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ON THE OPTIMIZATION OF NON PERIODIC HOMOGENIZED MICROSTRUCTURES (*)

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Abstract — *In this work, based on some ideas of N Kikuchi and M P Bendsøe ([10]) and using some preliminary results by D Chenais ([4], [5], [6] and [7]), by M L Mascarenhas and D Polisevski ([13]), we present an analytical study of the optimization of a domain, with respect to a broad class of admissible holes, (microstructures), in order to maximize, or minimize, some functionals This includes, for instance, the maximization of the torsion constant of the cross section of a rod and the cases treated in [10] and in [18] For the sake of simplicity and without loss of generality all the calculations will be done for the torsion constant We introduce some improvements of [13], in order to specify the class of admissible microstructures*

Résumé — *Dans ce travail, nous présentons des résultats théoriques concernant l'optimisation d'un domaine dans une large classe de domaines perforés, analysés par des techniques d'homogénéisation Une large classe de fonctionnelles à optimiser est considérée Le cas de la maximisation de la constante de torsion d'une barre est un exemple possible particulièrement étudié Ce travail est basé sur des idées d'abord introduites par N Kikuchi et M P Bendsoe ([10]), et utilise des résultats préliminaires de D Chenais ([4], [5], [6]) et M L Mascarenhas D Polisevski ([13])*

INTRODUCTION

We are interested in the optimization of the shape of a domain on which a partial differential equation is given. We want the solution to be as good as possible with respect to a given criterium.

The now standard way to handle this problem is to search a shape which is the transformed of a reference shape by a well-chosen homeomorphism. So the shape which can be found has the same topology as the reference one. The optimization is done within a range of shapes which have a given topology. This is satisfying in several engineering problems, but not in others.

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In the last decade, the question of the topological optimization has been studied by several authors.

In 1985, Murat and Tartar [16] and [17] gave a way to handle this question using homogenization techniques. In their work the question is : how to mix two different materials. They use homogenization techniques in order to modelize the different mixtures which can appear. As a matter of fact, in this work, they can get rid of periodicity (which often interferes in homogenization), but void-material materials are not allowed.

The results have been pretty much improved since then, by Allaire and Francfort [1] and Allaire and Kohn [2]. There are very interesting numerical results including void-material mixtures. Yet there are two restrictions : only particular partial differential equations are possible (Laplace equation, elasticity) and the functional which is optimized has to be the observance. Also in the « void-material » mixture, the void is approached by a very soft material. There seems to be instabilities in this approximation.

Bendsøe and Kikuchi [10], using an idea of Kohn and Strang [11] and [12] also propose a way to deal with this problem using homogenization techniques. In their studies, for sake of simplicity in the numerical treatment, they consider that each finite element is formed by a periodic array. In each cell of one finite element, there is a rectangular hole of given size and orientation. They use a descent method in order to chose the parameters of the periodic structures as well as possible. They also get very interesting results which can closely by related to the results of Allaire, Francfort, Kohn, Murat and Tartar in the common examples they treated. Notice that the derivatives of the homogenized coefficients with respect to the three real design parameters (size and orientation of the rectangle) on each finite element are computed numerically. This still restricts the class of admissible solutions and prevents the possibility of a direct mathematical study.

The method we present here is strongly related to the Bendsøe and Kikuchi ideas. We deal with « quasi-periodic » structures, as defined and studied by Mascarenhas and Polisevski [13]. The basic idea is to use periodic cells on which non-periodic holes are included. For a given microstructure (which is a law giving the holes in each cell, for each size of the cell), Mascarenhas and Polisevski [13] give with a mathematical proof the homogenized equations for the problem of the torsion of a rod (real void-material structures are allowed, no approximation needs to be done). The homogenized equations can be solved theoretically and numerically for any microstructure belonging to a broad class.

The following work consists in the mathematical computation of the differential of a given criterium which depends on the microstructure through the solution of the homogenized equations. The aim is of course to use it in a descent algorithm. In the problem of the torsion of a rod, it is reasonable to optimize the torsion constant, which is nothing but the observance. Though the methodology can clearly be used for a large class of other functionals, and also for a whole class of partial differential equations.

This work gives a generalization of the work by Bendsøe and Kikuchi. The optimization is done in a wider class of admissible structures, and it is based on more precise mathematical results. Numerical experiments need to be done, in order to see wether really better results can be found.

It is noticeable that we chose concepts to work on, which transform the initial problem into another problem of shape optimization in $\mathbf{R}^4$ if the rod section is in $\mathbf{R}^2$.

A microstructure needs to be defined in a precise mathematical manner. This has been done in Mascarenhas and Polisevski [13]. As a matter of fact, the definition they had given did not seem exactly appropriate for the optimization problem. So in this paper we give a slightly different one, and some improvements of [13] which become necessary for the proofs to be complete.

Now we briefly summarize the content of the paper.

In Section 1, we give the general setting. In subsection 1.1 we give the definition of a microstructure as it will be used in the sequel. We give the proofs which are necessary to get the complete results analogous to those of [13], and give the homogenized equations for the torsion of a rod. In subsections 1.2 and 1.3 we recall the basic notions of optimal control and shape optimization which are used in the following sections.

In Section 2, we write down the optimization problem that we have to study and give some of its properties.

Section 3 is devoted to the study of the differentiability of the relevant quantities showing up in the homogenized torsion problem.

Using all the previous results, in Section 4, we present the explicit differentiation of the torsion constant with respect to the class of admissible microstructures and establish the optimality conditions.

The paper ends up with some final remarks concerning the numerical computation of the gradient of the torsion constant and with an appendix where two general and technical results are proved.

1. SOME PRELIMINARY RESULTS

1.1. Modelling of the torsion of a homogenized bar

Let Ω be a bounded, open, connected and lipschitz subset of $\mathbf{R}^2$, and set $Y =]0, 1[^2$.

Let $T \subset Y$ be the closure of a regular, open, connected subset of Y, and $Y^* = Y \backslash T$.

Consider χ the characteristic function of Y^* and, keeping the same notation, extend it, periodically, to all of $\mathbf{R}^2$.

For any small positive parameter ε, $\mathbf{R}^2$ is covered by squares $Y_{\varepsilon k} = \varepsilon Y + \varepsilon k$, where $k \in \mathbf{Z}^2$.

Let $\mathscr{Z}_\varepsilon$ denote the set of all $k \in \mathbf{Z}^2$ such that $Y_{\varepsilon k}$ is included in $\bar{\Omega}$. Define :

$$\chi_\varepsilon(x) = \begin{cases} \chi\left(\frac{x}{\varepsilon}\right), & x \in Y_{\varepsilon k}, \quad k \in \mathscr{Z}_\varepsilon, \\ 1, & x \in \Omega \cap Y_{\varepsilon k}, \; k \notin \mathscr{Z}_\varepsilon. \end{cases}$$

The subset $\Omega_\varepsilon \subset \Omega$, defined by the characteristic function χ_ε, corresponds to a εY-periodic perforation of Ω, all the holes having same size and shape : we say that Ω is periodically perforated.

Classical homogenization results allow us to treat asymptotically, i.e., as ε goes to zero, a wide class of P.D.E. problems, in particular the torsion problem (see [8]).

The case where the size and shape of the holes vary from cell to cell, is called quasi-periodic and has been considered in [13].

In this last case we consider, instead of a unique reference perforated cell Y^*, a familly of perforated cells $\{Y^*(x)\}_{x \in \Omega}$, i.e., the reference hole varies with the zone of the perforation.

We say that the function

$$x \in \bar{\Omega} \mapsto Y^*(x) \subset Y \tag{1.1}$$

is the *microstructure* of the perforation.

Since we are interested in the optimization of the *microstructure,* we briefly summarize here the homogenization results obtained in [13]. Our present setting is slightly different from the one in [13], but more appropriate for the classical methods of control by domain. This new setting requires some minor changes in the proofs given in [13] in order to get similar results. We give these new proofs here.

Instead of (1.1), we will define a *microstructure* as an element

$$B \in C = \mathscr{C}^1(\bar{\Omega}\,; \mathbf{\Phi}_0\,), \tag{1.2}$$

where $\mathbf{\Phi}_0 \subset W^{1,\infty}(Y\,; \mathbf{R}^2\,)$ is the set of all the bilipschitzian homeomorphims of Y into Y, that coincide with the identity on the boundary ∂Y of Y and such that the image of the fixed lipschitz subdomain Y_0^* of Y is still lipschitz (3). Y_0^* is a given open subset of Y. We suppose that $Y \backslash Y_0^*$ is connected and contained in the interior of Y. $\mathbf{\Phi}_0$ is endowed with the usual norm of $W^{1,\infty}(Y\,; \mathbf{R}^2\,)$.

(3) We note that the image of a lipschitz domain, by a bilipschitzian homeomorphism, is not necessarily a lipschitz domain (*cf.* [5] and [20]).

Then we set,

$$Y^*(x) = B(x)\,(Y_0^*) = \{B(x)\,(z) : z \in Y_0^*\}\,, \tag{1.3}$$

and we let χ be the characteristic function of $\bigcup_{x \in \Omega} [\{x\} \times B(x)\,(Y_0^*)]$, in $\Omega \times Y$, extended by periodicity in the second variable, to all of $\Omega \times \mathbf{R}^2$.

As in the periodic case, we define $\Omega_\varepsilon \subset \Omega$ by the following characteristic function :

$$\chi_\varepsilon(x) = \begin{cases} \chi\left(x, \frac{x}{\varepsilon}\right), & x \in Y_{\varepsilon k}, \quad k \in \mathscr{Z}_\varepsilon, \\ 1, & x \in \Omega \cap Y_{\varepsilon k}, \ k \notin \mathscr{Z}_\varepsilon. \end{cases} \tag{1.4}$$

Let us describe this set Ω_ε. Consider $Y^*_{\varepsilon k, 1} \subset Y$, defined by

$$Y^*_{\varepsilon k, 1} = \frac{1}{\varepsilon}\,(Y_{\varepsilon k} \cap \Omega_\varepsilon) - k\,, \quad k \in \mathscr{Z}_\varepsilon,$$

Notice that it does not coincide, in general, with any $Y^*(x)$, for $x \in Y_{\varepsilon k}$, $k \in \mathscr{Z}_\varepsilon$. In fact, let $x \in Y_{\varepsilon k}$, $k \in \mathscr{Z}_\varepsilon$ and $x = \varepsilon(y + k)$. One has

$$\chi_\varepsilon(x) = \chi\left(x, \frac{x}{\varepsilon}\right) = 1 \quad \text{iff} \quad y \in B(\varepsilon y + \varepsilon k)\,(Y_0^*)\,, \tag{1.5}$$

which means

$$Y^*_{\varepsilon k, 1} = B(x)\,(Y_0^*) = \{y \in Y : \exists z \in Y_0^*, y = B(\varepsilon y + \varepsilon k)\,(z)\}\,.$$

We notice that $Y^*_{\varepsilon k, 1}$ is, in a way, implicitely defined by the above equation. More precisely, we have :

LEMMA 1.1 : *Consider, for ε and for k fixed, such that $k \in Z_\varepsilon$,*

$$\Psi : Y \times Y \to \mathbf{R}^2$$

$$\Psi(y, z) = y - B(\varepsilon y + \varepsilon k)\,(z)\,.$$

For ε small enough, the relation $\Psi(y, z) = 0$ defines, implicitly, a global bilipschitzian homeomorphism $F_{\varepsilon k} : Y \to Y$, satisfying

$$Y^*_{\varepsilon k, 1} = F_{\varepsilon k}(Y_0^*)\,, \quad \text{for some} \quad F_{\varepsilon k} \in \mathbf{\Phi}_0$$

and

$$\forall \delta > 0\ \exists \varepsilon_0 > 0 : \varepsilon < \varepsilon_0 \Rightarrow \|F_{\varepsilon k} - B(\varepsilon k)\|_{W^{1,\infty}(Y, \mathbf{R}^2)} < \delta\,, \quad \forall k \in Z_\varepsilon. \tag{1.7}$$

