M2AN - Modélisation mathématique et analyse numérique

Guohui Zhou

A local L^2-error analysis of the streamline diffusion method for nonstationary convection-diffusion systems

M2AN - Modélisation mathématique et analyse numérique, tome 29, nº 5 (1995), p. 577-603

<http://www.numdam.org/item?id=M2AN_1995__29_5_577_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

(Vol. 29, n° 5, 1995, p. 577 à 603)

A LOCAL L^2-ERROR ANALYSIS OF THE STREAMLINE DIFFUSION METHOD FOR NONSTATIONARY CONVECTION-DIFFUSION SYSTEMS (*)

GUOHUI ZHOU ([1])

Communicated by V. THOMÉE

Abstract. — *We consider the discretization of linear, nonstationary, convection-dominated, convection-diffusion systems by the streamline diffusion finite element method and give local error estimates in the energy norm for both linear scalar equations in arbitrary dimensions and for systems in one space dimension. For piecewise linear shape functions in time-space that are continuous in space and discontinuous in time, we obtain optimal local error estimates of order* $O(h^{3/2})$ *in those strip regions parallel to the streamline direction in which the exact solution is smooth.*

Résumé. — *Dans cet article nous considérons la discrétisation par la méthode SDFEM (Streamline Diffusion Finite Element Method) d'équations de convection-diffusion linéaires instationnaires à convection dominante. On donne une estimation d'erreur locale dans la norme-énergie pour des équations de convection-diffusion scalaires linéaires instationnaires à convection dominante dans un espace de dimension arbitraire ainsi que pour des systèmes unidimensionnels. On obtient, pour les fonctions de base linéaires par morceaux en temps ainsi qu'en espace (continues en espace et discontinues en temps), une estimation d'erreur locale d'ordre optimal* $O(h^{3/2})$ *dans les bandes parallèles aux caractéristiques où la solution est lisse.*

Key words : Hyperbolic systems, convection-diffusion problems, finite element method, stream-line diffusion, local error estimates.

1. INTRODUCTION

We consider two kinds of nonstationary convection-dominated convection-diffusion (or, essentially hyperbolic) problems. The first sort are scalar problems of the form

$$u_t + \beta \cdot \nabla u + u - \varepsilon \Delta u = f, \quad \text{in} \quad Q := \mathrm{I} \times \Omega, \qquad (1.1.a)$$

$$u(0, .) = u_0, \quad \text{in} \quad \Omega, \qquad (1.1.b)$$

$$u = g, \quad \text{in} \quad I \times \partial\Omega, \qquad (1.1.c)$$

(*) Manuscript received september 5, 1994.

([1]) Institut für Angewandte Mathematik, Universität Heidelberg, Im Neuenheimer Feld 294, D-69120 Heidelberg, Germany, E-mail : zhou@gaia.iwr.uni-heidelberg.de

where Ω is a bounded polygonal domain in R^d, $I = (0, T)$, $\beta = (\beta_1, \dots, \beta_d)$ is a d-dimensional vector depending possibly on space and time, $\Delta u = \sum_{i=1}^{d} u_{x_i x_i}$, and ε is a non-negative parameter. Boundary condition (1.1.c) is imposed in the case $\varepsilon > 0$, whereas for $\varepsilon = 0$, we prescribe only inflow boundary conditions, see Smoller [14].

We also consider linear (essentially) hyperbolic systems in one space dimension,

$$\mathbf{u}_t + B\mathbf{u}_x + \mathbf{u} - \varepsilon \mathbf{u}_{xx} = \mathbf{f}\,, \quad \text{in} \quad Q := I \times \Omega\,, \tag{1.2.a}$$

$$\mathbf{u}(0, .) = \mathbf{u}_0\,, \quad \text{in} \quad \Omega := (0, 1)\,, \tag{1.2.b}$$

$$\mathbf{u}(\,.\,, 0) = \mathbf{g}_l, \mathbf{u}(\,.\,, 1) = \mathbf{g}_r\,, \quad \text{in} \quad I := (0, T)\,, \tag{1.2.c}$$

where $B(t, x)$ is a real $M \times M$ possibly unsymmetric matrix, while $\mathbf{u}$ and $\mathbf{f}$ are M-dimensional vector functions (bold letters are used to denote vectors). System (1.2) is said to be of (essentially) hyperbolic type if the matrix B is diagonalizable with real eigenvalues. The case of a multi-dimensional domain is considerably complicated. For instance, in two dimensions, the transport term looks like $B\mathbf{u}_x + C\mathbf{u}_y$ and the problem is hyperbolic if for every real pair α and γ, the matrix $\alpha B + \gamma C$ is diagonalizable. However, this does not imply that the matrices B and C are simultaneously diagonalizable, which causes difficulties. Therefore, we confine ourselves to one-dimensional systems in this paper.

In both the scalar case and the case of a system, the solution is usually discontinuous if the initial or boundary conditions are not smooth. In a system there is the additional complication in that the characteristics can intersect, which implies for instance that a discontinuity of the initial condition can cause M jumps in each component of the exact solution. In all these cases, therefore, it is not reasonable to assume global smoothness in estimating the error of a discretization method. On the other hand, the solution of a hyperbolic problem is usually not rough everywhere, since the discontinuities of the initial and boundary data propagate along the corresponding characteristics. More practically, one can consider the local convergence behavior of the discrete solution on those strip regions where the exact solution is smooth.

For the space discretization, we consider the streamline diffusion finite element method which was proposed in the engineering literature by Hughes *et al.* ([3], [4], [5]). Error analysis was first performed in Johnson *et al.* [8] and Nävert [11] (and other literature cited therein). The discontinuous Galerkin method was proposed in Eriksson *et al.* [2]. Combining this with the streamline diffusion method, one obtains a fully discrete scheme constructed on a time-space mesh (*cf.* Johnson [6]). For piecewise linear shape functions in both time and space, the error in the energy norm is $O(h^{3/2})$ if the solution is in $\mathbf{H}^2(Q)$.

Nävert [11] performed a local L^2-error analysis for stationary scalar convection problems, obtaining the convergence order $O(h^{3/2})$ with piecewise linear shape functions. In Zhou [16], a detailed error analysis gave a local pointwise error of $O(h^{3/2-d/8})$ for nonstationary (essentially) hyperbolic problems. The sub-optimal rate of the convergence resulted from insufficiently sharp estimates of the discrete Green functions.

In this paper, we prove that the streamline diffusion method for linear scalar hyperbolic equations in arbitrary dimensions converges locally with an order $O(h^{3/2})$ in those strip subregions that are parallel to the characteristics, provided that the exact solution possesses $\mathbf{H}^2$-regularity in slightly larger subregions. For hyperbolic systems in one dimension we show that the streamline diffusion method has the same local convergence property in subregions that satisfy certain « shape » conditions that reflect the general properties of solutions of hyperbolic systems.

2. THE STREAMLINE DIFFUSION METHOD

We describe the streamline diffusion method for (1.2) and indicate the modifications needed for (1.1). Without loss of generality, we assume homogeneous boundary conditions. We let $0 = t_0 < t_1 < \cdots < t_N = T$ be a partition of $[0, T]$, set $I_n = (t_n, t_{n+1})$ and $\Delta t_n = t_n - t_{n-1}$, and define time-space « slabs » $S_n = I_n \times \Omega$ and « intersecting surfaces » $L_n = \{t_n\} \times \Omega$. For $h > 0$ and $0 \leqslant n \leqslant N-1$, we let T_h^n be a quasi-uniform triangulation of S_n into time-space elements e with element diameters $h_e \approx h$ satisfying the minimal angle condition. Note that T_h^n and T_h^{n-1} may be chosen independently. We assume that there exist constants K_1, $K_2 > 0$ such that

$$K_1 h \leqslant \Delta t_n \leqslant K_2 h . \tag{2.1}$$

On the triangulation T_h^n, we define the finite element subspaces

$$\mathbf{V}_{hj}^n = \{v \in C(S_n) : v|_e \in \mathbf{P}_1(e), \forall e \in T_h^n, v(t, 0) = 0, v(t, 1) = 0\} ,$$

for $j = 1, \ldots, M$, where $\mathbf{P}_1$ denotes the space of linear polynomials. We let $\mathbf{V}_h^n$ denote the tensor product of $\mathbf{V}_{hj}^n$ for $j = 1, \ldots, M$ and $\mathbf{V}_h$, the tensor product of $\mathbf{V}_h^n$ for $n = 0, \ldots, N-1$. A function in $\mathbf{V}_h$ is continuous in each S_n, but may be discontinuous across the intersecting surfaces L_n. Hence, we denote by $\mathbf{W}_+^n$ and $\mathbf{W}_-^n$ the two values

$$\mathbf{W}_+^n = \lim_{t \to t_n +} \mathbf{W}(t, .) , \quad \mathbf{W}_-^n = \lim_{t \to t_n -} \mathbf{W}(t, .) ,$$

while by

$$[\mathbf{W}^n] = \mathbf{W}_+^n - \mathbf{W}_-^n$$

we denote the jump of the function $\mathbf{W}$ across the intersecting surface L_n. We define various inner products for the vector functions $\mathbf{u}$ and $\mathbf{v}$ by

$$(\mathbf{u}, \mathbf{v})_{S_n} = \sum_{j=1}^{M} \int_{S_n} u_j v_j \, dx \, dt \,, \quad (\mathbf{u}, \mathbf{v}) = \sum_{n=0}^{N-1} (\mathbf{u}, \mathbf{v})_{S_n} \,,$$

$$\langle \mathbf{u}^n, \mathbf{v}^n \rangle_{L_n} = \sum_{j=1}^{M} \int_{L_n} u_j v_j \, dx \,.$$

The streamline diffusion method for (1.2) is defined as follows : for $n = 0, \dots, N-1$, given $\mathbf{U}_-^n$, find $\mathbf{U}^n \in \mathbf{V}_h^n$ such that

$$\begin{aligned} &\delta(\mathbf{U}_t^n + B\mathbf{U}_x^n + \mathbf{U}^n, \mathbf{W}_t^n + B^T \mathbf{W}_x^n)_{S_n} + \varepsilon(\mathbf{U}_x^n, \mathbf{W}_x^n)_{S_n} + \\ &\qquad + (\mathbf{U}_t^n + B\mathbf{U}_x^n + \mathbf{U}^n, \mathbf{W}^n)_{S_n} + \langle \mathbf{U}_+^n, \mathbf{W}_+^n \rangle_{L_n} \\ &\qquad = (\mathbf{f}, \mathbf{W}^n + \delta(\mathbf{W}_t^n + B^T \mathbf{W}_x^n))_{S_n} + \langle \mathbf{U}_-^n, \mathbf{W}_+^n \rangle_{L_n}, \quad \forall \mathbf{W}^n \in \mathbf{V}_h^n, \end{aligned} \tag{2.2}$$

where B^T is the transpose of B, $\delta = Kh$ for an appropriate constant K, and $\mathbf{U}_-^0 = \mathbf{u}_0$. Note that $\mathbf{U}_{xx}$ vanishes in each element since we are dealing with piecewise linear functions.

