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HOMOGENIZATION AND TWO-SCALE CONVERGENCE OF THE COMPRESSIBLE REYNOLDS LUBRICATION EQUATION MODELLING THE FLYING CHARACTERISTICS OF A ROUGH MAGNETIC HEAD OVER A ROUGH RIGID-DISK SURFACE (*)

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Abstract — *The hard disk data storage surfaces of magnetic storage devices are artificially roughened, and this phenomena induces an equation with a rapidly oscillating coefficients*
In this work we give homogenization results for both continuum and slip flow problems Proofs are based on a double-scale convergence and a new homogenized problem is given for the slip flow problem The existence and uniqueness of the limit solutions are given and some numerical results are presented

Resumé — *Le contact aérodynamique entre une tête de lecture et un disque magnétique dépend de la microgéométrie des surfaces et conduit a un problème d'homogénéisation*
En utilisant la convergence a double échelle, on obtient les equations homogenes correspondant a des équations de Reynolds compressible non lineaire L'existence et l'unicité de la solution limite sont demontrees Quelques resultats numériques sont présentés

1. INTRODUCTION

Motivated by higher recording densities, the clearance or « flying height » between read/write head and the disk surface of magnetic hard disk drives has decreased constantly and continues to do so and is now approaching the 0 1 micron level While a drop in flying height can lead to possible contact, an increase in the gap between the head and disk severely degrades the data storage performance of the device Therefore, precise control of the flow flying height is paramount to reliability and optimal operation. The hard disk data storage surfaces are artificially roughened in order to control the interfacial

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static force condition that exists between recording head and disk surface during the rotational start-up. The amplitude of the roughness is typically less than 0.1 micron, which is of the same order as the flying height.

Since the first conception and application of the disk recording head element in 1956, the classical compressible Reynolds equation (1.1) (continuum problem) has been found to predict accurately the performance of the air bearing operating with film thicknesses on the order 8 – 10 μm [4] :

$$\begin{cases} \operatorname{div}\,(h_\varepsilon^3(x)\,p_\varepsilon\,\nabla p_\varepsilon) = 6\,\mu\,\operatorname{div}\,(h_\varepsilon p_\varepsilon V) & x \in \Omega \\ p_\varepsilon = p_a > 0 & \text{on } \partial\Omega\,. \end{cases} \tag{1.1}$$

The air bearing pressure and clearance variables are given by p_ε and h_ε respectively. The ambient pressure appears as p_a, while the disk velocity and fluid viscosity are expressed by $V = (V_1, V_2)$ and μ. $\Omega \subset R^2$ is the region (with smooth boundary, $\partial\Omega$) where the head and the magnetic disk are in proximity. ε is the roughness wave length.

When either the air gap or the ambient pressure is substantially reduced, Burgdorfer [5] found it is necessary to modify the classical equation by taking into account the molecular slip boundary conditions at the bearing surfaces.

Therefore the classical Reynolds equation must be replaced by the modified Reynolds equation which accounts for first order slip-flow effects [5] :

$$\begin{cases} \operatorname{div}\,((h_\varepsilon^3(x)\,p_\varepsilon + \lambda h_\varepsilon^2(x))\,\nabla p_\varepsilon) = 6\,\mu\,\operatorname{div}\,(h_\varepsilon p_\varepsilon V) & x \in \Omega \\ p_\varepsilon = p_a > 0 & \text{on } \partial\Omega\,. \end{cases} \tag{1.2}$$

where $\lambda = 6\,\lambda_a p_a$ ($\lambda_a > 0$ is the molecular mean free path of the gas at ambient pressure).

Surfaces roughness effects have as a result been given a great deal of attention among researchers in the tribology community. Several works have been published for the equation (1.2) of the one dimensional case [4, 9, 11, 15]. Probably due to the increased complexity of analysis, very little work has been reported in the influence of two-dimensional roughness patterns on air bearing performance [18].

In this paper, the two scale-convergence method, introduced by Nguetseng [14], is used to homogenize both the continuum equation (1.1) and the slip flow equation (1.2). In the first part of this work, we give the homogenized problem of the equation (1.1). When the roughness of the surfaces is longitudinal or transversal, the homogenized problem looks like the classical Christensen [8] formulas. We give the convergence theorem, existence and uniqueness of the homogenized problem. In the second part we give the homogenized problem of the equation (1.2) and the convergence theorem, existence and uniqueness solution are established. To be noticed is the fact that we can't use the work of Artola and Duvau [10] for the large of quasi linear problems as they need that the coefficient must be bounded with respect to the unknown. Finally, the validity of these new problems has been confirmed through numerical experiments.

2. GENERAL NOTATIONS AND PRELIMINARIES

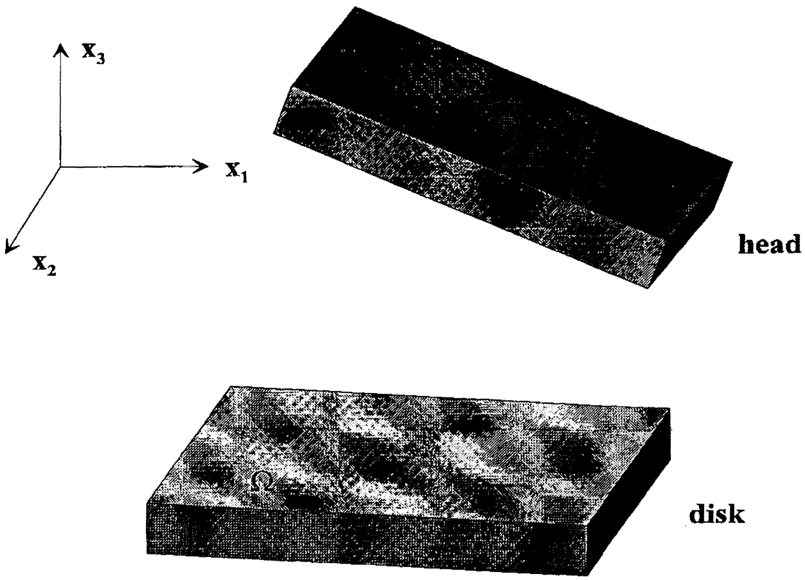

Figure 2.1. — Schematic of slider-disk interface.

Points in Ω are denoted by $x = (x_1, x_2)$ (the global variables) and $y = (y_1, y_2)$ with $y_i = \frac{x_i}{\varepsilon}$, $i = 1, 2$ are the local variables defined in the set

$$Y = \prod_{i=1}^{2}]0, Y_i[$$

a function $f : R^2 \to R$ is said Y-periodic if it admits a period Y_i in the direction Y_i, $i = 1, 2$.

For the sake of simplicity we assume that $Y_i = 1, i = 1, 2$.

In connection with periodic structure, let us introduce some specific spaces.

$$H_p^1(Y) = \{\phi / \phi \in H^1(Y),\ \phi \text{ takes equal values on opposite faces of } Y\}$$

which is a Hilbert space with the norm

$$\| \phi \|_{H^1(Y)} = \left(\| \phi \|^2_{L^2(Y)} + \sum_{i=1}^{2} \left\| \frac{\partial \phi}{\partial y_i} \right\|^2_{L^2(Y)} \right)^{1/2}$$

$$H_m^1(Y) = \left\{ \phi \in H_p^1(Y) / \int_Y \phi \, dy = 0 \right\}$$

the norm of which is

$$\| \phi \|_m = \left(\sum_{i=1}^{2} \left\| \frac{\partial \phi}{\partial y_i} \right\|^2_{L^2(Y)} \right)^{1/2}$$

which is equivalent to the above H_p^1-norm.

We suppose that the function h verifies the conditions :

$$\begin{cases} \bullet\ y \to h(x, y) \text{ is } Y-\text{periodic} \\ \bullet\ \text{There exist two positive constants } h_0 \text{ and } h_1 \text{ such that} \\ \qquad 0 < h_0 \leqslant h(x, y) \leqslant h_1 \quad \text{for} \quad (x, y) \text{ in } \Omega \times Y \\ \bullet\ h(x, y) \in W^{1, \infty}(\Omega \times Y) \quad \text{and} \quad h(x, \cdot\,) \in H_p^1(Y),\ \forall x \in \Omega\,. \end{cases} \tag{2.1}$$

Remark 2.1 : The last hypothesis in (2.1) means that the function h may be extended by periodicity to an element of $W^{1, \infty}(\Omega)$ (see P. Suquet [17], lemma 5).

For the two scale convergence method, we recall some definitions and theorems.

DEFINITION 2.2 : *The sequence $u_\varepsilon \in L^2(\Omega)$ is called two-scale converging to a limit $u \in L^2(\Omega \times Y)$ if for any $\psi \in D[\Omega\,;C_p^\infty(Y)]$, one has*

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x)\, \psi\left(x, \frac{x}{\varepsilon}\right) dx = \int_\Omega \int_Y u(x, y)\, \psi(x, y)\, dx\, dy\,. \tag{2.2}$$

DEFINITION 2.3 : *A function $\psi(x, y)$, Y-periodic in y, and satisfying*

$$\lim_{\varepsilon \to 0} \int_\Omega \psi\left(x, \frac{x}{\varepsilon}\right)^2 dx = \int_\Omega \int_Y \psi(x, y)^2\, dx\, dy \tag{2.3}$$

is called an « admissible » test function.

