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M2AN - Modélisation mathématique et analyse numérique, tome
30, n° 2 (1996), p. 123-155

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ON THE CONVERGENCE RATE OF SPECTRAL APPROXIMATION FOR THE EQUATIONS FOR NONHOMOGENEOUS ASYMMETRIC FLUIDS (*)

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Abstract. — *We study the convergence rate of solutions of spectral semi-Galerkin approximations for the equations for the motion of a nonhomogeneous incompressible asymmetric fluid in a bounded domain. We find error estimates that are optimal in the H^1 -norm as well as improved estimates in the L^2 -norm.*

Résumé. — *On étudie le taux de convergence d'une approximation de type semi-Galerkin spectrale vers la solution des équations du mouvement d'un fluide assymétrique incompressible non-homogène dans un domaine borné. On trouve des estimations d'erreur qui sont optimales dans la norme H^1 ainsi que des estimations améliorées dans la norme L^2 .*

1. INTRODUCTION

In this paper we will study the convergence rate of solutions of spectral semi-Galerkin approximations for the equations for the motion of a nonhomogeneous viscous incompressible asymmetric fluid. These equations are considered in a bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3 , with boundary Γ , in a time interval $[0, T]$. To describe them let $u(x, t) \in \mathbb{R}^n$, $w(x, t) \in \mathbb{R}^n$, $\rho(x, t) \in \mathbb{R}$ and $p(x, t) \in \mathbb{R}$ denote, respectively, the unknown velocity, angular velocity of rotation of the fluid particles, the density and the pressure at a point $x \in \Omega$, at a time $t \in [0, T]$. Then, the governing equations are

$$\left\{ \begin{array}{l} \rho \frac{\partial u}{\partial t} + \rho(u \cdot \nabla) u - (\mu + \mu_r) \Delta u + \text{grad } p = 2 \mu_r \text{rot } w + \rho f, \\ \text{div } u = 0, \\ \rho \frac{\partial w}{\partial t} + \rho(u \cdot \nabla) w - (C_a + C_d) \Delta w - (C_0 + C_d - C_a) \nabla \text{div } w \\ \quad + 4 \mu_r w = 2 \mu_r \text{rot } u + \rho g, \\ \frac{\partial \rho}{\partial t} + (u \cdot \nabla) \rho = 0, \end{array} \right. \quad (1.1)$$

Manuscript received February 11, 1993 ; revised March 7, 1994.

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together with the following boundary and initial conditions

$$\begin{cases} u = 0 & \text{on } \Gamma \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \Omega, \\ w = 0 & \text{on } \Gamma \times (0, T), \\ w(x, 0) = w_0(x) & \text{in } \Omega, \\ \rho(x, 0) = \rho_0(x) & \text{in } \Omega. \end{cases} \quad (1.2)$$

where, for simplicity of exposition we have taken homogeneous boundary conditions.

Here $f(x, t)$ and $g(x, t)$ are respectively known external sources of linear and angular momentum of particles. The positive constants $\mu, \mu_r, C_0, C_a, C_d$ characterize isotropic properties of the fluid; μ is the usual Newtonian viscosity; μ_r, C_0, C_a, C_d are new viscosities related to the asymmetry of the stress tensor, and in consequence related to the appearance of the field of internal rotation w ; these constants satisfy $C_0 + C_d > C_a$. The expressions $\text{grad}, \Delta, \text{div}$ and rot denote the gradient, Laplacian, divergence and rotational operators, respectively (we also denote the gradient by ∇ and $\frac{\partial u}{\partial t}$ by u_t); the i -th component of $(u \cdot \nabla)u$ and $(u \cdot \nabla)w$ in cartesian coordinates are given by

$$[(u \cdot \nabla)u]_i = \sum_{j=1}^n u_j \frac{\partial u_i}{\partial x_j} \quad \text{and} \quad [(u \cdot \nabla)w]_i = \sum_{j=1}^n u_j \frac{\partial w_i}{\partial x_j}$$

respectively; also $(u \cdot \nabla)\rho = \sum_{j=1}^n u_j \frac{\partial \rho}{\partial x_j}$.

For the derivation and physical discussion of equations (1.1) see Petrosyan [9] and Condiff, Dahler [2]. We observe that this model of fluid includes as a particular case the classical Navier-Stokes, which has been much studied (see, for instance, the classical books by Ladyzhenskaya [4] and Temam [15] and the references there in). It also includes the reduced model of the nonhomogeneous Navier-Stokes equations, which has been less studied than the previous case (see for instance Simon [14], Kim [3], Ladyzhenskaya and Solonnikov [5] and Salvi [13]).

Concerning the generalized model of fluids considered in this paper, Lukaszewicz [8] established the local existence of weak solutions for (1.1), (1.2) under certain assumptions by using linearization and an almost fixed

point theorem. In that same paper Lukaszewicz remarked about the possibility of proving the existence of strong solutions (under stronger hypothesis) by using the techniques of [6] and [7] (linearization and fixed point theorems ; [6] and [7] assume constant density).

More interested in techniques directly related with numerical applications, Boldrini and Rojas-Medar [1] established the local (and also global) existence of strong solutions of (1.1), (1.2) by using the spectral semi-Galerkin method (see Boldrini and Rojas-Medar [1] and also the next section for the precise statements of the results). Here, the word spectral is used in the sense that the eigenfunctions of the associated Stokes and Laplacian operators are used as the approximation basis.

Since Galerkin methods are much used in numerical simulations, it is important to derive error estimates for them, even in the case of spectral Galerkin method, as a preparation and guide for the more practical finite element Galerkin method.

In this paper we are interested in establishing such error estimates and the convergence rates of these spectral approximations in several norms. But, before we describe our results, let us briefly comment related results.

Rautmann in [10] gave a systematic development of error estimates for the spectral Galerkin approximations for the solutions of the classical Navier-Stokes equations. Salvi in [12] gave analogous error estimates for the reduced model of nonhomogeneous viscous incompressible fluids. However, although the statement of Theorem 3, p. 203, in [12] furnishes an optimal rate (λ_{n+1}^{-1} , where λ_{n+1} is the $(n+1)$ -th eigenvalue of the Stokes operator), this is not correct as it can be seen by the last inequality in the proof (p. 204 in [12]). The rate actually obtained was $\lambda_{n+1}^{-1/2}$.

In this paper we consider the convergence rate of the spectral semi-Galerkin approximations for the solutions of the more general fluid model (1.1). We show that there is optimal rate of convergence in the H^1 -norm (see Theorem 3.3), improving in the particular case of the reduced model the result in Salvi [12]. Differently as in the case of the classical Navier-Stokes equations, for which optimal L^2 -error estimates can be obtained (see Rojas-Medar and Boldrini [11]), in this case we are only able to obtain an improved L^2 -error estimates as compared to the trivial one that derives directly from the H^1 -estimate (see Theorem 4.2). Also, L^∞ and heigher order error estimates are proved (see Theorems 3.4 and 3.5).

Finally, we would like to mention that optimal H^1 and L^2 -error estimates for spectral Galerkin approximations for the Boussinesq and magnetohydrodynamic type equations can be obtained. These results will appear elsewhere.

2. PRELIMINARIES

Let $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3 , be a bounded domain with smooth boundary Γ (class C^3 is enough).

We will consider the usual Sobolev spaces

$$W^{m,q}(D) = \{f \in L^q(D), \|\partial^\alpha f\|_{L^q(D)} < +\infty, |\alpha| \leq m\},$$

$m = 0, 1, 2, \dots$, $1 \leq q \leq +\infty$, $D = \Omega$ or $\Omega \times (0, T)$, $0 < T < +\infty$, with the usual norm. When $q = 2$, we denote by $H^m(D) = W^{2,q}(D)$ and $H_0^m(D) = \text{closure of } C_0^\infty(D) \text{ in } H^m(D)$. If B is a Banach space, we denote by $L^q([0, T]; B)$ the Banach space of the B -valued functions defined in the interval $[0, T]$ that are L^q -integrable in the sense of Bochner. We shall consider the following spaces of divergence free functions

$$C_{0,\sigma}^\infty(\Omega) = \{v \in (C_0^\infty(\Omega))^n; \operatorname{div} v = 0 \text{ in } \Omega\},$$

$$H = \text{closure of } C_{0,\sigma}^\infty(\Omega) \text{ in } (L^2(\Omega))^n,$$

$$V = \text{closure of } C_{0,\sigma}^\infty(\Omega) \text{ in } (H^1(\Omega))^n.$$

Throughout the paper, P denotes the orthogonal projection from $(L^2(\Omega))^n$ onto H and $A = -PA$ is the Stokes operator. We will denote respectively by φ^k and λ_k the eigenfunctions and eigenvalues of the Stokes operator defined in $V \cap (H^2(\Omega))^n$. It is well known that $\{\varphi^k(x)\}_{k=1}^\infty$ form an orthogonal complete system in the spaces H , V and $V \cap (H^2(\Omega))^n$ with the usual inner product, (u, v) , $(\nabla u, \nabla v)$ and $(P \Delta u, P \Delta v)$, respectively. Here (\cdot, \cdot) denotes the inner product in $L^2(\Omega)$; also in this paper we will denote the L^2 -norm by $\|\cdot\|$.

For each $k \in \mathbb{N}$, we denote P_k the orthogonal projection from $(L^2(\Omega))^n$ onto $V_k = \text{span}[\varphi_1, \dots, \varphi_k]$.

It is easy to see that P and $P_k, P_m, k, m \in \mathbb{N}$ satisfy for all $f, g \in (L^2(\Omega))^n$

$$(P_k f, g) = (f, P_k g)$$

$$(P f, g) = (f, P g)$$

$$((P_m - P_k) f, g) = (f, (P_m - P_k) g)$$

$$((P - P_k) f, g) = (f, (P - P_k) g).$$

The following results can be found in Rautmann's paper [10]. If $v \in V$, then there holds

$$\|v - P_k v\|^2 \leq \frac{1}{\lambda_{k+1}} \|\nabla v\|^2. \quad (2.1)$$

Also, if $v \in V \cap (H^2(\Omega))^n$, we have

$$\|v - P_k v\|^2 \leq \frac{1}{\lambda_{k+1}^2} \|P \Delta v\|^2 \quad (2.2)$$

$$\|\nabla v - \nabla P_k v\|^2 \leq \frac{1}{\lambda_{k+1}} \|P \Delta v\|^2. \quad (2.3)$$

Now we observe that if $f \in (H^1(\Omega))^n$, from (2.2) we have

$$\|(I - P_k) P f\|^2 \leq \frac{1}{\lambda_{k+1}} \|\nabla P f\|^2. \quad (2.4)$$

Also, since $P : (H^1(\Omega))^n \rightarrow (H^1(\Omega))^n$ is a continuous operator (see [16]), we have

$$\|\nabla P f\|^2 \leq C \|f\|_{H^1}^2. \quad (2.5)$$

Thus, for all $f \in (H^1(\Omega))^n$, we have

$$\|(I - P_k) P f\| \leq \frac{C}{\lambda_{k+1}} \|f\|_{H^1}^2, \quad (2.6)$$

or equivalently, since $PP_k = P_k P = P_k$, we obtain

$$\|P f - P_k f\|^2 \leq \frac{C}{\lambda_{k+1}} \|f\|_{H^1}^2. \quad (2.7)$$

We observe that (2.4)-(2.7) also holds with any P_m , $m > k$, in place of P . Analogously, we have for any $f \in (H^2(\Omega))^n$

$$\|(I - P_k) P f\|^2 \leq \frac{C}{\lambda_{k+1}^2} \|f\|_{H^2}^2. \quad (2.8)$$

An easy consequence of the L^2 -orthogonality of the $\{\varphi_k\}_{k=1}^\infty$ is the following: let $m > k$, $m, k \in \mathbb{N}$, $f \in (L^2(\Omega))^n$ and $v_m \in V_m$, $v_k \in V_k$; then

$$((P_m - P_k) f, v_m - v_k) = (f, (I - P_k) v_m). \quad (2.9)$$

Now, let us denote $B = -\Delta : D(B) \subset (L^2(\Omega))^n \rightarrow (L^2(\Omega))^n$ where $D(B)$ denotes the domain of $-\Delta$ with the Dirichlet boundary conditions and $\phi^k(x)$, α_k be the eigen-functions and eigenvalues of B , respectively. As it is well known, all the above properties have a corresponding one for B .

