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A DOMAIN DECOMPOSITION METHOD FOR SOLVING A HELMHOLTZ-LIKE PROBLEM IN ELASTICITY BASED ON THE WILSON NONCONFORMING ELEMENT (*) (**)

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Résumé — Un procédé parallèle itératif fondé sur une technique de décomposition de domaine est proposé et analysé pour une suite des systèmes elliptiques avec des conditions aux limites absorbantes du premier ordre. Cette suite des systèmes, qui ne sont pas coercitives et ont des caractéristiques similaires à celles de l'équation de Helmholtz, décrit le mouvement d'un solide presque élastique dans le domaine de la fréquence. Comme application, le procédé est utilisé pour résoudre une approximation d'éléments finis aux systèmes elliptiques employant l'élément de Wilson non conforme. La convergence du procédé est démontrée et le taux de convergence est dérivé quand le domaine est divisé en sous-domaines dont chaque sous-domaine consiste en un élément individuel associé avec la méthode d'éléments finis de Wilson. L'hybridisation de l'élément fini de Wilson est essentielle pour la construction du procédé discret.

Abstract — A parallelizable iterative procedure based on a domain decomposition technique is proposed and analyzed for a sequence of elliptic systems with first order absorbing boundary conditions. This sequence of systems, which are not coercive and have characteristics similar to the Helmholtz equation, describe the motion of a nearly elastic solid in the frequency domain. As an application, the procedure is used to solve a finite element approximation to the elliptic systems using the Wilson nonconforming element. The convergence of the procedure is demonstrated and the rate of convergence is derived when the domain is decomposed into subdomains in which each subdomain consists of an individual element associated with the Wilson finite element method. The hybridization of the Wilson finite element is strongly used in the construction of the discrete procedure.

Key words Domain decomposition method, Wilson element, primal hybrid method, nearly elastic, absorbing boundary condition

1. INTRODUCTION

Iterative methods based on a domain decomposition technique for solving partial differential equations have been studied extensively in the past few years and have proven to be very efficient methods due to their parallelism and

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flexibility. In a domain decomposition method, the original problem is first divided into subdomain problems which are connected only through subdomain interfaces. Parallel or sequential iterative procedures are then constructed to decouple the whole domain problem into subdomain problems. During the iterative process, information must be transmitted between subdomains in order to guarantee convergence. This information transmission step is the key part of a domain decomposition method and it differs depending on the method. Domain decomposition methods were introduced to solve elliptic problems, and an abundance of literature is devoted to this subject. Among the vast literature, we refer to [6], [7], [8], [11], [22], [23], [30], and the references therein for recent developments in domain decomposition methods. We notice that, with the noticeable exceptions of [6], [7] and [11], most of these works have been directed at elliptic equations which are coercive, and the demonstration of convergence makes strong use of the coercivity of the equations.

Our objective is to introduce a nonoverlapping domain decomposition method for nearly-elastic wave equations in the frequency domain. The problems we consider here are noncoercive and have characteristics similar to those of the Helmholtz problem. One motivation for developing domain decomposition iterative methods for noncoercive problems is the fact that the relaxation methods such as Jacobi and SOR methods are not convergent for such problems. The other motivation is that these procedures can be naturally and easily implemented on parallel computers by assigning each subdomain to its own processor.

The iterative procedures constructed in this paper are closely related to one developed by Després, Joly and Roberts ([6], [7]) for the Helmholtz problem. The main idea here is to use a Robin-type boundary condition to transmit information between subdomains. Recently, more numerical experiments using these ideas have been performed to optimize the procedure for the Helmholtz problem by Kim [17]. Another related procedure, applicable to the approximate solution of second-order coercive elliptic and parabolic problems using mixed finite element methods, has also been developed by Douglas, Paes Leme, Roberts and Wang [8]. The generalization of their procedure to non-symmetric problems, especially to convection-dominated problems, is developed by Feng in [13].

The layout of this paper is as follows. In § 2, the statement of the problem and some preliminaries are presented. In § 3, the domain decomposition iterative procedure is introduced for the continuous differential problem based on its primitive variable weak formulation. The convergence of the iterative procedure is demonstrated in this section. In the last section, an application of the proposed iterative procedure is given for solving a finite element approximation of the differential system using the Wilson nonconforming element. The hybridization of the Wilson element plays an important role in defining the discrete procedure. The proof of convergence of the discrete procedure is

demonstrated in the case when the domain is decomposed into subdomains in which each subdomain consists of an individual element associated with the Wilson finite element method. For the same decomposition, we also show that the rate of convergence of the discrete procedure has an upper bound of the form $1 - Ch$. The paper is concluded with an appendix which contains a proof of error estimates for solving the differential system using the Wilson finite element method.

2. STATEMENT OF THE PROBLEM AND PRELIMINARIES

Following [28], the constitutive relation for a nearly-elastic material, which is given in the frequency domain, allows the inclusion of dissipative effects via the use of complex Lamé parameters. This then leads to the following frequency domain formulation for nearly elastic waves :

$$(2.1i) \quad - \omega^2 \underline{u} - \operatorname{div} \underline{\underline{\sigma}}(\underline{u}) = \underline{f}, \quad \text{in } \Omega,$$

$$(2.1ii) \quad \underline{\underline{\sigma}}(\underline{u}) \underline{\nu} + i\omega \underline{A} \underline{u} = \underline{g}, \quad \text{on } \Gamma = \partial\Omega$$

for each $\omega > 0$. (If a real-valued displacement is to result from inverse Fourier transforming \underline{u} , it suffices to consider $\omega > 0$). Here, Ω is assumed to be a polygonal domain in \mathbb{R}^N , $N = 2, 3$. In particular, we are interested in the case in which $\Omega = (0, 1)^N$. The outward normal vector is denoted by $\underline{\nu}$ and is assumed to exist almost everywhere on Γ . The displacement vector in the frequency domain is given by \underline{u} . The stress-strain relation in the frequency domain is given by

$$(2.2i) \quad \underline{\underline{\sigma}} = \lambda \operatorname{tr}(\underline{\underline{\varepsilon}}(\underline{u})) \underline{I} + 2 \mu \underline{\underline{\varepsilon}}(\underline{u}), \quad \text{in } \Omega,$$

$$(2.2ii) \quad \underline{\underline{\varepsilon}}(\underline{u}) = \frac{1}{2} (\nabla \underline{u} + \nabla \underline{u}^t), \quad \text{in } \Omega,$$

$$(2.2iii) \quad \lambda = \lambda_r + i\lambda_i, \quad \mu = \mu_r + i\mu_i,$$

where \underline{I} denotes the $N \times N$ identity matrix. It is assumed that λ_r and μ_r are strictly positive and that $\lambda_i \ll \lambda_r$ and $\mu_i \ll \mu_r$. The coefficients λ_i and μ_i are not measurable directly but are related to other parameters measuring attenuation. For their precise definitions and estimates, see [25] and [28]. Finally, \underline{f} denotes the external force vector in the frequency domain and \underline{A} is an $N \times N$, positive-definite, constant matrix. The boundary condition (2.1ii) with $\underline{g} = 0$ is a standard, first-order absorbing boundary condition which allows waves striking the boundary Γ normally to be completely annihilated ([11], [25]) and determines \underline{A} .

We remark that, when λ_t and μ_t vanish, the solid becomes an elastic material and (2.1) reduces to the Fourier-transformed (in time) equations of the following classical elastic wave propagation equations :

$$\begin{aligned} \rho \underline{U}_{tt} - \nabla \cdot \underline{\sigma}(\underline{U}) &= \underline{F}, & \text{in } \Omega \times [0, \infty), \\ \underline{A} \underline{U}_t + \underline{\sigma}(\underline{U}) \underline{\nu} &= \underline{G}, & \text{on } \Gamma \times [0, \infty), \\ \underline{U} = \underline{U}_t &= 0, & \text{in } \Omega \times \{0\}. \end{aligned}$$

Hence, the frequency domain formulation for elastic waves is included in (2.1) and can be regarded as the limit form of nearly elastic waves as λ_t and μ_t go to zero.

