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ON THE STABILITY OF A STELLAR STRUCTURE IN ONE DIMENSION II : The reactive case (*)

by B. DUCOMET [1]

Abstract — *We complete in this paper the study of stability of the interface in a free-boundary problem for a self-gravitating gas in one space dimension, with an external pressure P, and a Fourier coefficient λ, for the thermal flux, including a chemical, self consistent, reacting process*

In the non-radiative limit, we find different possible asymptotic behaviours if $\lambda > 0$, the gas tends to collapse, if $\lambda = 0$, we show that, when $P > 0$, the solution converges, for large time to the isothermal solution of the corresponding stationary problem, while for $P = 0$, under some additonal condition connecting the total energy and the mass of the structure, the system is unstable, and the gas tends to fill the space

In the limit of the photon gas, we show that anologous asymptotics hold

Résumé — *Nous poursuivons dans cet article l'étude de la stabilité de l'interface dans un problème à frontière libre concernant l'évolution d'un gaz autogravitant radiatif avec pression externe confinante et cinétique chimique du premier ordre pour la production d'énergie*

Dans les deux limites, non radiative et gaz de photons, nous identifions plusieurs états asymptotiques possibles aux grands temps, suivant la valeur des paramètres physiques convergence vers un état stationnaire, ou expansion du gaz dans tout l'espace

Mots Clés Astrophysique, Gravitation, Cinétique

1. INTRODUCTION

We study the evolution of a self-gravitating reacting gas in one dimension, compressible, viscous and heat-conducting, which can be considered as a simplified model for some large-scale structures described in the astrophysical litterature [2] under the name of « pancakes ».

A preliminary version of this model has been described in [13], but we want to allow some, more physical, self-consistent production of energy inside the star, producing some interesting dynamical phenomena.

In order to get a tractable problem, we introduce a simple reacting process with a first order kinetic.

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As in [13], we consider the free-boundary case, where the boundary is allowed to fluctuate : the dynamics of the interface is driven by a stress condition, including an external pressure.

The equations describing the model are those of reacting self-gravitating radiative hydrodynamics [4], [3] (Compressible reactive Navier-Stokes-Poisson system with radiation), we put a stress condition on the boundaries and a flux boundary condition for the temperature, together with a Neumann chemical condition for the mass fraction of reactant.

As classical in hydrodynamics (see [9]), our boundary problem can be transformed into a problem posed in a fixed domain, by considering Lagrange variables ([2]).

If x is the mass variable, $u(x, t)$ the specific volume, $v(x, t)$ the velocity, $\theta(x, t)$ the temperature, and $Z(x, t)$ the fraction of reactant, the system to be solved is :

$$\begin{cases} u_t = v_x \\ v_t + p_x = \nu\left(\dfrac{v_x}{u}\right)_x - G(x - 1/2) \\ C_v\,\theta_t + \theta\,p_\theta\,v_x = \left(\dfrac{\chi}{u}\theta_x\right)_x + \nu\,\dfrac{v_x^2}{u} + qK\phi(\theta)\,Z \\ Z_t + K\phi(\theta)\,Z = \left(\dfrac{d}{u^2}Z_x\right)_x, \end{cases} \tag{1}$$

for $t \geqslant 0$ and $x \in [0, M]$, where M is the mass of the slab. We suppose in the following that $M = 1$, and we use the notation $I = (0, 1)$.

We consider, for each x in I, the initial conditions :

$$(u, v, \theta, Z)(x, 0) = (u_0, v_0, \theta_0, Z_0)(x)\,. \tag{2}$$

We take, for each $t \geqslant 0$, the following dynamical boundary conditions :

$$\begin{cases} \left(-p + \nu\dfrac{v_x}{u}\right)(0, t) = -P \\ \left(-p + \nu\dfrac{v_x}{u}\right)(1, t) = -P\,, \end{cases} \tag{3}$$

where P is a pressure, modelling the external medium ($P = 0$ corresponds to the vacuum).

([2]) In this respect, the free-boundary character of our problem is quite different from what is called commonly a « free-boundary problem » in the litterature.

In this last case (the Stefan problem, or the obstacle problem, for example) the free boundary is an essential complication, the problem cannot be reduced to a fixed-boundary one and it leads generally to a variational inequality [6].

We consider also the following thermal boundary conditions :

$$\begin{cases} \left(\frac{\chi}{u}\theta_x + \lambda\theta\right)(1, t) = 0 \\ \left(\frac{\chi}{u}\theta_x + \lambda\theta\right)(0, t) = 0 , \end{cases} \tag{4}$$

where $\lambda \geqslant 0$ is a flux parameter. Then, we consider also the following chemical boundary conditions :

$$\begin{cases} Z_x(1, t) = 0 \\ Z_x(0, t) = 0 . \end{cases} \tag{5}$$

We suppose also that the data $(u_0, v_0, \theta_0, Z_0)(x)$ have sufficient regularity (see below), and that u_0, Z_0, and θ_0 are positive everywhere.

Moreover, we impose the following symmetry conditions, for $0 \leqslant x \leqslant 1/2$:

$$\begin{cases} (u, u_0, \theta, \theta_0, Z, Z_0)(1/2 + x, t) = (u, u_0, \theta, \theta_0, Z, Z_0)(1/2 - x, t) \\ (v, v_0)(1/2 + x, t) = -(v, v_0)(1/2 - x, t) . \end{cases} \tag{6}$$

We describe now the various terms in (1).

The gravitational $-G(x - 1/2)$ term has been chosen in such a way that $x = 1/2$ is a symmetry center for the slab (see below).

The pressure is given by the Stefan-Boltzmann law $p = Ru\theta + \frac{a}{3}\theta^4$ (R is the perfect gas constant, and a the Stefan constant). The conductivity is defined by $\chi = \alpha + \frac{4\,ac}{3\,\kappa} u\theta^3$, where α is the thermal conductivity, κ is the Rosseland opacity (taken here as a positive constant), and c is the speed of light. The specific heat at constant volume C_v is a positive function determined by thermodynamics (see [12]), and we call ν the viscosity coefficient.

The mass fraction of reactant is Z, and q is the difference in the heat of formation of the reactants. The positive constant d is the coefficient of species diffusion, and K the coefficient of rate of reactant. The rate function $\phi(\theta)$ is determined by the Arrhenius law :

$$\phi(\theta) = \begin{cases} \theta^\alpha e^{-\frac{E}{\theta}}, \theta > \theta_i > 0 , \\ 0, \theta < \theta_i, \end{cases} \tag{7}$$

where θ_i is the ignition temperature. We are going to show that the problem (1)-(6) has a unique global solution, and study its behaviour at large times, under various conditions on the physical parameters. Because of the simplification of the geometry, we hope to describe precisely these asymptotic states of the system for large time, covering the possibilities of the physical stellar evolution [5], [10] : asymptotically stable stationary state, expansion, and gravitational collapse, extending the analysis of [13].