The coefficient ε in this scheme can be small, or even zero (in which case, we add an outflow boundary term), and this causes difficulties in the error analysis. Therefore, following Johnson [10], we introduce an artificial viscosity ε_m :

$$\varepsilon_m = \max \{ K_0 h^{3/2}, \varepsilon \} \,. \tag{2.3}$$

Later, we will see that this choice allows optimal local error estimates. The modified scheme is thus

$$\begin{aligned} &\delta(\mathbf{U}_t^n + B\mathbf{U}_x^n + \mathbf{U}^n, \mathbf{W}_t^n + B^T \mathbf{W}_x^n)_{S_n} + \varepsilon_m(\mathbf{U}_x^n, \mathbf{W}_x^n)_{S_n} + \\ &\qquad + (\mathbf{U}_t^n + B\mathbf{U}_x^n + \mathbf{U}^n, \mathbf{W}^n)_{S_n} + \langle \mathbf{U}_+^n, \mathbf{W}_+^n \rangle_{L_n} \\ &\qquad = (\mathbf{f}, \mathbf{W}^n + \delta(\mathbf{W}_t^n + B^T \mathbf{W}_x^n))_{S_n} + \langle \mathbf{U}_-^n, \mathbf{W}_+^n \rangle_{L_n}, \quad \forall \mathbf{W}^n \in \mathbf{V}_h^n, \end{aligned}$$

for $\quad n = 0, \dots, N-1$.

Summing these equations for all n, we obtain a « global » formulation of the streamline diffusion method : find $\mathbf{U} \in \mathbf{V}_h$ such that

$$\delta(\mathbf{U}_t + B\mathbf{U}_x, \mathbf{W}_t + B^T \mathbf{W}_x) + \varepsilon_m(\mathbf{U}_x, \mathbf{W}_x) + (\mathbf{U}_t + B\mathbf{U}_x, \mathbf{W}) +$$

$$+ (\mathbf{U}, \mathbf{W}) + \delta(\mathbf{U}, \mathbf{W}_t + B^T \mathbf{W}_x) + \sum_{n=1}^{N-1} \langle [\mathbf{U}^n], \mathbf{W}_+^n \rangle_{L_n} + \langle \mathbf{U}_+^0, W_+^0 \rangle_{L_0}$$

$$= (\mathbf{f}, \mathbf{W} + \delta(\mathbf{W}_t + B^T \mathbf{W}_x)) + \langle \mathbf{u}_0, \mathbf{W}_+^0 \rangle_{L_0}, \quad \forall \mathbf{W} \in \mathbf{V}_h . \quad (2.4)$$

If we define the bilinear form $B(.,.)$

$$B(\mathbf{U}, \mathbf{W}) = \delta(\mathbf{U}_t + B\mathbf{U}_x, \mathbf{W}_t + B^T \mathbf{W}_x) + \varepsilon_m(\mathbf{U}_x, \mathbf{W}_x) + (\mathbf{U}_t + B\mathbf{U}_x, \mathbf{W}) +$$

$$+ (\mathbf{U}, \mathbf{W}) + \delta(\mathbf{U}, \mathbf{W}_t + B^T \mathbf{W}_x) + \sum_{n=1}^{N-1} \langle [\mathbf{U}^n], \mathbf{W}_+^n \rangle_{L_n} + \langle \mathbf{U}_+^0, W_+^0 \rangle_{L_0}, \quad (2.5)$$

and the linear functional $L(.)$

$$L(\mathbf{W}) = (\mathbf{f}, \mathbf{W} + \delta(\mathbf{W}_t + B^T \mathbf{W}_x)) + \langle \mathbf{u}_0, \mathbf{W}_+^0 \rangle_{L_0},$$

then the streamline diffusion method can be rewritten as : find $\mathbf{U} \in \mathbf{V}_h$ such that

$$B(\mathbf{U}, \mathbf{W}) = L(\mathbf{W}), \quad \forall \mathbf{W} \in \mathbf{V}_h . \quad (2.6)$$

It is easy to verify that the following quasi-orthogonality relation holds between $\mathbf{u}$ and $\mathbf{U}$:

$$B(\mathbf{u} - \mathbf{U}, \mathbf{W}) = Per(\mathbf{u}, \mathbf{W}), \quad \forall \mathbf{W} \in \mathbf{V}_h , \quad (2.7)$$

where

$$Per(\mathbf{u}, \mathbf{W}) := \varepsilon\delta(\mathbf{u}_{xx}, \mathbf{W}_t + B^T \mathbf{W}_x) - (\varepsilon_m - \varepsilon)(\mathbf{u}_x, \mathbf{W}_x) .$$

3. LOCAL ERROR ESTIMATES IN THE SCALAR CASE

In order to explain the main idea, we consider the scalar problem (1.1) in one dimension with constant coefficients,

$$u_t + u_x + u - \varepsilon u_{xx} = f, \quad \text{in } I \times \Omega , \quad (3.1.a)$$

$$u(0, .) = u_0 , \quad \text{in } \Omega , \quad (3.1.b)$$

$$u(0, .) = 0 , u(. ,1) = 0 , \quad \text{in } I , \quad (3.1.c)$$

where $\Omega = (0, 1)$ and $I = (0, T)$. The energy form simplifies to

$$B(U, W) = \delta(U_t + U_x, W_t + W_x) + \varepsilon_m(U_x, W_x) + (U_t + U_x, W) + (U, W) + \\ + \delta(U, W_t + W_x) + \sum_{n=1}^{N-1} \langle [U^n], W_+^n \rangle_{L_n} + \langle U_+^0, W_+^0 \rangle_{L_0}, \quad (3.2)$$

and

$$L(W) = (f, W + \delta(W_t + W_x)) + \langle u_0, W_+^0 \rangle_{L_0}. \quad (3.3)$$

The quasi-orthogonality relation (2.7) holds with

$$Per(u, W) \equiv \varepsilon\delta(u_{xx}, W_t + W_x) - (\varepsilon_m - \varepsilon)(u_x, W_x). \quad (3.4)$$

Integrating by parts in $B(U, W)$ yields

$$B(U, W) = \delta(U_t + U_x, W_t + W_x) + \varepsilon_m(U_x, W_x) + \\ + (U, W) - (1 - \delta)(U, W_t + W_x) - \sum_{n=1}^{N-1} \langle U_-^n, [W^n] \rangle_{L_n} + \langle U_-^N, W_-^N \rangle_{L_N}. \quad (3.5)$$

Taking $W = U$ in both (3.2) and (3.5), we find that

$$B(W, W) \geq \frac{1}{2}\delta(W_t + W_x, W_t + W_x) + \varepsilon_m(W_x, W_x) + \frac{1}{2}(W, W) + \\ + \frac{1}{2}\sum_{n=1}^{N-1} \langle [W^n], [W^n] \rangle_{L_n} + \frac{1}{2}\langle W_-^N, W_-^N \rangle_{L_N} + \frac{1}{2}\langle W_+^0, W_+^0 \rangle_{L_0},$$

which implies that (2.6) has a unique solution. Defining the L_2-norms by

$$\| W \|^2 = \sum_{n=0}^{N-1} \int_{S_n} W^2 \, dx \, dt \quad \text{and} \quad | W^n |_{Ln}^2 = \int_\Omega (W^n)^2 \, dx,$$

and the energy norm by

$$|||W|||^2 = \delta \| W_t + W_x \|^2 + \varepsilon_m \| W_x \|^2 + \| W \|^2 + \\ + \sum_{n=1}^{N-1} | [W^n] |_{L_n}^2 + | W_-^N |_{L_N}^2 + | W_+^0 |_{L_0}^2, \quad (3.6)$$

we obtain

$$B(W, W) \geq \frac{1}{2} |||W|||^2, \quad \forall W \in \mathbf{V}_h,$$

and get the global stability estimate,

$$|||U||| \leq 2(\|f\|^2 + |u_0|_{L_0}^2)^{1/2}. \tag{3.7}$$

We consider a given trapezoidal subdomain $\hat{Q}$ with its horizontal sides being the boundaries of Q, (*see fig.* 1). In addition, θ_d denotes the angle between the characteristic direction and the x-axis, θ_l represents the angle between the left side of $\hat{Q}$ and x-axis, while θ_r is the angle between the right side of $\hat{Q}$ and the x-axis. Our goal is to find conditions on θ_l and θ_r so as to be able to estimate the error in $\hat{Q}$.

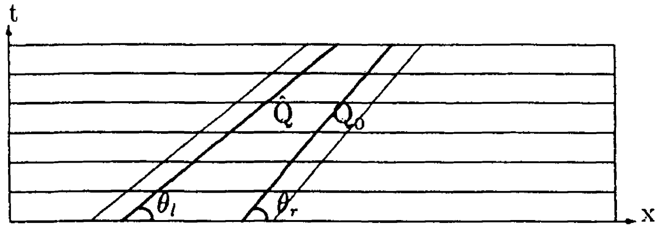

Figure 1. — Given trapezoid $\hat{Q}$ and its enlargement Q_0.