Standard space notation will be used in this paper. Thus, $H^k(\Omega)$ and $\|\cdot\|_{k,\Omega}$ ($k = -1, 0, 1, 2$) denotes the usual complex Sobolev space and its norm, and $H^s(\Gamma)$ ($s = 0, \pm \frac{1}{2}$) and $\|\cdot\|_{s,\Gamma}$ denotes the usual Sobolev space and its norm on the boundary Γ of Ω . For more descriptive details of these Sobolev spaces, we refer to [1], [3] and [21].

Now, for each $\omega > 0$, we define the sesquilinear form $a(\cdot, \cdot)_\Omega : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{C}$ by

$$\begin{aligned} a(\underline{u}, \underline{v})_\Omega &\equiv (\underline{\sigma}(\underline{u}), \underline{\varepsilon}(\underline{v}))_\Omega - \omega^2(\underline{u}, \underline{v})_\Omega \\ &= (\lambda_r \nabla \cdot \underline{u}, \nabla \cdot \underline{v})_\Omega + 2(\mu_r \underline{\varepsilon}(\underline{u}), \underline{\varepsilon}(\underline{v}))_\Omega - \omega^2(\underline{u}, \underline{v})_\Omega \\ (2.3) \quad &+ i[(\lambda_t \nabla \cdot \underline{u}, \nabla \cdot \underline{v})_\Omega + 2(\mu_t \underline{\varepsilon}(\underline{u}), \underline{\varepsilon}(\underline{v}))_\Omega], \end{aligned}$$

where $(\cdot, \cdot)_\Omega$ denotes the complex $L^2(\Omega)$ product. In addition, we will use $\langle \cdot, \cdot \rangle_\Gamma$, $(\cdot, \cdot)_\Gamma$ and $\langle \cdot, \cdot \rangle_\Omega$ to denote the duality between $H^{\frac{1}{2}}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$, the $L^2(\Gamma)$ product, and the duality between $H^1(\Omega)$ and $H^{-1}(\Omega)$, respectively.

DEFINITION 2.1 : A vector-valued function $\underline{u} \in H^1(\Omega)$ is said to be a weak solution of (2.1) if it satisfies the following equation :

$$(2.4) \quad a(\underline{u}, \underline{v})_\Omega + i\omega \langle \underline{A} \underline{u}, \underline{v} \rangle_\Gamma = \langle \underline{f}, \underline{v} \rangle_\Omega + \langle \underline{g}, \underline{v} \rangle_\Gamma, \quad \forall \underline{v} \in H^1(\Omega).$$

To prove the unique solvability of (2.1), we need to recall Korn's well-known second inequality. Different proofs of this inequality can be found in Duvaut and Lions [10] and Nitsche [24].

LEMMA 2.1 : *There exists a constant $C > 0$ (dependent only on Ω) such that*

$$(2.5) \quad \|\underline{u}\|_{1,\Omega}^2 \leq C[\|\underline{\varepsilon}(\underline{u})\|_{0,\Omega}^2 + \|\underline{u}\|_{0,\Omega}^2],$$

for any $\underline{u} \in H^1(\Omega)$.

THEOREM 2.1 : *Problem (2.1) has a unique weak solution for each $\omega > 0$.*

Proof: We prove uniqueness first. It suffices to show that $\underline{u} \equiv 0$ is the only solution of (2.1) for $\underline{f} = 0$, $\underline{g} = 0$. Therefore, set $\underline{f} = 0$ and $\underline{g} = 0$. The choice of $\underline{v} = \underline{u}$ in (2.4) yields

$$-\omega^2 \|\underline{u}\|_{0,\Omega}^2 + (\underline{\sigma}(\underline{u}), \underline{\varepsilon}(\underline{u}))_{\Omega} + i\omega \langle \underline{A}\underline{u}, \underline{u} \rangle_{\Gamma} = 0.$$

Taking the imaginary part of both sides we obtain

$$\text{Im} (\underline{\sigma}(\underline{u}), \underline{\varepsilon}(\underline{u}))_{\Omega} + \omega \langle \underline{A}\underline{u}, \underline{u} \rangle_{\Gamma} = 0,$$

or

$$(2.6) \quad (\lambda_i \nabla \cdot \underline{u}, \nabla \cdot \underline{u})_{\Omega} + 2(\mu_i \underline{\varepsilon}(\underline{u}), \underline{\varepsilon}(\underline{u}))_{\Omega} + \omega \langle \underline{A}\underline{u}, \underline{u} \rangle_{\Gamma} = 0.$$

In the nearly-elastic case, this implies that

$$(2.7i) \quad \nabla \cdot \underline{u} = 0, \quad \underline{\varepsilon}(\underline{u}) = 0 \quad \text{in } \Omega,$$

$$(2.7ii) \quad \underline{u} = 0, \quad \text{on } \Gamma.$$

Hence,

$$-\omega^2 \|\underline{u}\|_{0,\Omega} = 0,$$

so that

$$\underline{u} \equiv 0 \quad \text{in } \Omega.$$

In the elastic case, since $\lambda_i = 0$ and $\mu_i = 0$, (2.6) implies only that $\underline{u} = 0$ on Γ . To continue the proof, it follows from (2.4) that

$$(2.8) \quad \frac{\partial \underline{u}}{\partial \underline{v}} = 0, \quad \text{on } \Gamma.$$

Therefore, \underline{u} is a solution of the Cauchy problem (2.1i), (2.7ii) and (2.8). By the Unique Continuation Principle ([2], [6], [15]), we have $\underline{u} \equiv 0$ in Ω . This finishes the proof of uniqueness.

To prove existence of the solution, we notice that it follows from (2.2) and Korn's inequality (2.5) that

$$\operatorname{Re} a(\underline{u}, \underline{u})_{\Omega} \geq C_1 \|\underline{u}\|_{1, \Omega}^2 - C_2 \|\underline{u}\|_{0, \Omega}^2 ;$$

i.e., $a(\cdot, \cdot)_{\Omega}$ satisfies Gårding's inequality ([2], [10]). Then, an application of the Fredholm Alternative Theorem ([2], [31]) shows that we have existence. The proof is complete.

Remark 2.1 : It is not difficult to see that Theorem 2.1 still holds in the case in which Ω is an open bounded domain with piecewise smooth boundary and the density $\rho = \rho(x)$ is a function of x . It also holds if $i\omega$ is replaced by any number α having a nonzero complex part in (2.1ii).

We conclude this section by proving the following regularity results for the solution of problem (2.1), which will be useful in the next section for constructing the domain decomposition procedure for problem (2.1).

THEOREM 2.2 : *Suppose $\underline{f} \in \underline{L}^2(\Omega)$ and $\underline{g} \in \underline{L}^2(\Gamma)$. Then the weak solution \underline{u} of problem (2.1) satisfies $\underline{u} \in \underline{H}^{\frac{3}{2}}(\Omega)$.*

Proof : The proof of this theorem is a direct application of a recent result of Dahlberg, Kenig and Verchota [4] (also see [16]) for the Lamé system on a Lipschitz domain, and it proceeds as follows.

First, we rewrite system (2.1) as

$$(2.9i) \quad -\mu \Delta \underline{u} - (\lambda + \mu) \nabla(\operatorname{div} \underline{u}) = \underline{f} + \omega^2 \underline{u}, \quad \text{in } \Omega,$$

$$(2.9ii) \quad \lambda(\operatorname{div} \underline{u}) \underline{\nu} + \mu(\nabla \underline{u} + \nabla \underline{u}^T) \underline{\nu} = \underline{g} - i\omega \underline{A} \underline{u}, \quad \text{on } \Gamma.$$

Next, since $\underline{u} \in \underline{H}^1(\Omega)$ and $\underline{u}|_{\Gamma} \in \underline{H}^{\frac{1}{2}}(\Gamma) \subset \underline{L}^2(\Gamma)$,

$$(2.10) \quad \underline{f} + \omega^2 \underline{u} \in \underline{L}^2(\Omega) \quad \text{and} \quad \underline{g} - i\omega \underline{A} \underline{u} \in \underline{L}^2(\Gamma).$$

Finally, from a result of Kenig [16] (see Remark 2.2 below), we conclude that $\underline{u} \in \underline{H}^{\frac{3}{2}}(\Gamma)$ and can be represented as a single layer potential plus a volume potential. The proof is complete, even when Ω is an arbitrary Lipschitz domain.