The plan of the paper is the following : in section 2, we check the existence of a global (in time) solution, relying on the analysis of [1], then we study (Section 3) the asymptotic behaviour of the solution, in the pure gaseous limit $(a = 0)$. We end the paper by a brief analysis of the radiative limit $(R = 0)$, and by a number of remarks.

2. GLOBAL EXISTENCE PROBLEM

We denote by $H^l(I)$ the Sobolev space of order l, $\|\cdot\|_l$ its norm, and $\mathscr{B}^{l+\sigma}(I)$ the space of functions which are, together with their derivatives of order l, Holder with exponent σ, $0 < \sigma < 1$, with the norm :

$$|f|_{l+\sigma} = |f|_l + \sup_{x, y \in I, x \neq y} \left\{ \frac{D^l f(x) - D^l f(y)}{|x-y|^\sigma} \right\},$$

where $|f|_l$ is the norm in $\mathscr{B}_l(I)$, the space of bounded functions, with the l first derivatives continuous in I :

$$|f|_l = \sum_{k \leq l} \sup_{x \in I} \left| \frac{\partial^k f(x)}{\partial x^k} \right| .$$

For each $T > 0$, we note $I_T = [0, T] \times I$, and $\mathscr{B}^\sigma(I_T)$ the space of Holder functions $u(x, t)$, with exponent $\frac{\sigma}{2}$ (resp. σ) with respect to the variable t (resp. x).

The corresponding norm is :

$$\|u\|_{\sigma, T} = \|u\|_{0, T} + \sup_{(x,t), (x',t') \in I_T} \left\{ \frac{|u(t,x) - u(t',x')|}{|t-t'|^{\sigma/2} + |x-x'|^\sigma} \right\},$$

where $\|u\|_{0,T} = \sup_{(t,x) \in [0,T] \times I} |u(t,x)|$.

We write also :

$$\mathscr{B}^{1+\sigma}(I_T) = \{u \in \mathscr{B}^\sigma(I_T), u_t \in \mathscr{B}^\sigma(I_T), u_x \in \mathscr{B}^\sigma(I_T)\},$$

with the norm $\|u\|_{1+\sigma, T} = \|u\|_{0,T} + \|u_t\|_{\sigma,T} + \|u_x\|_{\sigma,T}$, and :

$$\mathscr{B}^{2+\sigma}(I_T) = \{u \in \mathscr{B}^\sigma(I_T), u_{xx} \in \mathscr{B}^\sigma(I_T)\},$$

with the norm $\|u\|_{2+\sigma, T} = \|u\|_{0,T} + \|u_t\|_{\sigma,T} + \|u_x\|_{\sigma,T} + \|u_{xx}\|_{\sigma,T}$.

First, we have a local result,in the spirit of [8], under a smoothing hypothesis of [1] for the positive reacting rate function.

2.1. Local existence

We consider a regularized rate function $\phi_\epsilon \in \mathbf{C}^1 (R^+)$, satisfying :

$$\phi_\epsilon(\theta) = \begin{cases} \theta^\alpha e^{-\frac{E}{\theta}}, \theta > \theta_i + \epsilon > 0 \,, \\ 0, \theta < \theta_i - \epsilon \,, \end{cases} \tag{8}$$

Then, we have :

THEOREM 1 : *If the data satisfy :*

$$u_0 \in \mathscr{B}^{1+\sigma}(I), (v_0, \theta_0, Z_0) \in \mathscr{B}^{2+\sigma}(I), \textit{with } 0 < \sigma < 1 \,,$$

if there is a positive constant $C_0 > 0$, such that :

$$\min_{x \in I} u_0(x) \geq \frac{1}{C_0}, \min_{x \in I} \theta_0(x) \geq \frac{1}{C_0}, \max_{x \in I} u_0(x) \leq C_0, \max_{x \in I} \theta_0(x) \leq C_0,$$

and if the rate function ϕ satisfies (8), then for any $C' > C_0$, there exist positive constants T_1 and $C_1 > C_0$, depending on C' such that the problem (1)-(6) has a unique solution on $[0, T_1] \times I$, satisfying :

$$u \in \mathscr{B}^{1+\sigma}(I_{T_1}) \,, \quad (v, \theta, Z) \in \mathscr{B}^{2+\sigma}(I_{T_1}) \,.$$

Moreover :

$$\frac{1}{C'} \leq u \leq C' , \quad \frac{1}{C_1} \leq \theta \leq C' ,$$

and one has the estimates :

$$\| u \|_{1+\sigma, T_1} < C_1, \ \| v \|_{2+\sigma, T_1} < C_1, \ \| \theta \|_{2+\sigma, T_1} < C_1, \textit{ and } \| Z \|_{2+\sigma, T_1} < C_1.$$

Sketch of the proof:

As [8], supposing that the functions (u, v, θ, Z) are known, we consider the associated linear parabolic system :

$$\begin{cases} V_t - \frac{\nu}{u} V_{xx} - \frac{\nu}{u^2} u_x V_x = - p_x - G(x - 1/2) \\ C_v \, \Theta_t - \frac{\chi}{u} \Theta_{xx} + \left[\frac{\chi}{u^2} u_x - \frac{4\,ac}{3\,\kappa} \right] \Theta_x + p_\theta v_x \Theta - qK\phi(\theta) Z = - \nu \frac{v_x^2}{u} \\ \mathbf{Z}_t - \frac{d}{u^2} \mathbf{Z}_{xx} + \frac{2\,d}{u^3} u_x \mathbf{Z}_x + K\phi(\theta) \mathbf{Z} = 0 , \end{cases} \tag{9}$$

together with the integrated form of the mass conservation :

$$U(x, t) = u_0(x) + \int_0^t v_x(\tau, x) \, d\tau , \tag{10}$$

with the notations : $C_v = C_v(u, \theta)$, $p = p(u, \theta)$, $\chi = \chi(u, \theta)$.
The initial conditions are :

$$(U, V, \Theta, \mathbf{Z})(x, 0) = (u_0, v_0, \theta_0, Z_0)(x) \tag{11}$$

and the boundary conditions :

$$\begin{cases} V_x(0, t) = \frac{1}{\nu} P - \frac{1}{\nu} p(0, t) \\ V_x(1, t) = \frac{1}{\nu} P - \frac{1}{\nu} p(1, t) , \\ \left(\frac{\chi}{u} \Theta_x + \lambda \Theta \right)(1, t) = 0 \\ \left(\frac{\chi}{u} \Theta_x - \lambda \Theta \right)(0, t) = 0 \\ \mathbf{Z}_x (1, t) = 0 \\ \mathbf{Z}_x (0, t) = 0 . \end{cases} \tag{12}$$

The system may be written more compactly :

$$\begin{cases} \mathcal{L}(X') X = H(X), \text{ for } x \in I \\ X(0) = X_0, \\ \mathcal{N}(X') X = 0 \text{ for } x \in \partial I \end{cases} \tag{13}$$

where : X' is the vector $X' = (v, \theta, Z)$, X is the vector $X = (V, \Theta, \mathbf{Z})$, with $X_0 = (v_0, \theta_0, Z_0)$, $\mathcal{L}(X')$ is the differential operator in the lhs of (7), $H(X')$ is the vector in the rhs of the same equation, and $\mathcal{N}(X')$ is the boundary differential operator of Robin type, defined by the boundary conditions.