First, we define a distance function

$$d(t, x) = \text{dist}\,((t, x), \hat{Q})$$

and a weight function

$$\varphi(t, x) = \exp(-d(t, x)/\sigma), \tag{3.8}$$

where

$$\sigma = \gamma \sqrt{\varepsilon_m}, \tag{3.9}$$

with $\gamma > 0$ to be determined. Obviously, $\varphi(t, x) \equiv 1$ for $(t, x) \in \hat{Q}$, and it decreases exponentially away from $\hat{Q}$. We define the weighted norms

$$\|W\|_\varphi^2 = \sum_{n=0}^{N-1} \int_{S_n} \varphi W^2 \, dx \, dt \quad \text{and} \quad |W^n|_{\varphi, L_n}^2 = \int_{L_n} \varphi^n (W^n)^2 \, dx.$$

Using the definitions of the weighted norms and the bilinear form $B(\,.\,,\,.\,)$, we obtain

$$B(W, \varphi W) = \delta \| W_t + W_x \|_\varphi^2 + \varepsilon_m \| W_x \|_\varphi^2 + \| W \|_\varphi^2 + \\ + \delta(W_t + W_x, (\varphi_t + \varphi_x) W) + \varepsilon_m (W_x, \varphi_x W) + \\ + \frac{1}{2} \left(\sum_{n=1}^{N-1} |[W^n]|_{\varphi, L_n}^2 + |W_-^N|_{\varphi, L_N}^2 + |W_+^0|_{\varphi, L_0}^2 \right) + \delta(W, \varphi(W_t + W_x)) + \\ + \delta(W, (\varphi_t + \varphi_x) W) - (W, (\varphi_t + \varphi_x) W) . \quad (3.10)$$

In order to prove that the weighted bilinear form $B(\,.\,, \varphi\,.\,)$ is positive definite, we will see that the last term on the right side must be non-negative, or equivalently

$$\varphi_t + \varphi_x \leqslant 0 , \quad \forall (t, x) \in Q . \quad (3.11)$$

This holds automatically in the interior of $\hat{Q}$. On the left side of $\hat{Q}$, $\varphi_t = \varphi' \cos \theta_l$ and $\varphi_x = -\varphi' \sin \theta_l$, thus

$$\varphi_t + \varphi_x = \varphi' (\cos \theta_l - \sin \theta_l) = -\varphi' \sin (\theta_l - \pi/4) .$$

Since $\varphi' \leqslant 0$ for all (t, x), θ_l: $0 \leqslant \theta_l \leqslant \pi$, must satisfy

$$\theta_l \leqslant \frac{\pi}{4} = \theta_d .$$

Similarly

$$\theta_r \geqslant \frac{\pi}{4} = \theta_d ,$$

and therefore, the subdomain $\hat{Q}$ must be chosen so that

$$0 \leqslant \theta_l \leqslant \theta_d \leqslant \theta_r \leqslant \pi . \quad (3.12)$$

We can explain condition (3.12) in the following way. We want to estimate the error in $\hat{Q}$ in terms of a norm on the exact solution in a slightly larger region. This is possible only when all points in $\hat{Q}$ are influenced exclusively from this larger region. In other words, all points in $\hat{Q}$ must originate from the larger region along the characteristic direction and, therefore, both sides of $\hat{Q}$ must be outflow boundaries.

Note that if the side boundaries of region $\hat{Q}$ or the characteristics are not straight lines, (3.12) must hold at every point on the side boundaries of $\hat{Q}$, which implies once again that the side boundaries of $\hat{Q}$ cannot be inflow boundaries.

We now assume that (3.12) holds and therefore φ satisfies (3.11). Since

$$|\varphi_t + \varphi_x| \leqslant \frac{1}{\sigma}\varphi \quad \text{and} \quad |\varphi_x| \leqslant \frac{1}{\sigma}\varphi , \quad \forall (t, x) \in Q , \tag{3.13}$$

the terms in (3.10) can be majorized as follows :

$$\delta|(W, \varphi(W_t + W_x))| \leqslant \frac{1}{4}\delta\| W_t + W_x\|_\varphi^2 + \delta\| W\|_\varphi^2 ,$$

$$\delta|(W, (\varphi_t + \varphi_x) W)| \leqslant \frac{\delta}{\sigma} \| W\|_\varphi^2 ,$$

$$\delta|(W_t + W_x, (\varphi_t + \varphi_x) W)| \leqslant \frac{\delta}{4} \| W_t + W_x\|_\varphi^2 + \frac{\delta}{\sigma} \left\|\sqrt{|\varphi_t + \varphi_x|}\, W\right\|^2 ,$$

$$\varepsilon_m|(W_x, \varphi_x W)| \leqslant \frac{\varepsilon_m}{2} \| W_x\|_\varphi^2 + \frac{\varepsilon_m}{\sigma^2} \| W\|_\varphi^2 .$$

Taking into account that $\delta/\sigma \leqslant C/\gamma$ and $\varepsilon_m/\sigma^2 = 1/\gamma$, we have from (3.10), for γ sufficiently large,

$$\begin{aligned} B(W, \varphi W) \geqslant \frac{1}{2}\delta\| W_t + W_x\|_\varphi^2 + \frac{1}{2}\varepsilon_m\| W_x\|_\varphi^2 + \frac{1}{2} \| W\|_\varphi^2 + \\ + \frac{1}{2}\sum_{n=1}^{N-1} |[W^n]|_{\varphi, L_N}^2 + \frac{1}{2} |W_-^N|_{\varphi, L_n}^2 + \frac{1}{2} |W_+^0|_{\varphi, L_0}^2 + \frac{1}{2} \left\|\sqrt{|\varphi_t + \varphi_x|}\, W\right\|^2 . \end{aligned}$$

We define the weighted energy norm by

$$\begin{aligned} |||W|||_\varphi^2 = \delta\| W_t + W_x\|_\varphi^2 + \varepsilon_m\| W_x\|_\varphi^2 + \| W\|_\varphi^2 + \left\|\sqrt{|\varphi_t + \varphi_x|}\, W\right\|^2 + \\ + \sum_{n=1}^{N-1} |[W^n]|_{\varphi, L_n}^2 + |W_-^N|_{\varphi, L_N}^2 + |W_+^0|_{\varphi, L_0}^2 , \end{aligned}$$

so that the inequality derived above can be rewritten as

$$B(W, \varphi W) \geqslant \frac{1}{2} |||W|||_\varphi^2 , \quad \forall W \in \mathbf{V}_h . \tag{3.14}$$

We define the enlargement Q_0 of a given subdomain $\hat{Q}$ by

$$Q_0 = \left\{ \begin{array}{l} \text{Union of time - space elements entirely contained in} \\ \{(t, x) : \text{dist}((t, x), \hat{Q}) \leqslant K\sigma|\log h|\} \cap Q \end{array} \right\} \tag{3.15}$$

with K to be determined and σ given in (3.9). We set $Q_c := Q \backslash Q_0$. Since, for ε very small, $\varepsilon_m = K_0 h^{3/2}$, the enlargement Q_0 is only $O(h^{3/4} |\log h|)$ wider than $\hat{Q}$ (*cf. fig.* 1).

We now prove a local error estimate in the energy norm. We denote by $|||\cdot|||_{\hat{Q}}$ the norm defined in (3.6) with integrals taken over $\hat{Q}$.

THEOREM 3.1 : *Given a subregion $\hat{Q}$ satisfying* (3.12), *for any fixed number $\nu \geqslant 3/2$, we can determine the constant $K > 0$ in* (3.15) *to fix the enlargement Q_0. Assume that*

$$u \in \mathbf{H}^2(Q_0), \quad u \in \mathbf{L}^\infty(Q) \quad \text{and} \quad f \in \mathbf{L}^1(Q_c),$$

and that $\varepsilon \leqslant \varepsilon_m = K_0 h^{3/2}$, then there exists a constant $C > 0$, independent of ε, such that

$$|||u - U|||_{\hat{Q}} \leqslant C h^{3/2} \| u \|_{H^2(Q_0)} + C h^\nu (\| u \|_{L^\infty(Q)} + \| f \|_{L^1(Q_c)}) . \tag{3.16}$$

Proof: We split the error into two parts,

$$(u - U) = (u - I_h u) + (I_h u - U) \equiv \eta + \xi ,$$

where $I_h u$ is the linear piecewise interpolant of u in $\mathbf{V}_h$. Since $\varphi(t, x) \equiv 1$ for $(t, x) \in \hat{Q}$ and $\hat{Q} \subset Q_0 \subset Q$,

$$|||u - U|||_{\hat{Q}} \leqslant |||\eta|||_{Q_0} + |||\xi|||_\varphi . \tag{3.17}$$

Standard interpolation theory implies that

$$|||\eta|||^2_{Q_0} \leqslant C h^3 \| u \|^2_{H^2(Q_0)} ,$$

thus the crux is to prove that (3.16) holds with $u - U$ replaced by ξ.

Since $\xi \in \mathbf{V}_h$, we use (3.14) and (3.4) to get

$$\frac{1}{2} |||\xi|||^2_\varphi \leqslant B(\xi, \varphi\xi) = B(\xi, \varphi\xi - W) + Per(u, W) - B(\eta, W) =$$

$$= B(\xi, E) - Per(u, E) + Per(u, \varphi\xi) + B(\eta, E) - B(\eta, \varphi\xi) , \tag{3.18}$$

where $W = I_h(\varphi\xi) \in \mathbf{V}_h$ and $E = \varphi\xi - W$. Let $0 < \theta < 1$ be a generic constant. Given $\nu \geqslant 3/2$, we claim that for K in (3.15) sufficiently large,

$$\delta \| E_t + E_x \|^2 + \varepsilon_m \| E_x \|^2 + h^{-1} \| E \|^2 + \sum_{n=0}^{N-1} | E_+^n |_{L_n}^2 \leqslant \theta \, |||\xi|||_\varphi^2 , \quad (3.19a)$$

$$\delta \| E_t + E_x \|_{L^\infty(Q_c)} + \varepsilon_m \| E_x \|_{L^\infty(Q_c)} + \| E \|_{L^\infty(Q_c)} \leqslant C h^\nu |||\xi|||_\varphi , \quad (3.19b)$$

$$\| \varphi(\xi_t + \xi_x) \|_{L^\infty(Q_c)} + \| \varphi\xi_x \|_{L^\infty(Q_c)} + \| \varphi\xi \|_{L^\infty(Q_c)} \leqslant C h^\nu |||\xi|||_\varphi . \quad (3.19c)$$