THEOREM 2.4 : *Let u_ε be a sequence of function in $L^2(\Omega)$ which two-scale converges to a limit $u_0(x, y) \in L^2(\Omega \times Y)$ and assume that*

$$\lim_{\varepsilon \to 0} \| u_\varepsilon \|_{L^2(\Omega)} = \| u_0 \|_{L^2(\Omega \times Y)}\,. \tag{2.4}$$

Then, for any sequence v_ε which two-scale converges to a limit $v_0(x, y) \in L^2(\Omega \times Y)$, we have

$$u_\varepsilon v_\varepsilon \rightarrow \int_Y u_0(x, y)\, v_0(x, y)\, dy \quad \text{in} \quad D'(\Omega)\,. \qquad \square$$

Proof : see Allaire [1].

Remark 2.5 : In the definition 2.2 of the two-scale convergence, the class of test functions $\psi(x, y)$ can be enlarged [1]. So $D[\Omega\,;\, C_p^\infty(Y)]$ can be replaced by $L^2[\Omega\,;\, C_p(Y)]$. Also the validity of (2.2) is extended to all « admissible » test functions ψ in the sense of definition 2.3.

THEOREM 2.6 : *Let u_ε be a bounded sequence in $H^1(\Omega)$ which converges weakly to a limit u_0 in $H^1(\Omega)$. Then u_ε two-scale converges to $u_0(x)$, and there exists a function $u_1(x, y)$ in $L^2(\Omega\,;\, H_m^1(Y))$ such that, up to a subsequence, ∇u_ε two scale converges to $\nabla_x u_0(x) + \nabla_y u_1(x, y)$.* □

Proof : see Allaire [1].

3. HOMOGENIZATION OF THE CONTINUUM EQUATION

If p_ε is a nonnegative solution to (1.1) then the dependent variable

$$u_\varepsilon = \frac{1}{2} p_\varepsilon^2$$

is a nonnegative solution to

$$\begin{cases} \operatorname{div}\,(h_\varepsilon^3(x)\, \nabla u_\varepsilon) = 6\,\mu\, \operatorname{div}\,(h_\varepsilon\, \beta(u_\varepsilon)\, V) & x \in \Omega \\ u_\varepsilon = \frac{1}{2} p_a^2 = u_a & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

where

$$\beta(u_\varepsilon) = \begin{cases} \sqrt{2\, u_\varepsilon}\,, & u_\varepsilon \geqslant 0 \\ 0\,, & u_\varepsilon \geqslant 0 \end{cases}$$

The problem (3.1) has a unique weak solution u_ε [6] which is nonnegative and verifies

$$\begin{cases} u_\varepsilon \in u_a H_0^1(\Omega) \\ \displaystyle\int_\Omega h_\varepsilon^3\, \nabla u_\varepsilon\, \nabla\phi\, dx = 6\,\mu \int_\Omega h_\varepsilon \sqrt{u_\varepsilon}\; V \cdot \nabla\phi\, dx \quad \phi \in H_0^1(\Omega)\,. \end{cases} \tag{3.2}$$

Remark 3.1 : Let u_ε be the solution of (3.2) then $p_\varepsilon = \sqrt{2\, u_\varepsilon}$ is a solution of problem (1.1).

3.1 *A priori* estimates and homogenized problem

PROPOSITION 3.2 : *Let u_ε be the solution of* (3.2). *Then there exists a constant C such that*

$$\| u_\varepsilon \|_{H^1(\Omega)} \leqslant C . \qquad \square$$

Proof: By setting $\phi = u_\varepsilon - u_a$ in (3.2) and by using the Cauchy-Schwarz inequality and conditions (2.1) we obtain the bound :

$$\| \nabla u_\varepsilon \|_{L^2(\Omega)} \leqslant C \cdot \| \sqrt{u_\varepsilon} \|_{L^2(\Omega)} . \tag{3.3}$$

Here and in what follows, C will denote a positive constant which can change from equation to equation. Now we have :

$$\| \sqrt{u_\varepsilon} \|^2_{L^2(\Omega)} = \int_\Omega u_\varepsilon \, dx \leqslant C \cdot \| u_\varepsilon \|_{L^2(\Omega)} \tag{3.4}$$

and Poincare inequality yields

$$\| u_\varepsilon - u_a \|_{L^2(\Omega)} \leqslant C \cdot \| \nabla (u_\varepsilon - u_a) \|_{L^2(\Omega)}$$

from which we deduce

$$\begin{aligned} \| u_\varepsilon \|_{L^2(\Omega)} &\leqslant \| u_\varepsilon - u_a \|_{L^2(\Omega)} + \| u_a \|_{L^2(\Omega)} \\ &\leqslant C \cdot (\| \nabla u_\varepsilon \|_{L^2(\Omega)} + 1) . \end{aligned} \tag{3.5}$$

we obtain from (3.4), (3.3) and (3.5)

$$\| \nabla u_\varepsilon \|_{L^2(\Omega)} \leqslant C \cdot (\| \nabla u_\varepsilon \|^{1/2}_{L^2(\Omega)} + 1) .$$

Then $\| u_\varepsilon \|_{H^1(\Omega)} \leqslant C.$ ■

PROPOSITION 3.3 : *Let u_ε be a solution of* (3.2). *Then, there exist functions $u_0(x) \geqslant 0$ in $u_a + H^1_0(\Omega)$ and $u_1(x, y)$ in $L^2(\Omega ; H^1_m(Y))$ such that, up to a subsequence, we have*

$$u_\varepsilon \underset{\varepsilon \to 0}{\rightarrow} u_0 \qquad \textit{weakly in } H^1(\Omega)$$

$$u_\varepsilon \underset{\varepsilon \to 0}{\rightarrow} u_0 \qquad \textit{in the two-scale sense}$$

$$\sqrt{u_\varepsilon} \underset{\varepsilon \to 0}{\rightarrow} \sqrt{u_0} \qquad \textit{strongly in } L^2(\Omega)$$

and for an other subsequence, we have

$$\nabla u_\varepsilon \to \nabla u_0(x) + \nabla_y u_1(x, y) \quad \textit{in the two-scale sense}\,.$$

Proof: From proposition 3.2, we have

$$\| u_\varepsilon - u_a \|_{H_0^1(\Omega)} \leqslant C$$

then, there exists u_0 in $u_a + H_0^1(\Omega)$ ($u_0(x) \geqslant 0$ since $u_\varepsilon(x) \geqslant 0$) and a subsequence u_ε such that u_ε converges weakly to u_0 in $H^1(\Omega)$. Theorem 2.6 applies and we get the existence of $u_1(x, y)$ in $L^2(\Omega\,;H_m^1(Y))$ such that, up to a subsequence, u_ε converges to u_0 and $\nabla u_\varepsilon \to \nabla u_0(x) + \nabla_y u_1(x, y)$ in the two-scale sense. Now from the inequality $(\sqrt{A} - \sqrt{B})^2 \leqslant |A - B|$, $\forall A, B \geqslant 0$, we have

$$\int_\Omega \left(\sqrt{u_\varepsilon} - \sqrt{u_0} \right)^2 dx \leqslant \int_\Omega |u_\varepsilon - u_0|\, dx \leqslant \sqrt{|\Omega|}\, \| u_\varepsilon - u_0 \|_{L^2(\Omega)}$$

so, from the strong convergence of u_ε to u_0, we infer the strong convergence of $\sqrt{u_\varepsilon}$ to $\sqrt{u_0}$. ■

THEOREM 3.4 : *The limits u_0 and u_1 obtained by the two-scale convergence are solutions of the following two scale homogenized system :*

$$\begin{cases} \operatorname{div}_y \left\{ h^3(x, y) \left(\nabla_x u_0(x) + \nabla_y u_1(x, y) \right) \right\} \\ \quad = 6\,\mu \operatorname{div}_y \left\{ h(x, y) \sqrt{u_0}\; V \right\} \textit{ in } \Omega \times Y \\ \operatorname{div}_x \left\{ \displaystyle\int_Y h^3(x, y) \left(\nabla_x u_0(x) + \nabla_y u_1(x, y) \right) dy \right\} \\ \quad = 6\,\mu \operatorname{div}_x \left\{ \sqrt{u_0}\; V \displaystyle\int_Y h(x, y)\, dy \right\} \textit{ in } \Omega \\ u_0(x) = u_a \\ y \mapsto u_1(x, y) \textit{ is Y-periodic}\,. \end{cases} \tag{3.6}$$

□

Proof: By setting $\phi(x) = \varphi(x) + \varepsilon \varphi_1\left(x, \frac{x}{\varepsilon}\right)$ in (3.2), with $\varphi(x) \in D(\Omega)$ and $\varphi_1(x, y) \in D(\Omega\,;C_p^\infty(Y))$, we obtain :

$$\int_\Omega h^3\left(x, \frac{x}{\varepsilon}\right) \nabla u_\varepsilon \cdot \left(\nabla\varphi(x) + \nabla_y \varphi_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon\, \nabla_x \varphi_1\left(x, \frac{x}{\varepsilon}\right) \right) dx =$$

$$= 6\,\mu \int_\Omega h\left(x, \frac{x}{\varepsilon}\right) \sqrt{u_\varepsilon}\; V \cdot \left(\nabla\varphi(x) + \nabla_y \varphi_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon\, \nabla_x \varphi_1\left(x, \frac{x}{\varepsilon}\right) \right) dx\,. \tag{3.7}$$

From conditions (2.1) h is continuous on $\Omega \times Y$. Then we have :

$$\lim_{\varepsilon \to 0} \int_\Omega h^6\left(x, \frac{x}{\varepsilon}\right)\left(\frac{\partial \varphi}{\partial x_i}(x) + \frac{\partial \varphi_1}{\partial y_i}\left(x, \frac{x}{\varepsilon}\right)\right)^2 dx = \int_\Omega \int_Y h^6(x, y)\left(\frac{\partial \varphi}{\partial x_i}(x) + \right.$$
$$\left. + \frac{\partial \varphi_1}{\partial y_i}(x, y)\right)^2 dx\, dy\,. \quad (3.8)$$

And, from proposition 3.2, ∇u_ε two-scale converges to $\nabla u_0 + \nabla_y u_1(x, y)$. Now the function

$$\psi\left(x, \frac{x}{\varepsilon}\right) = h^3\left(x, \frac{x}{\varepsilon}\right)\left(\nabla \varphi(x) + \nabla_y \varphi_1\left(x, \frac{x}{\varepsilon}\right)\right),$$

is an « admissible » test function from (3.8), and according to Theorem 2.4 (condition (2.4) is satisfied thanks to (3.8)), we can pass to the two-scale limit in the left hand of (3.7).