One observes that the initial problem is then obtained by putting $X' = X$ into (13), getting :

$$\begin{cases} \mathscr{L}(X)\, X = H(X), \text{ for } (x,t) \in I \times (0,\tau) \\ X(0) = X_0, \\ \mathscr{N}(X)\, X = 0 \text{ for } (x,t) \in \partial I \times (0,\tau)\,. \end{cases} \tag{14}$$

Let us now consider the initial boundary problem, for τ small enough :

$$\begin{cases} \mathscr{L}(X_0)\, X_1 = H(X_0), \text{ for } (x,t) \in I \times (0,\tau) \\ X_1(0) = X_0, \\ \mathscr{N}(X_0)\, X_1 = 0 \text{ for } (x,t) \in \partial I \times (0,\tau)\,. \end{cases} \tag{15}$$

To transform this problem into an homogeneous fixed point problem, we substract (15) from (14), and we denote by Y the vector $Y = X - X_1$, which satisfies :

$$\begin{cases} \mathscr{L}(X_1)\, Y = \mathscr{F}(X_0, X_1, Y), \text{ for } (x,t) \in I \times (0,\tau) \\ Y(0) = 0\,, \\ \mathscr{N}(X_0)\, X_1 = \mathscr{G}(X_0, X_1, Y) \text{ for } (x,t) \in \partial I \times (0,\tau) \end{cases} \tag{16}$$

where :

$$\mathscr{F} = [\mathscr{L}(X_1) - \mathscr{L}(X_1 + Y)](X_1 + Y) + H(X_1 + Y) - H(X_1) + \\ + [\mathscr{L}(X_0) - \mathscr{L}(X_1)]\, X_1 + H(X_1) - H(X_0)\,,$$

and :

$$\mathscr{G} = [\mathscr{N}(X_1) - \mathscr{N}(X_1 + Y)](X_1 + Y) + [\mathscr{N}(X_0) - \mathscr{N}(X_1)]\, X_1.$$

Then (16) can be inverted :

$$Y = \mathbf{P}(\tau)\, Y\,, \tag{17}$$

where the operator $\mathbf{P}(\tau)$ acts on the space of vectors $W \in \mathscr{B}^{\alpha}(I_\tau)$ in the following way : if we note $\Phi = \mathbf{P}(\tau)\, W \in \mathscr{B}^{2+\alpha}(I_\tau)$, Φ is the solution of the linear parabolic problem :

$$\begin{cases} \mathscr{L}(X_1)\, \Phi = W, \text{ for } (x, t) \in I \times (0, \tau) \\ \Phi(0) = 0\,, \\ \mathscr{N}(X_1)\, \Phi = 0 \text{ for } (x, t) \in \partial I \times (0, \tau)\,. \end{cases} \tag{18}$$

To achieve the proof, it remains to use standard parabolic estimates, to show that $\mathbf{P}(\tau_1)$ is a contraction on a sufficiently small ball of $\mathscr{B}^{2+\alpha}(I_{\tau_1})$, following the lines of [8]. □

To get a global in time result, we must obtain now a number of estimates.

For technical reasons, we restrict ourselves to the non-radiative case in the sequel of the paper $(a = 0)$.

2.2. *A priori* estimates

LEMMA 1 : *The following conservation laws hold, for any* $t \geqslant 0$:

$$\int_0^1 Z(x, t)\, dx + \int_0^t \int_0^1 K\phi(\theta)\, Z(x, s)\, ds\, dx = \int_0^1 Z_0(x)\, dx \tag{19}$$

$$\int_0^1 \left[\frac{1}{2} v^2 + C_v\, \theta + qZ + f(x)\, u\right] dx$$

$$= \int_0^1 \left[\frac{1}{2} v_0^2 + C_v\, \theta_0 + qZ_0 + f(x)\, u_0\right] dx - 2\,\lambda \int_0^t \theta(0, s)\, ds\,, \tag{20}$$

$$\int_0^t \int_0^1 [v^2 + (p - f(x))\, u]\, dx\, ds$$

$$= \nu \int_0^1 u(x, t)\, dx - \int_0^1 u(x, t) \int_0^x v(\xi, t)\, d\xi\, dx$$

$$- \nu \int_0^1 u_0\, dx + \int_0^1 u_0(x) \int_0^x v_0(\xi)\, d\xi\, dx\,, \tag{21}$$

where $f(x) \equiv P + \frac{1}{2} Gx(1 - x)$,

$$\Phi(t) + \int_0^t \Psi(t)\, dt \leqslant C\,, \tag{22}$$

where :

$$\Phi(t) = \int_0^1 \Big[\frac{1}{2} v^2 + R(u - Log\, u - 1) + C_v(\theta - Log\, \theta - 1)\Big]\, dx\,,$$

and :

$$\Psi(t) = \int_0^1 \left(\frac{v_x^2}{u\theta} + \chi\,\frac{\theta_x^2}{u\theta^2}\right) dx\,.$$

Proof : The relation (19) is obtained by integrating the fourth relation (1) on $I \times (0, t)$, using (5).

Multiplying by v the second relation in (1) gives :

$$\Big(\frac{1}{2} v^2 + C_v\,\theta + qZ + G(x - 1/2)\, r\Big)_t = \Big[-(pv)_x + \nu v\,\frac{v_x}{u} + \frac{\chi}{u}\,\theta_x + \frac{qd}{u^2} Z_x\Big]_x,$$

where $r = r(x, t)$ is the Lagrangian position, defined by :

$$\frac{\partial}{\partial t} r(x, t) = v(x, t)\,.$$

From this last relation, one computes easily, using (1) :

$$\int_0^1 G(x - 1/2)\, r(x, t)\, dx = \int_0^1 f(x)\, u(x, t)\, dx\,.$$

Now, integrating on [0,1], and using (2)-(5), we obtain (20).

For (21), we integrate on $[0, x]$ the second relation in (1), using (3) :

$$\partial_t \int_0^x v(\xi, t)\, d\xi = -p + P + \nu\,\frac{v_x}{u} + f(x)\,.$$

Then, multiplying by u and integrating on $(0, 1) \times [0, t]$, we obtain (12) after standard integrations by parts.

