} - U_{2N-2} . \end{aligned} \tag{5.36}$$

We note that these are second order IUs even for the points near the boundary as one can easily check it by using the Taylor expansion. They are in fact of the order of the operator considered here as we have seen it above.

b) The fourth order IUs case

In the same way as in the homogeneous Dirichlet case, we can build fourth order IUs attached to homogeneous Neumann boundary conditions using appropriate interpolation compact schemes. The definition of these IUs is identical to that of the dirichlet IUs for inner points (as it is explicited above for the second order case). The difference between these two types of IUs appears in the closure formulas which contain implicitly the boundary conditions.

The closure formula can be expressed as:

$$\hat{f}_1 + \alpha \hat{f}_3 = a f_2 + b f_4 + c f_6 ,$$

where of course the parameter α is the same as in (4.17), that is $\alpha = \frac{1}{6}$. The Taylor expansion gives:

$$\begin{aligned} f_1 &= f_0 + h \,.\, \partial + \frac{h^2}{2\,!}\partial^2 + \frac{h^3}{3\,!}\partial^3 + O(h^4) , \\ f_2 &= f_0 + 2 \,.\, h\partial + \frac{(2h)^2}{2\,!}\partial^2 + \frac{(2h)^3}{3\,!}\partial^3 + O(h^4) , \\ f_3 &= f_0 + 3 \,.\, h\partial + \frac{(3h)^2}{2\,!}\partial^2 + \frac{(3h)^3}{3\,!}\partial^3 + O(h^4) , \\ f_4 &= f_0 + 4 \,.\, h\partial + \frac{(4h)^2}{2\,!}\partial^2 + \frac{(4h)^3}{3\,!}\partial^3 + O(h^4) , \\ f_6 &= f_0 + 6 \,.\, h\partial + \frac{(6h)^2}{2\,!}\partial^2 + \frac{(6h)^3}{3\,!}\partial^3 + O(h^4) , \end{aligned}$$

where $\partial^q = \frac{\partial^q f}{\partial x^q}\big|_{x=0}$ (by an obvious symmetry argument, a similar system is verified for the definition of $\hat{f}_{2N-1}$). The boundary conditions imply that $\partial = 0$. Consequently, we deduce from the previous system that the coefficients a, b and c must verify the following system:

$$\begin{cases} 1 + \alpha &= a + b + c \qquad (\text{first order}) , \\ 1 + 9\,\alpha &= 4\,.\,a + 16\,.\,b + 36\,.\,c \quad (\text{third order}) , \\ 1 + 27\,\alpha &= 8\,.\,a + 64\,.\,b + 216\,.\,c \quad (\text{fourth order}) . \end{cases}$$

We find $a = \frac{537}{352}$, $b = -\frac{41}{88}$ and $c = \frac{113}{1\,056}$.