We will denote R_k , $k \in \mathbb{N}$, the orthogonal projection of $(L^2(\Omega))^n$ onto $\text{span} [\phi_1, \dots, \phi_k]$.

It will be also necessary the following variant of the Gronwall's inequality (see Rautmann [10]).

LEMMA 2.1 : *Let a function $a(t) \geq 0$ be absolutely continuous with $a'(t) \geq 0$ and $b(t) \geq 0$ summable in $[0, T]$. Assume the integral inequality*

$$\zeta(t) + \int_0^t \zeta^*(s) ds \leq \frac{a(t)}{\lambda} + \int_0^t b(s) \zeta(s) ds$$

holds for the positive continuous functions ζ and ζ^ on $[0, T]$ with a constant $\lambda > 0$.*

Then we have

$$\zeta(t) + \int_0^t \zeta^*(s) ds \leq \frac{A(t)}{\lambda}$$

with

$$A(t) = \left\{ 1 + \int_0^t b(s) dx \right\} \chi(t), \quad \chi(t) = a(t) \exp \int_0^t b(s) ds .$$

Concerning the existence of solutions for equations (1.1), (1.2), they can be obtained by using a semi-Galerkin approximation. That is, we consider a Galerkin approximations

$$u^k(x, t) = \sum_{i=1}^k C_{ik}(t) \phi^i(x), \quad w^k(x, t) = \sum_{i=1}^k d_{ik}(t) \phi^i(x)$$

for the velocity and rotation of particles respectively and an infinite dimensional approximation $\rho^k(x, t)$ for the density satisfying the following equations :

$$\left\{ \begin{array}{l} P_k(\rho^k u_t^k + \rho^k u^k \nabla u^k - \rho^k f - 2 \mu_r \operatorname{rot} w^k) + (\mu + \mu_r) A u^k = 0, \\ R_k(\rho^k w_t^k + \rho^k u^k \nabla w^k - \rho^k g - 2 \mu_r \operatorname{rot} u^k - (C_0 + C_d - C_a) \nabla \operatorname{div} w^k + \\ \quad + 4 \mu_r w^k) + (C_a + C_d) B w^k = 0, \\ \frac{\partial \rho^k}{\partial t} + u^k \nabla \rho^k = 0, \\ u^k(0) = P_k u_0, \\ w^k(0) = R_k w_0, \\ \rho^k(0) = \rho_0. \end{array} \right. \quad (2.10)$$

As before P_k and R_k are the orthogonal projections onto the spaces spanned by $\{\phi_1, \dots, \phi^k\}$ and $\{\phi^1, \dots, \phi^k\}$, respectively.

It can be proved that (u^k, w^k, ρ^k) converges in appropriate sense to a solution (u, w, ρ) of (1.1), (1.2). As we said in the Introduction, in this paper we are interested in deriving error bounds, that is, estimates for $\|u - u^k\|$, $\|w - w^k\|$, $\|\rho - \rho^k\|$ in suitable norms in terms of powers of $1/\lambda_{k+1}$ and $1/\alpha_{k+1}$.

These error estimates will be derived in the following sections and will be based on the next result. To ease the notation, in the rest of this paper the functions which are \mathbb{R} or \mathbb{R}^n valued will not be notationally distinguished ; the distinction will be clear from the context.

THEOREM 2.2 : (*Boldrini and Rojas-Medar*[1]). *Let the initial values satisfy $u_0 \in V \cap (H^2(\Omega))^n$, $w_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $\rho_0 \in W^{1,\infty}(\Omega)$ and the external fields $f, g \in L^2(0, T; H^1(\Omega))$ with $f_t, g_t \in L^2(0, T; L^2(\Omega))$. Then, on a (possibly small) time interval $[0, T]$ the problem (1.1) and (1.2) has a unique strong solution (u, w, ρ) . That is, there are functions u, w, ρ such that*

$$P(\rho u_t + \rho u \nabla u - 2 \mu_r \operatorname{rot} w - \rho f - (\mu + \mu_r) \Delta u) = 0$$

holds a.e. in $\Omega \times [0, T]$;

$$\begin{aligned} \rho w_t + \rho u \nabla w - 2 \mu_r \operatorname{rot} u - \rho g - (C_a + C_d) \Delta w - \\ - (C_0 + C_d - C_a) \nabla \operatorname{div} w + 4 \mu_r w = 0 \end{aligned}$$

holds a.e. in $\Omega \times [0, T]$;

$$\frac{\partial \rho}{\partial t} + u \nabla \rho = 0$$

holds in the $L^2(\Omega \times [0, T])$ sense. Moreover,

$$\rho \in W^{1,\infty}(\Omega \times [0, T]) ,$$

$$u \in C([0, T] ; H^2(\Omega) \cap V) \cap L^2(0, T ; H^3(\Omega)) \cap L^2([0, T] ; L^\infty(\Omega)) \\ \cap L^p([0, T] ; H^{3-\varepsilon}(\Omega)) \cap L^\infty_{\text{Loc}}([0, T] ; H^{3-\varepsilon}(\Omega)) ,$$

$$u_t \in C([0, T] ; L^2(\Omega) \cap V) \cap L^2(0, T ; H^{2-\varepsilon}(\Omega)) \cap L^p([0, T] ; H^{1-\varepsilon}(\Omega)) \\ \cap L^2_{\text{Loc}}([0, T] ; H^2(\Omega)) \cap L^\infty_{\text{Loc}}([0, T] ; H^1(\Omega)) ,$$

$$u_{tt} \in L^2_{\text{Loc}}(0, T ; H)$$

$$w \in C([0, T] ; H^2(\Omega) \cap H^1_0(\Omega)) \cap L^2(0, T ; H^3(\Omega)) \cap L^2([0, T] ; L^\infty(\Omega)) \\ \cap L^p(0, T ; H^{3-\varepsilon}(\Omega)) \cap L^\infty_{\text{Loc}}([0, T] ; H^{3-\varepsilon}(\Omega)) ,$$

$$w_t \in C([0, T] ; L^2(\Omega) \cap H^1_0(\Omega)) \cap L^2(0, T ; H^{2-\varepsilon}(\Omega)) \\ \cap L^p([0, T] ; H^{1-\varepsilon}(\Omega)) \cap L^2_{\text{Loc}}(0, T ; H^2(\Omega)) \cap L^\infty_{\text{Loc}}([0, T] ; H^1(\Omega)) ,$$

$$w_{tt} \in L^2_{\text{Loc}}(0, T ; L^2(\Omega))$$

for all $\varepsilon > 0$ and $1 < p < +\infty$.

Remark : Actually it is possible to prove that the strong solution of Theorem 2.2 is global either if $n = 2$ or if we take small enough initial data when $n = 3$ (Boldrini and Rojas-Medar [1]).

The above result depends on the certain estimates for the approximations (u^k, w^k, ρ^k) , and since these estimates will be also necessary in this paper, we describe them in the following.

LEMMA 2.3 : (*Boldrini and Rojas-Medar* [1]). Let (u^k, w^k, ρ^k) be the solution of (2.8). Then, they satisfies

$$\|\nabla u^k(t)\|^2 + \|\nabla w^k(t)\|^2 + \|\operatorname{div} w^k(t)\|^2 \leq F_1(t)$$

$$\int_0^t \{ \|\Delta w^k(s)\|^2 + \|P \Delta u^k(s)\|^2 \} ds \leq F_2(t),$$

$$\int_0^t \{ \|w_t^k(s)\|^2 + \|u_t^k(s)\|^2 \} ds \leq F_3(t),$$

$$\|w_t^k(t)\|^2 + \|u_t^k(t)\|^2 + \int_0^t \{ \|\nabla w_t^k(s)\|^2 + \|\nabla u_t^k(s)\|^2 + \|\operatorname{div} w_t^k(s)\| \} ds \leq F_4(t),$$

$$\|P \Delta u^k(t)\|^2 + \|\Delta w^k(t)\|^2 \leq F_5(t),$$

$$\int_0^t \{ \|u^k(s)\|_{H^3}^2 + \|w^k(s)\|_{H^3}^2 \} ds \leq F_6(t),$$

$$\int_0^t \{ \|\nabla u^k(s)\|_{L^\infty}^2 + \|\nabla w^k(s)\|_{L^\infty}^2 \} ds \leq F_7(t),$$

$$\int_0^t \sigma(s) \{ \|u_s^k(s)\|^2 + \|w_s^k(s)\|^2 \} ds + \sigma(t) \{ \|\nabla u_t^k(t)\|^2 + \|\nabla w_t^k(t)\| \} \leq F_8(t),$$

$$\sigma(t) \{ \|u^k(t)\|_{H^3}^2 + \|w^k(t)\|_{H^3}^2 \} \leq F_9(t),$$

$$\sigma(t) \{ \|\nabla u^k(t)\|_{L^\infty}^2 + \|\nabla w^k(t)\|_{L^\infty}^2 \} \leq F_{10}(t),$$

$$\int_0^t \sigma(s) \{ \|P \Delta u_t^k(s)\|^2 + \|\Delta w_t^k(s)\|^2 \} ds \leq F_{11}(t),$$

$$\alpha \leq \rho^k \leq \beta, \quad (0 < \alpha = \operatorname{ess\,inf} \rho_0, \beta = \operatorname{ess\,sup} \rho_0)$$

$$\|\nabla \rho^k(t)\|_{L^\infty}^2 \leq F_{12}(t),$$

$$\|\rho_t^k(t)\|_{L^\infty}^2 \leq F_{13}(t),$$

Here, $\sigma(t) = \min \{1, t\}$.

The same estimates hold for (u, w, ρ) .

In the following we assume the bounds on the right-hand sides of the above estimates are chosen in such a way that they are monotonously increasing in time.