To get (22), we have, by (1) :

$$C_v\,\theta_t + \Big(\frac{R\theta}{u}\Big)\, u_t = \nu\,\frac{v_x^2}{u} + \Big(\chi\,\frac{\theta_x}{u}\Big)_x + qK\phi(\theta)\, Z\,.$$

Dividing by θ and integrating on $[0, 1] \times [0, t]$, we have :

$$\int_0^1 (R\, Log\, u + C_v\, Log\, \theta)\, dx = \int_0^t \Psi(\tau)\, d\tau + \int_0^t\int_0^1 qK\,\frac{\phi(\theta)}{\theta}\, Z\, dx\, d\tau + C_1. \quad (23)$$

Now by (20), we have :

$$\int_0^1 \left(C_v \, \theta + Ru + \frac{1}{2} v^2 + qZ \right) dx < C_2, \tag{24}$$

where C_2 depends only on P, G, and the initial data.

So, by subtracting (23) from (24) :

$$\int_0^1 \left(\frac{1}{2} v^2 + R(u - 1 - Log\, u) + C_v(\theta - 1 - Log\, \theta) \right) dx +$$

$$+ \int_0^t \Psi(\tau)\, d\tau \leqslant C_3 + \int_0^t \int_0^1 qK \frac{\theta - 1}{\theta} \phi(\theta)\, Z \, dx \, d\tau .$$

But, using the maximum principle for the parabolic equation satisfied by Z, it is clear that $Z(x, t) \geqslant 0$, for $(x, t) \in [0,1] \times (0, \infty)$, so, by (19), there exists a constant C_4 such that :

$$\int_0^t \int_0^1 qK\phi(\theta)\, Z \, dx \, d\tau \leqslant C_4,$$

and, as $\frac{\theta - 1}{\theta} < 1$ if $\theta > 1$, (22) follows. □

Using the classical methods of [9], we obtain the following bounds :

LEMMA 2 :

$$0 \leqslant Z(t, x) \leqslant 1 ,$$

$$\exists C_1 > 0 : C_1^{-1} \leqslant u(t, x) \leqslant C_1,$$

$$\exists C_2 > 0 : 0 < C_2^{-1} \leqslant \theta(t, x) .$$

The first part is obtained in the same manner as Chen [1], and the second and third part rely on the estimates of Nagazawa, already used in [13], adapted to the reactive case. □

Now, using the above estimates, we can extend the result of [1] :

THEOREM 2 : *If the data satisfy :*

$$u_0 \in \mathscr{B}^{1+\sigma}(I), (v_0, \theta_0, Z_0) \in \mathscr{B}^{2+\sigma}(I), \text{ with } 0 < \sigma < 1 ,$$

and :

$$\| v_0, \theta_0, Z_0 \|_{H^1(I)} \leqslant C_0,$$

and if the rate function ϕ satisfies (8), then there exists a unique solution of the problem (1)-(6) such that, for any $T > 0$, there are positive constants $M_1(C_0)$ and $M_2(C_0, \epsilon, T)$, such that :

$$\begin{cases} \frac{1}{M_1} \leqslant u \leqslant M_1, \frac{1}{M_2} \leqslant \theta \leqslant M_2, |v| \leqslant M_1, 0 \leqslant Z(t,x) \leqslant 1\,, \\ \| v_t, v_x \|^2(t) + \int_0^t \| v_t, v_x \|^2(s,x)\, dx \leqslant M_1, \\ \| u_x, \theta_x, Z_x \|^2(t) + \int_0^t \| u_{xx}, \theta_{xx}, Z_{xx}, u_x, \theta_x, Z_x, u_t, Z_t \|^2(s,x)\, ds \leqslant M_1, \\ \| u \|_{1+\alpha, T}, \| v, \theta, Z \|_{2+\alpha, T} \leqslant M_2, \end{cases} \tag{25}$$

3. ASYMPTOTIC BEHAVIOUR

We assume in this section that the external pressure satisfies : $P \geqslant 0$.

We need first the following classical representation formula (see [13]) for the specific volume :

LEMMA 3 : *The following identity holds :*

$$u(x,t) = \frac{1}{B(x,t)\, Y(x,t)} \left[u_0(x) + \int_0^t \frac{R}{\nu} \theta(x,s)\, B(x,s)\, Y(x,s)\, ds \right], \tag{26}$$

where :

$$B(x,t) = \exp\left(\frac{1}{\nu} \int_0^x (v_0(\xi) - v(\xi,t))\, d\xi \right),$$

and :

$$Y(x,t) = \exp\left(\frac{t}{\nu} f(x) \right).$$

Now we compute the stationary solution :

LEMMA 4 : *Any stationary solution of (1)-(6) exists only if $P \geqslant 0$, and if $\lambda = 0$. It is given by the two parameters family :*

$$\begin{cases} \bar{u}(x) = \frac{R\bar{\theta}}{f(x)}, \\ \bar{v}\,(x) = 0\,, \\ \bar{\theta}(x) = \bar{\theta}\,, \\ \bar{Z}(x) = \bar{Z}\,. \end{cases} \tag{27}$$

Proof: Let us suppose first that $\lambda = 0$. The stationary system under consideration is :

$$\begin{cases} \bar{p}_x = -G(x-1/2) \\ \left(\frac{\chi}{\bar{u}}\bar{\theta}_x\right)_x + qK\phi(\bar{\theta})\,\bar{Z} = 0 \\ \left(\frac{d}{\bar{u}^2}\bar{Z}_x\right)_x - K\phi(\bar{\theta})\,\bar{Z} = 0\,, \end{cases} \tag{28}$$

An elementary resolution of the first equation :

$$\bar{p}_x + G(x-1/2) = 0\,,$$

with $\bar{p}(x) = \frac{R\bar{\theta}}{\bar{u}(x)}$, and $P = \frac{R\bar{\theta}}{\bar{u}(0)}$ gives the first relation (27).

Now we observe that, by integrating the second relation (28), we have the equality :

$$\int_0^1 \phi(\bar{\theta}(x))\,\bar{Z}(x)\,dx = 0\,.$$

From this, as ϕ and $\bar{Z}$ are positive valued functions, we deduce that the limit regime is constrained by : $\phi(\bar{\theta})\,\bar{Z} = 0$. Now putting into (28), we get :

$$\frac{\chi}{\bar{u}}\bar{\theta}_x = C_2,$$

and :

$$\frac{d}{\bar{u}^2}\bar{Z}_x = C_3.$$

Now, by the boundary conditions, we have $C_2 = C_3 = 0$, then, one more integration gives the two last relations (27).

We just remark that if $P = 0$, the corresponding solution (27) is singular at the boundary (where the density is zero), which is the signature of an external vacuum.