Remark 2.2 : Only the homogeneous Lamé system was considered in [4] and the solution is sought as a single layer potential for the traction problem. For the nonhomogeneous system the standard remedy is to add a volume potential to the homogeneous solution (*cf.* [18]). We also remark that the results of [4] were proved only for the Lamé system with real parameters and real datum functions, but it is not hard to see that the results still hold for the complex Lamé system (the parameters and datum functions are complex numbers and functions) because the fundamental solutions have exactly the same form for both the real and complex cases (*cf.* [4], [11]).

3. DOMAIN DECOMPOSITION FOR THE DIFFERENTIAL PROBLEM

In this section we introduce a nonoverlapping domain decomposition iterative procedure for (2.1) based on the weak formulation (2.4). The usefulness of this procedure is established by proving its convergence. For heuristic and physical considerations we will assume that $g = 0$.

Let $\Omega_1, \Omega_2, \dots, \Omega_M$ be a partition of Ω into Lipschitz subdomains such that

$$\overline{\Omega} = \bigcup_{k=1}^M \overline{\Omega}_k, \quad \Omega_k \cap \Omega_j = \emptyset, \quad \text{if } k \neq j;$$

then set

$$\Gamma_k = \Gamma_{k0} = \Gamma \cap \partial\Omega_k \quad \text{and} \quad \Gamma_{kj} = \partial\Omega_k \cap \partial\Omega_j.$$

Thus, it is well-known ([20]) that, under some conditions, (2.1) is equivalent to the following split subdomain problem in the sense that $\underline{u}|_{\Omega_k} = \underline{u}_k$, where \underline{u}_k is defined by

(3.1i)
$$-\omega^2 \underline{u}_k - \nabla \cdot \underline{\underline{\sigma}}(\underline{u}_k) = f \text{ in } \Omega_k,$$

(3.1ii)
$$\underline{\underline{\sigma}}(\underline{u}_k) \underline{\underline{\nu}} + i\omega A \underline{u}_k = 0, \quad \text{on } \Gamma_k,$$

(3.1iii)
$$\underline{u}_k = \underline{u}_j, \quad \text{on } \Gamma_{kj},$$

(3.1iv)
$$\underline{\underline{\sigma}}(\underline{u}_k) \underline{\underline{\nu}}_k = -\underline{\underline{\sigma}}(\underline{u}_j) \underline{\underline{\nu}}_j, \quad \text{on } \Gamma_{kj}.$$

We remark that (3.1) gives an overdetermined problem on each individual subdomain Ω_k . In order to formulate a well-posed problem on each Ω_k , we observe that the transmission conditions (3.1iii) and (3.1iv) are equivalent to the following Robin boundary conditions on Γ_{kj} :

(3.2i)
$$\underline{\underline{\sigma}}(\underline{u}_k) \underline{\underline{\nu}}_k + \alpha \underline{u}_k = -\underline{\underline{\sigma}}(\underline{u}_j) \underline{\underline{\nu}}_j + \alpha \underline{u}_j,$$

(3.2ii)
$$\underline{\underline{\sigma}}(\underline{u}_j) \underline{\underline{\nu}}_j + \alpha \underline{u}_j = -\underline{\underline{\sigma}}(\underline{u}_k) \underline{\underline{\nu}}_k + \alpha \underline{u}_k,$$

for any nonzero complex number α . In this paper we will always choose $\alpha = -\alpha_r + i\alpha_i$ with $\alpha_r \geq 0$ and $\alpha_i > 0$. The reason for such a choice will be clear later in the paper. Then, the problem given by (3.1i), (3.1ii) and (3.2i) is equivalent to

$$\begin{aligned} (3.3i) \quad & -\omega^2 \underline{u}_k - \nabla \cdot \underline{\sigma}(\underline{u}_k) = \underline{f}_k, & \text{in } \Omega_k, \\ (3.3ii) \quad & \underline{\sigma}(\underline{u}_k) \underline{\nu}_k + i\omega A \underline{u}_k = 0, & \text{on } \Gamma_k, \\ (3.3iii) \quad & \underline{\sigma}(\underline{u}_k) \underline{\nu}_k + \alpha \underline{u}_k = -\underline{\sigma}(\underline{u}_j) \underline{\nu}_j + \alpha \underline{u}_j, & \text{on } \Gamma_{kj}, \end{aligned}$$

for $k = 1, \dots, M$.

Therefore, in order to solve the original problem (2.1) it suffices to solve (3.3). There are many ways to decouple (3.3) into M subdomain problems (see [6], [23], and the references therein). In this section we introduce the following iterative localization. On each subdomain Ω_k , evaluate the quantities in (3.3) related to Ω_k at the new iterate level and those related to neighboring subdomains Ω_j at the old level. Thus, the iterative algorithm is as follows :

$$\begin{aligned} (3.4i) \quad & \forall \underline{u}_k^0 \in \underline{H}^1(\Omega_k) \text{ such that } \underline{\sigma}(\underline{u}_k^0) \underline{\nu}_k \in \underline{L}^2(\partial\Omega_k) \quad \forall k, \forall n \geq 1, \\ (3.4ii) \quad & -\omega^2 \underline{u}_k^n - \nabla \cdot \underline{\sigma}(\underline{u}_k^n) = \underline{f}_k, & \text{in } \Omega_k, \\ (3.4iii) \quad & \underline{\sigma}(\underline{u}_k^n) \underline{\nu}_k + i\omega A \underline{u}_k^n = 0, & \text{on } \Gamma_k, \\ (3.4iv) \quad & \underline{\sigma}(\underline{u}_k^n) \underline{\nu}_k + \alpha \underline{u}_k^n = -\underline{\sigma}(\underline{u}_j^{n-1}) \underline{\nu}_j + \alpha \underline{u}_j^{n-1}, & \text{on } \Gamma_{kj}. \end{aligned}$$

LEMMA 3.1 : *The functions $\{\underline{u}_k^n\}$, $k = 1, \dots, M$, $n \geq 1$, are well defined. Moreover, $\underline{\sigma}(\underline{u}_k^n) \underline{\nu}_k \in \underline{L}^2(\partial\Omega_k)$.*

Proof : By Theorem 2.2, $\underline{u}_k^1 \in \underline{H}^{\frac{3}{2}}(\Omega_k)$. Since the trace space of $\underline{H}^{\frac{3}{2}}(\Omega_k)$ is $\underline{H}^1(\partial\Omega_k)$, the traction vector $\underline{\sigma}(\underline{u}_k^1) \underline{\nu}_k \in \underline{L}^2(\partial\Omega_k)$, so that the right hand side of (3.4iv) is in $\underline{L}^2(\Gamma_{kj})$ when $n = 2$. This then implies that $\underline{u}_k^2 \in \underline{H}^{\frac{3}{2}}(\Omega_k)$. The lemma follows by repeating this recursion argument.

Remark 3.1 : The stationary point, if any, of the sequence $\{\underline{u}_k^n\}$ coincides with the unique solution of (2.1).

To establish the usefulness of this iterative process, it is necessary to demonstrate the convergence of the iterative sequence.

Let $e_k^n = u_k^n = u_k$, where $u_k = \underline{u}|_{\Omega_k}$ and \underline{u} is the solution of (2.1). Linearity implies that

$$(3.5i) \quad -\omega^2 \underline{e}_k^n - \nabla \cdot \underline{\sigma}(e_k^n) = 0, \quad \text{in } \Omega_k,$$

$$(3.5ii) \quad \underline{\sigma}(e_k^n) \nu_k + i\omega A e_k^n = 0, \quad \text{on } \Gamma_k,$$

$$(3.5iii) \quad \underline{\sigma}(e_k^n) \nu_k + \alpha e_k^n = -\underline{\sigma}(e_j^{n-1}) \nu_j + \alpha e_j^{n-1}, \quad \text{on } \Gamma_{kj}.$$

LEMMA 3.2 : *The following equalities hold :*

$$(3.6) \quad \begin{aligned} |\underline{\sigma}(e_k^n) \nu_k \pm \alpha e_k^n|_{0, \partial\Omega_k}^2 &= |\underline{\sigma}(e_k^n) \nu_k|_{0, \partial\Omega_k}^2 + |\alpha|^2 |e_k^n|_{0, \partial\Omega_k}^2 \\ &\quad \pm 2 \alpha_t \operatorname{Im} a(e_k^n, e_k^n)_{\Omega_k} \mp 2 \alpha_r \operatorname{Re} a(e_k^n, e_k^n)_{\Omega_k}. \end{aligned}$$