For example, to prove (3.19*a*), we use (2.3) to get (the index e refers to the integral over element e)

$$\delta \| E_t + E_x \|^2 \leqslant \delta \sum_e \| E_t + E_x \|_e^2 \leqslant C h^3 \sum_e \{ \| (\varphi\xi)_{xx} \|_e^2 + \| (\varphi\xi)_{xt} \|_e^2 + \| (\varphi\xi)_{tt} \|_e^2 \}$$

$$\leqslant C h^3 \sum_e \left\{ \max_e \varphi \left(\frac{1}{\sigma^4} \| \xi \|_{\varphi, e}^2 + \frac{1}{\sigma^2} \| \xi_t + \xi_x \|_{\varphi, e}^2 + \frac{1}{\sigma^2} \| \xi_x \|_{\varphi, e}^2 \right) \right\}$$

$$\leqslant \frac{C}{\gamma^2} \left(\frac{h^3}{\varepsilon_m^2} \| \xi \|_\varphi^2 + \frac{h^2}{\varepsilon_m} \delta \| \xi_t + \xi_x \|_\varphi^2 + \frac{h^3}{\varepsilon_m^2} \varepsilon_m \| \xi_x \|_\varphi^2 \right)$$

$$\leqslant \frac{C}{\gamma^2} |||\xi|||_\varphi^2 \leqslant \theta |||\xi|||_\varphi^2 ,$$

where the last step holds for γ sufficiently large. The other terms on the left side of (3.19*a*) can be bounded similarly. To get a bound at the first term in (3.19*b*), we use the fact that $\varphi(t, x)$ decreases exponentially away from Q_0. For any $\nu \geqslant 3/2$, we take K in (3.15) sufficiently large so that

$$\varphi(t, x) \leqslant C h^{2\nu + 4}, \quad \text{and} \quad | \nabla \varphi(t, x) | \leqslant C h^{2\nu + 4} , \quad \forall (t, x) \in Q_c .$$

Once K is fixed, Q_0 is determined for the given subdomain $\hat{Q}$. On the grid subdomain Q_c, we apply the inverse inequality of finite elements several times to obtain

$$\| E_t + E_x \|_{L^\infty(Q_c)} \leqslant C h \| \nabla^2 (\varphi\xi) \|_{L^\infty(Q_c)} \leqslant C h^{-1} \| \varphi^{1/2} \|_{L^\infty(Q_c)} |||\xi|||_\varphi \leqslant C h^\nu |||\xi|||_\varphi .$$

The other estimates in (3.19*b*) and (3.19*c*) are obtained in the same way.

We now examine (3.18). Applying (3.19) to $B(\xi, E)$, we obtain

$$|B(\xi, E)| \leq \theta |||\xi|||_{\varphi}^2 . \tag{3.20}$$

For $Per(u, E)$, we write

$$|Per(u, E)| \leq \varepsilon\delta |(u_{xx}, E_t + E_x)| + |\varepsilon_m - \varepsilon| \, |(u_x, E_x)| .$$

Integrating by parts over each element in Q_c and applying (3.19), we have for the first term

$$\begin{aligned} \varepsilon\delta |(u_{xx}, E_t + E_x)| &\leq \varepsilon\delta |(u_{xx}, E_t + E_x)_{Q_0}| + \delta |(u_t + u_x + u - f, E_t + E_x)_{Q_c}| \\ &\leq \theta |||\xi|||_{\varphi}^2 + Ch\varepsilon^2 \|u\|_{H^2(Q_0)}^2 + Ch^{2\nu}(\|u\|_{L^\infty(Q)}^2 + \|f\|_{L^1(Q_c)}^2) . \end{aligned}$$

Since $|\varepsilon_m - \varepsilon| \leq Ch^{3/2}$, we get similarly

$$|\varepsilon_m - \varepsilon| \, |(u_x, E_x)| \leq \theta |||\xi|||_{\varphi}^2 + Ch^3 \|u\|_{H^2(Q_0)}^2 + Ch^{2\nu} \|u\|_{L^\infty(Q)}^2 .$$

Therefore,

$$|Per(u, E)| \leq \theta |||\xi|||_{\varphi}^2 + Ch^3 \|u\|_{H^2(Q_0)}^2 + Ch^{2\nu}(\|u\|_{L^\infty(Q)}^2 + \|f\|_{L^1(Q_c)}^2) . \tag{3.21}$$

In the same way, we obtain

$$|Per(u, \varphi\xi)| \leq \theta |||\xi|||_{\varphi}^2 + Ch^3 \|u\|_{H^2(Q_0)}^2 + Ch^{2\nu}(\|u\|_{L^\infty(Q)}^2 + \|f\|_{L^1(Q_c)}^2) . \tag{3.22}$$

The treatment of each of the terms in $B(\eta, E)$ is very similar. For example, we consider $\delta(\eta_t + \eta_x, E_t + E_x)$. Using the interpolation results on Q_0 and (3.19), we obtain

$$\begin{aligned} \delta |(\eta_t + \eta_x, E_t + E_x)| &\leq \delta |(\eta_t + \eta_x, E_t + E_x)_{Q_0}| + \delta |(\eta_t + \eta_x, E_t + E_x)_{Q_c}| \\ &\leq \delta \|\eta_t + \eta_x\|_{Q_0} \|E_t + E_x\| + \delta \sum_{e \subseteq Q_c} |(\eta, E_t + E_x)_{\partial e}| \\ &\leq \theta |||\xi|||_{\varphi}^2 + Ch^3 \|u\|_{H^2(Q_0)}^2 + Ch^{2\nu} \|u\|_{L^\infty(Q)}^2 , \end{aligned}$$

and therefore

$$|B(\eta, E)| \leq \theta |||\xi|||_{\varphi}^{2} + Ch^{3} \| u \|_{H^{2}(Q_0)}^{2} + Ch^{2\nu} \| u \|_{L^{\infty}(Q)}^{2} . \tag{3.23}$$

We have an identical result for the last term $B(\eta, \varphi\xi)$ in (3.18)

$$|B(\eta, \varphi\xi)| \leq \theta |||\xi|||_{\varphi}^{2} + Ch^{3} \| u \|_{H^{2}(Q_0)}^{2} + Ch^{2\nu} \| u \|_{L^{\infty}(Q)}^{2} . \tag{3.24}$$

Summing up (3.20)-(3.24) completes the estimate for $|||\xi|||_{\varphi}$. Together with (3.17) we get (3.16) for the case $d = 1$. Since the interpolation results for finite elements also hold in three dimensions, we conclude that estimate (3.16) is also valid for $d = 2$. The interpolation results remain also true for $d \geq 3$ with proper choice of a generalized interpolation function in the finite element spaces (*cf.* Scott and Zhang [13]). □

If the characteristic is not a straight line, i.e., $\beta(t, x)$ is not constant, the analysis still holds under some reasonable conditions on $\beta(t, x)$ and $\hat{Q}$. We begin with determining the shape of $\hat{Q}$ that guarantees that $B(\,\cdot\,, \cdot\,)$ is positive definite. Given $\hat{Q}$, we define the weight function $\varphi(t, x)$ as in (3.8). Instead of (3.10), we now have

$$\begin{aligned} B(W, \varphi W) = {} & \delta \| W_t + \beta W_x \|_{\varphi}^{2} + \varepsilon_m \| W_x \|_{\varphi}^{2} + ((1 - \beta_x) W, \varphi W) \\ & + \frac{1}{2} \sum_{n=1}^{N-1} |[W^n]|_{\varphi, L_n}^{2} + \frac{1}{2} |W_-^N|_{\varphi, L_N}^{2} + \frac{1}{2} |W_+^0|_{\varphi, L_0}^{2} \\ & + \delta(W, \varphi(W_t + \beta W_x)) + \delta(W, (\varphi_t + \beta\varphi_x) W) \\ & + \delta(W_t + W_x, (\varphi_t + \beta\varphi_x) W) + \varepsilon_m (W_x, \varphi_x W) \\ & - (W, (\varphi_t + \beta\varphi_x) W) . \end{aligned}$$

To guarantee the positive definiteness of the weighted bilinear form $B(\,\cdot\,, \varphi\,\cdot\,)$, we therefore assume that

$$\beta_x(t, x) \leq 0 , \quad \forall (t, x) \in Q . \tag{3.25}$$

As before, we also require that

$$\varphi_t + \beta\varphi_x \leq 0 , \quad \forall (t, x) \in Q . \tag{3.26}$$

In the case of variable coefficients, the angles θ_d, θ_l and θ_r are also functions of (t, x) and (3.11) is equivalent to

$$0 \leq \theta_l \leq \theta_d \leq \theta_r \leq \pi , \quad \forall (t, x) \in \widehat{\partial\hat{Q}} , \tag{3.27}$$

where $\widehat{\partial \hat{Q}}$ stands for the side boundaries of $\hat{Q}$. This implies that the side boundaries of $\hat{Q}$ are not inflow boundaries of the subdomain $\hat{Q}$. One admissible shape of $\hat{Q}$, for example, has two characteristics of equation (1.1) as side boundaries.

In view of (3.25), we define a new energy norm by

$$|||W|||^2 = \delta \| W_t + \beta W_x \|^2 + \varepsilon_m \| W_x \|^2 + \left\| \sqrt{1 - \beta_x}\, W \right\|^2 + \sum_{n=1}^{N-1} |[W^n]|^2_{L_n} + |W_-^N|^2_{L_N} + |W_+^0|^2_{L_0} . \tag{3.28}$$

It is easy to check that

$$B(W, W) \geqslant \frac{1}{2} |||W|||^2 .$$

Due to (3.26), we define the weighted norm

$$|||W|||^2_\varphi = \delta \| W_t + \beta W_x \|^2_\varphi + \varepsilon_m \| W_x \|^2 + \left\| \sqrt{1 - \beta_x}\, W \right\|^2_\varphi + \left\| \sqrt{|\varphi_t + \beta \varphi_x|}\, W \right\|^2 + \sum_{n=1}^{N-1} |[W^n]|^2_{\varphi, L_n} + |W_-^N|^2_{\varphi, L_N} + |W_+^0|^2_{\varphi, L_0} . \tag{3.29}$$

Now, it follows that (3.14) is valid, and we obtain the following theorem in a straightforward fashion.