Remark: The above estimates implies that the approximations (u^k, w^k, ρ^k) converges to the solution (u, w, ρ) in the senses indicated below

(i) $u^k \rightarrow u, w^k \rightarrow w$ strongly in $L^p(0, T; H^{3-\varepsilon}(\Omega))$ and weakly- \star in $L^\infty_{\text{Loc}}(0, T; H^3(\Omega))$

(ii) $u_t^k \rightarrow u_t, w_t^k \rightarrow w_t$ weakly in $L^\infty_{\text{Loc}}(0, T; H^1(\Omega))$ and weakly in $L^2(0, T; H^{2-\varepsilon}(\Omega))$ in $L^p(0, T; H^{1-\varepsilon}(\Omega))$ and in $L^2_{\text{Loc}}(0, T; H^2(\Omega))$

(iii) $u_{tt}^k \rightarrow u_{tt}, w_{tt}^k \rightarrow w_{tt}$ weakly in $L^2_{\text{Loc}}(0, T; H^2(\Omega))$

(iv) $\rho^k \rightarrow \rho$ strongly in $L^p(0, T; C^{0,\gamma}(\Omega))$ $0 \leq \gamma < 1$

(v) $\nabla \rho^k \rightarrow \nabla \rho$ weakly- \star in $L^\infty(\Omega \times [0, T])$

(vi) $\rho_t^k \rightarrow \rho_t$ weakly- \star in $L^\infty(\Omega \times [0, T])$.

Here, as before, the above is true for all $\varepsilon > 0$ and $1 < p < +\infty$.

Finally, we would like to say that as is usual we will denote by C a generic constant depending at most on Ω and the fixed parameters in the problem $(\mu, \mu_r, C_a, C_d, C_0$ and the initial conditions, and also f, g and T). This will appear in most of the estimates to be obtained. When for any reason we want to emphasize the dependence of a certain constant on a given parameter we will denote this constant with a subscript.

3. ERROR BOUNDS FOR THE APPROXIMATIONS

Let $[0, T]$ be a time interval as in Theorem 2.2; u^k, w^k, ρ^k the k -th approximations of u, w, ρ respectively. We begin by considering the following.

THEOREM 3.1: *Suppose the assumptions of Theorem 2.2 hold. Then, the approximations u^k, w^k, ρ^k satisfy*

$$\|\rho(t) - \rho^k(t)\|^2 \leq G_0(t) \left\{ \frac{1}{\alpha_{k+1}} + \frac{1}{\lambda_{k+1}} \right\}, \tag{3.1}$$

$$\|u(t) - u^k(t)\|^2 + \|w(t) - w^k(t)\|^2 + \int_0^t \|\nabla u(s) - \nabla u^k(s)\|^2 ds$$

$$+ \int_0^t \|\nabla w(s) - \nabla w^k(s)\|^2 ds + \int_0^t \|\text{div } w(s) - \text{div } w^k(s)\|^2 ds$$

$$\leq G_1(t) \left\{ \frac{1}{\alpha_{k+1}} + \frac{1}{\lambda_{k+1}} \right\} \tag{3.2}$$

for any $t \in [0, T]$. The continuous functions $G_0(t)$, $G_1(t)$ depend on t and on the functions $F_i(t)$ in Lemma 2.3. (3.1) and (3.2) hold also with any u^m , w^m , ρ^m instead of u , w , ρ for $m > k$.

Proof: First we suppose (3.2) true. The difference $\rho^m - \rho^k$ with $m > k$ satisfies

$$(\rho^m - \rho^k)_t + u^m \nabla(\rho^m - \rho^k) = - (u^m - u^k) \nabla \rho^k$$

and

$$\rho_0^m - \rho_0^k = 0.$$

Let $z^m(t, s, x)$ be the solution of the Cauchy problem

$$z_t^m = u^m(z^m, s)$$

$$z^m = x \quad \text{for } t = s.$$

Then, by using the characteristic method, we obtain

$$\rho^m(x, t) - \rho^k(x, t) = - \int_0^t \varphi_{m,k}(z^m(s, t, x), s) ds$$

where

$$\varphi_{m,k}(z^m, t) = (u^m(z^m, t) - u^k(z^m, t)) \nabla \rho^k(z^m, t).$$

Bearing in mind properties of z^m (see [5, pp. 93-96]), we get

$$\|\rho^m - \rho^k\| \leq \int_0^t \|u^m - u^k\| \|\nabla \rho^k\|_{L^\infty} ds \leq C \int_0^t \|u^m - u^k\| ds$$

thanks to the estimates in Lemma 2.3. Hence by taking the limit as m goes to infinity (see the Remark after Lemma 2.3), we get

$$\|\rho - \rho^k\| \leq C \int_0^t \|u - u^k\| ds.$$

Consequently, by using (3.2), we get (3.1).

Now, we prove (3.2). We consider the following equations ($m > k$)

$$R_m(\rho^m w_t^m + \rho^m u^m \nabla w^m - \rho^m g + 4 \mu_r w^m - 2 \mu_r \operatorname{rot} u^m) + (C_a + C_d) B w^m \\ - (C_0 + C_d - C_a) R_m \nabla \operatorname{div} w^m = 0 \quad (3.3)$$

$$R_k(\rho^k w_t^k + \rho^k u^k \nabla w^k - \rho^k g + 4 \mu_r w^k - 2 \mu_r \operatorname{rot} u^k) + (C_a + C_d) B w^k \\ - (C_0 + C_d - C_a) R_k \nabla \operatorname{div} w^k = 0. \quad (3.4)$$

Subtracting (3.3) from (3.4), the differences

$$\xi = w^m - w^k \quad \text{and} \quad \eta = u^m - u^k$$

satisfy

$$R_m \rho^m w_t^m - R_k \rho^k w_t^k + R_m \rho^m u^m \nabla w^m - R_k \rho^k u^k \nabla w^k - \\ - R_m \rho^m g + R_k \rho^k g + 4 \mu_r R_m \\ - 4 \mu_r R_k w^k - 2 \mu_r R_m \operatorname{rot} u^m + 2 \mu_r R_k \operatorname{rot} u^k - (C_0 + C_d - C_a) R_m \nabla \operatorname{div} w^m \\ + (C_0 + C_d - C_a) R_k \nabla \operatorname{div} w^k + (C_a + C_d) B \xi = 0, \\ \xi(0) = (R_m - R_k) w_0. \quad (3.5)$$

We take the inner product in $L^2(\Omega)$ of (3.5) with ξ , after some computation, we obtain

$$(C_a + C_d) \|\nabla \xi\|^2 + 4 \mu_r \|\xi\|^2 + (C_0 + C_d - C_a) \|\operatorname{div} \xi\|^2 + \\ + \frac{1}{2} \frac{d}{dt} \|(\rho^m)^{1/2} \xi\|^2 \\ = ((R_m - R_k) (4 \mu_r w^k - \rho^m g - 2 \mu_r \operatorname{rot} u^k + \rho^k w_t^k + \rho^k u^k \nabla w^k + \\ + (C_0 + C_d - C_a) \\ \nabla \operatorname{div} w^k), \xi) + (R_m(\rho^m - \rho^k) (g + w_t^k + u^m \nabla w^m), \xi) + \frac{1}{2} (\rho_t^m \xi, \xi) \\ + (R_m(\rho^k \eta \nabla w^m - 2 \mu_r \operatorname{rot} \eta + \rho^k u^k \nabla \xi), \xi). \quad (3.6)$$

By using the Young's inequality, we get

$$\begin{aligned} \frac{1}{2} |(\rho_t^m \xi, \xi)| &\leq \frac{1}{2} \|\rho_t^m\|_{L^\infty}^2 \|\xi\|^2 \leq C \|\xi\|^2 \\ |-2 \mu_r (R_m \operatorname{rot} \eta, \xi)| &\leq \bar{\varepsilon} \|\nabla \eta\|^2 + C_{\bar{\varepsilon}} \|\xi\|^2. \end{aligned} \quad (3.7)$$

Also, we observe that

$$\begin{aligned} &|(R_m(\rho^m - \rho^k)(g + w_t^k + u^m \nabla w^m), \xi)| \\ &\leq \|\rho^m - \rho^k\|^2 \{C_\varepsilon \|g\|_{H^1}^2 + C_\varepsilon \|\nabla w_t^k\|^2 + C_\varepsilon \|u^m\|_{L^\infty}^2 \|\nabla w^m\|_{L^4}^2\} + 3 \varepsilon \|\nabla \xi\|^2. \end{aligned} \quad (3.8)$$

Consequently, by virtue the estimates of Theorem 2.2, and (2.7), we get

$$\begin{aligned} &|(R_m(\rho^m - \rho^k)(g + w_t^k + u^m \nabla w^m), \xi)| \\ &\leq C \int_0^t \|\eta\|^2 ds \{C_\varepsilon \|g\|_{H^1}^2 + C_\varepsilon \|\nabla w_t^k\|^2 + C_\varepsilon \|P \Delta u^m\|^2 \|\Delta w^m\|^2\} + 3 \varepsilon \|\nabla \xi\|^2 \\ &\leq C \int_0^t \|\eta\|^2 ds \{C_1 + C_\varepsilon \|\nabla w_t^k\|^2\} + 3 \varepsilon \|\nabla \xi\|^2. \end{aligned} \quad (3.9)$$

Moreover, bearing in mind the property (2.9) we have

$$\begin{aligned} &|(R_m - R_k)(4 \mu_r w^k + 2 \mu_r \operatorname{rot} u^k + \rho^k w_t^k + \rho^m g + \rho^k u^k \nabla w^k \\ &\quad + (C_0 + C_d - C_a) \nabla \operatorname{div} w^k, \xi)| \\ &\leq \{4 \mu_r \|w^k\| + 2 \mu_r C \|\nabla u^k\| + \beta \|w_t^k\| + \beta \|g\| + \beta \|u^k \nabla w^k\| \\ &\quad + (C_0 + C_d - C_a) \|\nabla \operatorname{div} w^k\|\} \frac{\|\Delta w^k\|}{\alpha_{k+1}} \\ &\leq \frac{C}{\alpha_{k+1}} \end{aligned} \quad (3.10)$$

by virtue the estimates in Lemma 2.2.

Now, we have

$$\begin{aligned}
 & |(R_m(\rho^k \eta \nabla w^m - \rho^k u^k \nabla \xi), \xi)| \\
 & \leq C_\varepsilon \beta^2 \|\Delta w^m\|^2 \|\eta\|^2 + C_\varepsilon \beta^2 \|P \Delta u^k\|^2 \|\xi\|^2 + 2\varepsilon \|\nabla \xi\|^2 \\
 & \leq C\{\|\eta\|^2 + \|\xi\|^2\} + 2\varepsilon \|\nabla \xi\|^2 \tag{3.11}
 \end{aligned}$$

again thanks to the estimates in Lemma 2.3.