Proof: For all $y \in Y$ there exists a unique $z \in Y$ satisfying $\Psi(y, z) = 0$, namely

$$z = [B(\varepsilon y + \varepsilon k)]^{-1}(y) .$$

Conversely, let $z \in Y$ be given, and fix $y_0 \in Y$. Define, for $n \geqslant 1$,

$$y_n = B(\varepsilon y_{n-1} + \varepsilon k)(z) .$$

By compactness there exists a subsequence of (y_n), converging in Y:

$$y_{n_k} \to y \quad \text{and then} \quad y = B(\varepsilon y + \varepsilon k)(z) .$$

Suppose that $\tilde{y}$ also satisfies $\Psi(\tilde{y}, z) = 0$, with $y \neq \tilde{y}$. Since the maps $x \mapsto [B(x)]^{-1}(y)$ and $x \mapsto [B(x)]^{-1}(\tilde{y})$ are continuous from $\bar{\Omega}$ into Y (4) and $y \neq \tilde{y}$, one has :

$$\inf_{x \in \Omega} \| [B(x)]^{-1}(y) - [B(x)]^{-1}(\tilde{y}) \| = \delta_0 > 0 , \tag{1.8}$$

and, for ε small enough,

$$\| [B(\varepsilon y + \varepsilon k)]^{-1}(\tilde{y}) - [B(\varepsilon \tilde{y} + \varepsilon k)]^{-1}(\tilde{y}) \| < \delta < \delta_0. \tag{1.9}$$

From (1.8) one obtains that

$$\| [B(\varepsilon y + \varepsilon k)]^{-1}(y) - [B(\varepsilon y + \varepsilon k)]^{-1}(\tilde{y}) \| \geqslant \delta_0, \tag{1.10}$$

and, from (1.9) and (1.10),

$$0 = \| [B(\varepsilon y + \varepsilon k)]^{-1}(y) - [B(\varepsilon \tilde{y} + \varepsilon k)]^{-1}(\tilde{y}) \| \geqslant \delta_0 - \delta > 0 ,$$

which is a contradiction.

Defining, then,

$$y = F_{\varepsilon k}(z) \quad \text{iff} \quad y = B(\varepsilon y + \varepsilon k)(z) ,$$

(4) If F, G are bilipschitzian homeomorphisms of Y onto Y, so are F^{-1}, G^{-1} and, for all $z \in Y$,

$$\|F^{-1}(z) - G^{-1}(z)\| \leqslant (1/\alpha)\|z - F(G^{-1}(z))\| = (1/\alpha)\|G(G^{-1}(z)) - F(G^{-1}(z))\| ,$$

so that $\|F^{-1} - G^{-1}\|_{L^\infty(Y, \mathbf{R}^2)} \leqslant (1/\alpha)\|F - G\|_{L^\infty(Y, \mathbf{R}^2)}$

we prove (1.7). For $\delta > 0$, using the continuity of B, in $\bar{\Omega}$, one has, for ε small enough,

$$\| B(\varepsilon k) - B(\varepsilon y + \varepsilon k) \|_{W^{1,\infty}(Y;\mathbf{R}^2)} < \delta/2 , \quad \forall y \in Y ,$$

which means

$$\sup_{z \in Y} \| B(\varepsilon k)(z) - B(\varepsilon y + \varepsilon k)(z) \| < \delta/2 , \tag{1.11}$$

and

$$\sup_{z, z' \in Y} \frac{\| B(\varepsilon k)(z) - B(\varepsilon y + \varepsilon k)(z) - B(\varepsilon k)(z') + B(\varepsilon y + \varepsilon k)(z') \|}{\| z - z' \|} < \delta/2 , \tag{1.12}$$

for all $y \in Y$.

From (1.11) one obtains $\| B(\varepsilon k)(z) - B(\varepsilon y + \varepsilon k)(z) \| < \delta/2$, $\forall z$, $y \in Y$, and, in particular,

$$\sup_{z \in Y} \| B(\varepsilon k)(z) - F_{\varepsilon k}(z) \| < \delta/2 .$$

Analogously one obtains. From (1.12), that

$$\sup_{z, z' \in Y} \frac{\| B(\varepsilon k)(z) - F_{\varepsilon k}(z) - B(\varepsilon k)(z') + F_{\varepsilon k}(z') \|}{\| z - z' \|} < \delta/2 ,$$

and, consequently,

$$\| B(\varepsilon k) - F_{\varepsilon k} \|_{W^{1,\infty}(Y;\mathbf{R}^2)} < \delta .$$

Since the set of all bilipschitzian homeomorphims of Y is open in $W^{1,\infty}(Y;\mathbf{R}^2)$ (5), one concludes that, for ε small enough, $F_{\varepsilon k}$ is a bilipschitzian homeomorphism. Besides it is clear that it coincides with the identity, on the boundary of Y. The fact that $F_{\varepsilon k} \in \mathbf{\Phi}_0$, will be a consequence of Proposition 1.4. □

Now, in order to apply the homogenization results stated in [13], we need more than the lipschitz property stated in Lemma 1.1. We will prove that, for ε small enough, the familly $\{Y^*_{\varepsilon k}\}_{\varepsilon, k \in Z_\varepsilon}$ is in $Lip(L, r)$, for some L and r in $\mathbf{R}^+$. We recall the definition of $Lip(L, r)$ (see [4] or [13]).

(5) $W^{1,\infty}(Y;\mathbf{R}^2) = \mathscr{C}^{0,1}(Y;\mathbf{R}^2)$: $W^{1,\infty}(Y;\mathbf{R}^2)$ coincides with the set of lipschitz continuous maps defined on Y.

DEFINITION 1.2 : *Let Y be as introduced before. For each $x \in \mathbf{R}^2$ and L, $r \in \mathbf{R}^+$, define*

$$P_{L,r}(x) = \{y \in R^2 : |y_1 - x_1| < r \quad \text{and} \quad |y_2 - x_2| < Lr\}$$

and $Lip(L, r)$ as the set of all the open subsets ω of Y such that : for all $x \in \partial\omega$, there exists a local coordinate system and a function

$$\phi_x :]x_1 - r, x_1 + r[\to \mathbf{R},$$

with has lischitz constant L, satisfying

$$y \in P_{L,r}(x) \cap \omega \quad \text{iff} \quad y \in P_{L,r}(x) \text{ and } y_2 > \phi_x(y_1) . \qquad \square$$

We recall that for each lipschitz subdomain $\omega \subset Y$ there exists L, $r \in \mathbf{R}^+$ such that $\omega \in Lip(L, r)$.

The following auxiliary result holds :

LEMMA 1.3 : *Let L, r, $\delta \in \mathbf{R}^+$. If $\omega \in Lip(L, r)$, and $\Psi \in \mathbf{\Phi}_0$ is such that $\| \Psi - I \|_{W^{1,\infty}} \leq \delta$, then*

$$\Psi^{-1}(\omega) \in Lip(L', r'),$$

where $L' = \dfrac{L + \delta(L+1)}{1 - \delta(L+1)}$ *and* $r' = r\dfrac{L[1 - \delta(L+1)]}{L + 2\delta(L+1)}$, *if* $\delta < \dfrac{1}{L+1}$. $\square$

The proof, due to M. Zerner [20], is presented in the Appendix.

PROPOSITION 1.4 : *There exist three constants in $\mathbf{R}^+$, L, r and ε_0, such that, for $\varepsilon < \varepsilon_0$,*

$$\{Y^*_{\varepsilon k}\}_{\varepsilon,\, k \in Z_\varepsilon} \subset Lip(L, r) . \tag{1.13}$$

Proof: Let $x \in \bar{\Omega}$ and $Y^*(x) = B(x)(Y^*_0) \in Lip(L(x), r(x))$. Since the map $F \mapsto F^{-1}$ is bounded in a neighborhood of $B(x)$, in $W^{1,\infty}(Y ; \mathbf{R}^2)$ (see [15]), let $\bar{\delta}$ and c be two positive constants such that

$$\| F - B(x) \|_{W^{1,\infty}} < \bar{\delta} \Rightarrow \| F^{-1} \|_{W^{1,\infty}} \leq c .$$

Suppose that F is a bilipschitzian homeomorphism, satisfying

$$\| F - B(x) \|_{W^{1,\infty}} < \min\left\{\bar{\delta}, \frac{\delta}{c}\right\}, \quad \delta < \frac{1}{L(x) + 1} . \tag{1.14}$$

Since

$$Y^*(x) = [I + (B(x) - F) F^{-1}] F(Y_0^*),$$

using Lemma 1.3,

$$F(Y_0^*) \in Lip(L'(x,\delta), r'(x,\delta)),$$

where

$$L'(x,\delta) = \frac{L(x) + \delta(L(x)+1)}{1 - \delta(L(x)+1)}$$

and

$$r'(x,\delta) = r(x) \min \left\{ \frac{1}{1+\delta(L'+1)}, \frac{L}{L'+\delta(L'+1)} \right\} = r(x) \frac{L[1-\delta(L+1)]}{L+2\delta(L+1)}.$$

From the compactness of $B(\bar{\Omega})$, we obtain the existence of a finite number n_0 such that

$$\{Y^*(x)\}_{x \in \Omega} = \{B(x)(Y_0^*) : x \in \bar{\Omega}\} \subset \bigcup_{i=1}^{n_0} Lip(L_i, r_i).$$

Defining $L = \max\{L_i\}$ and $r = \min\left\{r_i \frac{L_i}{L}\right\}$, we have, then, $\{Y^*(x)\}_{x \in \bar{\Omega}} \subset Lip(L, r)$.

Finally, since $Y_{\varepsilon k}^*$ satisfies (1.5) and (1.6), for ε small enough, we obtain (1.13). □

In order to ensure that $\mathbf{\Phi}_0$ is adapted to the variational calculus, we prove the following lemma :

LEMMA 1.5 : *Let $\Psi \in \mathbf{\Phi}_0$; then there exists ε_0 such that*

$$\xi \in W_0^{1,\infty}(Y; \mathbf{R}^2), \quad \|\xi\|_{W^{1,\infty}} < \varepsilon < \varepsilon_0 \Rightarrow \Psi + \xi \in \mathbf{\Phi}_0. \tag{1.15}$$

Proof: Since the set of all bilipschitzian homeomorphisms of Y into Y is open in $W^{1,\infty}(Y; \mathbf{R}^2)$, for ε small enough $\Psi + \xi$ also is a bilipschitzian homeomorphism. Using the argument presented in Proposition 1.4 we also conclude that if $\Psi(Y_0^*)$ is lipschitz continuous, the same holds true for $(\Psi + \xi)(Y_0^*)$, for ε small enough.

Let us prove that

$$(\Psi + \xi)(y) \in Y, \quad \forall y \in Y. \tag{1.16}$$

If $y \in \partial Y$ then $\Psi(y) + \xi(y) = \Psi(y) \in Y$. Let $y \in \operatorname{int} Y$, $\delta = d(\Psi(y), \partial Y) > 0$ and $y_0 \in \partial Y$ be such that $\delta = \| \Psi(y) - y_0 \|$. We have

$$\| \xi(y) \| = \| \xi(y) - \xi(y_0) \| \leqslant \varepsilon \| y - y_0 \| \leqslant \frac{\varepsilon}{\alpha} \| \Psi(y) - \Psi(y_0) \| = \frac{\varepsilon}{\alpha} \delta .$$

If $\varepsilon < \alpha$, then $\Psi(y) + \xi(y)$ is in the ball centered at $\Psi(y)$ and radius δ, which is included in Y. □

Remark 1.6 : The previous lemma says that, although $\mathbf{\Phi}_0$ is not an open subset of the Banach space $W^{1,\infty}(Y ; \mathbf{R}^2)$ it is an open subset of the affine subspace $I + W_0^{1,\infty}(Y ; \mathbf{R}^2)$, where I stands for the identity of Y. This fact allows us to apply the general classical setting presented in the sequel. □

We now recall the homogenized torsion equation (see [13]).