Since h is continuous on $\Omega \times Y$, we have

$$h\left(x, \frac{x}{\varepsilon}\right)\left(\nabla \varphi(x) + \nabla_y \varphi_1\left(x, \frac{x}{\varepsilon}\right)\right) \underset{\varepsilon \to 0}{\rightarrow} \int_Y h(x, y)\,(\nabla \varphi(x) + \nabla_y \varphi_1(x, y))\, dy$$

weakly in $L^2(\Omega)$.

And by proposition 3.3, $\sqrt{u_\varepsilon}$ converges to $\sqrt{u_0}$ strongly in $L^2(\Omega)$. Then we can pass to the limit in the right hand side of (3.7) and we obtain :

$$\int_\Omega \int_Y h^3(x, y)\,(\nabla u_0(x) + \nabla_y u_1(x, y))\,(\nabla \varphi(x) + \nabla_y \varphi_1(x, y))\, dx\, dy$$
$$= \int_\Omega \int_Y h(x, y) \sqrt{u_0}\; V \cdot (\nabla \varphi(x) + \nabla_y \varphi_1(x, y))\, dx\, dy\,. \quad (3.9)$$

By density, (3.9) holds true for any (φ, φ_1) in $H_0^1(\Omega) \times L^2(\Omega\,; H_m^1(Y))$. An easy integration by parts shows that (3.9) is a variational formulation associated to (3.6).

3.2. Study of the homogenized problem

In this section we begin by decoupling the two-scale homogenized system (3.6).

Let A_1 be the operator defined on $H_p^1(Y)$ by

$$A_1\, v = -\operatorname{div}_y\,(h^3(x, y)\, \nabla_y v)$$

and $a^{\#}$ be the bilinear form

$$a^{\#}(u, v) = \int_Y h^3(x, y)\, \nabla_y u\, \nabla_y v\, dy \quad \forall (u, v) \in (H_p^1(Y))^2 .$$

The first equation of (3.6) yields

$$A_1 u_1 = \frac{\partial u_0}{\partial x_1}\frac{\partial h^3}{\partial y_1} + \frac{\partial u_0}{\partial x_2}\frac{\partial h^3}{\partial y_2} - 6\mu\sqrt{u_0}\, V_1 \frac{\partial h}{\partial y_1} - 6\mu\sqrt{u_0}\, V_2\frac{\partial h}{\partial y_2} . \quad (3.10)$$

We define $\omega_i(x, y)$ and $\chi_i(x, y)$ as the Y-periodic solution (up to an additive constant) of :

$$A_i\,\omega_i = \frac{\partial h^3}{\partial y_i} \quad i = 1, 2 \quad (3.11)$$

$$A_i\,\chi_\iota = \frac{\partial h}{\partial y_i} \quad i = 1, 2 \quad (3.12)$$

ω_i and χ_i exist since

$$\int_Y \frac{\partial h^3}{\partial y_i}\, dy = \int_Y \frac{\partial h}{\partial y_i}\, dy = 0 .$$

Therefore (3.10) gives

$$u_1(x, y) = \frac{\partial u_0}{\partial x_i}\,\omega_1 + \frac{\partial u_0}{\partial x_2}\,\omega_2 - 6\mu\sqrt{u_0}\, V_1\chi_1 - 6\mu\sqrt{u_0}\, V_2\chi_2 .$$

By introducing u_1 in the second equation of (3.6), we obtain the following problem :

$$\begin{cases} \operatorname{div}_x\ (A^*(x)\,\nabla u_0) = \operatorname{div}_x\ (\Theta^*(x)\sqrt{u_0}\,) \text{ in } \Omega \\ u_0 = u_a \text{ on } \partial\Omega \end{cases} \quad (3.13)$$

where A^* and Θ^* are given by

$$\begin{cases} A^*(x) = \begin{bmatrix} a_{11}^0(x) & a_{12}^0(x) \\ a_{21}^0(x) & a_{22}^0(x) \end{bmatrix} \\ \Theta^*(x) = 6\mu \int_Y (V_1 h^3 \nabla_y \chi_1 + V_2 h^3 \nabla_y \chi_2 + Vh\ )\, dy \\ a_{ii}^0(x) = \int_Y h^3(x, y)\, dy + \int_Y h^3(x, y)\,\frac{\partial \omega_i}{\partial y_i}\, dy \quad i = 1, 2 \\ a_{12}^0(x) = \int_Y h^3(x, y)\,\frac{\partial \omega_2}{\partial y_1}\, dy\ ;\ a_{21}^0(x) = \int_Y h^3(x, y)\,\frac{\partial \omega_1}{\partial y_2}\, dy . \end{cases} \quad (3.14)$$

Using (3.11) and (3.12) we have classically :

$$\begin{cases} a_{ii}^0(x) = a^\#(\omega_i + y_i, \omega_i + y_i) & i = 1, 2 \\ a_{12}^0(x) = a_{21}^0(x) = a^\#(\omega_1 + y_1, \omega_2 + y_2) . \end{cases} \tag{3.15}$$

The variational formulation of problem (3.13) is

$$\begin{cases} u_0 \in u_a + H_0^1(\Omega) \\ \int_\Omega A^*(x)\, \nabla u_0\, \nabla\phi\, dx = \int_\Omega \sqrt{u_0}\; \Theta^*(x) \cdot \nabla\phi\, dx \qquad \forall\phi \in H_0^1(\Omega) . \end{cases} \tag{3.16}$$

Remark 3.5 : The matrix A^* is the same as the one introduced in the homogenized problem associated to the equation :

$$\operatorname{div}\,(h_\varepsilon^3\, \nabla p_\varepsilon) = \operatorname{div}\,(h_\varepsilon\, V)$$

which was studied by Bayada and Faure [2]. They were shown that A^* is a positive definite symmetric matrix, i.e. :

$$\forall\xi = (\xi_1, \xi_2) \in R^2, {}^t\xi A^*\, \xi \geqslant c|\xi|^2 . \tag{3.17}$$

LEMMA 3.6 : *The functions* ω_i *and* χ_i $(i = 1, 2)$ *solutions of* (3.11) *and* (3.12) *verify*

$$\|\omega_i\|_m \leqslant C, \quad \|\chi_i\|_m \leqslant C . \qquad \square$$

Proof: Multiplying (3.11) by ω_i $(i = 1, 2)$ and integrating by parts, we obtain :

$$\int_Y h^3(x, y)\, (\nabla_y\, \omega_i)^2\, dy = -\int_Y h^3(x, y)\, \frac{\partial\omega_i}{\partial y_i}\, dy .$$

Conditions (2.1) provide a constant C such that $\|\omega_i\|_m \leqslant C$.

Arguing in the same way as for equation (3.12), we get the estimate $\|\chi_i\|_m \leqslant C$.

For the next lemma we need the following proposition (Brezis [3], proposition IX.3, p. 153).

PROPOSITION 3.7 : *Let* G *be a function in* $L^p(\Omega)$, $1 < p \leqslant \infty$. *Then* $G \in W^{1,p}(\Omega)$ *iff there exists a constant* C *such that for all open set* $\Omega' \subset\subset \Omega$ *and* $t \in R^2$ *with* $|t| < \operatorname{dist}(\Omega', C\Omega)$, *we have*

$$\|G(\,\cdot + t) - G(\,\cdot\,)\|_{L^p(\Omega')} \leqslant C|t| . \qquad \square$$

LEMMA 3.8 : *The matrix A^* and the vector Θ^* defined by* (3.14) *verify :*

$$A^*(x) \in (W^{1,\infty}(\Omega))^4, \Theta^*(x) \in (W^{1,\infty}(\Omega))^2 . \qquad \square$$

Proof : We will show that $a_{11}^0(x) \in W^{1,\infty}(\Omega)$ (the proof is the same for the other terms). From (3.14) we have :

$$a_{11}^0(x) = \int_Y h^3(x, y)\, dy + \int_Y h^3(x, y) \frac{\partial \omega_1}{\partial y_1} dy . \tag{3.18}$$

Since $h(x, y) \in W^{1,\infty}(\Omega \times Y)$ the first term of the right hand side of (3.18) lies in $W^{1,\infty}(\Omega)$. Now to show that the second term lies in $W^{1,\infty}(\Omega)$, we use proposition 3.7. Let $\Omega' \subset\subset \Omega$ and $t \in R^2$ with $|t| < \text{dist}(\Omega', C\Omega)$. By considering the variational formulation associated with problem (3.11) with the two values x and $x + t$ and by subtraction, we obtain :

$$\int_Y h^3(x+t, y)\,(\nabla_y(\omega_1(x+t, y) - \omega_1(x, y)))\,\nabla_y \phi\, dy =$$

$$= \int_Y (h^3(x+t, y) - h^3(x, y)) \frac{\partial \phi}{\partial y_1} dy$$

$$- \int_Y (h^3(x+t, y) - h^3(x, y))\, \nabla_y \omega_1(x, y)\, \nabla_y \phi\, dy .$$