From the estimates (3.7)-(3.11) we get the differential inequality :

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \|(\rho^m)^{1/2} \xi\|^2 + (C_a + C_d) \|\nabla \xi\|^2 + 4\mu_r \|\xi\|^2 + \\
 & \qquad \qquad \qquad + (C_0 + C_d - C_a) \|\operatorname{div} \xi\|^2 \\
 & \leq C\{\|\eta\|^2 + \|\xi\|^2\} + C \int_0^t \|\eta\|^2 ds \{C + \|\nabla w_t^k\|^2\} + \frac{C}{\alpha_{k+1}} + \bar{\varepsilon} \|\nabla \eta\|^2.
 \end{aligned}$$

By integrating this last inequality, we get for any $\bar{\varepsilon} > 0$

$$\begin{aligned}
 & \|(\rho^m)^{1/2} \xi\|^2 + (C_a + C_d) \int_0^t \|\nabla \xi\|^2 ds + 4\mu_r \int_0^t \|\xi\|^2 ds \\
 & \qquad \qquad \qquad + (C_0 + C_d - C_a) \int_0^t \|\operatorname{div} \xi\|^2 ds \\
 & \leq C \int_0^t \{\|\eta\|^2 + \|\xi\|^2\} ds + \frac{Ct}{\alpha_{k+1}} + C \int_0^t \|\eta\|^2 ds \left\{ Ct + \int_0^t \|\nabla w_t^k\|^2 \right\} ds \\
 & \qquad \qquad \qquad + \|\rho_0^{1/2} \xi_0\|^2 + \bar{\varepsilon} \int_0^t \|\nabla \eta\|^2 ds \\
 & \leq C \int_0^t \{\|\eta\|^2 + \|\xi\|^2\} ds + \frac{Ct}{\alpha_{k+1}} + \|\rho_0^{1/2} \xi_0\|^2 + \bar{\varepsilon} \int_0^t \|\nabla \eta\|^2 ds \tag{3.12}
 \end{aligned}$$

in virtue of the estimates in Lemma 2.3.

Similarly, for $\eta = u^m - u^k$, we have for any $\delta > 0$

$$\begin{aligned} & (\mu + \mu_r) \int_0^t \|\nabla \eta\|^2 ds + \|\eta\|^2 \\ & \leq \|\eta_0\|^2 + C \int_0^t \|\eta\|^2 ds + \frac{Ct}{\alpha_{k+1}} + \delta \int_0^t \|\nabla \xi\|^2 ds. \end{aligned} \quad (3.13)$$

Adding inequalities (3.12) and (3.13), and taking $\bar{\varepsilon} > 0$ and $\delta > 0$ in such way that $(C_a + C_d) - \delta > 0$ and $\mu + \mu_r - \bar{\varepsilon} > 0$, we obtain the integral inequality

$$\begin{aligned} & \|\xi\|^2 + \|\eta\|^2 + C_1 \int_0^t \{\|\nabla \xi\|^2 + \|\nabla \eta\|^2\} ds + C_2 \int_0^t \|\operatorname{div} \xi\|^2 ds \\ & \leq C_3 \int_0^t \{\|\eta\|^2 + \|\xi\|^2\} ds + \frac{C_4}{\lambda_{k+1}} + \frac{C_5}{\alpha_{k+1}} + C \|\xi_0\|^2 + \|\eta_0\|^2 \end{aligned} \quad (3.14)$$

We observe that

$$\|\xi_0\|^2 = \|(R_m - R_k) w_0\|^2 \leq \frac{C}{\alpha_{k+1}} \quad (3.15)$$

and

$$\|\eta_0\|^2 = \|(P_m - P_k) u_0\|^2 \leq \frac{C}{\lambda_{k+1}}. \quad (3.16)$$

Using (3.15) and (3.16) in (3.14), we get :

$$\begin{aligned} & \|\xi\|^2 + \|\eta\|^2 + C_1 \int_0^t \{\|\nabla \xi\|^2 + \|\nabla \eta\|^2\} ds + C_2 \int_0^t \|\operatorname{div} \xi\|^2 ds \\ & \leq C_3 \int_0^t \{\|\eta\|^2 + \|\xi\|^2\} ds + C \left\{ \frac{1}{\alpha_{k+1}} + \frac{1}{\lambda_{k+1}} \right\}. \end{aligned}$$

Now applying Gronwall’s inequality (Lemma 2.1) we obtain

$$\begin{aligned} & \|\xi\|^2 + \|\eta\|^2 + C_1 \int_0^t \{ \|\nabla \xi\|^2 + \|\nabla \eta\|^2 \} ds + C_2 \int_0^t \|\operatorname{div} \xi\|^2 ds \\ & \leq C \exp(C_3 t) \left\{ \frac{1}{\alpha_{k+1}} + \frac{1}{\lambda_{k+1}} \right\}. \end{aligned}$$

Now, by taking the limit on m goes to infinity (using the Remark after Lemma 2.3) on the left side we obtain (3.2). Thus Theorem 3.1 proved. ■

LEMMA 3.2 : *Under the hypotheses of Theorem 2.2, the approximation ρ^k satisfy :*

$$\|\rho(t) - \rho^k(t)\|_{L^r}^2 \leq G_2(t) \left\{ \frac{1}{\alpha_{k+1}} + \frac{1}{\lambda_{k+1}} \right\} \tag{3.17}$$

with $2 \leq r \leq 6$ for any $t \in [0, T]$. The continuous function $G_2(t)$ depend on t and on the functions $F_i(t)$ in Lemma 2.3. Also, (3.17) holds with any ρ^m instead of ρ for $m \geq k$.

Proof: First, we observe that since Ω is a bounded domain, $L^{r_1}(\Omega) \subset L^{r_2}(\Omega)$ with continuous inclusions if $r_1 \geq r_2 \geq 1$. Therefore, it is enough to take $r \in \mathbb{R}$ such that $3 \leq r \leq 6$. Let $m > k$, $m, k \in \mathbb{N}$; then

$$\rho_t^m + u^m \nabla \rho^m = 0 \tag{3.18}$$

$$\rho_t^k + u^k \nabla \rho^k = 0 \tag{3.19}$$

$$\rho^m(0) = \rho^k(0) = \rho_0.$$

Subtracting (3.19) from (3.18), the difference $\pi = \rho^m - \rho^k$ satisfies

$$\pi_t = -\eta \nabla \rho^m - u^k \nabla \pi$$

$$\pi(0) = 0$$

where $\eta = u^m - u^k$. Now, multiplying by $|\pi|^{r-1}$ and integrating over Ω , we get

$$\frac{1}{r} \frac{d}{dt} \|\pi\|_{L^r}^r = - \int_{\Omega} \eta \nabla \rho^m |\pi|^{r-1} dx \leq \|\nabla \rho^m\|_{L^\infty} \|\eta\|_{L^r} \|\pi\|_{L^r}^{r-1},$$

thus,

$$\frac{d}{dt} \|\pi\|_{L^r} \leq C \|\nabla \rho^m\|_{L^\infty} \|\nabla \eta\| \leq C \|\nabla \eta\|.$$

since $3 \leq r \leq 6$ the estimates after formula (3.16) furnishes

$$\|\pi\|_{L^r}^2 \leq C \int_0^t \|\nabla \eta\|^2 ds \leq CG_1(t) \left\{ \frac{1}{\alpha_{k+1}} + \frac{1}{\lambda_{k+1}} \right\}.$$

Thus, we have

$$\|\rho^m(t) - \rho^k(t)\|_{L^r}^2 \leq CG_1(t) \left\{ \frac{1}{\alpha_{k+1}} + \frac{1}{\lambda_{k+1}} \right\}.$$

Finally by taking the limit as m goes to infinity (see the Remark after Lemma 2.3) on the left side, we obtain (3.17). Thus, the Lemma is proved. ■

Now, we have

THEOREM 3.3 : *Under the hypotheses of Theorem 1, we have*

$$\begin{aligned} & \|\nabla(u - u^k)(t)\|^2 + \|\nabla(w - w^k)(t)\|^2 + \int_0^t \|u_t - u_t^k\|^2 ds \\ & + \int_0^t \|w_t - w_t^k\|^2 ds \leq CG_3(t) \left\{ \frac{1}{\alpha_{k+1}} + \frac{1}{\lambda_{k+1}} \right\} \end{aligned} \quad (3.20)$$

for any $t \in [0, T]$. The continuous functions $G_3(t)$ depends on t and on the functions $F_i(t)$ of Lemma 2.3. Also, (3.20) holds with any u^m, w^m instead of u, w for $m > k$.

Proof: By taking the inner product in $L^2(\Omega)$ of (3.5) with ξ_t , after some computations, we obtain :

$$\begin{aligned} & \|(\rho^m)^{1/2} \xi_t\|^2 + \frac{(C_a + C_d)}{2} \frac{d}{dt} \|\nabla \xi\|^2 + \frac{(C_0 + C_d - C_a)}{2} \frac{d}{dt} \|\operatorname{div} \xi\|^2 \\ & = ((R_m - R_k) (-\rho^k w_t^k + 2\mu_r \operatorname{rot} u^k + \rho^k g - 4\mu_r w^k + \\ & + (C_0 + C_d - C_a) \nabla \operatorname{div} w^k \\ & - \rho^k u^k \nabla w^k, \xi_t) + (R_m(\rho^m - \rho^k) (-w_t^k + g - u^m \nabla w^m), \xi_t) \\ & + (R_m(-2\mu_r \operatorname{rot} \eta - 4\mu_r \xi + \rho^k \xi \nabla w^m + \rho^k u^k \nabla \xi, \xi_t)). \end{aligned} \quad (3.21)$$

Now, we observe that

$$\begin{aligned}
& |(R_m - R_k) (-\rho^k w_t^k + 2\mu_r \operatorname{rot} u^k + \rho^k g - 4\mu_r w^k) \\
& \quad + (C_0 + C_d - C_a) \nabla \operatorname{div} w^k - \rho^k u^k \nabla w^k, \xi_t)| \\
& \leq \frac{C_\varepsilon}{\alpha_{k+1}} \{ \|\rho^k w_t^k\|_{H^1}^2 + 4\mu_r^2 \|\operatorname{rot} u^k\|_{H^1}^2 + \|\rho^k g\|_{H^1}^2 + 16\mu_r^2 \|w^k\|_{H^1}^2 \\
& \quad + \|\rho^k u^k \nabla w^k\|_{H^1}^2 + (C_0 + C_d - C_a) \|\nabla \operatorname{div} w^k\|_{H^1}^2 \} + \varepsilon \|\xi_t\|^2 \\
& \leq \frac{C_\varepsilon}{\alpha_{k+1}} \{ C_1 + C_2 \|\nabla w_t^k\|^2 + C_3 \|w^k\|_{H^3}^2 \} + \varepsilon \|\xi_t\|^2, \tag{3.22}
\end{aligned}$$

thanks to the estimates of Lemma 2.3 and 2.7. Also,

$$\begin{aligned}
& |(R_m(\rho^m - \rho^k) (-w_t^k + g - u^m \nabla w^m), \xi_t)| \\
& \leq C_\varepsilon G_1(t) \left\{ \frac{1}{\alpha_{k+1}} + \frac{1}{\lambda_{k+1}} \right\} \{ C + \|\nabla w_t^k\|^2 \} + \varepsilon \|\xi_t\|^2 \tag{3.23}
\end{aligned}$$

in virtue of Lemma 3.2 and again the estimates of Lemma 2.3. Similarly we get

$$\begin{aligned}
& |(R_m(-2\mu_r \operatorname{rot} \eta - 4\mu_r \xi + \rho^k \xi \nabla w^n + \rho^k u^k \nabla \xi), \xi_t)| \\
& \leq C \|\nabla \xi\|^2 + C \|\nabla \eta\|^2 + \varepsilon \|\xi_t\|^2. \tag{3.24}
\end{aligned}$$