Let $H^1_{\#}(Y^*(x))$ stand for the set of the H^1 functions φ, defined on $Y^*(x)$ which are periodic in the sense that :

$$\varphi|_{y_i = 0} = \varphi|_{y_i = 1}, \quad \forall \varphi \in Y, \quad \forall i = 1, 2$$

where, for example, $\varphi|_{y_i = 0}$ denotes the trace of φ on $\partial Y \cap \{ y \in Y ; y_i = 0 \}$.

Let $\{ e_1, e_2 \}$ be the unit euclidean basis of $\mathbf{R}^2$. For each $x \in \bar{\Omega}$ and each $i = 1, 2$, we consider the function τ_i^x, the unique solution of the following elliptic problem :

$$\begin{cases} \tau_i^x \in H^1_{\#}(Y^*(x)), \quad \displaystyle\int_{Y^*(x)} \tau_i^x(y)\, dy = 0, \\ \displaystyle\int_{Y^*(x)} \langle \nabla \tau_i^x, \nabla \varphi \rangle (y)\, dy = - \int_{Y^*(x)} \langle \nabla \varphi, e_i \rangle (y)\, dy, \\ \forall \varphi \in H^1_{\#}(Y^*(x)) \quad \text{such that} \quad \displaystyle\int_{Y^*(x)} \varphi(y)\, dy = 0, \end{cases} \tag{1.17}$$

where $\langle \,.\,,\,.\, \rangle$ denotes the usual inner product in $\mathbf{R}^2$.

Let, for fixed $x \in \bar{\Omega}$,

$$m_{ij}(x) = \delta_{ij} | Y^*(x) | + \int_{Y^*(x)} \langle \nabla \tau_i^x, e_j \rangle (y)\, dy , \tag{1.18}$$

define the (i, j) term of a matrix $M(x)$, which is positive definite (see [13]), representing by $|Y^*(x)|$ the usual Lebesgue measure of $Y^*(x)$. The homogenized torsion is the function ψ defined by the following elliptic problem :

$$\begin{cases} \psi \in H_0^1(\Omega), \\ \int_\Omega \left\langle \frac{M(x)}{\det M(x)} \nabla\psi, \nabla\phi \right\rangle (x)\, dx = 2 \int_\Omega \phi(x)\, dx, \quad \forall \phi \in H_0^1(\Omega). \end{cases} \tag{1.19}$$

It is clear that the coefficients of the equation (1.19) depend on the chosen microstructure B, introduced in (1.2) and (1.3), and so does the solution ψ.

Our aim will be the optimization of a functional depending explicitly on B and also through ψ. One example could be :

$$j(B) = 2 \int_\Omega \psi ,$$

1.2. Setting of the optimization problem

Our aim is the following : let

$$J_0 : \mathbf{C} \times H_0^1(\Omega) \to \mathbf{R} : (B, \Phi) \mapsto J_0(B, \Phi)$$

and

$$J_i : \mathbf{C} \times H_0^1(\Omega) \to \mathbf{R} : (B, \Phi) \mapsto J_i(B, \Phi) , \quad i = 1, \dots, n$$

be $n + 1$ given functionals. For each $i \in \{0, \dots, n\}$, we define :

$$j_i(B) = J_i(B, \psi)$$

where ψ is the solution of (1.19). In this paper, we investigate the problem :

$$\min_{\mathbf{B} \in \mathscr{C}^1(\bar{\Omega}\,;\, \mathbf{\Phi}_0)} j_0(B)$$

under the constraints

$$j_i(B) \leqslant 0 , \quad i = 1, \dots, n .$$

We would like to find a solution B^* which is as good as possible. This means that all the constraints have to be satisfied, and the quantity $j_0(B^*)$ has to be as small as effectively possible.

In particular, in this paper, as very often in this type of problem, we will not study existence and uniqueness of solutions. What we plan to do is to try an arbitrary B_0 and follow a descent algorithm for j_0, under the constraints :

$$j_i(B) \leqslant 0\,, \quad i = 1, \ldots, n\,.$$

In the most usual algorithms, if the functionals j_i, $i = 0, \ldots, n$ are differentiable, the crucial point is the computation of their differential or, more generally, approximations of these differentials, which become gradients in finite dimensional spaces. If this can be done, these gradients together with appropriate approximations of the functionals j_i, $i = 0, \ldots, n$, are given as input parameters to a standard optimization algorithm (whith linear or non-linear constraints).

We see here that the mathematical work which needs to be done is the same for the functional j_0 and for the constraints j_i : we need a computation of their differential. So, in what follows, we give one fucntional j and compute its differential.

We can make one remark : each of the functionals J_i may or may not depend on Φ. If it does, its differentiation has to be done according to optimal control techniques, which are recalled in the following section. If not, then $j_i(B) = J_i(B)$ can be differentiated directly.

Examples

1. **The functional $\mathbf{J_0}$** : in solid mechanics, the usual quantities which are interesting to optimize are related to displacements or stresses. In a rod, both are proportional to $1/\int_\Omega \psi(B)$. So the most interesting problem is :

$$\max_B J_0(\psi(B)) \quad \text{with} \quad J_0(\varphi) = \int_\Omega \varphi\, dx\,.$$

In other problems, for example in solid mechanics the following functionals are classically considered :

$$J(u) = \int_\Omega \|u(x)\|^2\, dx \quad \text{or} \quad J(u) = \int_\Omega \|\sigma(u)(x)\|\, dx\,,$$

where u is the displacement, and σ the Cauchy stress field associated with the structure. It is also interesting to optimize

$$\max_{x \in \Omega} \|u(x)\| \quad \text{or} \quad \max_{x \in \Omega} \|\sigma(u)(x)\|\,.$$

These are not Fréchet-differentiable, but they can be treated by non-smooth optimization techniques, which also require the Fréchet-differentiation of appropriate functions.

Examples of constraints

1. A usual constraint consists in giving an upper bound for the total quantity of material available for the optimal solid body. In the problem stated before, this can be written as :

$$j_1(B) = \int_\Omega |Y^*(x)| \, dx \leqslant \gamma$$

(for a given positive γ) which is equivalent to writting :

$$j_1(B) = \int_\Omega \int_{Y_0^*} B(x)(y) \, dy \, dx \leqslant \gamma .$$

This functional is clearly linear and continuous in $B \in \mathscr{C}^1(\bar{\Omega}\,;\Phi_0)$ (with the norm $\mathscr{C}^1(\bar{\Omega}\,; W^{1,\infty}(Y\,;\mathbf{R}^2))$, so its differentiability is obvious.

2. Another natural constraint is

$$0 < \alpha \leqslant B(x)(y) \leqslant \beta < 1, \quad \forall x \in \Omega, \quad \forall y \in Y_0^*$$

for given positive constants α and β.

This is a max-type function which is not Fréchet-differentiable. The admissible directions can be found using subgradient techniques, which are obtained through Fréchet-differentiation of pointwise functionals.

In what follows, we treat the following question, which constitutes the first step in this type of problems. Let $J : \mathbf{C} \times H_0^1(\Omega) \to \mathbf{R}$ be a given $\mathscr{C}^1$ functional, and let j be defined by :

$$j(B) = J(B, \psi)$$

where ψ is the solution of (1.19). The question is to prove the differentiability of j, and find a way to compute its differential, which can give an effective way of using it in a numerical computation.

1.3. A standard result in optimal control

Let $\mathbf{V}$ be a Hilbert space (state space), Λ be a Banach space and Φ be an open subset of Λ (control space). We are given the functionals :

$$\bullet\ a : \begin{array}{ccc} \Phi \times \mathbf{V} \times \mathbf{V} & \to & \mathbf{R} \\ (\varphi, u, v) & \mapsto & a(\varphi\,; u, v) \end{array}$$

$$\bullet\ L : \begin{array}{ccc} \Phi \times \mathbf{V} & \to & \mathbf{R} \\ (\varphi, v) & \mapsto & L(\varphi\,; v) \end{array}$$

$$\bullet\ J : \begin{array}{ccc} \Phi \times \mathbf{V} & \to & \mathbf{R} \\ (\varphi, v) & \mapsto & J(\varphi\,; v) \end{array}$$

For each φ, $a(\varphi\,;.,.)$ is supposed to be bilinear, continuous, coercive in u and v ; L is supposed to be linear and continuous in v. Both are supposed to be of class $\mathscr{C}^1$ with respect to φ, in the *spaces of continuous bilinear functionals and continuous linear functionals,* respectively. As for J, it is supposed to be of class $\mathscr{C}^1$ with respect to the pair (φ, v).

The problem we are studying is the following. We consider $u^\varphi \in \mathbf{V}$ uniquely defined by the state equation :

$$a(\varphi\,;u^\varphi, v) = L(\varphi\,;v)\,,\quad \forall v \in \mathbf{V}\,,$$

and $j(\varphi) = J(\varphi, u^\varphi)$. We wish to compute :

$$\max_{\varphi \in \Phi} j(\varphi)\,.$$

More precisely, in order to use descent type methods we want to differentiate $j(\varphi)$ with respect to φ. We have the following classical result (see for instance [6]) :

THEOREM 1.7 : *Under the above conditions, the functions $\varphi \mapsto u^\varphi$ and $\varphi \mapsto j(\varphi)$ from Φ into $\mathbf{V}$ and $\mathbf{R}$, respectively, are of class $\mathscr{C}^1$. Moreover, for any $\delta\varphi \in \Lambda$:*

$$\frac{dj}{d\varphi}(\varphi)\,.\,\delta\varphi = \frac{\partial J}{\partial \varphi}(\varphi\,;u^\varphi)\,.\,\delta\varphi - \frac{\partial a}{\partial \varphi}(\varphi\,;u^\varphi, p^\varphi)\,.\,\delta\varphi + \frac{\partial L}{\partial \varphi}(\varphi\,;p^\varphi)\,.\,\delta\varphi\,,$$

where p^φ is the adjoint state variable, which is given as the unique solution of the equation :

$$p^\varphi \in \mathbf{V},\quad a(\varphi\,;w, p^\varphi) = \frac{\partial J}{\partial v}(\varphi\,;u^\varphi)\,.\,w,\quad \forall w \in \mathbf{V}\,. \qquad \square$$

1.4. Summary of some basic techniques in optimal design

The problem we are interested in is close to the problem of the preceding section. It is still an optimal control problem, but the control now is the domain on which the partial differential equation associated to the bilinear form a is posed. So, consider Π some appropriate family of open bounded subsets Z of $\mathbf{R}^n$:

- a Hilbert space $\mathbf{V}(Z)$ *depending on the control variable* Z,

$$\begin{array}{llcl} \bullet\ a^Z: & \mathbf{V}(Z)\times\mathbf{V}(Z) & \to & \mathbf{R} \\ & (u,v) & \mapsto & a^Z(u,v) \\ \bullet\ L^Z: & \mathbf{V}(Z) & \to & \mathbf{R} \\ & (v) & \mapsto & L^Z(v) \\ \bullet\ J^Z: & \mathbf{V}(Z) & \to & \mathbf{R} \\ & (v) & \mapsto & J^Z(v) \end{array}$$

For each Z, a^Z is supposed to be bilinear, continuous, symmetric and coercive on $\mathbf{V}(Z)$, L^Z is linear continuous, and J^Z is $\mathscr{C}^1$. Consequently, the equation :

$$u^Z \in \mathbf{V}(Z),\quad a^Z(u^Z,v) = L^Z(v),\quad \forall v \in \mathbf{V}(Z)\,, \tag{1.20}$$

has exactly one solution. Our problem now is the following :

$$\max_{Z\in\Pi} j(Z)\,, \tag{1.21}$$

where $\quad j(Z) = J^Z(u^Z)$.