If $\lambda > 0$, one sees that the only possible « stationary » solution is $(\bar{u} = 0,\ \bar{v} = 0,\ \bar{\theta} = 0,\ \bar{Z} = 0)$. It is degenerate and corresponds to the gravitational collapse of the slab into a plane, with an infinite specific volume. □

The asymptotic behaviour of our system is described as follows :

THEOREM 3 : *1. If $\lambda = 0$, and if $P > 0$, the solution of the problem (1)-(6) converge toward a stable stationary state $(\bar{u}, 0, \bar{\theta}, \bar{Z})$ in $H^1(0,1) \cap C^0(0,1)$, when t tends to infinity, where $(\bar{u}(x), \bar{\theta}, \bar{Z})$ are uniquely defined by (27), the mass conservation (18), together with the relation :*

$$\bar{\theta} + \frac{q}{R + C_v} \bar{Z} = \frac{1}{R + C_v} \int_0^1 \left(\frac{1}{2} v_0^2 + C_v \theta_0 + q Z_0 + f(x) u_0 \right) dx , \tag{29}$$

and the constraint :

$$\phi(\bar{\theta}) \bar{Z} = 0 . \tag{30}$$

Moreover the rate of convergence is exponential :

$$\| u = \bar{u}, v, \theta - \bar{\theta}, Z - \bar{Z} \|_{1,2} \leqslant \mathscr{C} e^{-\omega t}, \tag{31}$$

where $\mathscr{C}$, ω depend only on R, C_v, χ, q, K, and P.

2. If $\lambda = 0$, $P = 0$, and if the total energy E and the total mass M satisfy the inequality :

$$E \leqslant qM ,$$

the stationary solution (26) is unstable in the following sense, if $R(t)$ is the eulerian thickness of the gaseous slab (i.e. the width of the support of $\{y : \rho(y,t) \neq 0\}$, where ρ is the eulerian density), then :

$$R(t) \geqslant \mathscr{A} \, . \, t , \tag{32}$$

where $\mathscr{A}$ is a positive constant depending only on the data.

3. If $\lambda > 0$, the solution of (1)-(6) tends to the singular limit $(\bar{u}(x) \equiv 0,\ \bar{v} \equiv 0,\ \bar{\theta} \equiv 0,\ \bar{Z} \equiv 0)$, corresponding to the collapse of the slab into a plane.

Proof : 1. The first part is an extension of the method of [13].

First, due to the assumption $P > 0$, one can check that the system (1) is uniformly parabolic up to the boundary, so one can use lemma 2 to get uniform bounds for the specific volume, and the temperature :

$$0 < C_1 \leqslant u(x,t), \theta(x,t) \leqslant C_2 < \infty . \tag{33}$$

Now, using formula (26), (see lemma 3), we decompose $u(t,x)$ as follows :

$$u(t,x)=\frac{u_0(x)}{B(t,x)\,Y(t,x)}+\frac{1}{Y(t,x)}\int_0^t\frac{R}{\nu}\left(\frac{B(x,\tau)}{B(x,t)}-1\right)Y(x,\tau)\,\theta(x,\tau)\,d\tau$$

$$+\frac{1}{Y(t,x)}\int_0^t\frac{R}{\nu}Y(x,\tau)\left[\theta(x,\tau)-\int_0^1\theta(y,\tau)\,dy\right]d\tau$$

$$+\frac{1}{Y(t,x)}\int_0^t\frac{R}{\nu}Y(x,\tau)\int_0^1\theta(y,\tau)\,dy\,d\tau\,. \tag{34}$$

After the bounds (33) and the explicit form of Y, we see that the first termin the rhs of (34) tends to zero, as $t\to\infty$. Using [7], one shows that the second and third term go also to zero.

The fourth contribution may be decomposed as follows, using (20) :

$$\frac{1}{Y(x,t)}\int_0^t\frac{R}{\nu}Y(x,\tau)\int_0^1\theta(y,\tau)\,dy\,d\tau=\sum_{j=1}^4\Omega_j,$$

with :

$$\Omega_1=\frac{1}{Y(x,t)}\int_0^t\frac{R}{\nu}Y(x,\tau)\left[-\frac{1}{2\,C_v}\int_0^1 v^2\,dy\right]d\tau\,,$$

$$\Omega_2=\frac{1}{Y(x,t)}\int_0^t\frac{R}{\nu}Y(x,\tau)\times$$

$$\times\left[+\frac{1}{C_v}\int_0^1\left(\frac{1}{2}v_0^2(y)+C_v\,\theta_0(y)+qZ_0(y)+f(y)\,u_0(y)\right)dy\right]d\tau\,,$$

$$\Omega_3=\frac{1}{Y(x,t)}\int_0^t\frac{R}{\nu}Y(x,\tau)\left[-\frac{q}{C_v}\int_0^1 Z(y,\tau)\,dy\right]d\tau\,,$$

$$\Omega_4=\frac{1}{Y(x,t)}\int_0^t\frac{R}{\nu}Y(x,\tau)\left[-\frac{1}{C_v}\int_0^1 f(y)\,u(y,\tau)\,dy\right]d\tau\,.$$

The first term Ω_1 tends to zero when $t\to+\infty$, as in [7], let us consider the others.

First, by (29), we have :

$$\frac{R+C_v}{C_v}\bar{\theta}+\frac{1}{C_v}\bar{Z}=\frac{1}{C_v}\int_0^1\left(\frac{1}{2}v_0^2+C_v\,\theta_0+qZ_0+f(y)\,u_0\right)dy$$

So, putting the explicit expression of Y into the quantity Ω_2 :

$$\Omega_2(x,t) = \left(\frac{R + C_v}{C_v}\bar{\theta} + \frac{1}{C_v}\bar{Z}\right) \cdot \frac{1}{Y(x,t)} \int_0^t \frac{R}{\nu} Y(x,\tau)\, d\tau ,$$

we find :

$$\Omega_2(x,t) = \left(\frac{R + C_v}{C_v}\bar{\theta} + \frac{q}{C_v}\bar{Z}\right) \cdot \frac{R}{f(x)} \left[1 - \frac{1}{Y(x,t)}\right].$$

So, by (27) :

$$\lim_{t \to \infty} \Omega_2(x,t) = \frac{R + C_v}{C_v}\bar{u}(x) + \frac{q}{C_v \bar{\theta}}\bar{Z}\bar{u}(x) \tag{35}$$

For Ω_3, we use (19) :

$$\Omega_3(x,t) = \frac{-Rq}{\nu C_v} \frac{1}{Y(x,t)} \int_0^t Y(x,\tau) \int_0^1 Z(y,\tau)\, dy\, d\tau .$$

Inserting $\bar{Z}$ into the integral, we have :

$$\Omega_3(x,t) = \frac{-Rq}{\nu C_v}\bar{Z} \frac{1}{Y(x,t)} \int_0^t Y(x,\tau)\, d\tau$$

$$- \frac{Rq}{\nu C_v} \frac{1}{Y(x,t)} \int_0^t Y(x,\tau) \int_0^1 (Z(y,\tau) - \bar{Z})\, dy\, d\tau .$$

The first contribution in the rhs is $-\dfrac{q}{C_v \bar{\theta}}\bar{Z}\bar{u}(x)$.

For the second one, we first observe that (19) gives :

$$\frac{d}{dt}\int_0^1 Z(x,t)\, dx \leqslant - \int_0^\infty \int_0^1 K\phi(\theta)\, Z(x,s)\, ds\, dx \leqslant 0 .$$

So we see that the positive quantity $\int_0^1 Z(x,t)\, dx$ tends to a positive limit denoted by $\bar{Z}$, for large t :

$$\lim_{t \to \infty} \int_0^1 Z(x,t)\, dx = \bar{Z} ,$$

so we have :

$$\int_0^1 |Z(x,t) - \bar{Z}|\, dx \leqslant \epsilon ,$$

for any arbitrarily small ϵ, if t is large enough.