By setting $\phi = \omega_1(x+t, y) - \omega_1(x, t) \in H_p^1(Y)$ in the previous equation and using lemma 3.6, we get :

$$\| \omega_1(x+t, \cdot\,) - \omega_1(x, \cdot\,) \|_m \leq C|t| . \tag{3.19}$$

Let $G(x)$ be the second term of the right hand side of (3.18). Then G lies in $L^\infty(\Omega)$ and we have :

$$G(x+t) - G(x) = \int_Y (h^3(x+t, y) - h^3(x, y)) \frac{\partial \omega_1}{\partial y_1}(x+t, y)\, dy +$$

$$+ \int_Y h^3(x, y) \frac{\partial(\omega_1(x+t, y) - \omega_1(x, y))}{\partial y_1} dy .$$

As $x \mapsto h(x, y) \in W^{1,\infty}(\Omega)$, then by proposition 3.7, applied to $h^3(\,\cdot\,, y)$, and by the inequality (3.19), we have :

$$|G(x+t) - G(x)| \leq C \cdot |t| \quad \forall x \in \Omega'$$

Now, from proposition 3.7, $G \in W^{1,\infty}(\Omega)$ and thus $a_{11}^0 \in W^{1,\infty}(\Omega)$. ■

THEOREM 3.9 : *The solution of problem* (3.16) *is unique. Further, suppose that* u_0^1 *is a positive solution to* (3.16) *corresponding to boundary data* u_a^1 *and* u_0^2 *is a positive solution to* (3.16) *corresponding to boundary data* u_a^2. *If* $u_a^1 \geq u_a^2$ *on* $\partial\Omega$, *then* $u_0^1 \geq u_0^2$ *in* Ω. □

Proof : The uniqueness of positive weak solutions follows from the monotonicity result. We will use here an argument due to Carillo and Chipot [7].

It follows from subtracting (3.16) with $u_0 = u_0^2$ from (3.16) with $u_0 = u_0^1$ that

$$\int_\Omega A^*(x)\,\nabla(u_0^2 - u_0^1)\cdot\nabla\phi - \left(\sqrt{u_0^2} - \sqrt{u_0^1}\right)\Theta^*(x)\cdot\nabla\phi\,dx = 0\,.$$

Let $\xi \in C^\infty(\Omega)$, $\xi > 0$ and $\phi = \min\left(\frac{(u_0^2 - u_0^1)^+}{\varepsilon}, \xi\right) \in H_0^1(\Omega)$.

Then

$$\int_{[u_0^2 - u_0^1 > \varepsilon\xi]} A^*(x)\,\nabla(u_0^2 - u_0^1)\cdot\nabla\xi - \left(\sqrt{u_0^2} - \sqrt{u_0^1}\right)\Theta^*(x)\cdot\nabla\xi\,dx =$$

$$= -\frac{1}{\varepsilon}\int_{[0 < u_0^2 - u_0^1 \leq \varepsilon\xi]} A^*(x)\,\nabla(u_0^2 - u_0^1)\cdot\nabla(u_0^2 - u_0^1)\,dx + \frac{1}{\varepsilon}I \quad (3.20)$$

where

$$I = \int_{[0 < u_0^2 - u_0^1 \leq \varepsilon\xi]} \left(\sqrt{u_0^2} - \sqrt{u_0^1}\right)\Theta^*(x)\cdot\nabla(u_0^2 - u_0^1)\,dx\,.$$

Then

$$I = \int_{[0 < u_0^2 - u_0^1 \leq \varepsilon\xi]} \left(\sqrt{u_0^2} - \sqrt{u_0^1}\right)^t (A^{*-1/2}\,\Theta^*)\,A^{*1/2}\,\nabla(u_0^2 - u_0^1)\,dx\,\cdot$$

By using the Cauchy Schwarz inequality, we obtain

$$|I| \leq \left(\int_{[0 < u_0^2 - u_0^1 \leq \varepsilon\xi]} \left(\sqrt{u_0^2} - \sqrt{u_0^1}\right)^2 |A^{*1/2}\,\Theta^*|^2\,dx\right)^{1/2}$$

$$\times \left(\int_{[0 < u_0^2 - u_0^1 \leqslant \varepsilon\xi]} |A^{*1/2} \nabla(u_0^2 - u_0^1)|^2 \, dx \right)^{1/2}$$

$$\leqslant \frac{1}{4} \int_{[0 < u_0^2 - u_0^1 \leqslant \varepsilon\xi]} \left(\sqrt{u_0^2} - \sqrt{u_0^1} \right)^2 |A^{*1/2} \Theta^*|^2 \, dx$$

$$+ \int_{[0 < u_0^2 - u_0^1 \leqslant \varepsilon\xi]} A^* \nabla(u_0^2 - u_0^1) \cdot \nabla(u_0^2 - u_0^1) \, dx .$$

Introducing the previous estimation in (3.20) we obtain

$$\int_{[u_0^2 - u_0^1 > \varepsilon\xi]} A^*(x) \nabla(u_0^2 - u_0^1) \cdot \nabla\xi - \left(\sqrt{u_0^2} - \sqrt{u_0^1} \right) \Theta^*(x) \cdot \nabla\xi \, dx \leqslant$$

$$\leqslant \frac{1}{4\varepsilon} \int_{[0 < u_0^2 - u_0^1 \leqslant \varepsilon\xi]} \left(\sqrt{u_0^2} - \sqrt{u_0^1} \right)^2 |A^{*-1/2} \Theta^*|^2 \, dx . \quad (3.21)$$

From lemma 3.8 and remark 3.5, we have

$$|A^{*-1/2} A^*|^2 \in L^\infty(\Omega)$$

and thus (3.21) gives

$$\int_{[u_0^2 - u_0^1 > \varepsilon\xi]} A^*(x) \nabla(u_0^2 - u_0^1) \cdot \nabla\xi - \left(\sqrt{u_0^2} - \sqrt{u_0^1} \right) \Theta^*(x) \cdot \nabla\xi \, dx \leqslant$$

$$\frac{M}{4} \|A^{*-1/2} \Theta^*\|_\infty \int_{[0 < u_0^2 - u_0^1 \leqslant \varepsilon\xi]} dx \quad (3.22)$$

where $M = \max_{\Omega} \xi$. Now the measure of the set $[0 < u_0^2 - u_0^1 \leqslant \varepsilon\xi]$ goes to zero as $\varepsilon \to 0$ and passing to the limit in (3.22), the following estimate holds

$$\int_{[u_0^2 - u_0^1 > 0]} A^*(x) \nabla(u_0^2 - u_0^1) \cdot \nabla\xi - \left(\sqrt{u_0^2} - \sqrt{u_0^1} \right) \Theta^*(x) \cdot \nabla\xi \, dx \leqslant 0 . \quad (3.23)$$

Now it follows from integration by parts that

$$\int_{[u_0^2 - u_0^1 > 0]} A^*(x) \nabla(u_0^2 - u_0^1) \cdot \nabla\xi \, dx$$

$$= - \int_{[u_0^2 - u_0^1 > 0]} (u_0^2 - u_0^1) \operatorname{div} (A^*(x) \nabla\xi) \, dx$$

using the last equality in (3.23) we obtain

$$\int_{[u_0^2 - u_0^1 > 0]} (u_0^2 - u_0^1)\,(-\operatorname{div}(A^*(x)\,\nabla\xi))\,dx \leqslant - \int_{[u_0^2 - u_0^1 > 0]} \left(\sqrt{u_0^2} - \sqrt{u_0^1}\right) \Theta^* \cdot \nabla\xi\,dx\,. \quad (3.24)$$

Let us now introduce a vector $\mathbf{n} = (\mathbf{n_1}, \mathbf{n_2})$ such that $\Theta^*(x) \cdot n \leqslant 0$ for all $x \in \Omega$. This is possible as $\Theta^*(x) \in L^\infty(\Omega)^2$ from lemma 3.7.

For any positive s, chose W a constant large enough so that $\xi(x) = W - e^{s(n_1 x_1 + n_2 x_2)}$ is strictly positive. So

$$-\operatorname{div}(A^*(x)\,\nabla\xi) = \Bigg[(a_{11}^0(x)\,n_1^2 + 2\,a_{12}^0(x)\,\mathbf{n_1}\,\mathbf{n_2} + a_{22}^0(x)\,n_2^2)\,s^2 + \left(\frac{\partial a_{11}^0}{\partial x_1} n_1 + \frac{\partial a_{12}^0}{\partial x_1} n_2 + \frac{\partial a_{12}^0}{\partial x_2} n_1 + \frac{\partial a_{22}^0}{\partial x_2} n_2\right) s\Bigg] \cdot e^{s(n_1 x_1 + n_2 x_2)}\,. \quad (3.26)$$

From lemma 3.8, the coefficients of s and s^2, in (3.26), are in $L^\infty(\Omega)$. A^* is a positive definite symmetric matrix (remark 3.5), thus the coefficient of s^2 is positive. Hence, it follows that for a sufficiently large s, we have :

$$-\operatorname{div}(A^*(x)\,\nabla\xi) > 0 \quad \forall x \in \Omega\,. \quad (3.27)$$

Finally, the inequalities (3.24), (3.25) and (3.27) allow to conclude that $(u_0^2 - u_0^1)^+ = 0$. This finishes the proof of theorem 3.9. ∎