As in the previous section, we are interested in the *differential* of j with respect to Z. This needs first to be defined because the variable Z does not belong to a vector space. We work here in the very classical following setting (*cf.* [3], [7], [9], [14], [15], [19] and [21]) :

A regular bounded part $Z_0 \in \mathbf{R}^n$ is given as well as an open part $\mathscr{F}$ of $W^{k,\infty}(Z_0\,;\mathbf{R}^n)$ consisting of homeomorphisms of $\mathbf{R}^n$ with $W^{k,\infty}$ regularity. Let us define :

$$\Pi = \{Z = F(Z_0)\,;\, F\in\mathscr{F}\}\,.$$

The spaces $\mathbf{V}(Z)$ we have in mind are Sobolev spaces. We choose k such that the change of variable

$$z = F(z_0)$$

induces an isomorphism from $\mathbf{V}(Z_0)$ *onto* $\mathbf{V}(Z)$. Then the design variable is now $F \in \mathscr{F} \subset W^{k,\infty}(Z_0\,;\mathbf{R}^n)$. It belongs to a vector space. We are back to the setting of the previous section and therefore in a position to use its results.

Let us add in this section a very classical result which will be used several times in what follows. One often has to deal with the bilinear form :

$$a^Z(u, v) = \int_Z \langle \nabla u(z), \nabla v(z) \rangle \, dz ,$$

where ∇u denotes the gradient of u with respect to z. When one makes the change of variable :

$$z = F(z_0), \quad u(z) = \hat{u}(z_0) ,$$

the gradients of $\hat{u}$ and $\hat{v}$ have to be taken with respect to z_0 instead of z. After some computations one gets :

$$(\nabla_z u) (F(z_0)) = [D(F^{-1}) \circ F(z_0)]^t \, . \, \nabla_{z_0} \hat{u}(z_0) ,$$

so that :

$$a^Z(u, v) = \hat{a}(F ; \hat{u}, \hat{v}) \tag{1.22}$$

$$= \int_{Z_0} \langle [D(F^{-1}) \circ F]^t \, . \, \nabla \hat{u}, [D(F^{-1}) \circ F]^t \, . \, \nabla \hat{v} \rangle (z_0) \, |\det (D\varphi)(z_0)| \, dz_0 ,$$

where we have written ∇ instead ∇_{z_0} for simplicity. One then has :

LEMMA 1.9 : *Let $\mathscr{F}$ be an open subset of the set of bilipschitz homeomorphisms of $Z_0 \subset \mathbf{R}^n$ on their image. Then, the mapping :*

$$\mathscr{G} : \mathscr{F} \to [L^\infty(Z_0)]^{n^2}$$

$$F \mapsto D(F^{-1}) \circ F$$

is $\mathscr{C}^1$ and its differential at the point $F = I$ (where I stands for the identity of $\mathbf{R}^n$) is :

$$\left[\frac{d}{dF} \mathscr{G}(I) \, . \, V \right] (z_0) = - DV(z_0), \quad a.e. \ z_0 . \qquad \square$$

LEMMA 1.10 : *In the same setting as in the previous lemma, let us consider the mapping :*

$$\mathscr{H} : \mathscr{F} \to L^\infty(Z_0)$$

$$F \mapsto \det DF .$$

This map is of class $\mathscr{C}^1$, and its differential at the point $F = I$ is :

$$\left[\frac{d}{dF}\mathscr{H}(I) \,.\, V\right](z_0) = \operatorname{div} V(z_0), \quad a.e.\ z_0 . \qquad \square$$

Remark 1.8 : In what follows (Section 4.1), we will show how (as is usual in shape differentiation), one can get the differential of a functional at a current point F using the differential of another one evaluated only at $F = I$. So, with the previous lemmas, one proves immediately that the bilinear form $\hat{a}$ is continuously Fréchet-differentiable with respect to $F \in W^{1,\infty}(Z_0 ; \mathbf{R}^n)$ and its differential can be obtained by partially differentiating with respect to F under the integral sign. The differentials can be computed with the formulas given in these lemmas. $\square$

2. SETTING OF THE OPTIMAL TORSION PROBLEM

2.1. General setting

Let us now rewrite the homogenized torsion equations with these notations. We get them from Section 1.1. The set $\mathbf{C} = \mathscr{C}^1(\bar{\Omega} ; \mathbf{\Phi}_0)$ is now chosen. For any $F \in \mathbf{\Phi}_0$, we define $\theta_i^F \in H^1_{\#}(F(Y_0^*))$ as the unique solution of the elliptic problem :

$$\begin{cases} \theta_i^F \in H^1_{\#}(F(Y_0^*)), \quad \displaystyle\int_{F(Y_0^*)} \theta_i^F = 0, \\ \displaystyle\int_{F(Y_0^*)} \langle \nabla \theta_i^F, \nabla \varphi \rangle (y)\, dy = -\int_{F(Y_0^*)} \langle \nabla \varphi, e_i \rangle (y)\, dy, \\ \forall \varphi \in H^1_{\#}(F(Y_0^*)) \quad \text{such that} \quad \displaystyle\int_{F(Y_0^*)} \varphi(y)\, dy = 0, \end{cases} \tag{2.1}$$

which is similar to (1.17). For any $B \in \mathbf{C}$ and for any $x \in \bar{\Omega}$, we have $B(x) \in \mathbf{\Phi}_0$. Since we denote $Y^*(x) = B(x)(Y_0^*)$ (see (3)), this equation defines $\theta_i^{B(x)} \in H^1_{\#}(Y^*(x))$.

We emphasize here that the space $H^1_{\#}(Y^*(x)) = H^1_{\#}(B(x)(Y_0^*))$ depends on B. Since we want to study variations when B moves in $\mathbf{C}$, this has to be taken with great care.

Then, in the same way as $m_{ij}(x)$ has been defined in (1.18), we define :

$$a_{ij}(F) = \delta_{ij} \int_{F(Y_0^*)} dy + \int_{F(Y_0^*)} \langle \nabla \theta_i^F, e_j \rangle (y)\, dy ; \quad A = [a_{ij}] \tag{2.2}$$

so that $a_{ij}(B(x))$ is defined for each $x \in \Omega$.

Finally, we define :

$$\begin{cases} \psi(B) \in H_0^1(\Omega), \\ \displaystyle\int_\Omega \left\langle \frac{A(B(x))}{\det A(B(x))} \nabla \psi(B)(x), \nabla \phi(x) \right\rangle dx = 2 \int_\Omega \phi(x)\, dx, \forall \phi \in H_0^1(\Omega). \end{cases} \tag{2.3}$$

where $A(B(x))$ is the 2×2 matrix with coefficients $\{a_{ij}(B(x))\,;\, i, j = 1, 2\}$. Note that because of equation (2.1), we have

$$\int_{B(x)(Y_0^*)} \langle \nabla \theta_i^{B(x)}, e_j \rangle (y)\, dy = - \int_{B(x)(Y_0^*)} \langle \nabla \theta_i^{B(x)}, \nabla \theta_j^{B(x)} \rangle (y)\, dy,$$

so that $A(B(x))$ is a symetric matrix. Consequently, we have defined the function

$$\begin{aligned} \psi : C = \mathscr{C}^1(\bar{\Omega}\,;\,\Phi_0) &\to H_0^1(\Omega) \\ B &\mapsto \psi(B). \end{aligned}$$

Let us now give a function :

$$\begin{aligned} J : C \times H_0^1(\Omega) &\to \mathbf{R} \\ (B, \Phi) &\mapsto J(B, \Phi). \end{aligned}$$

The question we address is the following :

$$\max_{B \in \mathbf{C}} j(B),$$

where

$$j(B) = J(B, \psi(B)). \tag{2.4}$$

This question almost fits in the general framework of optimal control which has been recalled in Sections 1.2 and 1.3. There is one thing we have to deal with before directly using the results of these sections. For $F = B(x)$, the equation (2.1) defines the function $\theta_i^{B(x)}$ which belongs to the space $H^1_{\#}(Y^*(x)) = H^1_{\#}(B(x)(Y_0^*))$ and which has a zero mean value. We will have to differentiate $\theta_i^B(x)$ with respect to B. This function belongs to the space $H^1_{\#}(B(x)(Y_0^*))$, which depends on B. As described in Section 1.3, we can get rid of this dependence by using the change of variable

$$y = B(x)(z).$$

After this change of variable, the function $\theta_i^{B(x)} \circ B(x)$ belongs to the space $H^1_{\#}(Y_0^*)$, which does not depend on B. Nevertheless, this function has no longer zero mean. It is such that :

$$\int_{Y_0^*} [\theta_i^{B(x)} \circ B(x)](z)\, Det(D_z B(x)(z))\, dz = 0,$$

which is a condition depending on B. This condition could be treated as a constraint in the problem. We chose to address this question differently. We remark that this zero mean value condition is only necessary for the uniqueness of the solution θ_i^F. Since the function $\theta_i^{B(x)}$ is used to compute $a_{ij}(F)$ only through *its first derivatives,* we are not concerned with any constant which could be added to one particular solution. This question can be delt with by using the quotient space $H^1_{\#}(Y_0^*)/\mathbf{R}$. We prove in the following subsection that this is a way to proceed which fits in the setting of Sections 1.2 and 1.3.

2.2. The use of quotient spaces

Let F be chosen in $\mathbf{\Phi}_0$. We denote :

$$V(Y_0^*) = H^1_{\#}(Y_0^*)/\mathbf{R}, \quad \mathbf{V}(Y^*) = H^1_{\#}(Y^*)/\mathbf{R},$$

where $Y^* = F(Y_0^*)$.

They are endowed with the following norm :

$$\forall \bar{\psi} \in \mathbf{V}(Y_0^*), \quad \|\bar{\psi}\|_{\mathbf{V}} = \inf_{\psi \in \bar{\psi}} \|\psi\|_{H^1(Y_0^*)},$$

and similarly for $\mathbf{V}(Y^*)$. These are two Hilbert spaces, the first one does not depend on F. In order to be able to use the technique described in Section 1.4, one has to check that the following mapping $\bar{\imath} : \mathbf{V}(Y^*) \to \mathbf{V}(Y_0^*)$ defined by :

$$\forall \bar{\varphi} \in \mathbf{V}(Y^*), \quad \bar{\imath}(\bar{\varphi}) = \{\psi \circ F : \psi \in \bar{\varphi}\},$$

is a topological isomorphism. This requires checking that the image of $\bar{\varphi}$ by $\bar{\imath}$ is exactly one equivalence class in $\mathbf{V}(Y_0^*)$, which can be easily done. Moreover, it is easy to see that it is linear, one to one, and continuous.