Now, we observe that

$$\|(\rho^m)^{1/2}\|^2 \geq \alpha \|\xi_t\|^2,$$

and by taking $\varepsilon = \alpha/6$, we get from (3.21)

$$\begin{aligned}
& \alpha \|\xi_t\|^2 + (C_a + C_d) \frac{d}{dt} \|\nabla \xi\|^2 + (C_0 + C_d - C_a) \frac{d}{dt} \|\operatorname{div} \xi\|^2 \\
& \leq \frac{C}{\alpha_{k+1}} \{ C_1 + C_2 \|\nabla w_t^k\|^2 + C_3 \|w^k\|_{H^3}^2 \\
& \quad + CG_1(t) \left\{ \frac{1}{\alpha_{k+1}} + \frac{1}{\lambda_{k+1}} \right\} \{ C + \|\nabla w_t^k\|^2 \} + C \|\nabla \xi\|^2 + C \|\nabla \eta\|^2.
\end{aligned}$$

This differential inequality yields the integral inequality

$$\begin{aligned} & \int_0^t \|\xi_t\|^2 ds + \|\nabla\xi\|^2 + \|\operatorname{div}\xi\|^2 \\ & \leq \frac{C}{\alpha_{k+1}} \left\{ C_1 t + C_2 \int_0^t \|\nabla w_t^k\|^2 ds + C_3 \int_0^t \|w^k\|_{H^3}^2 ds \right\} \\ & \quad + C \sup G_1(t) \left\{ \frac{1}{\alpha_{k+1}} + \frac{1}{\lambda_{k+1}} \right\} \left\{ \int_0^t \|\nabla w_t^k\|^2 ds + ct \right\} \\ & \quad + C \int_0^t \{ \|\nabla\eta\|^2 + \|\nabla\xi\|^2 \} ds + C \|\nabla\xi(0)\|^2, \end{aligned}$$

which, together with

$$\begin{aligned} \|\nabla\xi_0\|^2 & \leq \frac{\|\Delta\xi(0)\|^2}{\alpha_{k+1}} \leq \frac{C}{\alpha_{k+1}}, \\ \int_0^t \{ \|\nabla\eta\|^2 + \|\nabla\xi\|^2 \} ds & \leq G_1(t) \left\{ \frac{1}{\alpha_{k+1}} + \frac{1}{\lambda_{k+1}} \right\}, \end{aligned}$$

the result of Theorem 3.1 and the estimates in Lemma 2.3 yields the stated result for w . The result for u can be obtained analogously. ■

THEOREM 3.4: *Under the hypothesis of Theorem 2.2, we have*

$$\int_0^t \|\Delta w - \Delta w^k\|^2 ds \leq L_1(t) \left\{ \frac{1}{\alpha_{k+1}} + \frac{1}{\lambda_{k+1}} \right\} \quad (3.25)$$

$$\int_0^t \|P \Delta u - P \Delta u^k\|^2 ds \leq L_2(t) \left\{ \frac{1}{\alpha_{k+1}} + \frac{1}{\lambda_{k+1}} \right\} \quad (3.26)$$

$$\|\rho(t) - \rho^k(t)\|_{L^\infty}^2 \leq L_3(t) \left\{ \frac{1}{\alpha_{k+1}} + \frac{1}{\lambda_{k+1}} \right\} \quad (3.27)$$

for any $t \in [0, T]$. The functions continuous $L_i(t)$ depend on t and the functions $F_i(t)$ in Lemma 2.3. (3.25), (3.26) and (3.27) hold with any u^m , w^m , ρ^m instead of u , w , ρ for $m > k$.

Proof: First we suppose (3.26) and we prove (3.27). We have

$$\|\rho - \rho^k\|_{L^\infty}^2 \leq \left\{ \int_0^t \|u - u^k\|_{L^\infty} \|\nabla \rho^k\|_{L^\infty} ds \right\}^2.$$

Consequently,

$$\begin{aligned} \|\rho - \rho^k\|_{L^\infty}^2 &\leq C \int_0^t \|u - u^k\|_{L^\infty}^2 \|\nabla \rho^k\|_{L^\infty}^2 ds \\ &\leq C \sup \|\nabla \rho^k\|_{L^\infty}^2 \int_0^t \|P \Delta u - P \Delta u^k\|^2 ds \\ &\leq CL_2(t) \left\{ \frac{1}{\alpha_{k+1}} + \frac{1}{\lambda_{k+1}} \right\} \end{aligned}$$

bearing in mind (3.26).

To prove (3.26), we consider the following equations with $m > k$:

$$P_m(\rho^m u_t^m + \rho^m u^m \nabla u^m - \rho^m f - 2 \mu_r \operatorname{rot} w^m - (\mu + \mu_r) \Delta u^m) = 0 \quad (3.28)$$

$$P_k(\rho^k u_t^k + \rho^k u^k \nabla u^k - \rho^k f - 2 \mu_r \operatorname{rot} w^k - (\mu + \mu_r) \Delta u^k) = 0. \quad (3.29)$$

Subtracting (3.29) from (3.28) and taking the inner product in $L^2(\Omega)$ of the result with $-P \Delta \eta$, we get

$$\begin{aligned} (\mu + \mu_r) \|P \Delta \eta\|^2 &\leq (P_m(\rho^m \eta_t + \rho^m u^m \nabla \eta + \rho^m \eta \nabla u^k - 2 \mu_r \operatorname{rot} \xi, P \Delta \eta)) \\ &\quad + (P_m(\rho^m - \rho^k)(u_t^k + u^k \nabla u^k - f), P \Delta \eta) \\ &\quad + ((P_m - P_k)(\rho^k u_t^k + \rho^k u^k \nabla u^k - 2 \mu_r \operatorname{rot} w^k - \\ &\quad - \rho^k f, P \Delta \eta) \\ &\leq C_\varepsilon \{ \|\rho^m \eta_t\|^2 + \|\rho^m u^m \nabla \eta\|^2 + \|\rho^m \eta \nabla u^k\|^2 + \\ &\quad + C 4 \mu_r^2 \|\nabla \xi\|^2 \} \\ &\quad + C_\varepsilon \|\rho^m - \rho^k\|_{L^3}^2 \{ \|\nabla u_t^k\|^2 + \|\nabla u^k \cdot \nabla u^k\|^2 + \\ &\quad + \|u^k \nabla^2 u^k\|^2 + \|\nabla f\|^2 \} \\ &\quad + \frac{C_\varepsilon}{\lambda_{k+1}} \{ \|\rho^k u_t^k\|_{H^1}^2 + \|\rho^k u^k \nabla u^k\|_{H^1}^2 + 2 \mu_r \|\operatorname{rot} w^k\|_{H^1}^2 \} \\ &\quad + \|\rho^k f\|_{H^1}^2 \} + 3 \varepsilon \|P \Delta \eta\|^2. \end{aligned}$$

Thus, by taking $\varepsilon = (\mu + \mu_r)/2$ and estimating the terms in the right-hand side using the results of Lemma 3.2 we have

$$\begin{aligned} \frac{1}{2}(\mu + \mu_r) \|P \Delta \eta\|^2 &\leq C \|\eta_t\|^2 + C \|\nabla \eta\|^2 + C \|\nabla \xi\|^2 + \\ &\quad + \frac{C}{\lambda_{k+1}} \{C + C \|\nabla u_t^k\|^2\} \\ &\quad + C \left\{ \frac{1}{\alpha_{k+1}} + \frac{1}{\lambda_{k+1}} \right\} \{C + \|\nabla u_t^k\|^2\} \end{aligned}$$

(we recall that C depends on the functions $F_i(t)$ of Lemma 3.2 and T).

By integrating the above inequality of 0 a t we get

$$\begin{aligned} \int_0^t \|P \Delta \eta\|^2 ds &\leq C \int_0^t \|\eta_t\|^2 ds + C \int_0^t \|\nabla \eta\|^2 ds + C \int_0^t \|\nabla \xi\|^2 ds \\ &\quad + \frac{C}{\lambda_{k+1}} \int_0^t \{C + C \|\nabla u^k\|^2\} ds + C \left\{ \frac{1}{\lambda_{k+1}} + \frac{1}{\alpha_{k+1}} \right\} \int_0^t \{C + \|\nabla u_t^k\|^2\} ds \\ &\leq L_2(t) \left\{ \frac{1}{\alpha_{k+1}} + \frac{1}{\lambda_{k+1}} \right\}. \end{aligned}$$

Analogously we prove (3.25). This complete the proof of Theorem 3.4. ■

Finally, we have

THEOREM 3.5 : *Under the hypotheses of Theorem 2.2, the approximations ρ^k, u^k, w^k satisfy :*

$$\sigma(t) \|(\rho^k)^{1/2}(u_t - u_t^k)\|^2 + \int_0^t \sigma(s) \|\nabla u_t - \nabla u_t^k\|^2 ds \leq R(t) \left\{ \frac{1}{\lambda_{k+1}} + \frac{1}{\alpha_{k+1}} \right\}$$

$$\sigma(t) \|(\rho^k)^{1/2}(w_t - w_t^k)\|^2 + \int_0^t \sigma(s) \|\nabla w_t - \nabla w_t^k\|^2 ds$$

$$+ \int_0^t \sigma(s) \|\operatorname{div} w_t - \operatorname{div} w_t^k\|^2 ds \leq R_1(t) \left\{ \frac{1}{\alpha_{k+1}} + \frac{1}{\lambda_{k+1}} \right\}$$

$$\sigma(t) \|u_t - u_t^k\|^2 \leq \frac{1}{\alpha} R(t) \left\{ \frac{1}{\alpha_{k+1}} + \frac{1}{\lambda_{k+1}} \right\}$$

$$\sigma(t) \|w_t - w_t^k\|^2 \leq \frac{1}{\alpha} R_2(t) \left\{ \frac{1}{\alpha_{k+1}} + \frac{1}{\lambda_{k+1}} \right\}$$

$$\sigma(t) \|P \Delta u - P \Delta u^k\|^2 \leq L(t) \left\{ \frac{1}{\alpha_{k+1}} + \frac{1}{\lambda_{k+1}} \right\}$$

$$\sigma(t) \|\Delta w - \Delta w^k\|^2 \leq L_1(t) \left\{ \frac{1}{\alpha_{k+1}} + \frac{1}{\lambda_{k+1}} \right\}$$

$$\sigma(t) \|u - u^k\|_{L^\infty}^2 \leq CL(t) \left\{ \frac{1}{\alpha_{k+1}} + \frac{1}{\lambda_{k+1}} \right\}$$

$$\sigma(t) \|w - w^k\|_{L^\infty}^2 \leq CL_1(t) \left\{ \frac{1}{\alpha_{k+1}} + \frac{1}{\lambda_{k+1}} \right\}$$

$$\int_0^t \sigma(s) \|\nabla u - \nabla u^k\|_{C^{0,\alpha}(\Omega)}^2 \leq CM(t) \left\{ \frac{1}{\alpha_{k+1}} + \frac{1}{\lambda_{k+1}} \right\}^{\frac{1}{2}}$$

$$\int_0^t \sigma(s) \|\nabla w - \nabla w^k\|_{C^{0,\alpha}(\Omega)}^2 \leq CM_1(t) \left\{ \frac{1}{\alpha_{k+1}} + \frac{1}{\lambda_{k+1}} \right\}^{\frac{1}{2}}$$

where $0 \leq \alpha < 1/4$ and $\sigma(t) = \min \{1, t\}$.