3.3. Transverse and longitudinal roughness

3.3.1. *Transverse roughness*

Here we consider the particular case where the roughness is independent of y_2. Hence the function h does not depend on y_2 and we obtain

$$\omega_2 = \chi_2 = a_{12}^0 = a_{21}^0 = 0$$

$$a_{22}^0(x) = \overline{h^3}(x)\,.$$

Moreover, the equation (3.11) with $i = 1$ becomes

$$-\frac{\partial}{\partial y_1}\left(h^3(x, y_1)\,\frac{\partial \omega_1}{\partial y_1}\right) - h^3(x, y_1)\,\frac{\partial^2 \omega_1}{\partial y_2^2} = \frac{\partial}{\partial y_1}\,(h^3(x, y_1))$$

An obvious solution may be found among y_1-dependent functions satisfying

$$-\frac{\partial}{\partial y_1}\left(h^3(x, y_1)\frac{\partial \omega_1}{\partial y_1}\right) = \frac{\partial}{\partial y_1}(h^3(x, y_1)) .$$

So

$$-\frac{\partial \omega_1}{\partial y_1} = 1 + \frac{k_1(x)}{h^3(x, y_1)} .$$

By using the periodicity of the function $y_1 \to \omega_1(x, y_1)$ and integrating the last equation on Y we obtain

$$k_1(x) = -\frac{1}{\overline{h^{-3}}(x)} .$$

Thus

$$a^0_{11}(x) = -k_1(x) = \frac{1}{\overline{h^{-3}}(x)} .$$

Now the equation (3.12) with $i = 1$ and since χ_1 is independent from y_2 gives

$$-\frac{\partial}{\partial y_1}\left(h^3(x, y_1)\frac{\partial \chi_1}{\partial y_1}\right) = \frac{\partial}{\partial y_1}(h(x, y_1)) .$$

So

$$-\frac{\partial \chi_1}{\partial y_1} = h^{-2}(x, y_1) + \frac{k_2(x)}{h^3(x, y_1)}$$

and by integration on Y we obtain

$$k_2(x) = -\frac{\overline{h^{-2}}(x)}{\overline{h^{-3}}(x)} .$$

Thus the components of Θ^* become

$$\Theta_1(x) = 6\mu V_1 \frac{\overline{h^{-2}}(x)}{\overline{h^{-3}}(x)}$$

$$\Theta^*_2(x) = 6\mu V_2 \bar{h}(x) .$$

Problem (3.13) now reduces to

$$\begin{cases} \frac{\partial}{\partial x_1}\left(\frac{1}{\overline{h^{-3}}(x)} \frac{\partial u_0}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\overline{h^3}(x) \frac{\partial u_0}{\partial x_2}\right) = 6\,\mu V_1 \frac{\partial}{\partial x_1}\left(\frac{\overline{h^{-2}}(x)}{\overline{h^{-3}}(x)} \sqrt{u_0}\right) + \\ \quad + 6\,\mu V_2 \frac{\partial}{\partial x_2}(\bar{h}(x)\sqrt{u_0}) \text{ in } \Omega \\ u_0 = u_a \text{ on } \Gamma . \end{cases} \tag{3.28}$$

3.3.2. *Longitudinal roughness*

In this case the function h is independent of y_1. The same calculations as in the transverse roughness gives the following homogenized problem

$$\begin{cases} \frac{\partial}{\partial x_1}\left(\overline{h^3}(x) \frac{\partial u_0}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{1}{\overline{h^{-3}}(x)} \frac{\partial u_0}{\partial x_2}\right) = 6\,\mu V_1 \frac{\partial}{\partial x_1}(\bar{h}(x)\sqrt{u_0}) + \\ \quad + 6\,\mu V_2 \frac{\partial}{\partial x_2}\left(\frac{\overline{h^{-2}}(x)}{\overline{h^{-3}}(x)} \sqrt{u_0}\right) \text{ in } \Omega \\ u_0 = u_a \text{ on } \Gamma . \end{cases} \tag{3.29}$$

4. HOMOGENIZATION OF THE SLIP FLOW EQUATION

To study problem (1.2), Chipot and Luskin [6] have introduced an auxiliary problem obtained by setting the new dependent variable

$$u_\varepsilon = \frac{1}{2} p_\varepsilon^2 + \frac{\lambda}{h_\varepsilon} p_\varepsilon \tag{4.1}$$

which leads to the problem

$$\begin{cases} \operatorname{div}(h_\varepsilon^3 \nabla u_\varepsilon) = \operatorname{div}(\beta_\varepsilon(x, u_\varepsilon)(\Lambda - \lambda \nabla h_\varepsilon)) \text{ on } \Omega \\ u_\varepsilon = \frac{1}{2} p_a^2 + \frac{\lambda}{h_\varepsilon} p_a \equiv u_a^\varepsilon \text{ in } \Gamma \end{cases} \tag{4.2}$$

where

$$\beta_\varepsilon(x, u_\varepsilon) = \begin{cases} -\lambda + \sqrt{\lambda^2 + 2\,h_\varepsilon^2 u_\varepsilon}\,, & u_\varepsilon \geqslant 0 \\ 0\,, & u_\varepsilon \geqslant 0 \end{cases} \tag{4.3}$$

and $\Lambda = 6\,\mu V$.

In this chapter we suppose that $h(x, y) \in W^{2,\infty}(\Omega \times Y)$, then $h_\varepsilon(x) \in W^{2,\infty}(\Omega)$.

The existence and uniqueness of a weak solution to problem (4.2)

$$\begin{cases} u_\varepsilon \in u_a^\varepsilon + H_1^0(\Omega) \\ \int_\Omega h_\varepsilon^3 \nabla u_\varepsilon \nabla \phi \, dx = \int_\Omega \beta_\varepsilon(x, u_\varepsilon)\,(\Lambda - \lambda \nabla h_\varepsilon)\, \nabla \phi \, dx \; \phi \in H_1^0(\Omega) \end{cases} \tag{4.4}$$

was proved in [6] and appears to be a nonnegative function. As λ is positive, the function given by

$$p_\varepsilon = -\frac{\lambda}{h_\varepsilon} + \sqrt{\frac{\lambda^2}{h_\varepsilon^2} + 2\, u_\varepsilon} \tag{4.5}$$

is a weak solution to problem (1.2) which is unique in the class of nonnegative functions.

4.1. Some estimations and regularity results

LEMMA 4.1 : *Let* u_ε *be the solution of* (4.4). *Then* $\beta_\varepsilon(x, u_\varepsilon) \in H^1(\Omega)$. □

Proof: Since

$$u_\varepsilon \geqslant 0,\ \beta_\varepsilon(x, u_\varepsilon) = -\lambda + \sqrt{\lambda^2 + 2\, h_\varepsilon^2\, u_\varepsilon} = \frac{2\, h_\varepsilon^2\, u_\varepsilon}{\lambda + \sqrt{\lambda^2 + 2\, h_\varepsilon^2\, u_\varepsilon}}$$

and since $\lambda > 0$ we deduce that $\beta_\varepsilon(x, u_\varepsilon) \in H^1(\Omega)$. ■

PROPOSITION 4.2 : *Let* u_ε *be the unique solution of* (4.4) *and* p_ε *given by* (4.5). *Then*

- $u_\varepsilon \in C^{1,\alpha}(\bar{\Omega}), p_\varepsilon \in C^{1,\alpha}(\bar{\Omega})$ *with* $0 < \alpha < 1$
- p_ε *and* $p_\varepsilon^2 \in H^1(\Omega)$. □

Proof: Since $\nabla h_\varepsilon \in (W^{1,\infty}(\Omega))^2$ and from lemma 4.1, the function

$$\beta_\varepsilon(x, u_\varepsilon)\,(\Lambda - \lambda \nabla h_\varepsilon) \in H^1(\Omega)^2\ (\subset L^q(\Omega)^2, 2 < q < \infty)$$

and since $h_\varepsilon \in W^{2,\infty}(\Omega) \subset C^0(\bar{\Omega})$, we have from theorem A.1(iii) $u_\varepsilon \in W^{1,q}(\Omega)\ (q > 2)$.

Thus $\beta_\varepsilon(x, u_\varepsilon) \in W^{1,q}(\Omega)$ and then $\beta_\varepsilon(x, u_\varepsilon)\,(\Lambda - \lambda \nabla h_\varepsilon) \in (W^{1,q}(\Omega))^2\ (\subset (C^{0,\alpha}(\Omega))^2, 0 < \alpha < 1)$.

Now from theorem A.2(i), $u_\varepsilon \in C^{1,\alpha}(\bar{\Omega})$.

The last property of proposition 4.2 is deduced since λ is strictly positive.