In these quotient spaces, the equation (2.1) can be rewritten :

$$\begin{cases} \bar{\theta}_i^F \in \mathbf{V}(Y^*), \\ \int_{Y^*} \langle \nabla \bar{\theta}_i^F, \nabla \bar{\varphi} \rangle (y)\, dy = - \int_{Y^*} \langle \nabla \bar{\varphi}, e_i \rangle (y)\, dy, \forall \bar{\varphi} \in \mathbf{V}(Y^*) . \end{cases} \tag{2.5}$$

Using the Poincaré-Wirtinger inequality, it can be proved that the space $\mathbf{V}(Y_0^*)$ can be endowed with the norm :

$$\| \bar{\varphi} \| = \int_{Y^*} |\nabla \varphi|^2 (y)\, dy, \quad \forall \varphi \in \bar{\varphi} ,$$

which is equivalent to the classical norm. Thus, the left hand-side of the equation (2.5) is associated to a coercive, bilinear form, and it has exactly one solution. Then, using the change of variable $y = F(z)$ in this equation, one gets the following corresponding equation, posed in the space $\mathbf{V}(Y_0^*)$ which does not depend anymore on F :

$$\begin{cases} \bar{\theta}_i(F) \in \mathbf{V}(Y_0^*), \\ \int_{Y_0^*} \langle [DF^{-1} \circ F]^t (z) \cdot \nabla_z \bar{\theta}_i(F)\,(z), [DF^{-1} \circ F]^t (z) \cdot \nabla_z \bar{\hat{\varphi}}(z) \rangle \det DF(z)\, dz \\ = - \int_{Y_0^*} \langle [DF^{-1} \circ F]^t (z) \cdot \nabla_z \bar{\hat{\varphi}}(z), e_i \rangle \det DF(z)\, dz, \forall \bar{\hat{\varphi}} \in \mathbf{V}(Y_0^*) , \end{cases} \tag{2.6}$$

where we have used the notation :

$$\hat{\varphi} = \varphi \circ F, \quad DF^{-1} \circ F(z) \cdot \nabla_z \bar{\hat{\varphi}} = \{ DF^{-1} \circ F(z) \cdot \nabla_z \hat{\varphi} \,;\, \hat{\varphi} \in \bar{\hat{\varphi}} \} .$$

the last set being reduced to one single element. The left-hand side of this new equation is also coercive.

We will now see that the optimization problem we have to deal with, fits in the general setting of optimal control, which has been recalled in Sections 1.2 and 1.3. Several questions arise : existence of a maximum, local or global, regularity, computation. In most of these problems not much is known about a *global optimum.* Even when existence is known, because of lack of concavity, one does not know wether a maximum is global or local. On the other hand, it occurs (and it does here) that the functional which has to be maximized is Fréchet-differentiable and its differential can be numerically computed. Therefore, gradient type algorithms can be used in order to make the objective functional j increase. This is of interest as soon as it makes it increase, even if it does not reach *the global maximum.*

In what follows, using the results of Sections 1.2 and 1.3, we prove that the functional j is Fréchet-differentiable and in Section 4 we are going to show how to compute its differential.

3. DIFFERENTIABILITY OF THE HOMOGENIZED TORSION

We use all the notations defined in Section 2. We want to differentiate the function $j(B)$ introduced in (2.4), with respect to $B \in \mathbf{C} = \mathscr{C}^1(\bar{\Omega}\,;\mathbf{\Phi}_0)$. This function is the result of the following composition :

$$B \mapsto \theta_\iota^B \mapsto \tilde{a}_{ij}(B) \mapsto \psi(B) \mapsto j(B)\,,$$

where we define

$$\tilde{a}_{ij}(B) = a_{ij} \circ B \quad \text{and} \quad \tilde{A}(B) = A \circ B\,, \tag{3.1}$$

so that equation (2.3) can be rewritten in the form :

$$\begin{cases} \psi(B) \in H_0^1(\Omega), \\ \displaystyle\int_\Omega \left\langle \frac{\tilde{A}(B)}{\det \tilde{A}(B)} \nabla\psi(B), \nabla\phi \right\rangle (x)\, dx = 2\int_\Omega \phi(x)\, dx, \quad \forall \phi \in H_0^1(\Omega)\,. \end{cases} \tag{3.2}$$

In this section, using the standard techniques of optimal design, we show that all the mappings of this sequence are $\mathscr{C}^1$ and we compute the differentials. In what follows, differentiable always means Fréchet-differentiable.

Let us denote, as previously, $\hat{\theta}_\iota(B(x)) = \theta_\iota^{B(x)} \circ B(x)$ and $\bar{\hat{\theta}}_\iota(B(x))$ its class of equivalence in the quotient space $\mathbf{V}(Y_0^*)$ which does not depend on $B(x)$ any longer. Thus, as for each x, $B(x) \in \mathbf{\Phi}_0$ strictly speaking, $\bar{\hat{\theta}}_\iota$ is a function from $\mathbf{\Phi}_0 \subset W^{1,\infty}(Y\,;\mathbf{R}^2)$ into $\mathbf{V}(Y_0^*)$, and we can study its regularity.

Let us now rewrite equation (2.2) using the change of variable. The formulas changing a gradient have been given in (1.22) of Section 1.3. We get, for each $F \in \mathbf{\Phi}_0$:

$$\begin{aligned} a_{ij}(F) = \delta_{ij} \int_{Y_0^*} \det DF(z)\, dz \\ + \int_{Y_0^*} \langle [D(F^{-1}) \circ F]^t \,.\, \nabla_z \hat{\theta}_\iota(F)\,(z) \,.\, e_j \rangle \det DF(z)\, dz\,. \end{aligned}$$

Note that $\nabla_z \hat{\theta}_i$ may as well be replaced by $\nabla_z \bar{\hat{\theta}}_i$ because there is only one function in this class of equivalence. We have :

PROPOSITION 3.1 : *The mappings* $\nabla\hat{\theta}_i$: $F \mapsto \nabla\hat{\theta}_i(F)$: $\mathbf{\Phi}_0 \subset W^{1,\infty}(Y;\mathbf{R}^2) \to L^2(Y_0^*)^2$ *are* $\mathscr{C}^1$.

Proof: We use the techniques described in Sections 1.2, 1.3, and 2.2. We recall that $\bar{\hat{\theta}}_i(F) \in \mathbf{V}(Y_0^*)$, and is uniquely defined by (2.6).

If we choose :

$$a(F;\bar{\hat{\theta}},\bar{\hat{\varphi}}) \tag{3.4}$$

$$= \int_{Y_0^*} \langle [D(F^{-1}) \circ F]^t(z) \, . \, \nabla_z \bar{\hat{\theta}}(F)(z),$$

$$[D(F^{-1}) \circ F]^t(z) \, . \, \nabla_z \bar{\hat{\varphi}}(z) \rangle \det DF(z)\, dz\,,$$

$$L(F;\bar{\hat{\varphi}}) = -\int_{Y_0^*} \langle [D(F^{-1}) \circ F]^t(z) \, . \, \nabla_z \bar{\hat{\varphi}}(z), e_i \rangle \det DF(z)\, dz\,, \tag{3.5}$$

for each F, $a(F;.,.)$ is bilinear, continuous, symmetric, coercive on $\mathbf{V}(Y_0^*) \times \mathbf{V}(Y_0^*)$. Its coefficients are $\mathscr{C}^1$ with respect to F with values in L^∞ (see Lemmas 1.9 and 1.10). Therefore, it is straightforward to prove that the hypotheses of Theorem 1.7 are fulfilled, thus the mapping :

$$F \mapsto \bar{\hat{\theta}}_i(F) : \mathbf{\Phi}_0 \to \mathbf{V}(Y_0^*)$$

is $\mathscr{C}^1$, as well as its gradient, with respect to z, in L^2. □

We immediately deduce the following :

PROPOSITION 3.2 : *The mappings* $F \mapsto a_{ij}(F)$: $\mathbf{\Phi}_0 \subset W^{1,\infty}(Y;\mathbf{R}^2) \to \mathbf{R}$ *are* $\mathscr{C}^1$.

Proof: We recall that (3.3), defines $a_{ij}(F)$ after the change of variable $y = F(z)$. The mappings (see Lemma 1.9) :

$$F \mapsto \det(DF) : \mathbf{\Phi}_0 \subset W^{1,\infty}(Y;\mathbf{R}^2) \to L^\infty(Y_0^*)\,,$$

$$F \mapsto D(F^{-1}) \circ F : \Phi_0 \subset W^{1,\infty}(Y;\mathbf{R}^2) \to [L^\infty(Y_0^*)]^{n^2},$$

$$F \mapsto \nabla\hat{\theta}_i(F) : \mathbf{\Phi}_0 \subset W^{1,\infty}(Y;\mathbf{R}^2) \to [L^2(Y_0^*)]^2,$$

are $\mathscr{C}^1$. One gets $a_{ij}(F)$ by composing these mappings with multilinear ones. This gives the result. □

We now prove that $\tilde{a}_y : B \mapsto \tilde{a}_y(B)$ maps $\mathbf{C} \subset \mathscr{C}^1(\bar{\Omega}\,;W^{1,\infty}(Y\,;\mathbf{R}^2\,))$ into $\mathscr{C}(\bar{\Omega})$ and is $\mathscr{C}^1$. We recall that $\tilde{a}_y$ is defined by (3.1) and that $a_y(F)$ is $\mathscr{C}^1$ with respect to F, as we just proved. The continuity of $\tilde{a}_y(B)$, with respect to x, and the differentiability of $\tilde{a}_y$, with respect to B, result from the following abstract proposition, whose proof is presented in the Appendix.

PROPOSITION 3.3 : *Let Λ be a Banach space, $\mathbf{\Phi}$ be an open subset of Λ and let :*

$$G : \mathbf{\Phi} \subset \Lambda \to \mathbf{R} : F \mapsto G(F)$$

be a $\mathscr{C}^1$ mapping. Moreover let $\mathbf{C}$ be a subset of $\mathscr{C}^1(\bar{\Omega}\,;\Lambda)$ such that :

$$\forall B \in \mathbf{C}, \quad \forall x \in \bar{\Omega} : \quad B(x) \in \mathbf{\Phi} .$$

Let $\tilde{G} : B \to \tilde{G}(B)$ be defined by $\tilde{G}(B)\,(x) = (G \circ B)\,(x)$. Then $\tilde{G}$ maps $\mathbf{C} \subset \mathscr{C}^1(\bar{\Omega}\,;\Lambda)$ into $\mathscr{C}(\bar{\Omega})$, $\tilde{G}$ is $\mathscr{C}^1$ with respect to B and, for each $H \in \mathscr{C}^1(\bar{\Omega}\,;\Lambda)$,

$$\left[\frac{d\tilde{G}}{dB}(B)\,.\,H\right](x) = \frac{dG}{dF}(B(x))\,.\,H(x)\,.$$

Proof : See the Appendix. □

From Propositions 3.2 and 3.3, and recalling Remark 1.6, we obtain :

COROLLARY 3.4 : *The mappings $\tilde{a}_y : \mathbf{C} \subset \mathscr{C}^1(\bar{\Omega}\,;W^{1,\infty}(Y\,;\mathbf{R}^2\,)) \to \mathscr{C}(\bar{\Omega})$ are $\mathscr{C}^1$ and, for all (i,j) and $H \in \mathscr{C}^1(\bar{\Omega}\,;W_0^{1,\infty}(Y\,;\mathbf{R}^2\,))$,*

$$\left[\frac{d\tilde{a}_y}{dB}(B)\,.\,H\right](x) = \frac{da_y}{dF}(B(x))\,.\,H(x)\,. \tag{3.6}$$

□

4. EXPLICIT DIFFERENTIATION OF j

We now give some lemmas which will help us to differentiate j :

LEMMA 4.1 : *The mapping $\tilde{\Gamma} : \mathbf{C} \subset \mathscr{C}^1(\bar{\Omega}\,;W^{1,\infty}(Y\,;\mathbf{R}^2\,)) \to [\mathscr{C}(\bar{\Omega})]^4$, defined by*

$$\tilde{\Gamma}(B) = \frac{\tilde{A}(B)}{\det \tilde{A}(B)}\,,$$

is $\mathscr{C}^1$ *and, for* $H \in \mathscr{C}^1(\bar{\Omega}\,;W_0^{1,\infty}(Y\,;\mathbf{R}^2))$:

$$\frac{d}{dB}\left(\frac{\tilde{A}}{\det \tilde{A}}\right)(B)\,.\,H =$$

$$[\det \tilde{A}(B)]^{-1}\left[\frac{d\tilde{A}}{dF}(B)\,.\,H - tr\left(\tilde{A}^{-1}(B)\,\frac{d\tilde{A}}{dF}(B)\,.\,H\right).\,\tilde{A}\right]. \quad (4.1)$$

Proof: As we already remarked in Section 1.1, for each $x \in \bar{\Omega}$ the matrix $A(B(x))$ is positive definite and, consequently, there exists $\alpha \in \mathbf{R}^+$, such that $\det A(B(x)) \geqslant \alpha > 0$.