THEOREM 3.2 : *Given a subregion satisfying* (3.27), *for any fixed number* $v > 3/2$, *we can determine the constant* $K > 0$ *in* (3.15) *sufficiently large to fix the enlargement* Q_0. *Assume that*

$$u \in \mathbf{H}^2(Q_0) , \quad u \in \mathbf{L}^\infty(Q) \quad \textit{and} \quad f \in \mathbf{L}^1(Q_c) ,$$

and that

$$\operatorname{div} \beta \leqslant 0 , \quad \operatorname{div} \beta \in \mathbf{L}^\infty(Q_0) , \quad \operatorname{div} \beta \in \mathbf{L}^1(Q) \quad \textit{and} \quad \beta \in \mathbf{L}^\infty(Q) ,$$

and $\varepsilon \leqslant \varepsilon_m = K_0 h^{3/2}$, *then there exists a constant* $C > 0$, *independent of* ε, *such that*

$$|||u - U|||_{\hat{Q}} \leqslant Ch^{3/2} \| u \|_{H^2(Q_0)} (\| \operatorname{div} \beta \|_{L^\infty(Q_0)} + \| \beta \|_{L^\infty(Q)}) + Ch^{\nu} (1 + \| \operatorname{div} \beta \|_{L^1(Q)}) (\| u \|_{L^\infty(Q)} + \| f \|_{L^1(Q_c)}) .$$

4. LOCAL ERROR ESTIMATES FOR HYPERBOLIC SYSTEMS

In this section, we prove that the result for the scalar problem is also true for hyperbolic systems (1.2), at least in one dimension. Roughly speaking, if the exact solution in a time-space subdomain belongs to $\mathbf{H}^2$, then the discrete solution of the streamline diffusion method converges in the energy norm with the order $O(h^{3/2})$. This means that the full convergence rate is achieved locally. However, we will see that the « shape » condition of the subdomain is very different from that in the scalar case.

As before, we first consider the simplest case, i.e., system (1.2) with a constant coefficient matrix B. The corresponding discrete solution satisfies equation (2.4). Since we have assumed that problem (1.2) is of (essentially) hyperbolic type, there exists a real invertible matrix D such that

$$D^{-1} B D = \Lambda \quad \text{and} \quad D^T B^T D^{-T} = \Lambda , \tag{4.1}$$

where $\Lambda = \operatorname{diag}(\lambda_i)$ is a real diagonal matrix and the i-th column of the matrix D is the eigenvector of the matrix B corresponding to the eigenvalue λ_i. Since B may be unsymmetric, D is generally not an orthogonal matrix.

Introducing function transformations,

$$\mathbf{V} = D\tilde{\mathbf{V}} \quad \text{and} \quad \mathbf{W} = D^{-T}\tilde{\mathbf{W}} , \tag{4.2}$$

and noticing the fact that D is a constant matrix and using (4.1), we obtain

$$\begin{aligned} B(\mathbf{V}, \mathbf{W}) = {} & \delta(\tilde{\mathbf{V}}_t + \Lambda \tilde{\mathbf{V}}_x, \tilde{\mathbf{W}}_t + \Lambda \tilde{\mathbf{W}}_x) + \varepsilon_m(\tilde{\mathbf{V}}_x, \tilde{\mathbf{W}}_x) + (\tilde{\mathbf{V}}_t + \Lambda \tilde{\mathbf{V}}_x, \tilde{\mathbf{W}}) \\ & + (\tilde{\mathbf{V}}, \tilde{\mathbf{W}}) + \delta(\tilde{\mathbf{V}}, \tilde{\mathbf{W}}_t + \Lambda \tilde{\mathbf{W}}_x) + \\ & + \sum_{n=1}^{N-1} \langle [\tilde{\mathbf{V}}^n], \tilde{\mathbf{W}}_+^n \rangle_{L_n} + \langle \tilde{\mathbf{V}}_+^0, \tilde{\mathbf{W}}_+^0 \rangle_{L_0} . \end{aligned}$$

This leads us to introduce an auxiliary bilinear form

$$\tilde{B}(\tilde{\mathbf{V}}, \tilde{\mathbf{W}}) := B(\mathbf{V}, \mathbf{W}) . \tag{4.3}$$

Since Λ is a diagonal matrix, the bilinear form $\tilde{B}(\,.\,,\,.\,)$ can be separated as a sum

$$\tilde{B}(\tilde{\mathbf{V}},\tilde{\mathbf{W}}) = \sum_{j=1}^{M} \tilde{B}_j(\tilde{V}_j, \tilde{W}_j)\,,$$

with

$$\begin{aligned}\tilde{B}_j(\tilde{V}_j, \tilde{W}_j) &= \delta(\tilde{V}_{jt} + \lambda_j \tilde{V}_{jx}, \tilde{W}_{jt} + \lambda_j \tilde{W}_{jx}) \\ &\quad + \varepsilon_m(\tilde{V}_{jx}, \tilde{W}_{jx}) + (\tilde{V}_{jt} + \lambda_j \tilde{V}_{jx}, \tilde{W}_j) \\ &\quad + (\tilde{V}_j, \tilde{W}_j) + \delta(\tilde{V}_j, \tilde{W}_{jt} + \lambda_j \tilde{W}_{jx}) \\ &\quad + \sum_{n=1}^{N-1} \langle [\tilde{V}_j^n], \tilde{W}_{j+}^n \rangle_{L_n} + \langle \tilde{V}_{j+}^0, \tilde{W}_{j+}^0 \rangle_{L_0}\,.\end{aligned}$$

As in the scalar case, we define the energy norm for $\tilde{B}_j(\,.\,,\,.\,)$ as

$$|||\tilde{W}_j|||_j^2 = \delta \| \tilde{W}_{jt} + \lambda_j \tilde{W}_{jx} \|^2 + \varepsilon_m \| \tilde{W}_{jx} \|^2 + \| \tilde{W}_j \|^2 + \sum_{n=1}^{N-1} |[\tilde{W}_j^n]|_{L_n}^2 + |\tilde{W}_{j-}^N|_{L_N}^2 + |\tilde{W}_{j+}^0|_{L_0}^2\,,$$

and we have the positive definiteness relation

$$\tilde{B}_j(\tilde{W}_j, \tilde{W}_j) \geq \frac{1}{2} |||\tilde{W}_j|||_j^2\,.$$

This leads to a natural definition of the energy norm associated with the bilinear form $\tilde{B}(\,.\,,\,.\,)$:

$$|||\tilde{\mathbf{W}}|||^2 = \sum_{j=1}^{M} |||\tilde{W}_j|||_j^2 = \delta \| \tilde{\mathbf{W}}_t + \Lambda \tilde{\mathbf{W}}_x \|^2 + \varepsilon_m \| \tilde{\mathbf{W}}_x \|^2 + \| \tilde{\mathbf{W}} \|^2 + \sum_{n=1}^{N-1} |[\tilde{\mathbf{W}}^n]|_{L_n}^2 + |\tilde{\mathbf{W}}_-^N|_{L_N}^2 + |\tilde{\mathbf{W}}_+^0|_{L_0}^2\,,$$

and the positive definiteness relation remains true due to the separability of $\tilde{B}(\,.\,,\,.\,)$:

$$\tilde{B}(\tilde{\mathbf{W}}, \tilde{\mathbf{W}}) \geq \frac{1}{2} |||\tilde{\mathbf{W}}|||^2\,. \tag{4.4}$$

Recalling the variable transformation (4.2), we have the relation

$$B(\mathbf{V}, \mathbf{W}) = \tilde{B}(D^{-1}\mathbf{V}, D^T\mathbf{W}) . \tag{4.5}$$

From these two relations, we obtain the positive definiteness inequality :

$$B(\mathbf{V}, D^{-T}D^{-1}\mathbf{V}) \geq \frac{1}{2} |||D^{-1}\mathbf{V}|||^2 , \quad \forall \mathbf{V} \in \mathbf{V}_h , \tag{4.6}$$

which gives the global stability estimate

$$|||\mathbf{U}||| \leq 2 \| D \| \; \| D^{-1} \| (\| \mathbf{f} \|^2 + |\mathbf{u}_0|^2_{L_0})^{1/2} . \tag{4.7}$$

For completeness, we introduce an additional perturbation term

$$\tilde{P}er(\mathbf{v}, \mathbf{W}) \equiv \varepsilon\delta(\mathbf{v}_{xx}, \mathbf{W}_t + \Lambda \mathbf{W}_x) - (\varepsilon_m - \varepsilon)(\mathbf{v}_x, \mathbf{W}_x) ,$$

it is also separable, i.e.,

$$\tilde{P}er(\mathbf{v}, \mathbf{W}) = \sum_{j=1}^{M} \tilde{P}er_j(v_j, W_j) ,$$

with

$$\tilde{P}er_j(v_j, W_j) = \varepsilon\delta(v_j, W_{jt} + \lambda_j W_{jx}) - (\varepsilon_m - \varepsilon)(v_{jx}, W_{jx}) .$$

Furthermore, we have the transformation relation

$$Per(\mathbf{u}, \mathbf{W}) = \tilde{P}er(D^{-1}\mathbf{u}, D^T\mathbf{W}) . \tag{4.8}$$

We now suppose that a subdomain $\hat{Q}$ is given on which we want to know the convergence behavior of the discrete solution. First, we assume that $\hat{Q}$ is a trapezoid with its horizontal sides to be the boundaries of Q (*see fig.* 2). We seek for conditions under which we can estimate the local convergence in the energy norm with the exact solution being measured on an larger subdomain. As in the last section, we define a distance function and a weight function as

$$d(t, x) = \operatorname{dist}((t, x), \hat{Q}) , \quad \varphi(t, x) = \exp(-d(t, x)/\sigma) , \quad \forall (t, x) \in Q ,$$

with σ given in (3.9). We define the weight matrix as

$$\Phi = \operatorname{diag}(\varphi(t, x)) ,$$

which has identical diagonal entries. Similarly as in (3.10), there holds

$$\tilde{B}(\mathbf{W}, \Phi\mathbf{W}) = \delta \| W_t + \Lambda \mathbf{W}_x \|_\varphi^2 + \varepsilon_m \| \mathbf{W}_x \|_\varphi^2 + \| \mathbf{W} \|_\varphi^2 + \delta(\mathbf{W}, (\Phi_t + \Lambda\Phi_x)\mathbf{W})$$

$$+ \varepsilon_m \| \mathbf{W}_x, \Phi_x \mathbf{W} \| + \frac{1}{2} \sum_{n=1}^{N-1} |[\mathbf{W}^n]|_{\varphi, L_n}^2$$

$$+ \frac{1}{2} |\mathbf{W}_-^N|_{\varphi, L_n}^2 + \frac{1}{2} |\mathbf{W}_+^0|_{\varphi, L_0}^2$$

$$+ \delta(\mathbf{W}, \Phi(\mathbf{W}_t + \Lambda\mathbf{W}_x)) + \delta(\mathbf{W}_t + \Lambda\mathbf{W}_x, (\Phi_t + \Lambda\Phi_x)\mathbf{W})$$

$$- (\mathbf{W}, (\Phi_t + \Lambda\Phi_x)\mathbf{W}) . \tag{4.9}$$

It is clear that, in order for $\tilde{B}(\mathbf{W}, \Phi\mathbf{W})$ to be positive definite, we have to impose conditions on Φ such that

$$- (\mathbf{W}, (\Phi_t + \Lambda\Phi_x)\mathbf{W}) \geq 0 ,$$

which is equivalent to

$$\varphi_t + \lambda_j \varphi_x \leq 0 , \quad \forall (t, x) \in Q , \quad j = 1, \dots, M .$$

Let $\theta_j = \arctan(1/\lambda_j)$ be the angle between the characteristic direction of λ_j and the x-axis and let θ_l and θ_r be defined as in the previous section. From the discussions above, we conclude that

$$\theta_l \leq \min_j \theta_j , \quad \theta_r \leq \max_j \theta_j .$$

An admissible subdomain is shown in *figure* 2.