If the dimension is two, we have that for all $\varepsilon > 0$, there is $C_\varepsilon > 0$:

$$\int_0^t \sigma(t) \|\nabla u - \nabla u^k\|_{L^\infty}^2 \leq C_\varepsilon N_1(t) \left\{ \frac{1}{\alpha_{k+1}^{1-\varepsilon}} + \frac{1}{\lambda_{k+1}^{1-\varepsilon}} \right\}$$

$$\int_0^t \sigma(t) \|\nabla w - \nabla w^k\|_{L^\infty}^2 \leq C_\varepsilon N_2(t) \left\{ \frac{1}{\alpha_{k+1}^{1-\varepsilon}} + \frac{1}{\lambda_{k+1}^{1-\varepsilon}} \right\}$$

for every $t \in [0, T]$. The continuous functions $R(t)$, $R_1(t)$, $L(t)$, $L_1(t)$, $M(t)$, $M_1(t)$, $N(t)$ and $N_1(t)$ depend on t and on the functions $F_i(t)$ in Lemma 2.3. The above estimates holds also with any u^m , w^m instead of u , w , $m > k$.

Proof: We only proof the first two estimates; the others are proved similarly.

Differentiating (3.28) and (3.29) with respect to t , subtracting the results and taking the inner product in $L^2(\Omega)$ with η_t , we obtain

$$(\mu + \mu_r) \|\nabla \eta_t\|^2 + \frac{1}{2} \frac{d}{dt} \|(\rho^k)^{1/2} \eta_t\|^2 = \frac{1}{2} (\rho_t^k \eta_t, \eta_t) + (P_m(\rho^m - \rho^k) u_{tt}^m, \eta_t)$$

$$+ ((P_m - P_k) \rho^k u_{tt}^k, \eta_t) + 2 \mu_r ((P_m - P_k) \operatorname{rot} w_t^m, \eta_t) +$$

$$+ 2 \mu_r (P_k \operatorname{rot} \xi_t, \eta_t) + (Z(t), \eta_t),$$

where

$$Z(t) = P_k \rho_t^k u_t^k - P_m \rho_t^m u_t^m + P_k (\rho^k u^k \nabla u^k)_t - P_m (\rho^m u^m \nabla u^m)_t$$

$$+ P_k (\rho^k f)_t - P_m (\rho^m f)_t.$$

The above inequality implies

$$(\mu + \mu_r) \|\nabla \eta_t\|^2 + \frac{1}{2} \frac{d}{dt} \|(\rho^k)^{1/2} \eta_t\|^2$$

$$\leq \frac{1}{2} \|\rho_t^k\|_{L^\infty} \|\eta_t\|^2 + \|\rho^m - \rho^k\|_{L^\infty} \|u_{tt}^m\| \|\eta_t\|$$

$$+ \|\rho^k\|_{L^\infty} \|u_{tt}^k\| \|(I - P_k) u_t^m\| + 2 \mu_r \|\operatorname{rot} w_t^m\| \|(I - P_k) u_t^m\|$$

$$+ 2 \mu_r \|\operatorname{rot} \xi_t\| \|\eta_t\| + |(Z(t), \eta_t)|$$

$$\leq C(t) \|\eta_t\|^2 + C(t) \left(\frac{1}{\lambda_{k+1}} + \frac{1}{\alpha_{k+1}} \right) \|u_{tt}^m\|^2$$

$$+ \frac{C(t)}{\lambda_{k+1}} \|u_{tt}^k\|^2 + \frac{C(t)}{\lambda_{k+1}} \|P \Delta u_t^m\|^2 + \frac{C(t)}{\lambda_{k+1}} \|\nabla w_t^m\|^2$$

$$+ \varepsilon \|\nabla \xi_t\|^2 + |(Z(t), \eta_t)| \quad (\varepsilon > 0) \tag{3.30}$$

where we use the young inequality (2.9), (2.1), Lemma 2.3 and Theorem 3.4.

Analogously, for every $\delta > 0$ we have

$$\begin{aligned}
& (C_a + C_d) \|\nabla \xi_t\|^2 + (C_0 + C_d - C_a) \|\operatorname{div} \xi_t\|^2 + 4\mu_r \|\xi_t\|^2 + \\
& + \frac{1}{2} \frac{d}{dt} \|(\rho^k)^{1/2} \xi_t\|^2 \\
& \leq C(t) \|\xi_t\|^2 + C(t) \left(\frac{1}{\lambda_{k+1}} + \frac{1}{\alpha_{k+1}} \right) \|w_t^m\|^2 \\
& + \frac{C(t)}{\alpha_{k+1}} \|\Delta w_t^m\|^2 + \frac{C(t)}{\alpha_{k+1}} \|w_t^m\|^2 + \frac{C(t)}{\alpha_{k+1}} \|w_t^m\|^2 \\
& + \frac{C(t)}{\alpha_{k+1}} \|\nabla \operatorname{div} w_t^k\|^2 + \frac{C(t)}{\alpha_{k+1}} \|\nabla u_t^k\|^2 + \delta \|\nabla \eta_t\|^2 + |(h(t), \eta_t)|, \quad (3.31)
\end{aligned}$$

where

$$\begin{aligned}
h(t) &= R_k \rho^k w_t^k - R_m \rho^m w_t^m + R_k (\rho^k u^k \nabla w^k)_t, \\
& - R_m (\rho^m u^m \nabla w^m)_t + R_m (\rho^m g)_t - R_k (\rho^k g)_t.
\end{aligned}$$

Adding the inequalities (3.30) and (3.31) and taking $\varepsilon > 0$ and $\delta > 0$ so that $(C_a + C_d) - \varepsilon > 0$ and $(\mu + \mu_r) - \delta > 0$, we get

$$\begin{aligned}
& (\mu + \mu_r) \|\nabla \eta_t\|^2 + (C_a + C_d) \|\nabla \xi_t\|^2 + (C_0 + C_d - C_a) \|\operatorname{div} \xi_t\|^2 \\
& + 4\mu_r \|\xi_t\|^2 + \frac{1}{2} \frac{d}{dt} \|(\rho^k)^{1/2} \eta_t\|^2 + \frac{1}{2} \frac{d}{dt} \|(\rho^k)^{1/2} \xi_t\|^2 \\
& \leq C(t) (\|\eta_t\|^2 + \|\xi_t\|^2) + C(t) \left(\frac{1}{\lambda_{k+1}} + \frac{1}{\alpha_{k+1}} \right) (\|u_t^m\|^2 + \|w_t^m\|^2) \\
& + \frac{C(t)}{\alpha_{k+1}} (\|u_t^k\|^2 + \|P \Delta u_t^m\|^2 + \|\nabla w_t^m\|^2) \\
& + \frac{C(t)}{\alpha_{k+1}} (\|\Delta w_t^m\|^2 + \|w_t^m\|^2 + \|w_t^k\|^2 + \|\nabla \operatorname{div} w_t^k\|^2 + \|\nabla u_t^k\|^2) \\
& + |(Z(t), \eta_t)| + |(h(t), \eta_t)|.
\end{aligned}$$

Multiplying the above inequality by $\sigma(t) = \min \{1, t\}$ integrating of 0 a t , using the Theorem 3.3 and Lemma 2.3 we obtain

$$\begin{aligned}
 & (\mu + \mu_r) \int_0^t \sigma(s) \|\nabla \eta_t\|^2 ds + (C_a + C_d) \int_0^t \sigma(s) \|\nabla \xi_t\|^2 ds \quad (3.32) \\
 & + (C_0 + C_d - C_a) \int_0^t \sigma(s) \|\operatorname{div} \xi_t\|^2 ds + 4 \mu_r \int_0^t \sigma(s) \|\xi_t\|^2 ds \\
 & + \sigma(t) \|(\rho^k)^{1/2} \eta_t\|^2 + \frac{\sigma(t)}{2} \|(\rho^k)^{1/2} \xi_t\|^2 \\
 & \leq C(t) \left(\frac{1}{\alpha_{k+1}} + \frac{1}{\alpha_{k+1}} \right) + \int_0^t \sigma(s) |(Z(s), \eta_t)| ds + \int_0^t \sigma(s) |h(s), \xi_t| ds,
 \end{aligned}$$

since $\sigma(0) \|(\rho^k)^{1/2} \eta_t\|^2 = 0$ and $\sigma(0) \|(\rho^k)^{1/2} \xi_t\|^2 = 0$.

Now, we estimate the first integral on the hand-right side of the above inequality; the second integral can be done similarly. We observe that

$$\begin{aligned}
 & \int_0^t \sigma(s) |(Z(s), \eta_t)| ds \leq \int_0^t \sigma(s) \|(Z(s))\| \|\eta_t\| ds \\
 & \leq \left(\int_0^t \sigma(s) \|(Z(s))\|^2 ds \right)^{1/2} \left(\int_0^t \sigma(s) \|\eta_t\|^2 ds \right)^{1/2} \\
 & \leq \left(\int_0^t \sigma(s) \|(Z(s))\|^2 ds \right)^{1/2} \left(\int_0^t \frac{\sigma(s)}{\lambda_{k+1}^2} \|P \Delta u_t^m\|^2 ds \right)^{1/2} \\
 & \leq \frac{C(t)}{\lambda_{k+1}} \left(\int_0^t \sigma(s) \|(Z(s))\|^2 ds \right)^{1/2}.
 \end{aligned}$$

Here we used (2.2) and the estimates of the Lemma 2.3. Now, we will prove that the integral on the hand-right side of the above inequality is finite. This is proved of in a standard manner by using the estimates of the Lemma 2.3. Consequently in (3.32), we have

$$\begin{aligned} & (\mu + \mu_r) \int_0^t \sigma(s) \|\nabla \eta_t\|^2 ds + (C_a + C_d) \int_0^t \sigma(s) \|\nabla \xi_t\|^2 ds \\ & + (C_0 + C_d - C_a) \int_0^t \sigma(s) \|\operatorname{div} \xi_t\|^2 ds + 4 \mu_r \int_0^t \sigma(s) \|\xi_t\|^2 ds \\ & + \frac{\sigma(t)}{2} \|(\rho^k)^{1/2} \eta_t\|^2 + \frac{\sigma(t)}{2} \|(\rho^k)^{1/2} \xi_t\|^2 \leq C(t) \left(\frac{1}{\alpha_{k+1}} + \frac{1}{\lambda_{k+1}} \right). \end{aligned}$$

Now, by taking the limit as m goes to infinity (using the remark after Lemma 2.3), on the left side we obtain the first two estimates of the Theorem. ■

4. IMPROVED L^2 -ERROR BOUNDS

The L^2 -estimates obtained in Theorem 3.1 are not optimal; in fact it is expected to obtain a rate of convergence of order $\frac{1}{\lambda_{k+1}^2} + \frac{1}{\alpha_{k+1}^2}$ instead of only $\frac{1}{\lambda_{k+1}} + \frac{1}{\alpha_{k+1}}$.