Again using Propositions 3.2 and 3.3, and recalling Remark 1.6, with, for each (i,j),

$$G(F) = \frac{a_{ij}(F)}{\det A(F)}, \quad A = [a_{ij}],$$

we conclude that $\tilde{G} : C \subset \mathscr{C}^1(\bar{\Omega}\,;\varLambda) \to \mathscr{C}(\bar{\Omega})$, defined by

$$\tilde{G}(B) = \frac{\tilde{a}_{ij}(B)}{\det \tilde{A}(B)}, \quad (4.2)$$

is $\mathscr{C}^1$ and that

$$\left[\frac{d\tilde{G}}{dB}(B)\,.\,H\right](x) = \frac{d}{dF}\left(\frac{a_{ij}}{\det A}\right)(B(x))\,.\,H(x)\,. \quad (4.3)$$

For each (i,j) and $h \in W_0^{1,\infty}(Y\,;\mathbf{R}^2)$, we have,

$$\frac{d}{dF}\left(\frac{a_{ij}}{\det A}\right)(F)\,.\,h$$

$$= [\det A(F)]^{-1}\left[\frac{da_{ij}}{dF}(F)\,.\,h - tr\left(A^{-1}(F)\,\frac{dA}{dF}(F)\,.\,h\right).\,a_{ij}\right], \quad (4.4)$$

noting that, in two dimensions, $\frac{d}{dA}(\det A)\,.\,B = (\det A)\,tr(A^{-1}B)$.

From (4.2)-(4.4), we obtain (4.1) and this concludes the proof. □

PROPOSITION 4.2 : *The mapping j, introduced in (2.4) is* $\mathscr{C}^1$, *and, for any* $H \in \mathscr{C}^1(\bar{\Omega}\,;W_0^{1,\infty}(Y\,;\mathbf{R}^2))$:

$$\frac{dj}{dB}(B)\,.\,H = \frac{\partial J}{\partial B}(B\,;\psi(B))\,.\,H - \frac{\partial a}{\partial B}(B\,;\psi(B),p(B))\,.\,H\,,$$

where, for all $u, v \in H_0^1(\Omega)$:

$$a(B ; u, v) = \int_\Omega \left\langle \frac{\tilde{A}(B)}{\det(\tilde{A}(B))} \cdot \nabla u, \nabla v \right\rangle (x)\, dx ,$$

$$\frac{\partial a}{\partial B}(B ; u, v) \,.\, H = \int_\Omega \left\langle \left[\frac{d}{dB}\left(\frac{\tilde{A}(B)}{\det(\tilde{A}(B))} \right) . H \right] . \nabla u, \nabla v \right\rangle (x)\, dx ,$$

and $p(B) \in H_0^1(\Omega)$ is the unique solution of

$$a(B ; w, p(B)) = \frac{\partial J}{\partial v}(B ; \psi(B)) \,.\, w \quad \forall w \in H_0^1(\Omega) .$$

Proof : This is a direct consequence of the general optimal control theorem (*cf.* Theorem 1.7). The functional $a(\, . \, ; . \, , . \,)$ fits in the general framework, and, using the previous lemma 4.1, we know that $\tilde{\Gamma}$ is $\mathscr{C}^1$, thus :

$$\frac{\partial a}{\partial B}(B ; u, v) \,.\, H = \int_\Omega \left\langle \left[\frac{d\tilde{\Gamma}}{dB}(B) \,.\, H \right] . \nabla u, \nabla v \right\rangle$$

$$\leqslant C(B) \;\|H\|_{\Lambda}\; \|u\|_{\mathbf{V}}\; \|v\|_{\mathbf{V}},$$

(with $\Lambda = \mathscr{C}^1(\bar{\Omega} ; W^{1,\infty}(Y ; \mathbf{R}^2))$ and $(\mathbf{V} = H_0^1(\Omega))$, and :

$$\left| a(B + H ; u, v) - a(B ; u, v) - \frac{\partial a}{\partial B}(B ; u, v) \,. \right|$$

$$\leqslant \epsilon(H) \;\|H\|_{\Lambda}\; \|u\|_{\mathbf{V}}\; \|v\|_{\mathbf{V}} .$$

Thus, the bilinear form a is differentiable with respect to B. Moreover it is clearly continuously differentiable.

Then, for $L(B ; v) = 2 \int_\Omega v(x)\, dx$, we have :

$$\frac{\partial L}{\partial B}(B ; v) \,.\, H = 0 .$$

Then we can conclude that j is $\mathscr{C}^1$. In order to compute its differential, we have to compute the solution of the adjoint equation which is given by :

$$\begin{cases} p(B) \in \mathbf{V}, \\ a(B ; w, p(B)) = \dfrac{\partial J}{\partial v}(B ; \psi(B)) \,.\, w, \quad \forall w \in \mathbf{V} . \end{cases}$$

Again using Theorem 1.7, we get the result. □

In this last section we studied the differentiability of the homogenized coefficients but we gave no explicit formulas. We now need to compute the differential of each a_{ij} with respect to F.

Recalling the definition of $a_{ij}(F)$ (see (2.2) and (2.5)), the situation fits exactly in the framework of Sections 1.2 and 1.3. What we have to do in order to get $\frac{da_{ij}}{dF}(F) \cdot H$ is to rewrite equation (2.2) and relation (2.5) in terms of integrals on the fixed domain Y_0^*, which means writing down the change of variables $y = F(z)$ (see (3.3) and (2.6)), and then using the formulas given in Theorem 1.7 with a_{ij} instead of the function j, a and L being given by (3.4) and (3.5), respectively.

Expression (3.6) allows us to compute $\frac{d\tilde{a}_{ij}}{dB}(B) \cdot H$, expression (4.1) to compute $\frac{d}{dB}\left(\frac{\tilde{A}(B)}{\det \tilde{A}(B)}\right)$ and, with Proposition 4.2, we finally get an expression for $\frac{dj}{dB}(B) \cdot H$. This is of course a heavy sequence of computations. Although this will not become very simple, one simplification can be done. It is classically done in shape optimization : the localisation of the differential.

4.1. Localisation and computation of the differential of $\tilde{a}_{ij}$

We want to see here how Remark 1.8 can be used in order to differentiate with respect to the shape, around the identity only.

We are looking for formulas in order to use them numerically in a gradient type algorithm. In such an algorithm, a sequence $(B_n)_n \subset \mathbf{C} \subset \mathscr{C}^1(\bar{\Omega} ; W^{1,\infty}(Y ; \mathbf{R}^2))$ is generated. When B_n is known, the differential of j is used to choose $H_n \in \mathscr{C}^1(\bar{\Omega} ; W_0^{1,\infty}(Y, \mathbf{R}^2))$ such that :

$$B_{n+1} = B_n + H_n,$$

gives to j an increment as big as possible. The n^{th} shape in $\bar{\Omega} \times Y$ is :

$$\{B_n(x)(y) ; x \in \bar{\Omega}, y \in Y_0^*\} = \bigcup_{x \in \Omega} B_n(x)(Y_0^*),$$

where Y_0^* has been fixed at once.

Let us choose a point $x \in \bar{\Omega}$. Recalling notations (1.3) and (2.2), we set

$$Y_n^*(x) = B_n(x)(Y_0^*) ;$$

$$a_{ij}(B_n(x)) = \delta_{ij} \int_{Y_n^*(x)} dy + \int_{Y_n^*(x)} \langle \nabla \theta_i^{B_n(x)}, e_j \rangle (y)\, dy . \tag{4.5}$$

Let us consider an increment $H(x)$ of $B_n(x)$ such that $B_n(x)+H(x)$ is still in $\Phi_0 \subset W^{1,\infty}(Y;\mathbf{R}^2)$. We define $V(x)$ by :

$$(B_n + H)(x) \circ [B_n(x)]^{-1} = (I+V)(x)$$

$$(\text{so that } V(x) = H(x) \circ [B_n(x)]^{-1}),$$

where I stands for the identity of Y. $V(x)$ is now a $W_0^{1,\infty}(Y;\mathbf{R}^2)$ mapping which is such that $I+V(x)$ is an homeomorphism of Y, and such that :

$$[(I+V)(x)]\,(Y_n^*(x)) = [B_n(x)+H(x)]\,(Y_0^*)\,.$$

Comparing expressions (2.2) and (4.5) we see that, in (4.5), F has been chosen as the homeomorphism which relates $Y_n^*(x)$ to Y_0^*. At the next step, we look for a good $H_n(x)$ giving

$$Y_{n+1}^*(x) = (B_n(x) + H_n(x))\,(Y_0^*) = (I + V_n(x))\,(Y_n^*)\,.$$

The main feature in the choice of F as a change of variable is to go back to a fixed domain when H moves. At this point, there is no reason to go back to Y_0^*. We can just as well go back to $Y_n^*(x)$ only in order to look for $Y_{n+1}^*(x)$ because $Y_n^*(x)$, at step n, is fixed when $H(x)$ moves.

So, let us define, for a given increment $H(x)$ of $B_n(x)$:

$$\alpha_{ij}^n((I+V)(x)) = a_{ij}(B_n(x)+H(x))\,,$$

(with $(V(x) = H(x) \circ [B_n(x)]^{-1})$. More generally, let $F \in \Phi_0$ be fixed and $V \in W_0^{1,\infty}(Y,\mathbf{R}^2)$ be an increment such that $I+V$ is still a homeomorphism. We define, in a neighbourhood of I,

$$\alpha_{ij}^F(I+V) = a_{ij}(F+h), \quad h = V \circ F\,. \tag{4.6}$$

As a straightforward consequence of the definition of differentiability and since $F(Y) = Y$, one can prove that $\alpha_{ij}^F(U)$ is differentiable at the point $U = I$, and :

$$\frac{d\alpha_{ij}^F}{dU}(I)\,.\,V = \frac{da_{ij}}{dF}(F)\,.\,h\,. \tag{4.7}$$

Defining

$$\alpha_{ij}^n((I+V)(x)) = a_{ij}(B_n(x)+H(x)) = \tilde{a}_{ij}(B_n+H)(x)\,, \tag{4.8}$$

we also obtain, from Corollary 3.4,

$$\left[\frac{d\tilde{a}_{ij}}{dB}(B_n) \cdot H\right](x) = \frac{d\alpha^n_{ij}}{dU}(I) \cdot V(x) . \tag{4.9}$$

We now compute $\frac{da_{ij}}{dF}(F) \cdot h = \frac{d\alpha^F_{ij}}{dU}(I) \cdot V$. In addition to the previous notation, we set :

$$\begin{aligned} &\bullet\quad x \text{ is fixed, as well as } F = B(x), \quad h = H(x) \in W_0^{1,\infty}(Y; \mathbf{R}^2), \\ &\bullet\quad Y^*_F = F(Y^*_0), \quad Y^* = (F+h)(Y^*_0) = (I+V)(Y^*_F), \end{aligned} \tag{4.9}$$

We recall that :

$$a_{ij}(F+h) = \alpha^F_{ij}(U) = \delta_{ij} \int_{Y^*} dt + \int_{Y^*} \langle \nabla \theta_i^{F+h}(t), e_j \rangle \, dt , \tag{4.10}$$

where

$$\begin{cases} \theta_i^{F+h} \in H^1_{\#}(Y^*)/\mathbf{R}, \\ \displaystyle\int_{Y^*} \langle \nabla \theta_i^{F+h}, \nabla \varphi \rangle (t) \, dt = - \int_{Y^*} \langle \nabla \varphi(t) \, e_i \rangle \, dt, \quad \forall \varphi \in H^1_{\#}(Y^*)/\mathbf{R} , \end{cases} \tag{4.11}$$

(we have omitted the bar on the elements of $(H^1_{\#}(Y^*)/\mathbf{R})$.