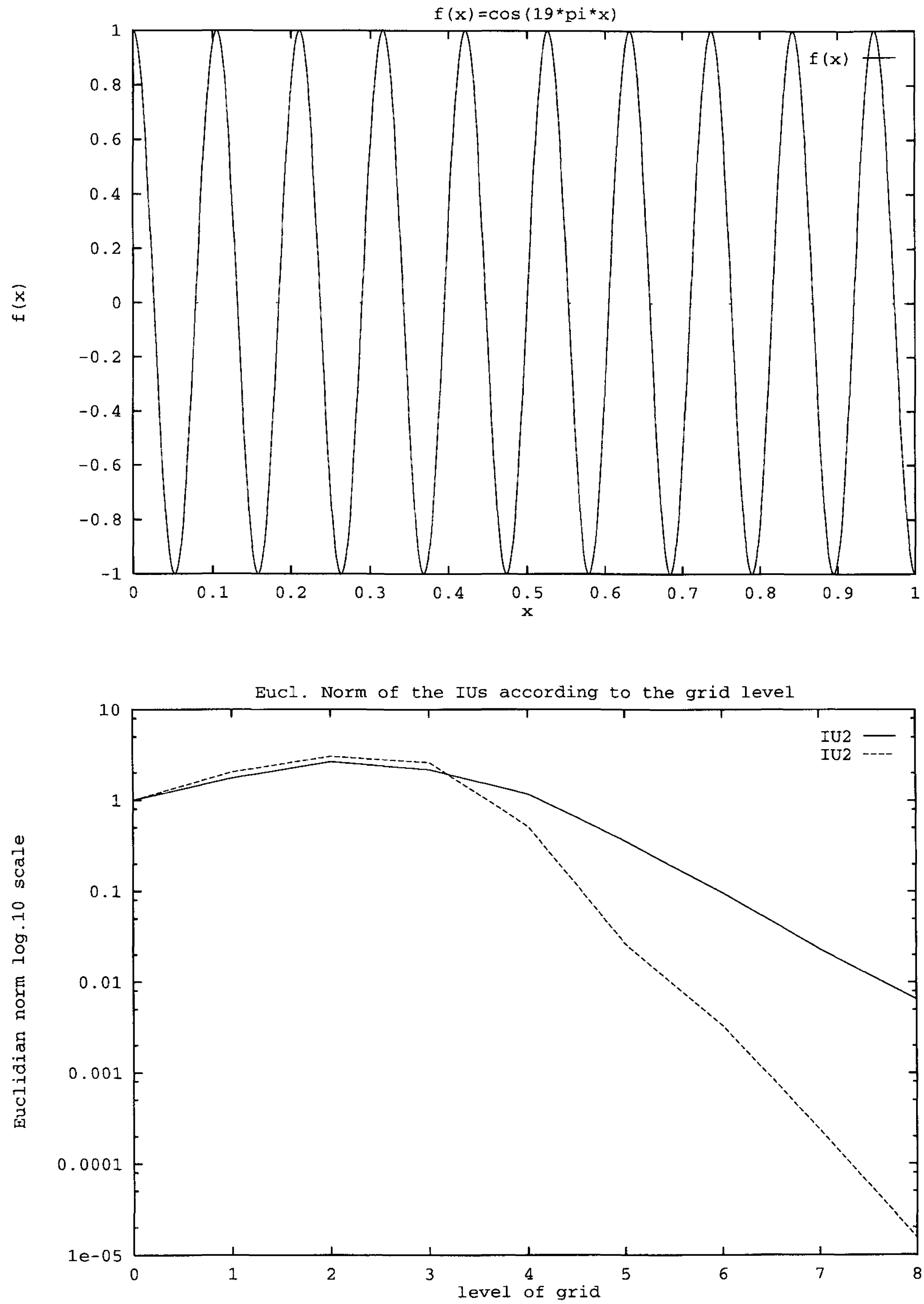
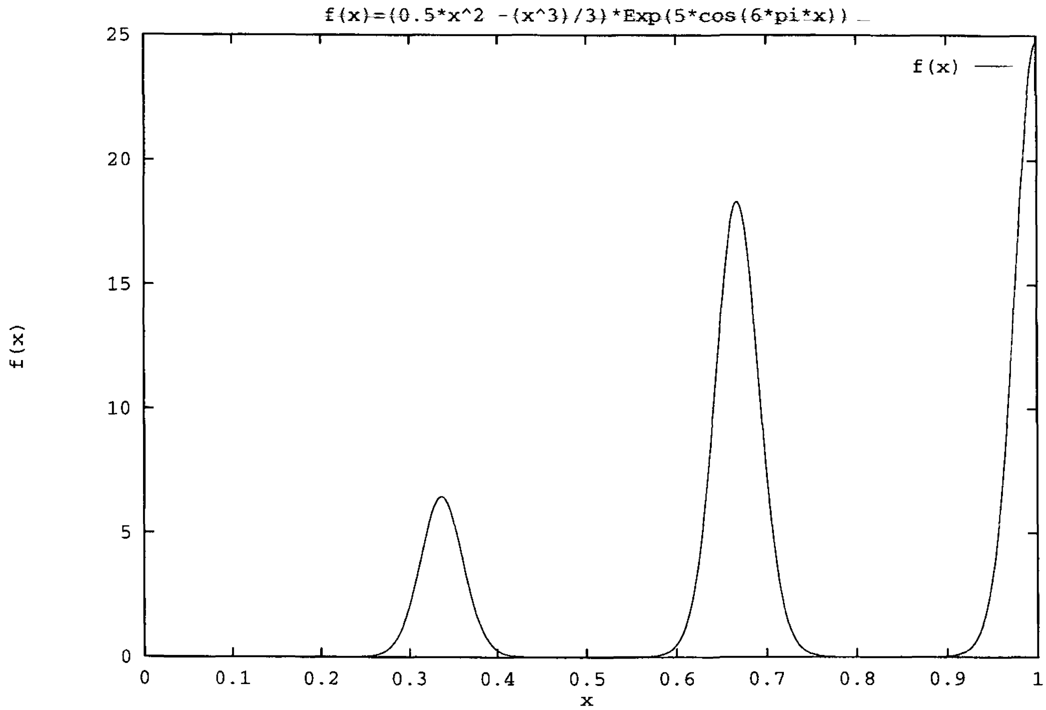


Figure 12. — Data compression in space dimension one. (a) The function $f(x) = \cos(19\pi x)$. (b) Decay of the structures according to the grid level to which they belong. Comparison between the second and the fourth order IUs. The grid is of type $C_{3,8}$.