Proof : By direct calculation,

$$\begin{aligned} |\underline{\sigma}(e_k^n) \nu_k \pm \alpha e_k^n|_{0, \partial\Omega_k}^2 &= \int_{\partial\Omega_k} [\underline{\sigma}(e_k^n) \nu_k \pm \alpha e_k^n] \overline{[\underline{\sigma}(e_k^n) \nu_k \pm \alpha e_k^n]} \\ &= |\underline{\sigma}(e_k^n) \nu_k|_{0, \partial\Omega_k}^2 + |\alpha|^2 |e_k^n|_{0, \partial\Omega_k}^2 \\ &\quad \pm 2 \operatorname{Re} \left[\overline{\alpha} \int_{\partial\Omega_k} \underline{\sigma}(e_k^n) \nu_k \cdot \bar{e}_k^n ds \right]. \end{aligned}$$

Next, test (3.5i) against \bar{e}_k^n , integrate by parts and take the real and imaginary parts of the resulting equality to obtain

$$(3.7i) \quad \begin{aligned} \operatorname{Re} \int_{\partial\Omega_k} \underline{\sigma}(e_k^n) \nu_k \cdot \bar{e}_k^n ds &= \operatorname{Re} a(e_k^n, e_k^n)_{\Omega_k} \\ &= \lambda_r \|\nabla \cdot e_k^n\|_{0, \Omega_k}^2 + \mu_r \|\underline{\varepsilon}(e_k^n)\|_{0, \Omega_k}^2 - \omega^2 \|e_k^n\|_{0, \Omega_k}^2 \end{aligned}$$

$$(3.7ii) \quad \begin{aligned} \operatorname{Im} \int_{\partial\Omega_k} \underline{\sigma}(e_k^n) \nu_k \cdot \bar{e}_k^n ds &= \operatorname{Im} a(e_k^n, e_k^n)_{\Omega_k} \\ &= \lambda_t \|\nabla \cdot e_k^n\|_{0, \Omega_k}^2 + \mu_t \|\underline{\varepsilon}(e_k^n)\|_{0, \Omega_k}^2. \end{aligned}$$

The proof is completed by combining the above equalities.

Define a « pseudo-energy » $E^n \equiv E(\{e_k^n\})$ by

$$(3.8) \quad E^n = \sum_k \sum_j \int_{\Gamma_{kj}} (|\underline{\sigma}(e_k^n) \nu_k + \alpha e_k^n|^2) ds.$$

LEMMA 3.3 : *Then,*

$$(3.9) \quad E^{n+1} = E^n - \sum_k \left\{ 4 \alpha_t [\text{Im } a(e_k^n, e_k^n)_{\Omega_k} + \omega \int_{\Gamma_k} A \underline{e}_k^n \cdot \bar{\underline{e}}_k^n ds - 4 \alpha_r \text{Re } a(e_k^n, e_k^n)_{\Omega_k}] \right\}.$$

Proof: The proof is a straightforward application of Lemma 3.2 and the Robin type transmission condition (3.5iii) :

$$\begin{aligned} E^{n+1} &= \sum_k \sum_j \int_{\Gamma_{kj}} |\underline{\sigma}(e_k^{n+1}) \nu_k + \alpha e_k^{n+1}|^2 ds \\ &= \sum_k \sum_j \int_{\Gamma_{kj}} |-\underline{\sigma}(e_j^n) \nu_j + \alpha e_j^n|^2 ds \\ &= \sum_j \int_{\partial\Omega_j} |-\underline{\sigma}(e_j^n) \nu_j + \alpha e_j^n|^2 ds - \sum_j \int_{\Gamma_j} |-\underline{\sigma}(e_j^n) \nu_j + \alpha e_j^n|^2 ds \\ &= \sum_j \int_{\partial\Omega_j} |\underline{\sigma}(e_j^n) \nu_j + \alpha e_j^n|^2 ds - \sum_j \int_{\Gamma_j} |\underline{\sigma}(e_j^n) \nu_j + \alpha e_j^n|^2 ds \\ &\quad - 4 \sum_k \left[\text{Re} \int_{\partial\Omega_k} \underline{\sigma}(e_k^n) \nu_k \cdot \overline{\alpha e_k^n} ds - \text{Re} \int_{\Gamma_k} \underline{\sigma}(e_k^n) \nu_k \cdot \overline{\alpha e_k^n} ds \right] \\ &= E^n + 4 \alpha_r \text{Re } a(e_k^n, e_k^n)_{\Omega_k} - 4 \alpha_t \left[\sum_k \text{Im } a(e_k^n, e_k^n)_{\Omega_k} + \omega \int_{\Gamma_k} A \underline{e}_k^n \cdot \bar{\underline{e}}_k^n ds \right]. \end{aligned}$$

The proof is complete.

We are now in the position to show the convergence of $\{u_k^n\}$. Specifically, we have the following convergence theorem.

THEOREM 3.1 : Suppose $\sigma(\underline{e}_k^0) \nu_k \in L^2(\partial\Omega_k)$ for $k = 1, \dots, M$, and choose the parameter $\alpha = -\alpha_r + i\alpha_t$ such that $\alpha_t \lambda_t - \alpha_r \lambda_r > 0$ and $\alpha_t \mu_t - \alpha_r \mu_r > 0$. Then, $\{\underline{u}_k^n\}$ converges to the solution \underline{u} of (2.1) in the following sense :

$$\underline{u}_k^n \rightarrow u_k \equiv \underline{u}_k|_{\Omega_k} \quad \text{in } H^1(\Omega_k), \quad \forall k.$$

Proof: Obviously, it suffices to show that

$$\underline{e}_k^n \rightarrow 0 \quad \text{in } H^1(\Omega_k), \quad \forall k.$$

Since A is positive definite, there is a constant $c_0 > 0$ such that

$$c_0 |\underline{e}_k^n|^2 \leq A \underline{e}_k^n \cdot \underline{e}_k^n \leq c_0^{-1} |\underline{e}_k^n|^2.$$

By Lemma 3.3 we have

(3.10)

$$\begin{aligned} E^{n+1} &\leq E^n - 4 \alpha_t \omega c_0 \sum_k \| \underline{e}_k^n \|_{0, \Gamma_k}^2 - 4 \sum_k [(\alpha_t \lambda_t - \alpha_r \lambda_r) \| \nabla \cdot \underline{e}_k^n \|_{0, \Omega_k}^2 \\ &\quad + 2(\alpha_t \mu_t - \alpha_r \mu_r) \| \varepsilon(\underline{e}_k^n) \|_{0, \Omega_k}^2 + \alpha_r \omega^2 \| \underline{e}_k^n \|_{0, \Omega_k}^2] \\ &\leq E^0 - 4 \alpha_t \omega c_0 \sum_{l=0}^n \sum_k \| \underline{e}_k^l \|_{0, \Gamma_k}^2 \\ &\quad - 4 \sum_{l=0}^n \sum_k [(\alpha_t \lambda_t - \alpha_r \lambda_r) \| \nabla \cdot \underline{e}_k^l \|_{0, \Omega_k}^2 \\ &\quad + (\alpha_t \mu_t - \alpha_r \mu_r) \| \varepsilon(\underline{e}_k^l) \|_{0, \Omega_k}^2 + \alpha_r \omega^2 \| \underline{e}_k^l \|_{0, \Omega_k}^2]. \end{aligned}$$

Choose α_r and α_t such that

$$\alpha_t \lambda_t - \alpha_r \lambda_r > 0, \quad \alpha_t \mu_t - \alpha_r \mu_r > 0,$$

then if $\alpha_r \neq 0$ we have

$$(3.11i) \quad \|e'_k\|_{0, \Omega_k} \xrightarrow{l \rightarrow \infty} 0, \quad \forall k \geq 1,$$

$$(3.11ii) \quad \|\nabla \cdot e'_k\|_{0, \Omega_k} \xrightarrow{l \rightarrow \infty} 0, \quad \forall k \geq 1,$$

$$(3.11iii) \quad \|\varepsilon_{\approx}(e'_k)\|_{0, \Omega_k} \xrightarrow{l \rightarrow \infty} 0, \quad \forall k \geq 1,$$

$$(3.11iv) \quad \|e'_k\|_{0, \Gamma_k} \xrightarrow{l \rightarrow \infty} 0, \quad \forall k \geq 1.$$

If $\alpha_r = 0$, we only get (3.11ii)-(3.11iv) from (3.10). But we recover (3.11i) from (3.7i), (3.11ii), (3.11iii), and the fact of that $\|\varepsilon_{\approx}(e'_k) \nu_k\|_{0, \Gamma_k}$ is bounded uniformly in n since E^n is bounded uniformly in n .