PROPOSITION 4.3 : *Let* p_ε *given by* (4.5). *Then there exists a constant* C *independent of* ε *such that*

$$\|p_\varepsilon^2\|_{H^1(\Omega)} \leqslant C \quad and \quad \|p_\varepsilon\|_{H^1(\Omega)} \leqslant C .$$

Proof: The variational formulation associated with problem (1.2) is

$$\int_\Omega (h_\varepsilon^3 p_\varepsilon + \lambda h_\varepsilon^2 p_\varepsilon) \nabla p_\varepsilon \cdot \nabla \phi \, dx = \int_\Omega h_\varepsilon p_\varepsilon \Lambda \cdot \nabla \phi \, dx \quad \phi \in H_1^0(\Omega) . \tag{4.6}$$

By setting $\phi = p_\varepsilon^2 - p_a^2$ in (4.6) we obtain

$$\int_\Omega h_\varepsilon^3 p_\varepsilon \nabla p_\varepsilon \cdot \nabla p_\varepsilon^2 \, dx + \lambda \int_\Omega h_\varepsilon^2 \nabla p_\varepsilon \cdot \nabla p_\varepsilon^2 \, dx = \int_\Omega h_\varepsilon p_\varepsilon \Lambda \cdot \nabla p_\varepsilon^2 \, dx$$

then

$$\frac{1}{2} \int_\Omega h_\varepsilon^3 (\nabla p_\varepsilon^2)^2 \, dx + 2 \lambda \int_\Omega h_\varepsilon^2 p_\varepsilon \cdot (\nabla p_\varepsilon)^2 \, dx = \int_\Omega h_\varepsilon p_\varepsilon \Lambda \cdot \nabla p_\varepsilon^2 \, dx .$$

Since $\lambda > 0, p_\varepsilon \geqslant 0$ and $0 < h_0 \leqslant h_\varepsilon \leqslant h_1$ we deduce

$$\|\nabla p_\varepsilon^2\|_{L^2(\Omega)} \leqslant \frac{2 |\Lambda| h_1}{h_0^3} \|p_\varepsilon\|_{L^2(\Omega)}$$

but

$$\int_\Omega p_\varepsilon^2 \, dx \leqslant \sqrt{|\Omega|} \left(\int_\Omega p_\varepsilon^4 \, dx \right)^{1/2}$$

then

$$\|\nabla p_\varepsilon^2\|_{L^2(\Omega)} \leqslant c \, \|p_\varepsilon^2\|_{L^2(\Omega)}^{1/2} \tag{4.7}$$

now using the Poincare inequality we have

$$\|p_\varepsilon^2\|_{L^2(\Omega)} \leqslant \|p_\varepsilon^2 - p_a^2\|_{L^2(\Omega)} + \|p_a^2\|_{L^2(\Omega)} \leqslant c \, \|\nabla p_\varepsilon^2\|_{L^2(\Omega)} + \|p_a^2\|_{L^2(\Omega)} . \tag{4.8}$$

We then deduce from (4.7) and (4.8) that $\|p_\varepsilon^2\|_{H^1(\Omega)} \leqslant C$.

By setting $\phi = p_\varepsilon - p_a$ in (4.6) we show the second inequality of proposition 4.3. ∎

PROPOSITION 4.4 : *Let* u_ε *be the unique solution of* (4.4) *and* p_ε *given by* (4.5). *Then there exists* $p_0 \in p_a + H_0^1(\Omega)$ *such that, up to a subsequence,*

$$p_\varepsilon \to p_0 \quad \textit{and} \quad p_\varepsilon^2 \to p_0^2 \quad \textit{weakly in } H^1(\Omega)\,. \qquad \square$$

Proof: From proposition 4.3, $\|p_\varepsilon\|_{H^1(\Omega)} \leqslant C$, then there exists $p_0 \in p_a + H_0^1(\Omega)$ such that, up to a subsequence p_ε converges to p_0 weakly in $H^1(\Omega)$.

By $\|p_\varepsilon^2\|_{H^1(\Omega)} \leqslant C$ there exists $\chi \in p_a^2 + H_0^1(\Omega)$ such that, up to another subsequence, p_ε^2 converges to χ weakly in $H^1(\Omega)$. Thus $\chi = p_0^2$ a.e. in Ω and then $p_0^2 \in H^1(\Omega)$. ■

THEOREM 4.5 : *Let* u_ε *be the unique solution of* (4.4). *Then the function* p_ε *given by* (4.5) *converges to* p_0 *weakly in* $H^1(\Omega)$ *and, up to a subsequence,* ∇p_ε *converges to* $\nabla p_0 + \nabla_y p_1(x, y)$ *in the two-scale sense where* $(p_0, p_1) \in (p_a + H_0^1(\Omega)) \times L^2(\Omega\,; H_m^1(Y))$ *is a solution of the following two scale homogenized system :*

$$\begin{cases} \operatorname{div}_y \left\{ (h^3(x,y)\,p_0 + \lambda h^2(x,y))\,(\nabla_x p_0(x) + \nabla_y p_1(x,y)) \right\} \\ \quad = \operatorname{div}_y \left\{ h(x,y)\,p_0\,\Lambda \right\} \text{ in } \Omega \times Y \\ \operatorname{div}_x \left\{ \displaystyle\int_Y (h^3(x,y)\,p_0 + \lambda h^2(x,y))\,(\nabla_x p_0(x) + \nabla_y p_1(x,y))\,dy \right\} \\ \quad = 6\,\mu \operatorname{div}_x \left\{ p_0\,\Lambda \displaystyle\int_Y h(x,y)\,dy \right\} \text{ in } \Omega \\ p_0(x) = p_a \\ y \mapsto p_1(x,y) \text{ } Y\text{-periodic}\,. \end{cases} \tag{4.9}$$

$\square$

Proof: From proposition 4.4 there exists $p_0 \in p_a + H_0^1(\Omega)$ such that.

$$p_\varepsilon \to p_0 \quad \text{and} \quad p_\varepsilon^2 \to p_0^2 \quad \text{weakly in } H^1(\Omega) \tag{4.10}$$

and by theorem 2.6 there exists $p_1(x, y) \in L^2(\Omega\,; H_m^1(Y))$ such that ∇p_ε converges to $\nabla p_0 + \nabla_y p_1(x, y)$ in the two-scale sense.

By setting $\phi(x) = \varphi(x) + \varepsilon\varphi_1\left(x, \frac{x}{\varepsilon}\right)$ in (4.4), with $\varphi(x) \in D(\Omega)$ and $\varphi_1(x, y) \in D(\Omega\,;\, C_p^\infty(Y))$, we obtain :

$$\int_\Omega \left(h^3\left(x, \frac{x}{3}\right) p_\varepsilon + \lambda h^2\left(x, \frac{x}{\varepsilon}\right)\right) \nabla p_\varepsilon \left(\nabla\varphi(x) + \nabla_y \varphi_1\left(x, \frac{x}{\varepsilon}\right) + \right.$$

$$\left. + \varepsilon \nabla_x \varphi_1\left(x, \frac{x}{\varepsilon}\right)\right) dx = \int_\Omega h\left(x, \frac{x}{\varepsilon}\right) p_\varepsilon A \cdot \left(\nabla\varphi(x) + \nabla_y \varphi_1\left(x, \frac{x}{\varepsilon}\right) + \right.$$

$$\left. + \varepsilon \nabla_x \varphi_1\left(x, \frac{x}{\varepsilon}\right)\right) dx \quad (4.11)$$

as for the homogenization of the equation (1.1) (theorem 3.4) we can establish :

$$\varepsilon \int_\Omega \left(h^3\left(x, \frac{x}{3}\right) p_\varepsilon + \lambda h^2\left(x, \frac{x}{\varepsilon}\right)\right) \nabla p_\varepsilon \nabla_x \varphi_1\left(x, \frac{x}{\varepsilon}\right) dx +$$

$$+ \lambda \int_\Omega h^2\left(x, \frac{x}{\varepsilon}\right) \nabla p_\varepsilon \left(\nabla\varphi(x) + \nabla_y \varphi_1\left(x, \frac{x}{\varepsilon}\right)\right) dx$$

$$\underset{\varepsilon \to 0}{\to} \lambda \int_\Omega \int_Y h^2(x, y)\, (\nabla p_0(x) + \nabla_y p_1(x, y))\, (\nabla\varphi(x) +$$

$$+ \nabla_y \phi_1(x, y))\, dx\, dy \quad (4.12)$$

$$\int_\Omega h\left(x, \frac{x}{\varepsilon}\right) p_\varepsilon A \cdot \left(\nabla\varphi(x) + \nabla_y \varphi_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon \nabla_x \varphi_1\left(x, \frac{x}{\varepsilon}\right)\right) dx$$

$$\underset{\varepsilon \to 0}{\to} \int_\Omega \int_Y h(x, y)\, p_0 A \cdot (\nabla\varphi(x) + \nabla_y \varphi_1(x, y))\, dx\, dy\,. \quad (4.13)$$

Now we show the last limit.

From proposition 4.4 p_ε^2 converges to p_0^2 weakly in $H^1(\Omega)$, then from theorem 2.6, p_ε^2 converges to p_0^2 in the two-scale sense.

The functions

$$h^3\left(x, \frac{x}{\varepsilon}\right)\left(\frac{\partial\varphi(x)}{\partial x_i} + \frac{\partial}{\partial x_i}\varphi_1\left(x, \frac{x}{\varepsilon}\right)\right) \quad i = 1, 2$$

are admissible test functions in the sense of definition 2.3, then

$$p_\varepsilon\left(h^3\left(x, \frac{x}{\varepsilon}\right)\left(\frac{\partial\varphi(x)}{\partial x_i} + \frac{\partial}{\partial x_i}\varphi_1\left(x, \frac{x}{\varepsilon}\right)\right)\right) \quad (4.14)$$

is an admissible test function and since ∇p_ε two-scale converges to $\nabla p_0 + \nabla_y p_1(x, y)$, we can set the function (4.15) in the definition of the two-scale convergence to obtain the limit

$$\int_\Omega h^3\left(x, \frac{x}{3}\right) p_\varepsilon \nabla p_\varepsilon\left(\nabla\varphi(x) + \nabla_y \varphi_1\left(x, \frac{x}{\varepsilon}\right)\right) dx$$

$$\underset{\varepsilon \to 0}{\rightarrow} \int_\Omega \int_Y h^3(x, y)\, p_0(\nabla p_0 + \nabla_y p_1(x, y))\,(\nabla\varphi(x) + \nabla_y \varphi_1(x, y))\, dx\, dy\,. \tag{4.15}$$

From the limits (4.12), (4.13) and (4.15) we obtain from (4.11) :

$$\int_\Omega \int_Y (h^3(x, y)\, p_0 + \lambda h^2(x, y))\,(\nabla p_0(x) + \nabla_y p_1(x, y)) \times$$
$$\times (\nabla\varphi(x) + \nabla_y \varphi_1(x, y))\, dx\, dy$$
$$= \int_\Omega \int_Y h(x, y)\, p_0\, \Lambda \cdot (\nabla\varphi(x) + \nabla_y \varphi_1(x, y))\, dx\, dy\,. \tag{4.16}$$

By density, (4.16) holds true for any (φ, φ_1) in $H_0^1(\Omega) \times L^2(\Omega\,;H_m^1(\Omega))$. An easy integration by parts shows that (4.16) is a variational formulation associated with (4.9).