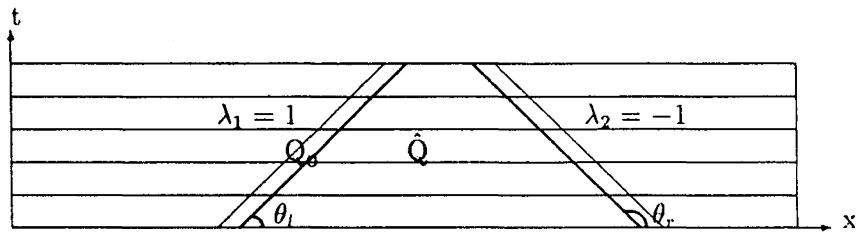

Figure 2. — An admissible subdomain $\hat{Q}$ and its enlargement Q_0.

Then, we assume that the given subdomain $\hat{Q}$ satisfies

$$0 \leqslant \min_j \theta_j \leqslant \theta_l \leqslant \theta_r \leqslant \max_j \theta_j \leqslant \pi . \tag{4.10}$$

Correspondingly, we define the weighted energy norm as

$$|||\mathbf{W}|||^2_\varphi = \delta\, |||W_t + A\mathbf{W}_x|||^2_\varphi = \varepsilon_m \| \mathbf{W}_x \|^2_\varphi + \| \mathbf{W} \|^2_\varphi - (\mathbf{W}, (\Phi_t + A\Phi_x)\, \mathbf{W})$$

$$+ \sum_{n=1}^{N-1} |[\mathbf{W}^n]|^2_{\varphi, L_n} + |\mathbf{W}^N_-|^2_{\varphi, L_N} + |\mathbf{W}^0_+|^2_{\varphi, L_0} .$$

Since the weight function $\varphi(t, x)$ satisfies the estimates (3.13), we conclude the desired positive definiteness from (4.9), i.e.,

$$\tilde{B}(\mathbf{W}, \Phi\mathbf{W}) \geqslant \frac{1}{2} |||\mathbf{W}|||^2_\varphi . \tag{4.11}$$

For the given subdomain $\hat{Q}$, we define its enlargement Q_0 as

$$Q_0 = \left\{ \begin{array}{c} \text{Union of time-space elements entirely contained in} \\ \{(t, x) : dist\,((t, x), \hat{Q}) \leqslant K\sigma |\log h|\} \cap Q \end{array} \right\}, \tag{4.12}$$

with K to be determined and σ given in (3.9). Furthermore, we set $Q_c := Q \backslash Q_0$. As usual, this enlargement Q_0 is only $O(h^{3/4} |\log h|)$ wider than $\hat{Q}$. Condition (4.10) seems to be necessary for the local error estimate. Later we will show that if (4.10) is not imposed on the subregion $\hat{Q}$, then the enlargement Q_0 must be increased significantly. With this definition, we state the first result in this section.

THEOREM 4.1 : *Given a subdomain $\hat{Q} \subseteq Q$ satisfying condition* (4.10) *and for any $\nu \geqslant 3/2$, we can determine the constant K in* (4.12). *Suppose that on this fixed subdomain Q_0 :*

$$\mathbf{u} \in \mathbf{H}^2(Q_0)^M, \quad \mathbf{u} \in \mathbf{L}^\infty(Q)^M, \quad \mathbf{f} \in \mathbf{L}^1(Q_c)^M ,$$

and that the diffusion coefficient satisfies $\varepsilon \leqslant \varepsilon_m = K_0 h^{3/2}$. Then, there exists a constant $C > 0$, independent of the diffusion coefficient ε, such that

$$|||\mathbf{u} - \mathbf{U}|||_{\hat{Q}} \leqslant C h^{3/2} \| \mathbf{u} \|_{H^2(Q_0)^M} + C h^\nu (\| \mathbf{u} \|_{L^\infty(Q)^M} + \| \mathbf{f} \|_{L^1(Q_c)^M}) .$$

Proof : We split the error into two parts

$$\mathbf{u} - \mathbf{U} = (\mathbf{u} - I_h \mathbf{u}) + (I_h \mathbf{u} - \mathbf{U}) \equiv \boldsymbol{\eta} + \boldsymbol{\xi} ,$$

and deduce

$$|||\mathbf{u}-\mathbf{U}|||_{\hat{Q}} \leqslant |||\boldsymbol{\eta}|||_{\hat{Q}} + |||\boldsymbol{\xi}|||_{\hat{Q}} \leqslant |||\boldsymbol{\eta}|||_{Q_0} + \|D\| \; |||D^{-1}\boldsymbol{\xi}|||_{\varphi},$$

with $\|D\|$ the usual euklidian matrix norm. As Q_0 consists of entire elements, we can apply the standard interpolation results and get

$$|||\mathbf{u}-\mathbf{U}|||_{\hat{Q}} \leqslant Ch^{3/2}\|\mathbf{u}\|_{H^2(Q_0)^M} + \|D\| \; |||\bar{\boldsymbol{\xi}}|||_{\varphi}, \tag{4.13}$$

with the abbreviation $\bar{\boldsymbol{\xi}} = D^{-1}\boldsymbol{\xi}$. In the rest of the proof we focus on estimating $|||\bar{\boldsymbol{\xi}}|||_{\varphi}$. Recalling the transformation relation (4.5) between the bilinear form $B(\cdot,\cdot)$ and the auxiliary bilinear form $\tilde{B}(\cdot,\cdot)$ as well as the positive definiteness relation (4.11), and using in addition the quasi-orthogonality relation (2.7), we have, setting $\mathbf{W} = I_h(\Phi\bar{\boldsymbol{\xi}})$,

$$\begin{aligned}\frac{1}{2}|||\bar{\boldsymbol{\xi}}|||_{\varphi}^2 &\leqslant \tilde{B}(\bar{\boldsymbol{\xi}}, \Phi\bar{\boldsymbol{\xi}}) = B(\boldsymbol{\xi}, D^{-T}\Phi D^{-1}\boldsymbol{\xi}) \\ &= B(\boldsymbol{\xi}, D^{-T}(\Phi D^{-1}\boldsymbol{\xi} - \mathbf{W})) \\ &\quad + Per(\mathbf{u}, D^{-T}\mathbf{W}) - B(\boldsymbol{\eta}, D^{-T}\mathbf{W}) \\ &= \tilde{B}(\bar{\boldsymbol{\xi}}, \Phi\bar{\boldsymbol{\xi}} - \mathbf{W}) + \tilde{Per}(\tilde{\mathbf{u}}, \Phi\bar{\boldsymbol{\xi}}) - \tilde{Per}(\tilde{\mathbf{u}}, \Phi\bar{\boldsymbol{\xi}} - \mathbf{W}) \\ &\quad + \tilde{B}(\tilde{\boldsymbol{\eta}}, \Phi\bar{\boldsymbol{\xi}} - \mathbf{W}) - \tilde{B}(\tilde{\boldsymbol{\eta}}, \Phi\bar{\boldsymbol{\xi}}),\end{aligned}$$

with $\tilde{\mathbf{u}} = D^{-1}\mathbf{u}$ and $\tilde{\boldsymbol{\eta}} = D^{-1}\boldsymbol{\eta}$. Here, we have used again the fact that $D^{-T}\mathbf{W}$ is also in $\mathbf{V}_h$. Setting $\mathbf{E} = \Phi\bar{\boldsymbol{\xi}} - \mathbf{W}$, we get

$$\frac{1}{2}|||\bar{\boldsymbol{\xi}}|||_{\varphi}^2 \leqslant \tilde{B}(\bar{\boldsymbol{\xi}}, \mathbf{E}) - \tilde{Per}(\tilde{\mathbf{u}}, \mathbf{E}) + \tilde{Per}(\tilde{\mathbf{u}}, \Phi\bar{\boldsymbol{\xi}}) + \tilde{B}(\tilde{\boldsymbol{\eta}}, \mathbf{E}) - \tilde{B}(\tilde{\boldsymbol{\eta}}, \Phi\bar{\boldsymbol{\xi}}). \tag{4.14}$$

Comparing this inequality with (3.18), we find that both are very similar with respect to the notation used. In fact, since the auxiliary bilinear form $\tilde{B}(\cdot,\cdot)$ and the auxiliary perturbation $\tilde{Per}(\cdot,\cdot)$ are separable, they can be viewed as a sum of M scalar inner products. Thus, we apply the result in the scalar case to (4.14) and obtain

$$|||\bar{\boldsymbol{\xi}}|||_{\varphi}^2 \leqslant Ch^3\|\tilde{\mathbf{u}}\|_{H^2(Q_0)^M}^2 + Ch^{2\nu}(\|\tilde{\mathbf{u}}\|_{L^\infty(Q)^M}^2 + \|\tilde{\mathbf{f}}\|_{L^1(Q_c)^M}^2),$$

with $\tilde{\mathbf{f}} = D^{-1}\mathbf{f}$. Clearly, there holds

$$\|\tilde{\mathbf{u}}\|_{H^2(Q_0)^M} \leq \|D^{-1}\| \; \|\mathbf{u}\|_{H^2(Q_0)^M},$$

$$\|\tilde{\mathbf{u}}\|_{L^\infty(Q)^M} \leq \max_{i,j} |\hat{d}_{i,j}| \; \|\mathbf{u}\|_{L^\infty(Q)^M},$$

$$\|\tilde{\mathbf{f}}\|_{L^1(Q_c)^M} \leq \|D^{-1}\| \; \|\mathbf{f}\|_{L^1(Q_c)^M},$$

with $D^{-1} = (\hat{d}_{i,j})$. Therefore, we have the estimate

$$|||\tilde{\xi}|||_\varphi^2 \leq Ch^3 \|\mathbf{u}\|^2_{H^2(Q_0)^M} + Ch^{2\nu}(\|\mathbf{u}\|^2_{L^\infty(Q)^M} + \|\mathbf{f}\|^2_{L^1(Q_c)^M}).$$

Combining this estimate with (4.13) completes the proof. □

The result in Theorem 4.1 is only valid for subdomains like the one shown in *figure* 2. This is because we want the measure of $Q_0 \backslash \hat{Q}$ to be very small. Certainly, if we do not confine ourselves to that case, such a shape condition becomes redundant.