Finally, it follows from (3.11i), (3.11iii), and Korn's inequality that

$$e'_k \xrightarrow{l \rightarrow \infty} 0 \quad \text{in } H^1_{\approx}(\Omega_k), \quad \forall k \geq 1.$$

This completes the proof.

Remark 3.2: If the material is elastic (i.e., $\lambda_i = \mu_i = 0$) and we choose $\alpha_r = 0$ and $\alpha_i > 0$, the conclusion of Theorem 3.1 still holds. But the proof is more complicated, and we refer to [11] for a detailed proof.

4. DOMAIN DECOMPOSITION FOR THE WILSON FINITE ELEMENT METHOD

In this section, we consider an iterative procedure for an approximation of (2.1) based on the Wilson nonconforming finite element (cf. [3], [29], [19]). For simplicity, we consider the case $N = 2$ and assume $\underline{g} = 0$. Let \mathcal{T}_h be a rectangular partition of Ω which aligns with the boundary of $\partial\Omega$. Let V_h denote the finite element space constructed by using the Wilson element. We recall that $v_h \in V_h$ if it satisfies the following conditions :

(i) $v_h|_T \in P_2(T)$, $\forall T \in \mathcal{T}_h$.

(ii) v_h is continuous at the vertices of \mathcal{T}_h , and $v_h|_T$, $\forall T \in \mathcal{T}_h$, is uniquely determined by its function values at the vertices of T and the values of its two second, nonmixed partial derivatives on T .

Note that $V_h \not\subset H^1(\Omega)$. Let

$$a_h(u_h, v_h)_{\Omega} \equiv \sum_{T \in \mathcal{T}_h} a(u_h, v_h)_T, \quad \|v_h\|_h = \left(\sum_{T \in \mathcal{T}_h} |v_h|_{1,T}^2 \right)^{\frac{1}{2}}.$$

We remark that it was proven in [19] that $\| \cdot \|_h$ is a norm on $V_h \equiv (V_h)^2$ and $a_h(\cdot, \cdot)$ is coercive on $V_h \times V_h$.

The Wilson finite element method for (2.1) is to seek $u_h \in V_h$ such that

$$(4.1) \quad a_h(u_h, v_h)_\Omega + i\omega \langle A_{\underline{z}} u_h, v_h \rangle_\Gamma = (f, v_h)_\Omega, \quad \forall v_h \in V_h.$$

THEOREM 4.1 : *There exists an $h_0 > 0$ such that, for all $h \in (0, h_0]$, (4.1) has a unique solution $u_h \in V_h$. Moreover, the following error estimate holds :*

$$(4.2) \quad \|u - u_h\|_{0, \Omega} + h \|u - u_h\|_h \leq C(\omega) h^2 |u|_{2, \Omega}.$$

A proof of Theorem 4.1 is given in the appendix.

The domain decomposition that interests us is the case in which each finite element is a subdomain ; i.e., $\{\Omega_j\}$ is a partition of Ω into individual rectangular elements. Larger subdomains, in which each subdomain is composed of more than one element, are permissible.

Next, we notice that, since both u_h and $\sigma(u_h)$ are discontinuous across the interface Γ_{jk} , we do not have Robin conditions analogous to (3.3) for the discrete problem (4.1), so that it is not possible to define an analogous domain decomposition iterative procedure directly for the discrete problem (4.1). However, this difficulty can be overcome by considering an equivalent (primal) hybrid formulation of (4.1), where a Lagrange multiplier is introduced. Hybridization was also used in [8] and [12] to handle similar difficulties. For notational brevity, we will abuse notation by omitting the index h for functions in finite element space for the remainder of this section.

Let $V_{j,h}$ denote the Wilson finite element space on Ω_j , and note that $V_{j,h} \subset H^1(\Omega_j)$. Let

$$V_{j,h} = (V_{j,h})^2, \quad W_h = \prod_j V_{j,h}, \quad \tau_{jk} = \text{sign}(j - k), \\ A_h = \prod_{j > k} P_{-1}(\Gamma_{jk}), \quad A_h^2 = \prod_{(j,k) \atop j \neq k} P_{-1}(\Gamma_{jk}).$$

Also, let I_{jk}^h be the standard linear interpolation operator which interpolates at the end points of Γ_{jk} . Then, (4.1) has the following equivalent hybrid formulation.

Seek $(u, \lambda) \in W_h \times A_h$ such that

$$(4.3i) \quad \sum_j a(u, v)_\Omega + i\omega \sum_j \langle A_{\underline{z}} u, v \rangle_{\Gamma_j} - \sum_{j,k} \langle \tau_{jk} \lambda, I_{jk}^h v \rangle_{\Gamma_{jk}} = (f, v)_\Omega, \quad \forall v \in W_h,$$

$$(4.3ii) \quad \sum_{j,k} \langle \tau_{jk} \chi, I_{jk}^h u \rangle_{\Gamma_{jk}} = 0, \quad \forall \chi \in A_h.$$

The equivalence of (4.1) and (4.3) is precisely described by the following lemma (cf. [27]).

LEMMA 4.1 : *Problem (4.3) possesses a unique solution $(\underline{u}, \underline{\lambda}) \in \underline{W}_h \times \underline{A}_h$ for all $0 < h \leq h_0$. Moreover, $\underline{u} \in \underline{V}_h$ and satisfies (4.1).*

On the other hand, if we let (u_j, λ_{jk}) be the solution of the split problem given by

$$(4.4i) \quad a(u_j, v_j)_{\Omega_j} + i\omega \langle \underline{A}u_j, v_j \rangle_{\Gamma_j} - \sum_k \langle \tau_{jk} \lambda_{jk}, I_{jk}^h v_j \rangle_{\Gamma_{jk}} = \\ = (f_j, v_j)_{\Omega_j}, \quad \forall v_j \in V_{j,h},$$

$$(4.4ii) \quad \tau_{jk} \lambda_{jk} + \alpha I_{jk}^h u_j = -\tau_{jk} \lambda_{kj} + \alpha I_{jk}^h u_k, \quad \text{on } \Gamma_{jk}, \quad \forall k,$$

then it is not hard to see that $u_j = u|_{\Omega_j}$ and $\lambda_{jk} = \lambda_{kj} = \lambda|_{\Gamma_{jk}}$, where (u, λ) is the solution of (4.3).

Based on (4.4) we define a domain decomposition iterative procedure analogous to the procedure defined by (3.5). Starting with arbitrary initial guesses $(u_j^0, \lambda_{jk}^0) \in V_{j,h} \times P_1(\Gamma_{jk})$ for all j and all relevant $\{j, k\}$ we define the iterate $(u_j^n, \lambda_{jk}^n) \in V_{j,h} \times P_1(\Gamma_{jk})$ recursively by solving on each Ω_j the following system of equations :

$$(4.5i) \quad a(u_j^n, v_j)_{\Omega_j} + i\omega \langle \underline{A}u_j^n, v_j \rangle_{\Gamma_j} - \sum_k \langle \tau_{jk} \lambda_{jk}^n, I_{jk}^h v_j \rangle_{\Gamma_{jk}} = \\ = (f_j, v_j)_{\Omega_j}, \quad \forall v_j \in V_{j,h},$$

$$(4.5ii) \quad \tau_{jk} \lambda_{jk}^n + \alpha I_{jk}^h u_j^n = -\tau_{kj} \lambda_{kj}^{n-1} + \alpha I_{kj}^h u_k^{n-1}, \quad \text{on } \Gamma_{jk}, \quad \forall k.$$