4.2. Study of the homogenized problem

Let (p_0, p_1) be a solution of problem (4.9) and let $A_1(p_0)$ be the operator defined on $H_p^1(Y)$ by

$$A_1(p_0)\, v = -\operatorname{div}_y\,((h^3(x, y)\, p_0 + \lambda h^2(x, y))\,\nabla_y v) \tag{4.17}$$

and $a^\#(p_0)$ be the bilinear form

$$a^\#(p_0)\,(u, v) = \int_Y (h^3(x, y)\, p_0 + \lambda h^2(x, y))\,\nabla_y u\, \nabla_y v\, dy \quad \forall u, v \in H_p^1(Y)\,. \tag{4.18}$$

The first equation of (4.9) yields

$$A_1(p_0)\, p_1 = \frac{\partial p_0}{\partial x_1}\frac{\partial}{\partial y_1}(h^3 p_0 + \lambda h^2) + \frac{\partial p_0}{\partial x_2}\frac{\partial}{\partial y_2}(h^3 p_0 + \lambda h^2) -$$
$$- p_0\, \Lambda_1 \frac{\partial h}{\partial y_1} - p_0\, \Lambda_2 \frac{\partial h}{\partial y_2}\,. \tag{4.19}$$

We define $\omega_i(p_0)(x, y)$ and $\chi_i(p_0)(x, y)$ as the Y-periodic solution (up to an additive constant) of :

$$A_1(p_0)\,\omega_i(p_0) = \frac{\partial}{\partial y_i}(h^3 p_0 + \lambda h^2) \quad i = 1, 2 \tag{4.20}$$

$$A_1(p_0)\,\chi_i(p_0) = \frac{\partial h}{\partial y_i} \quad i = 1, 2 \tag{4.21}$$

From (4.19) we deduce

$$p_1(x, y) = \frac{\partial p_0}{\partial x_1}\omega_1(p_0) + \frac{\partial p_0}{\partial x_2}\omega_2(p_0) - \Lambda_1 \chi_1(p_0) - \Lambda_2 \chi_2(p_0)\,.$$

By introducing p_1 in the second equation of (4.9), we obtain the following problem :

$$\begin{cases} \operatorname{div}_x\,(A^*(p_0)(x)\,\nabla p_0) = \operatorname{div}_x\,(\Theta^*(p_0)(x)\,p_0) \text{ in } \Omega \\ p_0 = p_a \text{ on } \partial\Omega \end{cases} \tag{4.22}$$

where the matrix $A^*(p_0)$ and the vector $\Theta^*(p_0)$ are defined by

$$\begin{cases} A^*(p_0)(x) = \begin{bmatrix} a^0_{11}(p_0)(x) & a^0_{12}(p_0)(x) \\ a^0_{21}(p_0)(x) & a^0_{22}(p_0)(x) \end{bmatrix} \\ \Theta^*(x) = \int_Y (h^3 p_0 + \lambda h^2)(\Lambda_1 \nabla_y \chi_1(p_0) + \Lambda_2 \nabla_y \chi_2(p_0))\,dy \\ \qquad + \int_Y \Lambda h\,dy \\ a^0_{ii}(p_0)(x) = \int_Y (h^3 p_0 + \lambda h^2)\,dy + \int_Y (h^3 p_0 + \lambda h^2)\frac{\partial \omega_i(p_0)}{\partial y_i}\,dy \; i = 1, 2 \\ a^0_{12}(p_0)(x) = \int_Y (h^3 p_0 + \lambda h^2)\frac{\partial \omega_2(p_0)}{\partial y_1}\,dy \; a^0_{21}(p_0)(x) \\ \qquad = \int_Y (h^3 p_0 + \lambda h^2)\frac{\partial \omega_1(p_0)}{\partial y_2}\,dy\,. \end{cases} \tag{4.23}$$

The variational formulation of the problem (4.22) is

$$\begin{cases} p_0 \in p_a + H^1_0(\Omega) \\ \int_\Omega A^*(p_0)(x)\,\nabla p_0\,\nabla\phi\,dx = \int_\Omega p_0\,\Theta^*(p_0)(x)\cdot\nabla\phi\,dx \quad \forall\phi \in H^1_0(\Omega)\,. \end{cases} \tag{4.24}$$

LEMMA 4.6 : *$A^*(p_0)$ is a positive definite symmetric matrix.* □

Proof : From (4.20) we have

$$a_{ii}^0(p_0)(x) = a^*(p_0)(\omega_i(p_0) + y_i, \omega_i(p_0) + y_i) \quad i = 1, 2$$

$$a_{12}^0(p_0)(x) = a_{21}^0(p_0)(x) = a^*(p_0)(\omega_1(p_0) + y_1, \omega_2(p_0) + y_2) .$$

Then $A^*(p_0)$ is symmetric. Now we will show that it is positive definite. For this we show that for all $\xi = (\xi_1, \xi_2) \in R^2$, the following inequality holds :

$$a_{11}^0(p_0)\,\xi_1^2 + 2\,a_{12}^0(p_0)\,\xi_1\,\xi_2 + a_{22}^0(p_0)\,\xi_2^2 \geq (h_0^3 p_0 + \lambda h_0^2)\,(\xi_1^2 + \xi_2^2)\,. \quad (4.25)$$

Denoting by $t_i = (\omega_i(p_0) + y_i)\,\xi_i (i = 1, 2)$ and from (4.23) we obtain

$$a_{11}^0(p_0)\,\xi_1^2 + 2\,a_{12}^0(p_0)\,\xi_1\,\xi_2 + a_{22}^0(p_0)\,\xi_2^2 \geq$$

$$\geq (h_0^3 p_0 + \lambda h_0^2) \int_Y (\nabla_y(t_1 + t_2))^2\,dy$$

on the other hand

$$\int_Y (\nabla_y(t_1 + t_2))^2\,dy = \int_Y (\xi_1^2 + \xi_2^2)\,dy + 2\int_Y \left[\xi_1\,\xi_2\left(\frac{\partial \omega_2}{\partial y_1} + \frac{\partial \omega_1}{\partial y_2}\right) + \right.$$

$$\left. \xi_1^2\,\frac{\partial \omega_1}{\partial y_1} + \xi_2^2\,\frac{\partial \omega_2}{\partial y_2}\right] dy + \int_Y \left(\xi_1\,\frac{\partial \omega_1}{\partial y_1} + \xi_2\,\frac{\partial \omega_2}{\partial y_1}\right)^2 dy +$$

$$+ \int_Y \left(\xi_1\,\frac{\partial \omega_1}{\partial y_2} + \xi_2\,\frac{\partial \omega_2}{\partial y_1}\right)^2 dy$$

from periodicity of $\omega_i(p_0)$ the second term is null and the positivity of the other terms leads to :

$$\int_Y (\nabla_y(t_1 + t_2))^2\,dy \geq \xi_1^2 + \xi_2^2$$

whence the inequality (4.25). ■

LEMMA 4.7 : *Let p_0 be a solution of* (4.24). *Then there exists a constant independent of p_0 such that*

$$\|\omega_i(p_0)\|_m \leq C\,;\ \|\chi_i(p_0)\|_m \leq C \quad i = 1, 2\,. \qquad \square$$

Proof: We will show the proof for $\omega_i(p_0)$ (the proof is the same for $(\chi_i(p_0))$. From (4.20) we deduce

$$\int_Y (h^3 p_0 + \lambda h^2)\, \nabla_y \omega_i(p_0)\, \nabla\varphi\, dy = -\int_Y (h^3 p_0 + \lambda h^2) \frac{\partial \varphi}{\partial y_i} dy$$

$$\forall \varphi \in H_p^1(Y.) \quad (4.26)$$

By setting $\varphi = \omega_i(p_0)$ in (4.26) we obtain

$$\int_Y (\nabla_y \omega_i(p_0))^2\, dy \leqslant \frac{h_1^3 p_0(x) + \lambda h_1^3}{h_0^3 p_0(x) + \lambda h_0^3} \int_Y \left| \frac{\partial \omega_i(p_0)}{\partial y_i} \right| dy$$

$$\leqslant \frac{h_1^3 p_0(x) + \lambda h_1^3}{h_0^3 p_0(x) + \lambda h_0^3} \left[\int_Y (\nabla_y \omega_i(p_0))^2\, dy \right]^{1/2}$$

thus

$$\| \omega_i(p_0) \|_m \leqslant \frac{h_1^3 p_0(x) + \lambda h_1^3}{h_0^3 p_0(x) + \lambda h_0^3}.$$

As the function $t \to \dfrac{h_1^3 t + \lambda h_1^3}{h_0^3 t + \lambda h_0^3}$ is increasing on $[0, +\infty[$ and bounded by $\dfrac{h_1^3}{h_0^3}$, we have

$$\| \omega_i(p_0) \|_m \leqslant \frac{h_1^3}{h_0^3}. \qquad \blacksquare$$

LEMMA 4.8 : *Let p_0^1 and p_0^2 be two solutions to problem* (4.22). *Then*

$$|a_{ij}^0(p_0^2) - a_{ij}^0(p_0^1)| \leqslant C|p_0^2 - p_0^1| \quad 1 \leqslant i, j \leqslant 2$$

$$|\Theta^*(p_0^2) - \Theta^*(p_0^1)| \leqslant C|p_0^2 - p_0^1|. \qquad \square$$