We were not able to do that, but in this Section we will improve the L^2 -estimates in Theorem 3.1 by using a bootstrap argument.

In order to do that, let $u = \sum_{i=1}^{\infty} a_i(t) \phi^i(x)$ and $w = \sum_{i=1}^{\infty} b_i(t) \phi^i(x)$ be the eigenfunction expansions of u and w . Let $v^k = \sum_{i=1}^k a_i(t) \phi^i(x)$ and $z^k = \sum_{i=1}^k b_i(t) \phi^i(x)$ be the k -th partial sum of the series for u and w , respectively, and let

$$e^k = u - v^k, \varepsilon^k = w - z^k, \psi^k = u^k - v^k \quad \text{and} \quad \sigma^k = w^k - z^k,$$

where u^k and w^k be the k -th Galerkin approximations of u and w , respectively. Then, we have $u - u^k = e^k - \psi^k$ and $w - w^k = \varepsilon^k - \sigma^k$.

With these notations, we state

LEMMA 4.1 :

$$\begin{aligned} & \|\psi^k(t)\|^2 + \|\sigma^k(t)\|^2 + \int_0^t (\|\nabla\psi^k\|^2 + \|\nabla\sigma^k\|^2 + \|\operatorname{div}\sigma^k\|^2) ds \\ & \leq C(t) \left\{ \frac{1}{\alpha_{k+1}^2} + \frac{1}{\lambda_{k+1}^2} + \frac{1}{\alpha_{k+1}^{3/2}} + \frac{1}{\lambda_{k+1}^{3/2}} + \frac{1}{\alpha_{k+1}\lambda_{k+1}^{1/2}} + \frac{1}{\alpha_{k+1}^{1/2}\lambda_{k+1}} \right\} \end{aligned}$$

for any $t \in [0, T]$. The continuous function $C(t)$ depend on t and on the functions, $F_i(t)$ in Lemma 2.3.

Proof: We observe that v^k and z^k satisfy

$$P_k(\rho u_t + \rho u \nabla u - \rho f - 2\mu_r \operatorname{rot} w) + (\mu + \mu_r) Av^k = 0 \quad (4.1)$$

$$\begin{aligned} R_k(\rho w_t + \rho u \nabla w - \rho g - 2\mu_r \operatorname{rot} u) + 4\mu_r z^k - \\ - (C_0 + C_d - C_a) R_k \nabla \operatorname{div} w + (C_a + C_d) Bz^k = 0 \end{aligned} \quad (4.2)$$

for all, $t \in [0, T]$.

Subtracting (4.1) of (2.8) (i) and (4.2) of (2.8) (ii) we obtain

$$\begin{aligned} & (\mu + \mu_r) A\psi^k + P_k(\rho^k u_t^k - \rho u_t) + P_k(\rho^k u^k \nabla u^k - \rho u \nabla u) \\ & + P_k(\rho - \rho^k) f + 2\mu_r P_k(\operatorname{rot}(w - w^k)) = 0, \end{aligned} \quad (4.3)$$

$$\begin{aligned} & (C_a + C_d) B\sigma^k + 4\mu_r \sigma^k + (C_0 + C_d - C_a) R_k(\nabla \operatorname{div}(w - w^k)) \\ & + R_k(\rho^k w_t^k - \rho w_t) + R_k(\rho^k u^k \nabla w^k - \rho u \nabla w) \\ & + R_k(\rho - \rho^k) g + 2\mu_r R_k(\operatorname{rot}(u - u^k)) = 0 \end{aligned} \quad (4.4)$$

By taking the inner product in L^2 of (4.3) with ψ^k and of (4.4) with σ^k and proceeding as in the previous Section, we obtain

$$\begin{aligned}
& (\mu + \mu_r) \|\nabla \psi^k\|^2 + \frac{1}{2} \frac{d}{dt} \|(\rho^k)^{1/2} \psi^k\|^2 - (\rho e_r^k, \psi^k) \\
&= \frac{1}{2} (\rho_t^k \psi^k, \psi^k) + ((\rho - \rho^k) v_r^k, \psi^k) \\
&\quad + ((\rho - \rho^k) u^k \nabla u^k, \psi^k) - (\rho \psi^k \nabla u^k, \psi^k) \\
&\quad + (\rho e^k \nabla u^k, \psi^k) - (\rho u \nabla \psi^k, \psi^k) + (\rho u \nabla e^k, \psi^k) \\
&\quad + (\text{rot } \sigma^k, \psi^k) - (\varepsilon^k, \text{rot } \psi^k) + ((\rho - \rho^k) f, \psi^k) \tag{4.5}
\end{aligned}$$

$$\begin{aligned}
& (C_a + C_d) \|\nabla \sigma^k\|^2 + \frac{1}{2} \frac{d}{dt} \|(\rho^k)^{1/2} \sigma^k\|^2 - \\
&\quad - (\rho \varepsilon_r^k, \sigma^k) + (C_0 + C_d - C_a) \|\text{div } \sigma^k\|^2 \\
&= \frac{1}{2} (\rho_t^k \sigma^k, \sigma^k) + ((\rho - \rho^k) z_r^k, \sigma^k) \\
&\quad + ((\rho - \rho^k) u^k \nabla w^k, \sigma^k) + (\rho u^k, \nabla \varepsilon^k, \sigma^k) \\
&\quad + (\rho e^k \nabla w, \sigma^k) + 2 \mu_r (\text{rot } \psi^k, \sigma^k) - 2 \mu_r (e^k, \text{rot } \sigma^k) \\
&\quad + (C_0 + C_d - C_a) (\nabla \text{div } \xi^k, \sigma^k) + ((\rho - \rho^k) g, \sigma^k) . \tag{4.6}
\end{aligned}$$

Also, we have

$$\rho - \rho^k = - \int_0^t (u - u^k) \nabla \rho^k - \int_0^t e^k \nabla \rho^k + \int_0^t \psi^k \nabla \rho^k .$$

From this, we observe that if χ is any function in $(H^1(\Omega))^n$, for any $\delta > 0$, there holds

$$\begin{aligned}
((\rho - \rho^k) \chi, \psi^k) &\leq C_\delta \left[\int_0^t \|e^k\|^2 \|\nabla \rho^k\|_{L^\infty}^2 ds \right] \|\chi\|_{L^4}^2 \\
&+ C_\delta \left[\int_0^t \|\psi^k\|^2 \|\nabla \rho^k\|_{L^\infty}^2 \right] \|\chi\|_{L^4}^2 + 2 \delta \|\nabla \psi^k\|^2 .
\end{aligned}$$

By using this with $\chi = f$, $\chi = u^k \cdot \nabla u^k$ and $\chi = v_t^k$ and the following facts

$$\|e^k\|^2 \leq \frac{C}{\lambda_{k+1}^2} \|u\|_{H^2} \leq \frac{C}{\lambda_{k+1}^2},$$

$$\|\xi^k\|^2 \leq \frac{C}{\alpha_{k+1}^2} \|w\|_{H^2} \leq \frac{C}{\alpha_{k+1}^2},$$

$$|(\rho u \nabla e^k, \psi^k)| = |(\rho u \nabla \psi^k, e^k)|,$$

for which we have used (2.2), together the estimates of the lemma 2.3 ; estimating as usual, (4.5) furnishes, for any $\delta > 0$, $\gamma > 0$,

$$\begin{aligned} & (\mu + \mu_r) \|\nabla \psi^k\|^2 + \frac{1}{2} \frac{d}{dt} \|(\rho^k)^{1/2} \psi^k\|^2 - \\ & - (\rho e_t^k, \psi^k) \leq \frac{C}{\lambda_{k+1}^2} + \frac{C}{\alpha_{k+1}^2} + C \|\psi^k\|^2 \\ & + \frac{C}{\lambda_{k+1}^2} \{ \|\nabla v_t^k\|^2 + \|\nabla(u^k \nabla u^k)\|^2 + \|f\|_{H^1}^2 \} \\ & + \gamma \|\nabla \sigma^k\|^2 + 13 \delta \|\nabla \psi^k\|^2 \\ & + C \int_0^t \|\psi^k\|^2 ds \{ \|\nabla v_t^k\|^2 + \|\nabla(u^k \nabla u^k)\|^2 + \|f\|_{H^1}^2 \}. \quad (4.7) \end{aligned}$$

Proceeding in the some way with (4.6), we obtain for any $\delta > 0$, $\gamma > 0$

$$\begin{aligned} & (C_0 + C_d) \|\nabla \sigma^k\|^2 + \frac{1}{2} \frac{d}{dt} \|(\rho^k)^{1/2} \sigma^k\|^2 - (\rho \varepsilon_t^k, \sigma^k) \\ & + \left(\frac{C_0 + C_d - C_a}{2} \right) \|\nabla \operatorname{div} \sigma^k\|^2 \leq \frac{C}{\alpha_{k+1}^2} + \frac{C}{\lambda_{k+1}^2} \\ & + C \|\sigma^k\|^2 + \frac{C}{\alpha_{k+1}^2} \|w\|_{H^3}^2 + C \int_0^t \|\psi^k\|^2 ds \{ \|g\|_{H^1}^2 + \|\nabla z_t^k\|^2 + \\ & + \|\nabla(u^k \nabla w^k)\|^2 \\ & + \frac{C}{\lambda_{k+1}^2} \{ \|g\|_{H^1}^2 + \|\nabla z_t^k\|^2 + \|\nabla(u^k \nabla w^k)\|^2 \} + \gamma \|\nabla \sigma^k\|^2 + \delta \|\nabla \psi^k\|^2. \quad (4.8) \end{aligned}$$

By adding (4.7) and (4.8), we are left with

$$\begin{aligned}
 & (\mu + \mu_r) \|\nabla \psi^k\|^2 + (C_0 + C_d) \|\nabla \sigma^k\|^2 + \frac{(C_0 + C_d - C_a)}{2} \|\nabla \operatorname{div} \sigma^k\|^2 \\
 & + \frac{1}{2} \frac{d}{dt} \|(\rho^k)^{1/2} \sigma^k\|^2 + \frac{1}{2} \frac{d}{dt} \|(\rho^k)^{1/2} \psi^k\|^2 - (\rho \varepsilon_t^k, \sigma^k) - (\rho \varepsilon_t^k, \psi^k) \\
 & \leq \frac{C}{\alpha_{k+1}^2} + \frac{C}{\lambda_{k+1}^2} + C(\|\psi^k\|^2 + \|\sigma^k\|^2) \\
 & + \frac{C}{\alpha_{k+1}^2} \|w\|_{H^3}^2 + \frac{C}{\lambda_{k+1}^2} \{\|\nabla u_t\|^2 + \|\nabla w_t\|^2\} + C \int_0^t \|\psi\|^2 ds \{\|\nabla w_t\|^2 + \\
 & + \|\nabla u_t\|^2\}.
 \end{aligned}$$