Obviously, the sequence $\{(u_j^n, \lambda_{jk}^n)\}$ of iterates is well-defined. To establish the usefulness of the above algorithm, we need to show the convergence of the sequence. To this end, we first define the error functions $(e_j^n, \xi^n) \in \underline{W}_h \times \underline{A}_h^2$ by

$$e_j^n|_{\Omega_j} \equiv e_j^n = u_j - u_j^n, \quad \xi^n|_{\Gamma_{jk}} \equiv \xi_{jk}^n = \lambda_{jk} - \lambda_{jk}^n,$$

where $\hat{\lambda}_{kj} = \hat{\lambda}_{jk}$ for $k > j$. It is easy to see from (4.4) and (4.5) that the error functions satisfy the following equations :

$$(4.6i) \quad a(e_j^n, v_j)_{\Omega_j} + i\omega \langle A_{\underline{z}} e_j^n, v_j \rangle_{\Gamma_j} - \sum_k \langle \tau_{jk} \xi_{jk}^n, I_{jk}^h v_j \rangle_{\Gamma_{jk}} = 0, \quad \forall v_j \in V_{j,h},$$

$$(4.6ii) \quad \tau_{jk} \xi_{jk}^n + \alpha I_{jk}^h e_j^n = -\tau_{kj} \xi_{kj}^{n-1} + \alpha I_{kj}^h e_k^{n-1}, \quad \text{on } \Gamma_{jk}, \quad \forall k.$$

Let $v_j = e_j^n$ in (4.6i) :

$$\sum_k \langle \tau_{jk} \xi_{jk}^n, I_{jk}^h e_j^n \rangle_{\Gamma_{jk}} = a(e_j^n, e_j^n)_{\Omega_j} + i\omega \langle A_{\underline{z}} e_j^n, e_j^n \rangle_{\Gamma_j}.$$

Then,

$$(4.7i) \quad \text{Im} \sum_k \langle \tau_{jk} \xi_{jk}^n, I_{jk}^h e_j^n \rangle_{\Gamma_{jk}} = \text{Im} a(e_j^n, e_j^n)_{\Omega_j} + \omega \langle A_{\underline{z}} e_j^n, e_j^n \rangle_{\Gamma_j}$$

$$= \lambda_i \|\nabla \cdot e_j^n\|_{0,\Omega_j}^2 + 2\mu_i \|\underline{\underline{\varepsilon}}(e_j^n)\|_{0,\Gamma_j}^2$$

$$+ \omega \langle A_{\underline{z}} e_j^n, e_j^n \rangle_{\Gamma_j},$$

$$(4.7ii) \quad \text{Re} \sum_k \langle \tau_{jk} \xi_{jk}^n, I_{jk}^h e_j^n \rangle_{\Gamma_{jk}} = \text{Re} a(e_j^n, e_j^n)_{\Omega_j}$$

$$= \lambda_r \|\nabla \cdot e_j^n\|_{0,\Omega_j}^2 + 2\mu_r \|\underline{\underline{\varepsilon}}(e_j^n)\|_{0,\Gamma_j}^2$$

$$- \omega^2 \|e_j^n\|_{0,\Omega_j}.$$

For any $(\underline{v}, \underline{\chi}) \in W_{\underline{h}} \times A_{\underline{h}}^2$, we define a « pseudo-energy » $\mathcal{E}(\underline{v}, \underline{\chi})$ by

$$(4.8) \quad \mathcal{E}(\underline{v}, \underline{\chi}) = \sum_{j,k} \int_{\Gamma_{jk}} |\alpha I_{jk}^h v_j + \tau_{jk} \chi_{jk}|^2 ds,$$

where $v_j = \underline{v}|_{\Omega_j}$ and $\chi_{jk} = \underline{\chi}|_{\Gamma_{jk}}$.

LEMMA 4.2 : Let $\mathcal{E}^n = \mathcal{E}(e_j^n, \xi_j^n)$. Then,

$$(4.9) \quad \mathcal{E}^n = \mathcal{E}^{n-1} - \sum_j \{ 4 \alpha_i [\operatorname{Im} a(e_j^{n-1}, e_j^{n-1})_\Omega] + \omega \langle A_{\bar{z}} e_j^{n-1}, e_j^{n-1} \rangle_{\Gamma_j} \} \\ - 4 \alpha_r \operatorname{Re} a(e_j^{n-1}, e_j^{n-1})_\Omega \}.$$

Proof: From (4.7), we have

$$(4.10) \quad \int_{\Gamma_{jk}} |\alpha I_{jk}^h e_j^n \pm \tau_{jk} \xi_{jk}^n|^2 ds = \int_{\Gamma_{jk}} (|\xi_{jk}^n|^2 + |\alpha|^2 |I_{jk}^h e_j^n|^2) ds \\ \pm 2 \alpha_i \operatorname{Im} \langle \tau_{jk} \xi_{jk}^n, I_{jk}^h e_j^n \rangle_{\Gamma_{jk}} \\ \mp 2 \alpha_r \operatorname{Re} \langle \tau_{jk} \xi_{jk}^n, I_{jk}^h e_j^n \rangle_{\Gamma_{jk}} \\ = \int_{\Gamma_{jk}} (|\xi_{jk}^n|^2 + |\alpha|^2 |I_{jk}^h e_j^n|^2) ds \\ \mp 2 \alpha_r \operatorname{Re} a(e_j^n, e_j^n)_\Omega \\ \pm 2 \alpha_i [\operatorname{Im} a(e_j^n, e_j^n)_\Omega + \omega \langle A_{\bar{z}} e_j^n, e_j^n \rangle_{\Gamma_j}].$$

It follows from the definition of \mathcal{E}^n , (4.7), (4.6ii) and (4.10) that

$$\mathcal{E}^n = \sum_{j,k} \int_{\Gamma_{jk}} |\alpha I_{jk}^h e_j^n + \tau_{jk} \xi_{jk}^n|^2 ds \\ = \sum_{j,k} \int_{\Gamma_{kj}} |\alpha I_{kj}^h e_k^{n-1} - \tau_{kj} \xi_{kj}^{n-1}|^2 ds \\ = \sum_{j,k} \int_{\Gamma_{kj}} (|\alpha|^2 |I_{kj}^h e_k^{n-1}|^2 + |\xi_{kj}^{n-1}|^2) ds + 2 \alpha_r \sum_j \operatorname{Re} a(e_j^{n-1}, e_j^{n-1})_\Omega \\ - 2 \alpha_i \sum_j [\operatorname{Im} a(e_j^{n-1}, e_j^{n-1})_\Omega + \omega \langle A_{\bar{z}} e_j^{n-1}, e_j^{n-1} \rangle_{\Gamma_j}] \\ = \mathcal{E}^{n-1} + 4 \alpha_r \sum_j \operatorname{Re} a(e_j^{n-1}, e_j^{n-1})_\Omega \\ - 4 \alpha_i \sum_j [\operatorname{Im} a(e_j^{n-1}, e_j^{n-1})_\Omega + \omega \langle A_{\bar{z}} e_j^{n-1}, e_j^{n-1} \rangle_{\Gamma_j}].$$

The proof is complete.

From Lemma 4.2 we know that the « pseudo-energy » of the error functions is decreasing for the iterative procedure (4.5). Next, we are going to show that the error functions converge to zero.

THEOREM 4.2 : *Let $\{u_j^n, \lambda_{jk}^n\}$ be defined by (4.5). If $\alpha_i \lambda_i - \alpha_r \lambda_r > 0$ and $\alpha_i \mu_i - \alpha_r \mu_r > 0$, then*

- (i) $u_j^n \rightarrow u_j \equiv \underline{u}|_{\Omega_j}$ strongly in $H^1(\Omega_j)$.
- (ii) $\lambda_{jk}^n, \lambda_{kj}^n \rightarrow \lambda_{jk} \equiv \underline{\lambda}|_{\Gamma_{jk}}$ strongly in $L^2(\Gamma_{jk})$.