Now, suppose an arbitrary subdomain $\hat{Q}$ is given. We can find a point (t_0, x_0) and constants μ_j $(j = 1, \ldots, M)$ such that the given subdomain $\hat{Q}$ is contained in the intersection of the following M subdomains (*see fig.* 3) :

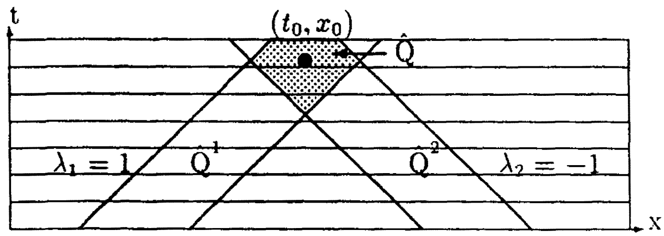

Figure 3. — Given arbitrary subdomain $\hat{Q}$ and its affecting subdomains $\hat{Q}^j$.

$$\hat{Q}^j = \{(t, x) : |(x - \lambda_j t) - (x_0 - \lambda_j t_0)| \leq$$
$$\leq \mu_j, (t, x) \in Q\}, \quad j = 1, \ldots, M, \quad (4.15)$$

i.e.,

$$\hat{Q} \subseteq \bigcap_{j=1}^{M} \hat{Q}^j . \quad (4.16)$$

We define the distance functions and the weight functions as

$$d_j(t,x) = \text{dist}\,((t,x), \hat{Q}^j),\ \varphi_j(t,x) = \exp(-d_j(t,x)/\sigma)\,, \quad \forall (t,x) \in Q\,,$$

for $j = 1, \dots, M$, and the weight matrix as

$$\Phi = \text{diag}\,(\varphi_j(t,x))\,.$$

We define the enlargement of the subdomain $\hat{Q}^j$ as

$$Q_0^j = \left\{ \begin{array}{c} \text{Union of time-space elements entirely contained in} \\ \{(t,x) : dist\,((t,x), \hat{Q}^j) \leqslant K\sigma |\log h|\} \end{array} \right\}, \tag{4.17}$$

with σ defined in (3.9) and K being an appropriately chosen constant. Further, we set $Q_c^j = Q \backslash Q_0^j$. With these definitions, we state the second result.

THEOREM 4.2 : *Given any subdomain $\hat{Q}$ and any fixed large $\nu \geqslant 3/2$, we can specify the constant K to fix the widths of the subdomains $\{Q_0^j\}$. Assume that the exact solution* $\mathbf{u}$ *and* $\tilde{\mathbf{u}} = D^{-1}\mathbf{u}$ *satisfy the following conditions*

$$\mathbf{u} \in \mathbf{L}^\infty(Q)^M, \quad \tilde{u}_j \in \mathbf{H}^2(Q_0^j)\,, \quad j = 1, \dots, M \quad \text{and} \quad \tilde{\mathbf{f}} \in \mathbf{L}^1(Q)^M\,,$$

and that the diffusion coefficient satisfies $\varepsilon \leqslant \varepsilon_m = K_0 h^{3/2}$. Then, there exists a constant $C > 0$, independent of the diffusion coefficient ε, but possibly dependent on the matrices D and Λ, such that the streamline diffusion method of problem (2.4) *admits the local error estimate*

$$|||\mathbf{u} - \mathbf{U}|||_{\hat{Q}} \leqslant Ch^{3/2} \sum_{j=1}^{M} \|\tilde{u}_j\|_{H^2(Q_0^j)} + Ch^{\nu}(\|\mathbf{u}\|_{L^\infty(Q)^M} + \|\mathbf{f}\|_{L^1(Q)^M})\,.$$

Proof: As in the proof of the previous theorem, we have

$$|||\mathbf{u} - \mathbf{U}|||_{\hat{Q}} \leqslant \|D\|\,(|||\tilde{\boldsymbol{\eta}}|||_{\hat{Q}} + |||\tilde{\boldsymbol{\xi}}|||_{\hat{Q}})\,.$$

Taking condition (4.16) into account, it follows that

$$|||\tilde{\boldsymbol{\eta}}|||_{\hat{Q}}^2 \leqslant \sum_{j=1}^{M} |||\tilde{\eta}_j|||_{Q_0^j}^2 \leqslant Ch^3 \sum_{j=1}^{M} \|\tilde{u}_j\|_{H^2(Q_0^j)}^2\,.$$

Since $\varphi_j(t,x) \equiv 1$ for all j in $\hat{Q}$, we have

$$|||\tilde{\boldsymbol{\xi}}|||_{\hat{Q}}^2 = \sum_{j=1}^{M} |||\tilde{\xi}_j|||_{\hat{Q}}^2 = \sum_{j=1}^{M} \|\tilde{\xi}_j\|_{\varphi_j, \hat{Q}}^2 \leqslant \sum_{j=1}^{M} |||\tilde{\xi}_j|||_{\varphi}^2 = |||\tilde{\boldsymbol{\xi}}|||_{\varphi}^2\,.$$

Now, inequality (4.14) is again useful for estimating $|||\xi|||$. Applying the separability of $\tilde{B}(\,.\,,\,.\,)$ and $\tilde{P}er(\,.\,,\,.\,)$, we again reduce the system to several scalar problems. Thus, we obtain

$$|||\xi|||_{\varphi}^{2} \leq Ch^{3} \sum_{j=1}^{M} \|\tilde{u}_{j}\|_{H^{2}(Q_{0}^{j})}^{2} + Ch^{2v}\left(\|\mathbf{u}\|_{L^{\infty}(Q)^{M}}^{2} + \|\mathbf{f}\|_{L^{1}(Q)^{M}}^{2} \right),$$

and the proof is complete. □

If the given subdomain does not satisfy condition (4.10), the result in Theorem 4.2 is different from that in Theorem 4.1 with respect to the right hand sides of the corresponding estimates. In Theorem 4.1, we estimate the local error with the exact solution on a *slightly* larger subdomain. However, in Theorem 4.2 we must control the local error with the single component of the solution on each strip. In a certain sense the result in Theorem 4.2 is sharper than that in Theorem 4.1. Usually the solution $\mathbf{u}$ of (1.2) is not globally smooth and has some (smeared) shocks (as diffusion is present), but the transformed solution $\tilde{\mathbf{u}}$ has its shocks separated. If every $\tilde{u}_j$ is smooth on the corresponding strip Q_0^j, then $\tilde{\mathbf{u}}$ or $\mathbf{u}$ must be smooth on the intersection $\hat{Q}$ of these subdomains.

After having treated the constant coefficient case, we naturally turn to systems with variable coefficients. Since almost all the proofs given in the previous theorems can be carried over to this situation, we do not give the proof in detail. Consider problem (1.2) with the coefficient matrix B depending on (t, x). Given the subdomain $\hat{Q}$, we first want to estimate the error on $\hat{Q}$ just with the exact solution on its enlargement. In order to do so, we must introduce some shape conditions like (4.10). For simplicity we assume that

$$\left\{ \begin{array}{c} \text{The left side of } \hat{Q} \text{ is a characteristic corresponding to } \lambda_{\max}\,. \\ \text{The right side of } \hat{Q} \text{ is a characteristic corresponding to } \lambda_{\min}\,. \\ \text{The other two horizontal sides are the boundaries of } Q\,. \end{array} \right\} \quad (4.18)$$

This coincides with the condition $\theta_l = \min \theta_j$ and $\theta_r = \max \theta_j$ in (4.10) in the constant convection case. We define the enlargement Q_0 as in (4.12). Clearly, Q_0 is larger than $\hat{Q}$ only by $O(h^{3/4}|\log h|)$. Then, we have a result similar to Theorem 4.1.