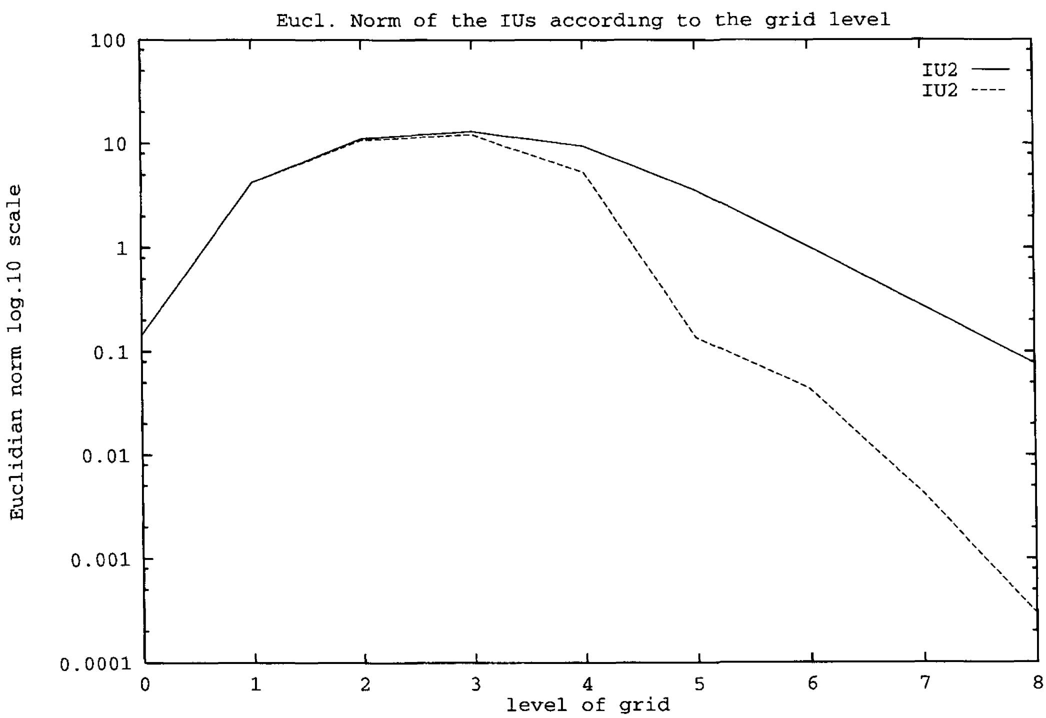


Figure 13. — Data compression in space dimension one. (a) The function $f(x) = \left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right) e^{5\cos(6\pi x)}$. (b) Decay of the structures according to the grid level to which they belong. Comparison between the second and the fourth order IUs. The grid is of type $C_{3,8}$.

As for the Dirichlet IU2, we present here some numerical results that illustrate the data compression property of the Neumann IU2.

c) Compression of the data using the Neumann IUs

The numerical results we present here illustrate the improvement of the data compression we obtained by using high order IU instead of second order ones for function that verify homogeneous Neumann boundary conditions. As in the homogeneous Dirichlet case, we compare the decay of the magnitude of the incremental unknownss according to the grid level to wich they belong, in the fourth and the second order case.

In (*fig. 12*) we consider the discretization of the oscillating function $f(x) = \cos(19\pi x)$. The associated vector is written in both the IU2 and the IU4 base. As in the (*fig. 5*), in the Dirichlet case, we see that when the number M of discretization is large enough, the decay of the IU4 is much more accentued that one of the IU2. Indeed in the finest grid we can observe that there is about a factor 400 between the magnitude of this two types of IUs.

In (*fig. 13*) the function considered has strong gradients near the points $x = \frac{1}{3}, \frac{2}{3}, 1$. As in the above illustration starting to a certain grid level, the decay of the magnitude of the IU4 is greather than that of the IU2. We find about a factor 200.

6. CONCLUSION

The link we propose here between Incremental Unknowns method and the compact schemes techniques is situated in the context of the double exploitation of the notion of the hierarchical preconditioner which is the basic idea of the IU methodology.

In improving the data compression we give a new tool for the implementation of Nonlinear Galerkin Method-like when finite differences are used. In this situation the compact schemes are used only for constructing high order hierarchical preconditioner aimed at generating several structures whose the finer are effectively smalls.

Concerning the preconditioning of elliptic selfadjoint operators, the "classical" IU2 give a very good preconditioner for the underlying matrices which are built by using a high order discretization compact scheme.

In constructing high order IUs associated to various types of boundary conditions, we have seen that the IU methodology is versatile, and, particulary, that it can be used for the solution of a large number of problems.

These results illustrate the flexibility of the IU methodology. They show also that it is thinkable to develop Nonlinear Galerkin Method-like in finite differences with a high spatial accuracy (comparable to the spectral one) and using a reasonable number of discretization points. Using the tools introduced here, we hope in a near future to illustrate the efficiency of our approach developping numerical schemes for solving dissipative evolution equations where the several structures generated by ours high order hierarchical preconditioners are treated differently.

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