Proof : The proof of (i) is similar to that of Theorem 3.3. From (4.9),

$$\begin{aligned}
 \mathcal{E}^n = \mathcal{E}^0 - 4 \alpha_i \omega \sum_{l=1}^{n-1} \sum_j \langle A_{\underline{z}} e_j^l, e_j^l \rangle \Gamma_j - \sum_{l=1}^{n-1} \sum_j [(\alpha_i \lambda_i - \alpha_r \lambda_r) \|\nabla \cdot e_j^l\|_{0, \Omega}^2, \\
 (4.11) \quad + (\alpha_i \mu_i - \alpha_r \mu_r) \|\varepsilon_{\underline{z}}(e_j^l)\|_{0, \Omega}^2 + \alpha_r \omega^2 \|e_j^l\|_{0, \Omega}^2] .
 \end{aligned}$$

If $\alpha_r \neq 0$, it follows from (4.11) and the assumptions on α that

$$(4.12i) \quad \|e_j^l\|_{0, \Omega} \xrightarrow{l \rightarrow \infty} 0, \quad \forall j \geq 1,$$

$$(4.12ii) \quad \|\nabla \cdot e_j^l\|_{0, \Omega} \xrightarrow{l \rightarrow \infty} 0, \quad \forall j \geq 1,$$

$$(4.12iii) \quad \|\varepsilon_{\underline{z}}(e_j^l)\|_{0, \Omega} \xrightarrow{l \rightarrow \infty} 0, \quad \forall j \geq 1,$$

$$(4.12iv) \quad \|e_j^l\|_{0, \Gamma_j} \xrightarrow{l \rightarrow \infty} 0, \quad \forall j \geq 1.$$

If $\alpha_r = 0$, we obtain only (4.12ii)-(4.12iv) from (4.11). But (4.12i) can be recovered from (4.7ii), (4.12ii), (4.12iii), and the fact that $\|e_j^n\|_{0, \Gamma_{jk}}$ is bounded uniformly in n , since \mathcal{E}^n is bounded uniformly in n . Now, (i) follows from (4.12i), (4.12iii), and Korn's inequality.

Next, for any $\underline{\lambda} \in A_{\underline{z}h}^2$, let

$$\begin{aligned}
 (4.13) \quad S_j(\underline{\lambda}) = \{v_j; v_j \in V_{j,h} \text{ such that } v_j = \tau_{jk} \lambda_{jk} \text{ at the end points of } \Gamma_{jk} \\
 \text{and } v_j|_{\Gamma_j} = 0\}.
 \end{aligned}$$

Then, let $v_j \in S_j(\xi^n)$ in (4.6i) and note that $I_{jk}^n v_j = \tau_{jk} \xi_{jk}^n$ on Γ_{jk} . Using Schwarz's inequality, we see that

$$\begin{aligned}
 (4.14i) \quad \sum_k \|\xi_{jk}^n\|_{0, \Gamma_{jk}}^2 &= \operatorname{Re} a(e_j^n, v_j)_{\Omega_j} \\
 &\leq |\operatorname{Re} a(e_j^n, e_j^n)_{\Omega_j}|^{\frac{1}{2}} |\operatorname{Re} a(v_j, v_j)_{\Omega_j}|^{\frac{1}{2}} \\
 &\leq \sqrt{C_j} |\operatorname{Re} a(e_j^n, e_j^n)_{\Omega_j}|^{\frac{1}{2}} \left(\sum_k \|\xi_{jk}^n\|_{0, \Gamma_{jk}}^2 \right)^{\frac{1}{2}},
 \end{aligned}$$

where

$$(4.14ii) \quad C_j = \sup_{\lambda \in \mathcal{A}_i} \inf_{v_j \in S_j(\lambda)} \frac{|\operatorname{Re} a(v_j, v_j)_{\Omega_j}|}{\sum_k \|\lambda_{jk}\|_{0, \Gamma_{jk}}^2}.$$

Hence,

$$(4.15) \quad \sum_k \|\xi_{jk}^n\|_{0, \Gamma_{jk}}^2 \leq C_j |\operatorname{Re} a(e_j^n, e_j^n)_{\Omega_j}|.$$

The proof is completed by letting $n \rightarrow \infty$ in (4.15).

Remark 4.1 : For elastic materials, $\lambda_i = \mu_i = 0$, and it is not clear if one can still get (4.12) from (4.11). By choosing $\alpha_r = 0$ and $\alpha_i > 0$, we obtain (4.12iv) from (4.11). This suggests that we try the idea used to prove the convergence for the differential problem (cf. [11]), but the proof for the differential problem is based on the Unique Continuation Principle, and there is no discrete counterpart available to the authors' knowledge. Therefore, the convergence of the proposed iterative method for elastic waves is still open. We also remark that the same situation happens for the Helmholtz problem (cf. [6]).

In the rest of this section, we would like to estimate the rate of convergence for the iterative procedure (4.5); i.e., to answer the question of how fast \mathcal{E}^n decays. More specifically, we have the following result.

THEOREM 4.3 : *If $\alpha_i \lambda_i > 2 \alpha_r \lambda_r$ and $\alpha_i \mu_i > 2 \alpha_r \mu_r$, then*

$$(4.16) \quad \mathcal{E}(\{e_{\bar{z}}^{n+1}, \xi_{\bar{z}}^{n+1}\}) \leq (1 - Ch) \mathcal{E}(\{e_{\bar{z}}^n, \xi_{\bar{z}}^n\})$$

for some positive constant C which is independent of h .

Proof: Let

$$\begin{aligned}
 (4.17) \quad \mathcal{R}(\{\underline{e}^n, \underline{\xi}^n\}) &= \sum_j \{ \alpha_i [\text{Im } a(e_j^n, e_j^n)_{\Omega_i} + \langle A_{\underline{z}} e_j^n, e_j^n \rangle_{\Gamma_i}] \\
 &\quad - \alpha_r \text{Re } a(e_j^n, e_j^n)_{\Omega_j} \} \\
 &= \sum_j \{ (\alpha_i \lambda_i - \alpha_r \lambda_r) \|\text{div } e_j^n\|_{0, \Omega_i}^2, \\
 &\quad + 2(\alpha_i \mu_i - \alpha_r \mu_r) \|\underline{\varepsilon}(e_j^n)\|_{0, \Omega_i}^2, \\
 &\quad + \alpha_r \omega^2 \|e_j^n\|_{0, \Omega_j}^2 + \alpha_i \omega \langle A_{\underline{z}} e_j^n, e_j^n \rangle \}.
 \end{aligned}$$

From (4.9), we know that it suffices to show that

$$(4.18) \quad \mathcal{E}(\{\underline{e}^n, \underline{\xi}^n\}) \leq (Ch)^{-1} \mathcal{R}(\{\underline{e}^n, \underline{\xi}^n\}).$$

First, since I_{jk}^h is a bounded map of $P_1(\Gamma_{jk})$ into itself in $L^2(\Gamma_{jk})$ and $|\underline{e}_j^n|_{0, \partial\Omega_j}^2 \leq Ch^{-1} \|e_j^n\|_{0, \Omega_j}^2$,

$$\begin{aligned}
 (4.19) \quad \sum_{j,k} \int_{\Gamma_{jk}} |\alpha I_{jk}^h e_j^n|^2 ds &\leq C \sum_{j,k} \int_{\Gamma_{jk}} |\alpha e_j^n|^2 ds \\
 &\leq C |\alpha|^2 h^{-1} \sum_j \|e_j^n\|_{0, \Omega_j}^2 \\
 &\leq C_1(\alpha, \omega) h^{-1} \mathcal{R}(\{\underline{e}^n, \underline{\xi}^n\}),
 \end{aligned}$$

given the assumptions on α_i and α_r .

Next, an essentially standard scaling argument yields

$$C_j \leq Ch^{-1},$$

where C_j is defined by (4.14ii). Hence, by (4.15) and the assumptions on α_i and α_r ,

$$(4.20) \quad \sum_{j,k} \int_{\Gamma_{jk}} |\underline{\xi}_{jk}|^2 ds \leq Ch^{-1} \sum_j |\text{Re } a(e_j^n, e_j^n)_{\Omega_j}| \leq C_2(\alpha) h^{-1} \mathcal{R}(\{\underline{e}^n, \underline{\xi}^n\}),$$

if α_i and α_r satisfy $\lambda_i \alpha_i > 2 \lambda_r \alpha_r$ and $\mu_i \alpha_i > 2 \mu_r \alpha_r$.