Integrating with respect to t , we have (with $C_1 > 0$)

$$\begin{aligned}
 C_1 \int_0^t (\|\nabla \psi^k\|^2 + \|\nabla \sigma^k\|^2 + \|\nabla \operatorname{div} \sigma^k\|^2) ds + \|\sigma^k\|^2 + \|\psi^k\|^2 - \\
 - \int_0^t (\rho \varepsilon_t^k, \sigma^k) ds \\
 - \int_0^t (\rho \varepsilon_t^k, \psi^k) ds \leq C \int_0^t (\|\psi^k\|^2 + \|\sigma^k\|^2) ds + C \left\{ \frac{1}{\lambda_{k+1}^2} + \frac{1}{\alpha_{k+1}^2} \right\}. \quad (4.9)
 \end{aligned}$$

By integration by parts with respect to t and recalling that $\sigma^k(0) = 0$, we can estimate as follows

$$\begin{aligned}
 & \left\| \int_0^t (\rho \varepsilon_t^k, \sigma^k) ds \right\| \leq \left\| \int_0^t (\rho_t \varepsilon^k, \sigma^k) ds \right\| + \left\| \int_0^t (\rho \varepsilon^k, \sigma_t^k) ds \right\| + \\
 & + |(\rho(t) \varepsilon^k(t), \sigma^k(t))| \\
 & \leq \int_0^t \|\rho_t\|_{L^\infty} \|\varepsilon^k\| \|\sigma^k\| ds + \int_0^t \|\rho\|_{L^\infty} \|\varepsilon^k\| (\|\varepsilon_t^k\| + \|(w - w^k)_t\|) ds \\
 & + \beta \|\varepsilon^k\| \|\sigma^k\|
 \end{aligned}$$

$$\begin{aligned}
&\leq C \int_0^t \|\varepsilon^k\|^2 ds + C \int_0^t \|\sigma^k\|^2 ds + C \int_0^t \|\varepsilon^k\| \|\varepsilon_t^k\| + \|\varepsilon^k\| \|(w - w^k)_t\| + \\
&\quad + \frac{C_\beta}{\alpha_{k+1}} \cdot \frac{1}{\alpha_{k+1}^{1/2}} \\
&\leq \frac{C}{\alpha_{k+1}^2} + \frac{C}{\alpha_{k+1}^{3/2}} + \frac{C}{\alpha_{k+1}} \left(\int_0^t \|w_t - w_t^k\|^2 ds \right)^{1/2} \\
&\leq C \left(\frac{1}{\alpha_{k+1}^2} + \frac{1}{\alpha_{k+1}^{3/2}} + \frac{1}{\alpha_{k+1} \cdot \lambda_{k+1}^{1/2}} \right).
\end{aligned}$$

Similarly, we estimate

$$\left\| \int_0^t (\rho e_t^k, \psi^k) dt \right\| \leq C \left(\frac{1}{\lambda_{k+1}^2} + \frac{1}{\lambda_{k+1}^{3/2}} + \frac{1}{\lambda_{k+1} \cdot \alpha_{k+1}^{1/2}} \right).$$

Thus, from (4.9) and the above, we are left with

$$\begin{aligned}
\|\psi^k(t)\|^2 + \|\sigma^k(t)\|^2 + C_1 \int_0^t \left(\|\nabla \psi^k\|^2 + \|\nabla \sigma^k\|^2 ds \leq C \int_0^t (\|\psi^k(t)\|^2 + \right. \\
\left. + \|\sigma^k(t)\|^2 ds) \right) \\
+ C \left(\frac{1}{\lambda_{k+1}^2} + \frac{1}{\lambda_{k+1}^2} + \frac{1}{\lambda_{k+1}^{3/2}} + \frac{1}{\alpha_{k+1}^{3/2}} + \frac{1}{\alpha_{k+1} \lambda_{k+1}^{1/2}} + \frac{1}{\lambda_{k+1} \alpha_{k+1}^{1/2}} \right).
\end{aligned}$$

From this and Gronwall's inequality, we conclude that

$$\begin{aligned}
&\|\psi^k(t)\|^2 + \|\sigma^k(t)\|^2 + \int_0^t (\|\nabla \psi^k(s)\|^2 + \|\nabla \sigma^k(s)\|^2 ds) \\
&\leq C e^{ct} \left(\frac{1}{\lambda_{k+1}^2} + \frac{1}{\lambda_{k+1}^2} + \frac{1}{\lambda_{k+1}^{3/2}} + \frac{1}{\alpha_{k+1}^{3/2}} + \frac{1}{\alpha_{k+1} \lambda_{k+1}^{1/2}} + \frac{1}{\lambda_{k+1} \alpha_{k+1}^{1/2}} \right)
\end{aligned}$$

and the Lemma 4.1 is proved. ■

Now we are ready to prove the following

THEOREM 4.2 :

$$\begin{aligned}
 & 1) \quad \|u(t) - u^k(t)\|^2 + \|w(t) - w^k(t)\|^2 \\
 & \leq \left\{ \frac{1}{\alpha_{k+1}^2} + \frac{1}{\lambda_{k+1}^2} + \frac{1}{\alpha_{k+1}^{3/2}} + \frac{1}{\lambda_{k+1}^{3/2}} + \frac{1}{\alpha_{k+1} \lambda_{k+1}^{1/2}} + \frac{1}{\alpha_{k+1}^{1/2} \lambda_{k+1}} \right\} \cdot C(t) \\
 & \text{ii) } \|\rho(t) - \rho^k(t)\|^2 \\
 & \leq \left\{ \frac{1}{\alpha_{k+1}^2} + \frac{1}{\lambda_{k+1}^2} + \frac{1}{\alpha_{k+1}^{3/2}} + \frac{1}{\lambda_{k+1}^{3/2}} + \frac{1}{\alpha_{k+1} \lambda_{k+1}^{1/2}} + \frac{1}{\alpha_{k+1}^{1/2} \lambda_{k+1}} \right\} \cdot C(t)
 \end{aligned}$$

for any, $t \in [0, T]$.

Proof : From estimates in Lemma 2.3 and (2.8) we conclude that

$$\|e^k\|^2 = \|u - v^k\|^2 \leq \frac{C}{\lambda_{k+1}^2} \|u\|_{H^2} \leq \frac{C(t)}{\lambda_{k+1}^2}.$$

Analogously, we have

$$\|\varepsilon^k\|^2 = \|w - z^k\|^2 \leq \frac{C}{\alpha_{k+1}^2} \|w\|_{H^2} \leq \frac{C(t)}{\alpha_{k+1}^2}.$$

Thus, since $\|u - u^k\| = \|e^k - \psi^k\| \leq \|e^k\| + \|\psi^k\|$ and $\|w - w^k\| = \|\varepsilon^k - \sigma^k\| \leq \|\varepsilon^k\| + \|\sigma^k\|$, Lemma 4.1 and the above imply (i).

Now, proceeding as we done in the begining of Theorem 3.1

$$\|\rho(t) - \rho^k(t)\|^2 \leq C(t) \int_0^t \|u(s) - u^k(s)\|^2 ds.$$

This and (i) imply (ii). ■

We observe that we cannot obtain the optimal rate due to the terms $\rho^k e_i^k$ and $\rho^k \varepsilon_i^k$. However, they do not appear in the classical Navier-Stokes equations, and also in the Boussinesq and mathetohydrodynamic type equations, and, consequently, for these equations it is possible to obtain optimal L^2 -error estimates.

Remark : If we consider the reduced model of nonhomogeneous fluids, like in Salvi [12], the above estimates are reduced to

$$\|u(t) - u^k(t)\|^2 \leq \frac{C(t)}{\lambda_{k+1}^{3/2}} \quad \text{and} \quad \|\rho(t) - \rho^k(t)\|^2 \leq \frac{C(t)}{\lambda_{k+1}^{3/2}}$$

with again of $\frac{1}{2}$ with respect to the convergence rate obtained in [12].

REFERENCES

- [1] J. L. BOLDRINI, M. A. ROJAS-MEDAR, Strong solutions of the equations for nonhomogeneous asymmetric fluids, to appear.
- [2] D. W. CONDIFF, J. S. DAHLER, 1964, Fluid mechanics aspects of antisymmetric stress, *Phys. Fluids*, **7**, number 6, pp. 842-854.
- [3] J. U. KIM, 1987, Weak solutions of an initial boundary value problem for an incompressible viscous fluid, *SIAM J. Math. Anal.*, **18**, pp. 890-96.
- [4] O. A. LADYZHENSKAYA, 1969, The Mathematical Theory of Viscous Incompressible Flow, Gordon and Breach, Second Revised Edition, New York.
- [5] O. A. LADYZHENSKAYA, V. A. SOLONNIKOV, 1978, Unique solvability of an initial and boundary value problem for viscous incompressible fluids, *Zap. Naučn Sem. Leningrado Otdel Math. Inst. Steklov*, **52**, 1975, pp. 52-109 ; English Transl., *J. Soviet Math.*, **9**, pp. 697-749.
- [6] G. LUKASZEWICZ, 1988, On nonstationary flows of asymmetric fluids, *Rendiconti Accademia Nazionale delle Scienze detta dei XL, Memorie di Matematica 106°*, **XII**, fasc. 3, pp. 35-44.
- [7] G. LUKASZEWICZ, 1989, On the existence, uniqueness and asymptotic properties of solutions of flows of asymmetric fluids, *Rendiconti Accademia Nazionale della Scienze detta dei XL, Memorie di Matematica 107°*, **XIII**, fasc. 6, pp. 105-120.
- [8] G. LUKASZEWICZ, 1990, On nonstationary flows of asymmetric fluids, *Math. Methods Appl. Sci.*, **19**, no. 3, pp. 219-232.
- [9] L. G. PETROSYAN, *Some Problems of Mechanics of Fluids with Antisymmetric Stress Tensor*, Erevan, 1984 (in Russian).
- [10] R. RAUTMANN, 1980, On convergence rate of nonstationary Navier-Stokes approximations, *Proc. IUTAM Symp. Approx. Meth., for Navier-Stokes Problem, Lecture Notes in Math.*, **771**, Springer-Verlag.
- [11] M. ROJAS-MEDAR, J. L. BOLDRINI, 1993, Spectral Galerkin approximations for the Navier-Stokes Equations : uniform in time error estimates, *Rev. Mat. Apl.*, **14**, pp. 1-12.
- [12] R. SALVI, 1989, Error estimates for the spectral Galerkin approximations of the solutions of Navier-Stokes type equation, *Glasgow Math. J.*, **31**, pp. 199-211.
- [13] R. SALVI, 1991, The equations of viscous incompressible nonhomogeneous fluid : on the existence and regularity, *J. Australian Math. Soc., Series B - Applied Mathematics*, **33**, Part 1, pp. 94-110.
- [14] J. SIMON, 1990, Nonhomogeneous viscous incompressible fluids : existence of velocity, density, and pressure, *SIAM J. Math. Anal.*, **21**, pp. 1093-1117.
- [15] R. TEMAM, 1979, *Navier-Stokes Equations, Theory and Numerical Analysis*, North-Holland, Amsterdam.
- [16] W. von WAHL, 1985, The equations of Navier-Stokes Equations and Abstract Parabolic Equations, *Aspects of Math.*, **58**, Vieweg, Braunschweig-Wiesbaden.