As an elementary computation shows : $\frac{1}{Y(x,t)} \int_0^t Y(x,\tau)\, d\tau \leqslant C$, where the constant C does not depend on t. So we find :

$$\lim_{t\to\infty} \Omega_3(x,t) = - \frac{q}{C_v \bar{\theta}} \bar{Z}\bar{u}(x) . \tag{36}$$

To evaluate the term Ω_4 we use the identity $\partial_t Y(x,t) = \frac{1}{\nu} f(x)\, Y(x,t)$. So if we integrate by parts :

$$\begin{aligned}
\Omega_4(x,t) &= - \frac{1}{Y(x,t)} \int_0^t Y(x,\tau) \left(\int_0^1 f(y)\, u(y,\tau)\, dy \right) d\tau \\
&= - \frac{\nu}{f(x)} \int_0^1 f(y)\, u(y,t)\, dy + \frac{\nu}{f(x)\, Y(x,t)} \int_0^1 f(y)\, u_0(y)\, dy \\
&\quad + \frac{\nu}{Y(x,t)} \int_0^t Y(x,\tau) \left(\int_0^1 f(y)\, v_y(y,\tau)\, dy \right) d\tau .
\end{aligned}$$

The second term in the rhs tends to zero for large t, and using (22), we can show, as in [7], that the last one tends also to zero. So we have finally :

$$\lim_{t\to\infty} \Omega_4(x,t) = \lim_{t\to\infty} \left(- \frac{1}{Y(x,t)} \int_0^t Y(x,\tau) \left(\int_0^1 f(y)\, u(y,\tau)\, dy \right) d\tau \right) . \tag{37}$$

Collecting the limits (34), (35), (36), and (37), we obtain, uniformly in x :

$$\lim_{t\to\infty} \left[u(x,t) + \frac{R}{f(x)\, C_v} \int_0^1 f(y)\, u(y,t)\, dy \right] = \frac{R + C_v}{C_v} \bar{u}(x) . \tag{38}$$

To evaluate the integral, we multiply by $f(x)$ in (38), and we integrate on [0,1] :

$$\lim_{t\to\infty} \left[\int_0^1 f(x)\, u(x,t)\, dx + \frac{R}{C_v} \int_0^1 f(y)\, u(y,t)\, dy \right] = \frac{R + C_v}{C_v} R\bar{\theta} .$$

So we find :

$$\lim_{t \to \infty} \int_0^1 f(x)\, u(x, t)\, dx = R\bar{\theta} .$$

Putting into (38), we get :

$$\lim_{t \to \infty} u(x, t) = \bar{u}(x) . \tag{39}$$

Now, as in [7], one can show that :

$$\lim_{t \to \infty} \int_0^1 v^2(x, t)\, dx = 0 . \tag{40}$$

Then, if we take the limit $t \to \infty$ into (11), using (39) and (40), we obtain :

$$\lim_{t \to \infty} \int_0^1 \theta(x, t)\, dx = \bar{\theta} .$$

The decay rate in (31) is now obtained by standard application of the Gronvall's lemma, as in [7].

2. As u is singular for $x = 0$ and $x = 1$, we come back to the eulerian version of (1), using an argument of [11] :

$$\begin{cases} \rho_t + (\rho v)_y = 0 \\ \rho(v_t + v v_y) = -p_y - \rho \phi_y + \nu v_{yy} \\ (\rho E)_t + (v(\rho E + p))_y = (\nu v v_y)_y + (\chi \theta_y)_y + (q d \rho Z_y)_y \\ (\rho Z)_t + (\rho v Z)_y = -K\phi(\theta)\, \rho Z + (d \rho Z_y)_y . \end{cases} \tag{41}$$

The unknown eulerian quantities are : the density $\rho(y, t)$, the velocity $v(y, t)$ the temperature $\theta(y, t)$, and the mass fraction of reactant $Z(y, t)$. Let us recall that we consider the non-radiative quantities : $p = \rho R \theta$, $C_v = C = \text{Cte}$, $\chi = \alpha$, (the gaseous contribution to the thermal conduction). The corresponding energy is :

$$E = \int \rho\Big(\frac{1}{2} v^2 + C_v\, \theta + qZ + \frac{1}{2} \Pi\Big)\, dx , \tag{42}$$

where Π is the density of gravitational energy, given, in one dimension, by :

$$\Pi(x, t) = G \int |x - y|\, \rho(y, t)\, dy .$$

Let us consider the second order moment :

$$I(t) = \int x^2 \rho \, dx .$$

We compute the derivatives :

$$I'(t) = 2 \int x \rho v \, dx ,$$

and :

$$I''(t) = \int (\rho v^2 + p) \, dx - \int x \rho \Pi_x \, dx .$$

Integrating by parts in the last integral, and using the symmetry of the kernel $|x - y|$, we find :

$$I''(t) = \int \left(\rho v^2 + p + \tfrac{1}{2} \rho \Pi \right) dx . \tag{43}$$

From (42) and (43), we obtain :

$$nI''(t) - E = \int \left(\left(n - \tfrac{1}{2} \right) \rho v^2 + (nR - C_v) \rho \theta + \left(\tfrac{n-1}{2} \right) \rho \Pi - q \rho Z \right) dx , \tag{44}$$

which is positive if n is large enough, provided that the rhs be positive.

Due to the Lagrangian bound $0 \leqslant Z \leqslant 1$, we have :

$$0 \leqslant \int q \rho Z \, dy \leqslant qM ,$$

where M is the mass of the slab.

So, if the inequality $E \leqslant qM$, connecting the total energy and the total mass is satisfied, the rhs of (44) is bounded from below by $-qM$.

Then, if this constraint holds, we have, by integrating twice (44) :

$$I(t) \geqslant I(0) + tI'(0) + \frac{1}{2n} (E - qM) t^2,$$

and for t large enough :

$$I(t) \geqslant At^2,$$

where A is a positive constant.

Now, if $R(t)$ is the thickness of the slab, we have the bound :

$$I(t) = \int x^2 \rho \, dx \leqslant R^2(t) \, . \, M \, .$$

So (32) is proved, and the corresponding stationary solution (with $P = 0$) is unstable.

3. If $\lambda > 0$, we see that, if $\mathscr{E}(t)$ denotes the lhs of (20), as θ has a positive lower bound :

$$\frac{d\mathscr{E}}{dt} < 0 \, ,$$

for t large enough. So, the positive quantity $\mathscr{E}$ is monotone decreasing, and tends to a finite limit $\bar{\mathscr{E}} \geqslant 0$. If the limit were strictly positive, the corresponding state would be a stationary state, which is impossible by lemma 4. So the limit is $\bar{\mathscr{E}} = 0$, corresponding to the collapsing state described above. □

As a conclusion, we briefly analyze the optimality of the bound (32), in the special case $K = 0$, where only a diffusion of species take place, in the absence of chemical reaction.