In these equations, let us make the change of variable $t = U(y) = (I+V)(y)$ which carries Y^* back to Y^*_F. We obtain, using the notation $\hat{\varphi} = \varphi \circ U$:

$$a_{ij}(F+h) = \alpha^F_{ij}(U)$$

$$= \delta_{ij} \int_{Y^*_F} \det DU(y) \, dy + \int_{Y^*_F} \langle [D(U^{-1}) \circ U]^t \, \nabla \hat{\theta}_i^{F+h}, e_j \rangle (y) \det DU(y) \, dy , \tag{4.12}$$

where :

$$\begin{cases} \hat{\theta}_i^{F+h} \in H^1_{\#}(Y_F^*)/\mathbf{R}, \\ \displaystyle\int_{Y_F^*} \langle [D(U^{-1}) \circ U]^t \, \nabla \hat{\theta}_i^{F+h}, [D(U^{-1}) \circ U]^t \, \nabla \varphi \rangle \,(y) \det DU(y)\, dy \\ = - \displaystyle\int_{Y_F^*} \langle [D(U^{-1}) \circ U\,]^t \, \nabla \varphi(y), e_i \rangle \det DU(y)\, dy, \quad \forall \hat{\varphi} \in H^1_{\#}(Y_F^*)/\mathbf{R} . \end{cases} \tag{4.13}$$

Let us define :

- $\Phi = \{ U = I + V \,;\, V \in W_0^{1,\infty}(Y \,;\, \mathbf{R}^2) \,;\, U \circ F \in \mathbf{\Phi}_0 \}$,

  $(U \circ F = (I + V) \circ F = F + h)$,

- $\mathbf{V} = H^1_{\#}(Y_F^*)/\mathbf{R}$,

- $A_{ij}^F : \Phi \times \mathbf{V} \to \mathbf{R} : (U \,;\, \hat{\theta}) \mapsto A_{ij}^F(U \,;\, \hat{\theta})$,

$$A_{ij}^F(U \,;\, \hat{\theta}) = \delta_{ij} \int_{Y_F^*} \det DU(y)\, dy + \int_{Y_F^*} \langle [D(U^{-1}) \circ U]^t \,.\, \nabla \hat{\theta}, e_j \rangle \,(y) \det DU(y)\, dy , \tag{4.14}$$

- $b : \Phi \times \mathbf{V} \times \mathbf{V} \to \mathbf{R} : (U, \hat{\theta}, \hat{\varphi}) \mapsto b(U \,;\, \hat{\theta}, \hat{\varphi})$,

$$b(U \,;\, \hat{\theta}, \hat{\varphi}) = \int_{Y_F^*} \langle [D(U^{-1}) \circ U]^t \,.\, \nabla \hat{\theta}, [D(U^{-1}) \circ U]^t \,.\, \nabla \hat{\varphi} \rangle \,(y) \det DU(y)\, dy , \tag{4.15}$$

- $\lambda_i : \Phi \times V \to R : (U, \hat{\varphi}) \mapsto \lambda_i(U \,;\, \hat{\varphi})$,

$$\lambda_i(U \,;\, \hat{\varphi}) = - \int_{Y_F^*} \langle [D(U^{-1}) \circ U]^t \,.\, \nabla \hat{\varphi}(y), e_i \rangle \det DU(y)\, dy . \tag{4.16}$$

We are precisely in the setting of Sections 1.2 and 1.3, so, by Theorem 1.7 :

$$\frac{d\alpha_{ij}^F}{dU}(I) \cdot V = \frac{\partial A_{ij}^F}{\partial U}(I ; \hat{\theta}_i^F) \cdot V - \frac{\partial b}{\partial U}(I ; \hat{\theta}_i^F, \hat{p}_i^F) \cdot V + \frac{\partial \lambda_i}{\partial U}(I ; \hat{p}_i^F) \cdot V, \quad (4.17)$$

$$\begin{cases} \hat{\theta}_i^F \in \mathbf{V}, \\ b(I ; \hat{\theta}_i^F, \hat{\varphi}) = \lambda_i(I ; \hat{\varphi}), \quad \forall \hat{\varphi} \in \mathbf{V}, \end{cases} \quad (4.18)$$

$$\begin{cases} \hat{p}_i^F \in \mathbf{V}, \\ b(I ; \hat{p}_i^F, \hat{\varphi}) = \dfrac{\partial A_{ij}^F}{\partial \hat{\theta}}(I ; \hat{\theta}_i^F) \cdot \hat{\varphi}, \quad \forall \hat{\varphi} \in \mathbf{V}. \end{cases} \quad (4.19)$$

THEOREM 4.3 : *Using the previous notations, one has :*

$$\frac{da_{ij}}{dF}(F) \cdot H = \frac{d\alpha_{ij}^F}{dU}(I) \cdot V$$

$$= \delta_{ij} \int_{Y_F^*} \operatorname{div} V(y)\, dy + \int_{Y_F^*} [\langle \nabla \hat{\theta}_i^F(y), \nabla \hat{\theta}_j^F(y) \rangle] \operatorname{div} V(y)\, dy$$

$$- \int_{Y_F^*} [\langle \nabla \hat{\theta}_i^F(y), DV(y) \cdot e_j \rangle + \langle \nabla \hat{\theta}_j^F(y), DV(y) \cdot e_i \rangle]\, dy$$

$$+ \int_{Y_F^*} [\langle \nabla \hat{\theta}_i^F(y), e_j \rangle + \langle \nabla \hat{\theta}_j^F(y), e_i \rangle] \operatorname{div} V(y)\, dy$$

$$- \int_{Y_F^*} \langle (DV + (DV)^t) \cdot \nabla \hat{\theta}_i^F, \nabla \hat{\theta}_j^F \rangle (y)\, dy.$$

where $Y_F^* = F(Y_0^*)$, $V = H \circ F^{-1}$.

Proof: From (4.14) and (4.16) we have :

$$A_{ij}^F(U ; \hat{\theta}) = \delta_{ij} \int_{Y_F^*} \det DU\, dy - \lambda_j(U, \hat{\theta}),$$

$$\frac{\partial A_{ij}^F}{\partial \hat{\theta}}(I ; \hat{\theta}) \cdot \hat{\varphi} = -\lambda_j(I, \hat{\varphi}),$$

so that :

$$\hat{p}_i^F = -\hat{\theta}_j^F. \quad (4.20)$$

Differentiating (4.14), (4.15) and (4.16), with the help of Lemmas 1.9 and 1.10, one obtains :

$$\frac{\partial A_{ij}^F}{\partial U}(I\,;\hat{\theta})\,.\,V=\delta_{ij}\int_{Y_F^*}\operatorname{div}V(y)\,dy-\frac{\partial \lambda_j}{\partial U}(I\,;\hat{\theta})\,.\,V\,,$$

$$\frac{\partial b}{\partial U}(I\,;\hat{\theta},\hat{\varphi})\,.\,V=-\int_{Y_F^*}\langle(DV+(DV)^t)\,.\,\nabla\hat{\theta},\nabla\hat{\varphi}\rangle(y)\,dy$$

$$+\int_{Y_F^*}\langle\nabla\hat{\theta},\nabla\hat{\varphi}\rangle\operatorname{div}V(y)\,dy\,,$$

$$\frac{\partial \lambda_j}{\partial U}(I\,;\hat{\theta})\,.\,V=\int_{Y_F^*}\langle\nabla\hat{\theta},DV(y)\,.\,e_j\rangle-\int_{Y_F^*}\langle\nabla\hat{\theta},e_j\rangle\operatorname{div}V(y)\,dy\,.$$

From (4.17) and (4.20) we obtain the result.

4.2. Calculation of the differential of the functional j

We are concerned here with the computation of $\frac{dj}{dB}(B)\,.\,H$ where $H\in\mathscr{C}^1(\bar{\Omega}\,;W_0^{1,\infty}(Y\,;\mathbf{R}^2))$. We want to get a formula, or a sequence of formulas, for this differential, which allow us to obtain its numerical approximation. In this section we give such a sequence of formulas.

As a matter of fact, all the formulas have already been previously computed. All that is left to us is to put them together in a proper way.

First we remind that the functions $\tilde{a}_{ij}(B)(x)$ are continuous with respect to $x\in\bar{\Omega}$ (Proposition 3.2), and the mappings $B\mapsto\tilde{a}_{ij}(B)$ are $\mathscr{C}^1$ (Corollary 3.4). So we can use Proposition 4.2 which gives the differential of j, in terms of the differential of $\tilde{\Gamma}$, which is in turn given in Lemma 4.1, in terms of $\frac{d\tilde{a}_{ij}}{dB}(B)\,.\,H$. Then, in Corollary 3.1, this last differential is given for each $x\in\bar{\Omega}$, in terms of $\frac{da_{ij}}{dF}(F)\,.\,h$, for $F=B(x)$, $h=H(x)$. Finally, $\frac{da_{ij}}{dF}(F)\,.\,h$ is given in Theorem 4.3.

Let us write down this sequence of derivatives. Let there be given

$$Y_0^*\subset Y\,;\quad B\in\mathscr{C}^1(\bar{\Omega}\,;\boldsymbol{\Phi}_0)\,;\quad H\in\mathscr{C}^1(\bar{\Omega}\,;W_0^{1,\infty}(Y\,;\mathbf{R}^2))\,.$$

THEOREM 4.4 : *The following list gives the differential of the functional j :*
- *For F in* Φ_0 *and* $h \in W_0^{1,\infty}(Y ; \mathbf{R}^2)$,

$$\frac{da_{ij}}{dF}(F) . h$$

$$= \delta_{ij} \int_{Y_F^*} \operatorname{div} V(y)\, dy + \int_{Y_F^*} [\langle \nabla\theta_i^F(y), \nabla\theta_j^F(y)\rangle] \operatorname{div} V(y)\, dy$$

$$- \int_{Y_F^*} [\langle \nabla\theta_i^F(y), DV(y) . e_j\rangle + \langle \nabla\theta_j^F(y), DV(y) . e_i\rangle]\, dy$$

$$+ \int_{Y_F^*} [\langle \nabla\theta_j^F(y), e_j\rangle + \langle \nabla\theta_j^F(y), e_j\rangle] \operatorname{div} V(y)\, dy$$

$$- \int_{Y_F^*} \langle (DV + (DV)^t) . \nabla\theta_i^F, \nabla\theta_j^F\rangle (y)\, dy .$$

where $Y_F^* = F(Y_0^*)$, $V = h \circ F^{-1}$ *(see Theorem 4.3).*

- $$\forall x \in \bar{\Omega} : \left[\frac{d\tilde{a}_{ij}}{dB}(B) . H\right](x) = \frac{da_{ij}}{dF}(F) . h ,$$

for $F = B(x)$, $h = H(x)$, *(see Corollary 3.4),*

- $$\frac{d\tilde{\Gamma}}{dB}(B) . H$$

$$= \frac{d}{dB}\left(\frac{\tilde{A}}{\det \tilde{A}}\right)(B) . H = [\det \tilde{A}(B)]^{-1}$$

$$\times \left[\frac{d\tilde{A}}{dF}(B) . H - tr\left(\tilde{A}^{-1}(B) \frac{d\tilde{A}}{dB}(B) . H\right) . \tilde{A}\right] .$$