Finally, it follows from (4.8), (4.19) and (4.20) that

$$\begin{aligned} \mathcal{E}(\{\underline{e}^n, \underline{\xi}^n\}) &= \sum_{j,k} \int_{\Gamma_{jk}} |\alpha I_{jk}^h e_J^n + \tau_{jk} \underline{\xi}_{jk}|^2 ds \\ &\leq 2 \sum_{j,k} \int_{\Gamma_{jk}} (|\alpha I_{jk}^h e_J^n|^2 + |\underline{\xi}_{jk}|^2) ds \\ &\leq 2 [C_1(\alpha, \omega) + C_2(\alpha)] h^{-1} \mathcal{R}(\{\underline{e}^n, \underline{\xi}^n\}). \end{aligned}$$

This completes the proof by choosing $C = \frac{1}{2} [C_1(\alpha, \omega) + C_2(\alpha)]^{-1}$.

APPENDIX

The purpose of this Appendix is to give a proof of Theorem 4.1. The argument to be given below represents a natural nonconforming analogue of the argument developed by Schatz [26] for treating Galerkin methods for the Dirichlet problem for non-coercive operators. We include a proof here since, to our knowledge, no proof is available in the literature. The analogue of the Schatz argument for mixed methods was given by Douglas and Roberts in [9].

To prove the theorem we need the following lemmas.

LEMMA A.1 : *If \underline{u} and \underline{u}_h are the solutions of (2.1) and (4.1), respectively, then*

$$(A.1) \quad \|\underline{u} - \underline{u}_h\|_h^2 \leq C[\omega^2 \|\underline{u} - \underline{u}_h\|_{0,\Omega}^2 + h^2(1 + \omega^2 h^2) |\underline{u}|_{2,\Omega}^2].$$

Proof : Let

$$\|\underline{v}_h\|_h = \sum_{K \in \mathcal{T}_h} \operatorname{Re}(\sigma(\underline{v}_h), \underline{\varepsilon}(\underline{v}_h))_K, \quad \forall \underline{v}_h \in V_h.$$

For any $\underline{v}_h \in V_h$,

$$\begin{aligned} & \|\underline{u}_h - \underline{v}_h\|_h^2 - \omega^2 \|\underline{u}_h - \underline{v}_h\|_{0,\Omega}^2 = \operatorname{Re} [a_h(\underline{u}_h - \underline{v}_h, \underline{u}_h - \underline{v}_h)_\Omega] \\ &= \operatorname{Re} [a_h(\underline{u} - \underline{v}_h, \underline{u}_h - \underline{v}_h)_\Omega + a_h(\underline{u}_h - \underline{u}, \underline{u}_h - \underline{v}_h)_\Omega] \\ &= \operatorname{Re} [a_h(\underline{u} - \underline{v}_h, \underline{u}_h - \underline{v}_h)_\Omega + (f, \underline{u}_h - \underline{v}_h)_\Omega - a_h(\underline{u}, \underline{u}_h - \underline{v}_h)_\Omega \\ & \quad - i\omega \langle \underline{A}_{\underline{v}_h}, \underline{u}_h - \underline{v}_h \rangle_\Gamma], \end{aligned}$$

from which we deduce that

$$\begin{aligned} & \left| \|u_h - v_h\|_h^2 - \omega^2 \|u_h - v_h\|_{0,\Omega}^2 \right| \\ & \leq C \|u - v_h\|_h \|u_h - v_h\|_h + \omega^2 \|u - v_h\|_{0,\Omega} \|u_h - v_h\|_{0,\Omega} \\ & \quad + |(f, u_h - v_h)_\Omega - a_h(u, u_h - v_h)_\Omega - i\omega \langle A_{\underline{z}} u_h, u_h - v_h \rangle_\Gamma|. \end{aligned}$$

On the other hand, since \underline{u} satisfies (2.1), integrating by parts on each element yields

$$\begin{aligned} & (f, w_h)_\Omega - a_h(\underline{u}, w_h)_\Omega - i\omega \langle A_{\underline{z}} u_h, w_h \rangle_\Gamma = \\ & = \sum_k \sum_j \int_{\Gamma_{kj}} -\sigma(\underline{u}) \nu_K \bar{w}_h ds + i\omega \langle A_{\underline{z}}(\underline{u} - u_h), w_h \rangle_\Gamma, \quad \forall w_h \in V_h, \end{aligned}$$

which measures the effect of the nonconformity of V_h .

For any $\underline{u} \in \underline{H}^1(\Omega)$ and $w_h \in V_h$, we set

$$E_h(\underline{u}, w_h) = \sum_{K \in \mathcal{T}_h} \int_{\partial K} -\sigma(\underline{u}) \nu_K \bar{w}_h ds + i\omega \langle A_{\underline{z}}(\underline{u} - u_h), w_h \rangle_\Gamma.$$

From [19], we have

$$(A.3i) \quad |E_h(\underline{v}, w_h)| \leq Ch |\underline{v}|_{2,\Omega} \|w_h\|_h,$$

$$(A.3ii) \quad |E_h(\underline{v}, w_h)| \leq Ch^2 |\underline{v}|_{2,\Omega} \left(\sum_{K \in \mathcal{T}_h} |w_h|_{2,K}^2 \right)^{\frac{1}{2}},$$

for all $\underline{v} \in \underline{H}^2(\Omega)$ and $w_h \in V_h$.

Finally, the proof is completed by using (A.2), (A.3i), Korn's inequality and the interpolation properties of the Wilson element.

The next lemma is the nonconforming generalization of the Aubin-Nitsche lemma ; a proof of it can be found in [19].

LEMMA A.2 : Let \underline{u} and \underline{u}_h be the solution of (2.1) and (4.1), respectively. Then,

$$(A.4) \quad \|\underline{u} - \underline{u}_h\|_{0,\Omega} \leq C^* \sup_{\varphi \in (H^2(\Omega))^2} \left(\inf_{\varphi_h \in \underline{V}_h} \frac{|D_h(\underline{u}, \underline{u}_h, \varphi, \varphi_h)|}{\|\varphi\|_{2,\Omega}} \right),$$

where

$$\begin{aligned} D_h(\underline{u}, \underline{u}_h, \varphi, \varphi_h) &= a_h(\underline{u} - \underline{u}_h, \varphi - \varphi_h)_\Omega + \\ &+ i\omega \langle \underline{u} - \underline{u}_h, \phi - \phi_h \rangle_\Gamma - E_h(\underline{u}, \varphi_h) + E_h(\varphi, \underline{u}_h). \end{aligned}$$

Remark A.1 : Lemma A.2 was proved by using a classical duality argument in which the estimate bounding the H^2 -norm of the solution of (2.1) by the L^2 -norm of the forcing term \underline{f} plays a crucial role. Such an estimate for the solution of (2.1) was derived in [11], and from this estimate we know that the constant C^* in (A.4) depends on the frequency, ω , and is $O(\omega^3 + \omega^{-1})$.

We now are ready to give the proof of Theorem 4.1.

Proof of Theorem 4.1 : By (A.3), (A.4), the approximation properties of the Wilson element (cf. [19]), and the following inequality,

$$|v|_{0,\Gamma} \leq C \|v\|_{\frac{1}{2},\Omega}^{\frac{1}{2}} \|v\|_{\frac{1}{2},\Omega}^{\frac{1}{2}}, \quad \forall v \in \underline{V}_h \cup \underline{H}^1(\Omega),$$

we see that

$$(A.5) \quad \|\underline{u} - \underline{u}_h\|_{0,\Omega} \leq C(C^*) [(h + h^3 \omega^2) \|\underline{u} - \underline{u}_h\|_h + h^2 |u|_{2,\Omega}].$$

If (A.5) is substituted into (A.1) and if $C(C^*) \omega h < 1$,

$$(A.6) \quad \|\underline{u} - \underline{u}_h\|_h^2 \leq \frac{Ch^2(1 + C(C^*)^2 h^2 \omega^2 + h^2 \omega^2)}{1 - C(C^*)^2 \omega^2 h^2} |u|_{2,\Omega}^2.$$

The proof is completed by substituting (A.6) into (A.5).

Remark A.2 : By Remark A.1 we know that $C^* = O(\omega^3 + \omega^{-1})$. So in order to satisfy (A.6) we need to require $\omega^4 h < 1$. Thus, for large frequencies, it is not efficient to solve (2.1) using the Wilson nonconforming finite element

method, unless the amplitude of the component of the wave associated with any large ω is small, as it normally is in practical applications. Nevertheless, the method is efficient for solving the problem when the frequency is not too large.

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