In fact, a simple argument from [7], is going to show us that, for large t :

$$R(t) \sim C \, . \, t \, . \tag{45}$$

First, we observe that, when $G = 0$, the system (1) decouples into a pure hydrodynamical system for u, v, θ, and a diffusion equation with a time-dependant diffusion coefficient for Z.

The hydrodynamical part admits the following time-dependant solution :

$$\begin{cases} u(x, t) = \bar{u}(1 + t) \, , \\ v(x, t) = \bar{u}(x - 1/2) \, , \\ \theta(x, t) = \bar{\theta} \, . \end{cases} \tag{46}$$

The fraction of reactant is solution of the equation :

$$Z_t = \left(\frac{d}{u^2} Z \right)_{xx} .$$

Taking into account the expression of u in (46), we have to solve, for $x \in [0,1]$ and $t > 0$:

$$\begin{cases} Z_t = \dfrac{d}{\bar{u}^2(1+t)^2} Z_{xx}, \\ Z(x,0) = Z_0(x), \\ Z_x(0,t) = 0, \\ Z_x(1,t) = 0. \end{cases} \tag{47}$$

The solution of (47) can be easily computed :

$$Z(x,t) = \sum_{n=0}^{+\infty} Z_n^0 \, e^{-\frac{n^2 \pi^2 d^2}{\bar{u}^2} \frac{t}{1+t}} \cos(n\pi x), \tag{48}$$

where Z_n^0 is the Fourier coefficient of Z_0, given by :

$$Z_n^0 = \int_0^1 Z_0(x) \cos(n\pi x)\, dx.$$

When $t \to +\infty$, using Lebesgue theorem, we define :

$$\bar{Z}(x) \equiv \lim_{t\to\infty} Z(x,t) = \sum_{n=0}^{+\infty} Z_n^0 \, e^{-\frac{n^2\pi^2 d^2}{\bar{u}^2}} \cos(n\pi x).$$

The thickness of the slab is :

$$R(t) - R(0) = \int_0^t v(1,\tau)\, d\tau = \frac{1}{2}\bar{u} \, . \, t,$$

which is a linear bound of type (45).

Now, we have the following asymptotic result :

THEOREM 4 : *Let $(\bar{u}, \bar{\theta})$ be the positive solution of the system :*

$$\begin{cases} \nu \bar{u} = R\bar{\theta}, \\ \displaystyle\int_0^1 \left(\frac{1}{2} v_0^2 + C_v \theta_0\right) dx = \int_0^1 \frac{1}{2} \bar{u}(x - 1/2)^2 \, dx + C_v \bar{\theta}. \end{cases} \tag{49}$$

Then, there exist constants $C > 1$ and $\mu > 0$ depending on R, ν, C_v, and initial data, such that the solution of (1)-(5) satisfies :

$$\left\| \frac{u(x,t)}{1+t}, v(x,t) - \bar{u}(x - 1/2), \theta(x,t) - \bar{\theta}, Z(x,t) - \bar{Z}(x) \right\|_1^2 \leqslant \frac{C}{(1+t)^\mu}.$$

Proof: The hydrodynamic part is due to Nagasawa [14], the behaviour of Z is an elementary study of the formulae (47) and (48), using the inequality : $\left|e^{-\frac{\lambda^2 t}{1+t}} - e^{-\lambda^2}\right| \leq \frac{\lambda^2}{t} e^{-\frac{\lambda^2 t}{1+t}}$, for any $\lambda > 0$.

Then :

$$\| Z(x, t) - \bar{Z}(x) \|_1^2 \leq \sum_{n=0}^{+\infty} (1 + n^2) |Z_n^0|^2 \frac{n^2 \pi^2 d^2}{t \cdot \bar{u}^2} e^{-\frac{n^2 \pi^2 d^2}{\bar{u}^2} \frac{t}{1+t}},$$

so, for t large enough, if C is a positive constant :

$$\| Z(x, t) - \bar{Z}(x) \|_1 \leq \frac{C}{1+t} \| Z_0 \|_1,$$

where C is a positive constant, depending on $\bar{u}$ and d. □

As the gravitation has a confining effect, we conclude that if the slab expands at a rate linear in t in absence of gravitation, and as this rate cannot be worse in the gravitational case, it actually expands at the same rate if $G \neq 0$, due to the estimate (24).

If $K \neq 0$ (and $G = 0$), the above decoupling does not applied, however we can verify that the linear bound is achieved for a particular solution.

First, we observe that, if θ and Z are only function t, the system (1) can be rewritten :

$$\begin{cases} u_t = v_x \\ v_t = \left(\nu \frac{v_x}{u} - \frac{R\theta}{u} \right) \\ C_v \, \theta_t = \left(\nu \frac{v_x}{u} - \frac{R\theta}{u} \right) v_x + qK\phi(\theta) Z \\ Z_t + K\phi(\theta) Z = 0 \, . \end{cases} \tag{50}$$

It admits the following time-dependant solution :

$$\begin{cases} u(x, t) = \frac{R}{\nu} \int_0^t \bar{\theta}(s) \, ds + \bar{u} \, , \\ v(x, t) = \frac{R}{\nu} \bar{\theta}(t)(x - 1/2) \, , \\ \theta(x, t) = \bar{\theta}(t) \, , \\ Z(x, t) = \frac{C_v}{q} (\tilde{\theta} - \bar{\theta}(t)) + \tilde{Z} \, , \end{cases} \tag{51}$$

where $\bar{\theta}$ is solutionof the non-linear Cauchy problem :

$$\begin{cases} \frac{d}{dt}\bar{\theta} = K\left[\tilde{\theta} - \bar{\theta}(t) + \frac{q}{C_v}\tilde{Z}\right]\phi(\bar{\theta}), \\ \bar{\theta}(0) = \tilde{\theta} \end{cases} \tag{52}$$

where $\tilde{u}$, $\tilde{\theta}$, and $\tilde{Z}$ are positive constants.

The solution of this last problem is given implicitly by the formula :

$$t = \frac{1}{K}\int_{\tilde{\theta}}^{\theta} \frac{d\beta}{\left[\tilde{\theta} + \frac{q}{C_v}\tilde{Z} - \beta\right]\phi(\beta)}. \tag{53}$$

This solution is clearly global due to the behaviour of ϕ : the maximal time of existence $T_{\max}$ is trivially $+\infty$, if $\tilde{\theta} < \theta_i$, because $\phi \equiv 0$ in this domain, and $T_{\max}$ is also $+\infty$, if $\tilde{\theta} > \theta_i$, because the integrand has a non-integrable singularity for $\beta = \tilde{\theta} + \frac{q}{C_v}\tilde{Z}$.