- *For all* Ψ, $\Phi \in H_0^1(\Omega)$,

$$\frac{\partial a}{\partial B}(B ; \Psi, \Phi) . H = \int_\Omega \left\langle \left[\frac{d\tilde{\Gamma}}{dB}(B) . H\right](x)\, \nabla\Psi(x), \nabla\Phi(x)\right\rangle dx ,$$

(see proof of Proposition 3.2).
- *Finally :*

$$\frac{dj}{dB}(B) . H = \frac{\partial J}{\partial B}(B, \psi(B)) . H = -\frac{\partial a}{\partial B}(B ; \psi(B), p(B)) . H ,$$

where :

$$\int_{\Omega} \langle \Gamma(B) \,.\, \psi(B), \Phi \rangle (x)\, dx = 2 \int_{\Omega} \Phi(x)\, dx, \quad \forall \Phi \in H_0^1(\Omega),$$

$$\int_{\Omega} \langle \Gamma(B) \,.\, p(B), \Phi \rangle (x)\, dx = \frac{\partial J}{\partial \Phi}(B ; \psi(B)) \,.\, \Phi, \quad \forall \Phi \in H_0^1(\Omega).$$

4.3. A few remarks about the numerical computation of the gradient of j

The methodology used here is basically the same as the one used in [7], [15], [19] and [21]. The basic principle of descent methods can be seen on the usual formula :

$$j(B+H) = j(B) + \frac{d}{dB} j(B) \,.\, H + o(\|H\|).$$

One chooses an appropriate subspace Λ_N of Λ and tries to optimize j in this subspace. Let $\{s_1, s_2, ..., s_N\}$ be a basis of Λ_N. If B is an element of Λ_N, we want to increment it in $B + H \in \Lambda_N$ such that $j(B+H)$ be as small (locally) as possible. As H belongs to Λ_N, there exist $\lambda_1, \lambda_2, ..., \lambda_N$ such that :

$$H = \sum_{k=1}^{N} \lambda_k s_k,$$

so the discrete unknowns are the coefficients λ_k. We have :

$$\frac{dj}{dB}(B) \,.\, H = j(B) + \sum_{k=1}^{N} \lambda_k \frac{dj}{dB}(B) \,.\, s_k + o(\|H\|),$$

and the best choice (at first order) is :

$$\lambda_k = \frac{dj}{dB}(B) \,.\, s_k, \quad k = 1, 2, ... N.$$

Thus, the question is to compute

$$\frac{dj}{dB}(B) \,.\, H,$$

for $H = s_1, s_2, \ldots, s_k, \ldots, s_N$. We see here that one has to write down a routine computing $\frac{dj}{dB}(B) \,.\, H$ where B and H are input parameters, and to call it for $B := B_n$ given by step n of the algorithm, and $H = s_1, s_2, \ldots, s_N$.

Such a routine can be written following the sequence of formulas given in Theorem 4.4. Let us mention where the approximations have to be done :

— the functions θ_i (which give the coefficients a_{ij}) are solutions to a variational equation which will certainly have to be approximated (using finite elements or any other appropriate process). This gives approximations θ_i, h, associated to a discretization of Y ;

— the function $\Psi(B)$ is also computed through an approximation process corresponding to a discretization of Ω. Let $\Psi_{h'}(B)$ be this approximation ;

— $\frac{da_{ij}}{dF}(F) \,.\, h$ is given by an integral :

$$\frac{da_{ij}}{dF}(F) \,.\, h = \int_{F(Y_0^*)} \mathcal{F}(\theta_i, F, h)(y)\, dy\,,$$

where $\mathcal{F}$ is a function coming from Theorem 4.4. This will certainly be computed by a quadrature formula of the form :

$$\frac{da_{ij}}{dF}(F) \,.\, h \simeq \sum_l \omega_l \mathcal{F}(\theta_i, F, h)(y_l)\,.$$

So, a routine needs to be written which has θ, F, h, y as input parameters, (F and h are FORTRAN type functions each), and which computes $\mathcal{F}(\theta, F, h)(y)$;

— $\frac{\partial a}{\partial B}(B\,; \Psi, \Phi) \,.\, H$ is also given by an integral which is likely to be approximated by :

$$\frac{\partial a}{\partial B}(B\,; \Psi \,.\, \Phi) \,.\, H = \int_\Omega \mathcal{G}(B, \Psi, \Phi, H)(x)\, dx$$

$$\simeq \sum_r \sigma_r \mathcal{G}(B, \Psi, \Phi, H)(x_r)\,.$$

Thus, a routine needs to be written which has B, Ψ, Φ, H (functions) and x (real) as input parameters, in order to compute $\mathcal{G}(B, \Psi, \Phi, H)(x)$.

Therefore, the formulas given in Theorem 4.4 can be rewritten in an appropriate software language, each one eventually calling the previous ones for appropriate values of the input parameters. Each integral is replaced by its approximation. Each time θ_i or Ψ are called, they are replaced by their approximation coming for instance from a finite element procedure.

This is a very brief description of the structure of a software which could be written in order to compute $\frac{dj}{dB}(B) \cdot H$. Of course it needs to be more detailed in order to be completely implemented. An error analysis would also be necessary.

5. CONCLUSION

The problem of the optimization of the cross section of a rod, in the framework of Saint Venant's torsion theory, has been considered by several authors both from the analytical and the numerical points of view.

In recent works, the use of homogenization theory allowed for a unification of both the shape and the topology optimization methods. However, this has been done essentially on a numerical basis, which makes it hard to study analytically and to generalize to other situations.

In this work, we defined the class of admissible perforations to be used with the homogenization technique and showed how to compute the differential of both the homogenized coefficients and the cost functional, with respect to the class of admissible perforations.

Moreover, the theoretical setting used makes it possible to generalize this type of analysis to other optimization problems governed by a state equation of elliptic type, including the linearized elasticity case.

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APPENDIX

Proof of Lemma 1.3

We first prove the Lemma for $\omega = \{x = (x_1, x_2) \in \mathbf{R}^2 : x_2 > \varphi(x_1)\}$, where $\varphi : \mathbf{R}^2 \to \mathbf{R}$, satisfies

$$|\varphi(x_1) - \varphi(x_1')| \leqslant L|x_1 - x_1'|, \quad \forall x_1, x_2 \in \mathbf{R}\,. \tag{1}$$

We consider, in $\mathbf{R}^2$, the norm $\|x\| = |x_1| + |x_2|$. Let $\Psi = I + \theta$, where $\theta = (\theta_1, \theta_2)$ satisfies $\|\theta\|_{1,\infty} \leqslant \delta$. We prove that

$$\Psi^{-1}(\omega) = \{x \in R^2 : x_2 + \theta_2(x) > \varphi(x_1 + \theta_1(x))\}$$

$$= \{x \in R^2 : x_2 > h(x_1)\}\,,$$

for some $h : \mathbf{R}^2 \to \mathbf{R}$, such that :

$$|h(x_1) - h(x_1')| \leqslant \frac{L + \delta(L+1)}{1 - \delta(L+1)} |x_1 - x_1'|, \quad \forall x_1, x_2 \in \mathbf{R}. \tag{2}$$

For each fixed x_1, $f(x_1, x_2) = x_2 + \theta_2(x) - \varphi(x_1 + \theta_1(x))$ is strictly increasing in x_2, if $\delta(L+1) < 1$. Since $f(x_1, .)$ changes sign at infinity, we conclude that it has only one zero. Define, then,

$$h(x_1) = x_2, \quad \text{if} \quad f(x_1, x_2) = 0 .$$

One has :

$$x_2 > h(x_1), \quad \text{iff} \quad f(x_1, x_2) > 0 ,$$

and (2) is satisfied. In fact, from

$$h(x_1) - h(x_1') + \theta_2(x) - \theta_2(x') - \varphi(x_1 + \theta_1(x)) + \varphi(x_1' + \theta_1(x')) = 0 ,$$

we deduce that

$$[1 - \delta(L+1)] \, |h(x_1) - h(x_1')| \leqslant [L + \delta(L+1)] \, |x_1 - x_1'| ,$$

which means that h is Lipschitz continuous, with constant L'.

Consider, now,

$$\omega = \{x \in R^2 : |x_1 - a| < r, |x_2 - b| < Lr, x_2 > \varphi(x_1), b = \varphi(a)\} ,$$

where φ and Ψ satisfy the previous conditions. Set $a = \xi_1 + \theta_1(\xi)$, $b = \xi_2 + \theta_2(\xi)$, $\xi_2 + \theta_2(\xi) = \varphi(\xi_1 + \theta_1(\xi))$, for some $\xi = (\xi_1, \xi_2)$, in $\mathbf{R}^2$. Then, in view of the definition of r',

$$\Psi^{-1}(\omega) = \{x \in \mathbf{R}^2 : |(x_1 - \xi_1) + (\theta_1(x) - \theta_1(\xi))| < r ,$$

$$|(x_2 - \xi_2) + (\theta_2(x) - \theta_2(\xi))| < Lr, x_2 + \theta_2(x) > \varphi(x_1 + \theta_1(x))\}$$

satisfies the inclusion

$$\{x \in \mathbf{R}^2 : |x_1 - \xi_1| < r', |x_2 - \xi_2| < L'r', x_2 > h(x_1)\} \subset \Psi^{-1}(\omega) ,$$

which, in view of Definition 1.2, allows us to conclude the proof. □

Proof of Proposition 3.3

$\tilde{G}(B)$ is the composition of two continuous functions and thus is continuous with respect to x. Then, it is enough to prove that

$$I = \frac{\sup_{x \in \bar{\Omega}} \left| \tilde{G}(B+H)(x) - \tilde{G}(B)(x) - \frac{dG}{dF}(B(x)) \cdot H(x) \right|}{\sup_{x \in \bar{\Omega}} \| H(x) \|_A} \to 0 ,$$

when $\sup_{x \in \bar{\Omega}} \| H(x) \|_A \to 0$. We know that for each $B \in \mathbf{C}$ and for each $H \in \mathscr{C}^1(\bar{\Omega}\,; A)$ such that $B + H$ also belongs to $\mathbf{C}$, $\tilde{G}(B)(x)$ and $\tilde{G}(B+H)(x)$ are continuous with respect to x. Moreover, as G is $\mathscr{C}^1$, $\frac{dG}{dF}(F)$ is a continuous linear operator from A into $\mathbf{R}$, continuous with respect to F. As B belongs to $\mathscr{C}^1(\bar{\Omega}\,; A)$, we know that $\frac{dG}{dF}(B(x)) \cdot H(x)$ is continuous with respect to x. So there exists $x_0 \in \bar{\Omega}$ such that :

$$\sup_{x \in \bar{\Omega}} \left| \tilde{G}(B+H)(x) - \tilde{G}(B)(x) - \frac{dG}{dF}(B(x)) \cdot H(x) \right|$$

$$= \left| \tilde{G}(B+H)(x_0) - \tilde{G}(B)(x_0) - \frac{dG}{dF}(B(x_0)) \cdot H(x_0) \right| .$$

So

$$I \leqslant \frac{\left| \tilde{G}(B+H)(x_0) - \tilde{G}(B)(x_0) - \frac{dG}{dF}(B(x_0)) \cdot H(x_0) \right|}{\| H(x_0) \|_A}$$

$$= \frac{\left| G((B+H)(x_0)) - G(B(x_0)) - \frac{dG}{dF}(B(x_0)) \cdot H(x_0) \right|}{\| H(x_0) \|_A} .$$

Using the differentiability of G at $B(x_0)$ we conclude the proof. □

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