THEOREM 4.3 : *Given a subdomain* $\hat{Q} \in Q$ *which satisfies condition* (4.18), *and for any fixed number* $v \geq 3/2$, *we can specify the constant* K *in* (4.12). *For this fixed subdomain* Q_0, *we assume that*

$$\mathbf{u} \in \mathbf{H}^2(Q_0)^M, \quad \mathbf{u} \in \mathbf{L}^{\infty}(Q)^M, \quad \mathbf{f} \in \mathbf{L}^1(Q)^M,$$

and that the diffusion coefficient satisfies $\varepsilon \leqslant \varepsilon_m = K_0 h^{3/2}$. *Moreover, assume that the coefficient matrix* $B(t, x)$ *satisfies the condition*

$$Y^T D^{-1} B_x DY \leqslant 0 , \quad \forall Y \in \mathbf{R}^M , \quad \forall (t, x) \in Q , \tag{4.19}$$

and that

$$\lambda_j \in \mathbf{L}^\infty(Q) , \quad \lambda_{jx} \in \mathbf{L}^\infty(Q_0) , \quad \lambda_{jx} \in \mathbf{L}^1(Q) ,$$

$$d_{i,j} \in \mathbf{C}^2(Q) , \quad i, j = 1, ..., M ,$$

with $d_{i,j}$ *being the entries of the matrix D. Then, there exists a constant* $C > 0$, *independent of the diffusion coefficient* ε, *such that the discrete solution* (2.4) *approximates the exact solution of* (1.2) *with a local error*

$$|||\mathbf{u} - \mathbf{U}|||_{\hat{Q}} \leqslant Ch^{3/2} \, \|\mathbf{u}\|_{H^2(Q_0)^M} + Ch^{\nu}(\|\mathbf{u}\|_{L^\infty(Q)^M} + \|\mathbf{f}\|_{L^1(Q_c)^M}) .$$

Proof: As for the constant convection case, we estimate the error with the help of the auxiliary bilinear form $\tilde{B}(. , .)$. Setting $\xi = I_h \mathbf{u} - \mathbf{U}$, $\boldsymbol{\eta} = \mathbf{u} - I_h \mathbf{u}$, $\tilde{\xi} = D^{-1} \xi$ and $\tilde{\boldsymbol{\eta}} = D^{-1} \boldsymbol{\eta}$, we split the error into two parts and obtain

$$|||\mathbf{u} - \mathbf{U}|||_{\hat{Q}} \leqslant |||\mathbf{u} - I_h \mathbf{u}|||_{\hat{Q}} + |||I_h \mathbf{u} - \mathbf{U}|||_{\hat{Q}} \leqslant Ch^{3/2} \|\mathbf{u}\|_{H^2(Q_0)^M} + |||\xi|||_{\hat{Q}} .$$

Setting Φ to be the weight function matrix similarly defined as in (3.8), we obtain the obvious estimates

$$|||\xi|||_{\hat{Q}} \leqslant \|D\| \; |||\tilde{\xi}|||_{\hat{Q}} \leqslant \|D\| \; |||\tilde{\xi}|||_{\varphi} .$$

It is easy to show that there holds the positive definiteness estimate

$$\frac{1}{2} |||\tilde{\xi}|||_{\varphi}^2 \leqslant \tilde{B}(\tilde{\xi}, \Phi\tilde{\xi}) .$$

From the transformation relation (4.5) and the quasi-orthogonality (2.7), we infer

$$\begin{aligned}
\tilde{B}(\tilde{\xi}, \Phi\tilde{\xi}) &= B(\xi, D^{-T} \Phi D^{-1} \xi) = \\
&= Per(\mathbf{u}, P^h(D^{-T} \Phi D^{-1} \xi)) + B(\boldsymbol{\eta}, D^{-T} \Phi D^{-1} \xi - P^h(D^{-T} \Phi D^{-1} \xi)) \\
&\quad + B(\xi, D^{-T} \Phi D^{-1} \xi - P^h(D^{-T} \Phi D^{-1} \xi)) - B(\boldsymbol{\eta}, D^{-T} \Phi D^{-1} \xi) \\
&= - Per(\tilde{\mathbf{u}}, D^T(D^{-T} \Phi\tilde{\xi} - P^h(D^{-T} \Phi\tilde{\xi}))) \\
&\quad + \tilde{B}(\tilde{\boldsymbol{\eta}}, D^T(D^{-T} \Phi\tilde{\xi} - P^h(D^{-T} \Phi\tilde{\xi}))) \\
&\quad + \tilde{B}(\tilde{\xi}, D^T(D^{-T} \Phi\tilde{\xi} - P^h(D^{-T} \Phi\tilde{\xi}))) - \tilde{B}(\tilde{\boldsymbol{\eta}}, \Phi\tilde{\xi}) + \tilde{P}er(\tilde{\mathbf{u}}, \Phi\tilde{\xi}) .
\end{aligned}$$

Finally, setting $\mathbf{E} = D^T(D^{-T}\Phi\xi - I_h(D^{-T}\Phi\xi))$, we see that

$$\frac{1}{2}|||\xi|||_\varphi^2 \leqslant \tilde{B}(\xi, \mathbf{E}) - \tilde{Per}(\tilde{\mathbf{u}}, \mathbf{E}) + \tilde{B}(\tilde{\eta}, \mathbf{E}) - \tilde{B}(\tilde{\eta}, \Phi\xi) + \tilde{Per}(\tilde{\mathbf{u}}, \Phi\xi).$$

This inequality is the same as in (4.14) for the constant convection case. Since we have assumed that $\tilde{B}(\,.\,,\,.\,)$ and $\tilde{Per}(\,.\,,\,.\,)$ are separable, we estimate just as before. We skip further details. □

If the given subdomain does not satisfy condition (4.18), we have a similar result as above. For any given subdomain $\hat{Q}$, we define $\hat{Q}^j$ as follows : It consists of complete characteristics originating from the boundaries of Q and encloses the given subdomain $\hat{Q}$. This definition coincides with (4.15) in the constant convection case. We define Q_0^j and Q_c^j just as in (4.17). A repetition of the proofs of the previous theorems gives

THEOREM 4.4 : *Given any subdomain $\hat{Q}$ and any fixed number $\nu \geqslant 3/2$, we can specify the constant K to fix the subdomains $\{Q_0^j\}$. Assume that the exact solution $\mathbf{u}$ of* (1.2) *and $\tilde{\mathbf{u}} = D^{-1}\mathbf{u}$ satisfy*

$$\mathbf{u} \in \mathbf{L}^\infty(Q)^M, \quad \tilde{u}_j \in \mathbf{H}^2(Q_0^j), \quad j = 1, \dots, M \quad \text{and} \quad \mathbf{f} \in \mathbf{L}^2(Q)^M,$$

and that the diffusion coefficient satisfies $\varepsilon \leqslant \varepsilon_m = K_0 h^{3/2}$. Moreover, we assume that the coefficient matrix $B(t, x)$ satisfies the condition (4.19) *and that*

$$\lambda_j \in \mathbf{L}^\infty(Q), \quad \lambda_{jx} \in \mathbf{L}^\infty(Q_0), \quad \lambda_{jx} \in \mathbf{L}^1(Q),$$

$$d_{i,j} \in \mathbf{C}^2(Q), \quad i, j = 1, \dots, M.$$

Then, there exists a constant $C > 0$, independent of ε, but possibly dependent on the matrices D and Λ, such that the streamline diffusion method for problem (1.2) *admits the local error estimate*

$$|||\mathbf{u} - \mathbf{U}|||_{\hat{Q}} \leqslant C h^{3/2} \sum_{j=1}^{M} \|\tilde{u}_j\|_{H^2(Q_0^j)} + C h^\nu (\|\mathbf{u}\|_{L^\infty(Q)^M} + \|\mathbf{f}\|_{L^1(Q)^M}).$$

ACKNOWLEDGEMENTS. — The author wishes to thank Professor R. Rannacher for suggesting the problem and for all of his advice and comments on this manuscript. This research has been supported by the Deutsche Forschungsgemeinschaft, SFB 359, Universität Heidelberg.

REFERENCES

[1] E. ERIKSSON, C. JOHNSON, Adaptive finite element methods for parabolic problems IV : Nonlinear problems, to appear.

[2] E. ERIKSSON, C. JOHNSON, V. THOMÉE, Time discretization of parabolic problems by the discontinuous Galerkin method, *Math. Modelling and Numer. Anal.,* **19,** 1985, 611-643.

[3] T. J. R. HUGHES, M. MALLET, A. MIZUKAMI, A new finite element formulation for computational fluid dynamics : II. Beyond SUPG, *Comput. Methods Appl. Mech. Engrg.,* **54,** 1986, 341-355.

[4] T. J. R. HUGHES, M. MALLET, A new finite element formulation for computational fluid dynamics : III. The general streamline operator for multidimensional advective-diffusive systems, *Comput. Methods Appl. Mech. Engrg.,* **58,** 1986, 305-328.

[5] T. J. R. HUGHES, M. MALLET, A new finite element formulation for computational fluid dynamics : IV. A discontinuity-capturing operator for multidimensional advective-diffusive systems, *Comput. Methods Appl. Mech. Engrg.,* **58,** 1986, 329-336.

[6] C. JOHNSON, Finite element methods for convection-diffusion problems, in : *Computing Methods in Engineering and Applied Sciences V,* (R. Glowinski and J. L. Lions, eds.), North-Holland, 1981.

[7] C. JOHNSON, *Numerical Solution of Partial Differential Equations by the Finite Element Method,* Cambridge University Press, 1987.

[8] C. JOHNSON, U. NÄVERT, An analysis of some finite element methods for advection-diffusion, in *Analytical and Numerical Approaches to Asymptotic Problems in Analysis,* (O. Axelsson, L. S. Frank and A. van der Sluis, eds.), North-Holland, 1981.

[9] C. JOHNSON, U. NÄVERT, J. PITKÄRANTA, Finite element methods for linear hyperbolic problems, *Comput. Methods Appl. Mech. Engrg.,* **45,** 1984, 285-312.

[10] C. JOHNSON, A. H. SCHATZ, L. B. WAHLBIN, Crosswind smear and pointwise errors in streamline diffusion finite element methods, *Math. Comp.,* **49,** 179, 1987, 25-38.

[11] U. NÄVERT, *A finite element method for convection-diffusion problems,* Thesis, Chalmers University of Technology, Göteborg, Sweden, 1982.

[12] R. RANNACHER, G. ZHOU, *Mesh adaptation via a predictor-corrector strategy in the streamline diffusion method for nonstationary hyperbolic systems.* Proceedings of the 9th GAMM-Seminar Kiel, Eds. W. Hackbusch and G. Wittum, Vieweg Verlag Stuttgart, 1993.

[13] L. R. SCOTT, S. Y. ZHANG, Finite Element Interpolation of Nonsmooth Functions Satisfying Boundary Conditions, *Math. Comp.,* **54,** 190, 1990, 483-493.

[14] J. SMOLLER, *Shock Waves and Reaction-Diffusion Equations,* Springer Heidelberg, 1983.

[15] G. ZHOU, *An adaptive streamline diffusion finite element method for hyperbolic systems in gas dynamics,* Thesis, Heidelberg University, Germany, 1992.

[16] G. ZHOU, Local pointwise error estimates for the streamline diffusion method applied to nonstationary hyperbolic problems, SFB 359 Preprint 93-64, Heidelberg University, 1993.

[17] G. ZHOU, R. RANNACHER, Mesh orientation and refinement in the streamline diffusion method, SFB 359 Preprint 93-57, Heidelberg University, 1993.