The thickness of the slab is :

$$R(t) - R(0) = \int_0^t v(1, \tau)\, d\tau = \frac{1}{2}\int_0^t \bar{\theta}(\tau)\, d\tau .$$

As $t \to \theta(t)$ is not decreasing, we have :

$$R(t) - R(0) \geqslant \frac{1}{2}\tilde{\theta} \,.\, t$$

which is once more a linear bound of type (44).

4. THE PHOTON GAS ($R = 0$) FOR IMPERMEABLY INSULATED BOUNDARIES ($\lambda = 0$)

Let us consider the radiative limit of the system (1), corresponding to $R = 0$.

In this limit case, due to a simple change of unknown, we are going to check that the preceding result can be used without any significant modification.

Let us denote by $\mathbf{T}$ the quantity : $\mathbf{T} = u\theta^4$, by $\mathbf{p}$ the new pressure : $\mathbf{p} = \mathbf{R}\frac{\mathbf{T}}{u}$, and, with $C_v \equiv 4\,au\theta^3$, we find :

$$C_v\,\theta_t = a(u\theta^4)_t - au_t\,\theta^4 = \mathbf{C}_v\,\mathbf{T}_t - 3\,\mathbf{p}v_x,$$

$$\theta p_\theta = \frac{4\,a}{3}\theta^4 = 4\,\mathbf{p}\,,$$

and :

$$\frac{\chi}{u}\theta_x = \frac{a}{3\kappa}(\theta^4)_x = \mathbf{K}T_x,$$

with : $\mathbf{R} = \frac{a}{3}$, $\mathbf{C}_v = a$, $\mathbf{K} = \frac{ac}{3\kappa}$. The rate function depends now on $\mathbf{T}$ and u :

$$\phi(\mathbf{T}, u) = \begin{cases} \left(\frac{\mathbf{T}}{u}\right)^{\alpha/4} e^{-\frac{E \cdot u^{1/4}}{\mathbf{T}^{1/4}}}, \mathbf{T} - \theta_i^4 u > 0, \\ 0, \mathbf{T} - \theta_i^4 u < 0, \end{cases} \tag{54}$$

So we find that the system (1) may be written :

$$\begin{cases} u_t = v_x \\ v_t + \mathbf{p}_x = \nu\left(\frac{v_x}{u}\right)_x - G(x - 1/2) \\ C_v \mathbf{T}_t + \mathbf{p}v_x = \left(\mathbf{K}\frac{\mathbf{T}}{u}\right)_{xx} + \nu\frac{v_x^2}{u} + qK\phi(\mathbf{T}, u) Z \\ Z_t + K\phi(\mathbf{T}, u) Z = \left(\frac{d}{u^2} Z_x\right)_x. \end{cases} \tag{55}$$

The initial conditions become :

$$(u, v, \mathbf{T}, Z)(x, 0) = (u_0, v_0, \mathbf{T}_0 = u_0\,\theta_0^4, Z_0)(x), \tag{56}$$

the dynamical boundary conditions :

$$\begin{cases} \left(-\mathbf{p} + \nu\frac{v_x}{u}\right)(0, t) = -P \\ \left(-\mathbf{p} + \nu\frac{v_x}{u}\right)(1, t) = -P, \end{cases} \tag{57}$$

and the thermal boundary conditions are simply Neumann conditions :

$$\begin{cases} \mathbf{T}_x(1, t) = 0 \\ \mathbf{T}_x(0, t) = 0. \end{cases} \tag{58}$$

The chemical boundary conditions (5) still hold.

The system (55)-(58), (5) is of the same type as the initial one (1)-(5). Then we find in the same way that above that the problem (55)-(58), (5) has a unique solution, satisfying :

$$(u, v, \mathbf{T}, Z) \in \bigcap_{T>0} [\mathscr{B}^{1+\sigma}(I_T) \times \mathscr{B}^{2+\sigma}(I_T) \times \mathscr{B}^{2+\sigma}(I_T) \times \mathscr{B}^{2+\sigma}(I_T)] ,$$

and we have the same asymptotics :

1. If $\lambda = 0$, and if $P > 0$, the solution of the problem (55)-(58) converges toward a stable stationary state $(\bar{u}, 0, \bar{\mathbf{T}}, Z)$, when t tends to infinity, moreover, the rate of convergence is exponential :

$$\| u - \bar{u}, v, \mathbf{T} - \bar{\mathbf{T}}, Z - \bar{Z} \|_{1,2} \leqslant \mathscr{C} e^{-\omega t}, \tag{59}$$

where ω is a positive number.

2. If $\lambda = 0$, $P = 0$, the stationary solution is unstable : if $R(t)$ is the eulerian thickness of the slab, then :

$$R(t) \geqslant \mathscr{A} \cdot t , \tag{60}$$

where $\mathscr{A}$ is a positive constant depending on the data.

The proof is similar to that of theorem 3.

Using this argument, we conclude that, for any small « gaseous » perturbation of the photon slab, the stationary state corresponding to an external vacuum is unstable. So, it seems physically unlikely that the total absence of gaseous internal pressure could stabilize the expansion of the slab. □

5. FINAL REMARKS

1. As we have considered a regularized version of the rate function ϕ_ϵ, (see formula (8)), our solution $(u_\epsilon, v_\epsilon, \theta_\epsilon, Z_\epsilon)$ depends on ϵ.
   However a limiting argument, as in [1], shows that there exists a strongly converging subsequence such that our results hold for the limit (u, v, θ, Z) and one has :

   $$\| (u_\epsilon - u, v_\epsilon - v, \theta_\epsilon - \theta, Z_\epsilon - Z) \|_{H^1} \to 0 ,$$

   when $\epsilon \to 0$.

2. As in [1], we have two different physical situations for the asymptotics corresponding to different initial data.

   If the initial temperature is sufficiently high :

$$\int_0^1 \left[\frac{1}{2} v^2 + C_v \theta + qZ_0 + f(x) u_0\right] dx > \theta_i + q ,$$

   then $\bar{Z} = 0$, so the reaction process is complete and the structure burns all its available combustible. In contrast, if θ_0 is too low :

$$\int_0^1 \left[\frac{1}{2} v^2 + C_v \theta + qZ_0 + f(x) u_0\right] dx < \theta_i,$$

   then, the asymptotic temperature satisfies : $\bar{\theta} < \theta_i$, and the process stops after a certain time.

3. In this paper, we have made rather strong regularity assumptions on the data. In the pure compressible Navier-Stokes system with finite mass, Hoff [15], [16], [17] and Serre [18], [19], [20] have shown that it is possible to relax these assumptions, to obtain weak solutions, as soon as the specific volume has a finite total variation. At least for a strictly positive external pressure, we can reasonnably expect that their results hold also in the above gravitational situation.

4. Another one-dimensional situation commonly considered in the astrophysical context [21] is the spherical symmetry, leading to interesting stability problems (see [22]), for which some global existence results are known in the viscous barotropic case [23] and also the non-viscous (Euler) isothermal case [24].

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