Proof: we prove lemma 4.8 in the case a_{ii} (the proof is the same for the other cases).

$$a_{ii}^0(p_0^2) - a_{ii}^0(p_0^1) = (p_0^2 - p_0^1) \left\{ \int_Y h^3\, dy + \int_Y h^3 \frac{\partial \omega_i(p_0^2)}{\partial y_i} dy \right\} +$$

$$+ \int_Y (h^3 p_0^1 + \lambda h^2)\, \nabla_y(\omega_i(p_0^2) - \omega_i(p_0^1)) \cdot \nabla_y y_i\, dy. \quad (4.27)$$

It follows from subtracting (4.26) with $p_0 = p_0^2$ from (4.26) with $p_0 = p_0^1$ that

$$-\int_Y (h^3 p_0^1 + \lambda h^2)\, \nabla_y(\omega_i(p_0^2) - \omega_i(p_0^1)) \cdot \nabla\phi\, dy =$$

$$= (p_0^2 - p_0^1) \left\{ \int_Y (h^3\, \nabla_y\, \omega_i(p_0^2) \cdot \nabla\phi + h^3\, \nabla\phi \cdot \nabla_y y_i)\, dy \right\}. \quad (4.28)$$

On the other hand, from equation (4.26) with $p_0 = p_0^1$ and $\varphi = \omega_i(p_0^2) - \omega_i(p_0^1)$ we have

$$\int_Y (h^3 p_0^1 + \lambda h^2)\, \nabla_y(\omega_i(p_0^2) - \omega_i(p_0^1)) \cdot \nabla y y_i\, dy =$$

$$= -\int_Y (h^3 p_0^1 + \lambda h^2)\, \nabla_y(\omega_i(p_0^2) - \omega_i(p_0^1)) \cdot \nabla_y\, \omega_i(p_0^1)\, dy. \quad (4.29)$$

Now equations (4.27) (4.28) and (4.29) give

$$a_{ii}^0(p_0^2) - a_{ii}^0(p_0^1) = (p_0^2 - p_0^1) \left\{ \int_Y \left(h^3 + h^3 \frac{\partial \omega_i(p_0^2)}{\partial y_i} + h^3\, \nabla_y(\omega_i(p_0^2) - \right.\right.$$

$$\left.\left. - \omega_i(p_0^1)) \cdot (\nabla_y\, \omega_i(p_0^1) + \nabla_y y_i) \right) dy \right\}$$

and from lemma 4.7, there exists a constant C such that

$$|a_{ii}^0(p_0^2) - a_{ii}^0(p_0^1)| \leqslant C |p_0^2 - p_0^1|. \quad \blacksquare$$

THEOREM 4.9 : *Let* p_0 *be a solution of problem* (4.24). *Then there exists* $\alpha \in]0, 1[$ *such that* $p_0 \in C^{1,\alpha}(\bar{\Omega})$. □

Proof : Problem (4.24) can be written as

$$\operatorname{div}\{(K_1(x, p_0)\, p_0 + \lambda K_2(x, p_0))\, \nabla p_0\} = \operatorname{div}\{p_0\, \Theta^*(p_0)\} \quad (4.30)$$

where

$$K_1(x, p_0) = \begin{bmatrix} \overline{h^3}(x) + \overline{h^3 \dfrac{\partial \omega_1(p_0)}{\partial y_1}} & \overline{h^3 \dfrac{\partial \omega_2(p_0)}{\partial y_1}} \\ \overline{h^3 \dfrac{\partial \omega_1(p_0)}{\partial y_2}} & \overline{h^3}(x) + \overline{h^3 \dfrac{\partial \omega_2(p_0)}{\partial y_2}} \end{bmatrix}$$

$$K_2(x, p_0) = \begin{bmatrix} \overline{\overline{h^2}(x) + h^2 \frac{\partial \omega_1(p_0)}{\partial y_1}} & \overline{h^2 \frac{\partial \omega_2(p_0)}{\partial y_1}} \\ \overline{h^2 \frac{\partial \omega_1(p_0)}{\partial y_2}} & \overline{\overline{h^2}(x) + h^2 \frac{\partial \omega_2(p_0)}{\partial y_2}} \end{bmatrix}.$$

Let

$$G(p_0) = p_0 + \frac{1}{2} p_0^2$$

as p_0 and $p_0^2 \in H^1(\Omega)$, $G(p_0) \in H^1(\Omega)$ and since G is non negative, $p_0 = -1 + \sqrt{1 + 2G} = r(G)$. Thus $G(p_0)$ is a solution of the following problem

$$\begin{cases} \operatorname{div}\{a(x, G)\nabla G\} = \operatorname{div}\{r(G)\,\Theta^*(r(G))\} \\ G = p_a + \frac{1}{2} p_a^2 \text{ on } \partial\Omega \end{cases} \tag{4.31}$$

where

$$a(x, G) = \frac{K_1(x, r(G))\, r(G) + \lambda K_2(x, r(G))}{\sqrt{1 + 2G}}.$$

The matrix K_1 and K_2 lie in $(W^{1,\infty}(\Omega))^4$ and the vector $r(G)\,\Theta^*(r(G))$ lies in $(H^1(\Omega))^2$. Then the function $a(x, G) \in (L^\infty(\Omega))^4$ and from theorem A.1(i) applied to (4.31), there exists $\alpha \in]0, 1[$ such that $G \in C^{0,\alpha}(\bar{\Omega})$ and then $p_0 \in C^{0,\alpha}(\bar{\Omega})$ and $K_1 p_0 + \lambda K_2 \in C^0(\bar{\Omega})$. Now by applying theorem A.2(i) to (4.30) there exists $\alpha \in]0, 1[$ such that $p_0 \in C^{1,\alpha}(\bar{\Omega})$.

THEOREM 4.10 : *The solution of problem* (4.22) *is unique. Furthermore, suppose that* p_0^1 *is a positive solution to* (4.22) *corresponding to boundary data* p_a^1 *and* p_0^2 *is a positive solution to* (4.22) *corresponding to boundary data* p_a^2. *If* $p_a^1 \geq p_a^2$ *on* $\partial\Omega$, *then* $p_0^1 \geq p_0^2$ *in* Ω. □

Proof : We use the same arguments as in theorem 3.9. We show as for (3.23) that for all $\xi \in C^\infty(\bar{\Omega})$ and $\xi > 0$ we have

$$\int_{[p_0^2 - p_0^1 > 0]} (A^*(p_0^2)\nabla p_0^2 - A^*(p_0^1)\nabla p_0^1) \cdot \nabla\xi \, dx -$$

$$- \int_{[p_0^2 - p_0^1 > 0]} (\Lambda^*(p_0^2)\, p_0^2 - \Theta^*(p_0^1)\, p_0^1) \cdot \nabla\xi \, dx < 0$$

which can be written

$$\int_{[p_0^2 - p_0^1 > 0]} A^*(p_0^1)\, \nabla(p_0^2 - p_0^1) \cdot \nabla\xi\, dx +$$

$$+ \int_{[p_0^2 - p_0^1 > 0]} (A^*(p_0^2) - A^*(p_0^1)) \cdot \nabla p_0^2\, \nabla\xi\, dx -$$

$$- \int_{[p_0^2 - p_0^1 > 0]} (p_0^2 - p_0^1)\, \Theta^*(p_0^2) \cdot \nabla\xi\, dx -$$

$$- \int_{[p_0^2 - p_0^1 > 0]} (\Theta^*(p_0^2) - \Theta^*(p_0^1)) \cdot \nabla\xi\, dx < 0\,. \quad (4.32)$$

By integrating by parts the first term of (4.32) we obtain

$$\int_{[p_0^2 - p_0^1 > 0]} (p_0^2 - p_0^1)\, [-\operatorname{div}(A^*(p_0^1)\, \nabla\xi)] + g(x)\, \nabla p_0^2 \cdot \nabla\xi\, dx -$$

$$- \Theta^*(p_0^2) \cdot \nabla\xi\, dx - p_0^1\, l(x) \cdot \nabla\xi\, dx < 0 \quad (4.33)$$

where $g(x)$ is the matrix defined by

$$g(x) = \begin{cases} \dfrac{A^*(p_0^2) - A^*(p_0^1)}{p_0^2 - p_0^1} & \text{if} \quad p_0^2(x) \neq p_0^1(x) \\ 0 & \text{ifnot} \end{cases}$$

and $l(x)$ the vector defined by

$$l(x) = \begin{cases} \dfrac{\Theta^*(p_0^2) - \Theta^*(p_0^1)}{p_0^2 - p_0^1} & \text{if} \quad p_0^2(x) \neq p_0^1(x) \\ 0 & \text{ifnot}\,. \end{cases}$$

From lemma 4.8 $g(x) \in (L^\infty(\Omega))^4$ and $l(x) \in (L^\infty(\Omega))^2$.

For any positive s, choose W a constant large enough so that $\xi = W - e^{sx_1}$ is strictly positive. So

$$-\operatorname{div}(A^*(p_0^1)\, \nabla\xi) = e^{sx_1}\left(a_{11}^0(p_0^1)\, s^2 + \frac{\partial a_{11}^0(p_0^1)}{\partial x_1}\, s \right)$$

then

$$- \operatorname{div}\,(A^*(p_0^1)\,\nabla\xi) + g(x)\,\nabla p_0^2\cdot\nabla\xi - p_0^1\, l(x)\cdot\nabla\xi =$$

$$e^{sx_1}\Bigg(a_{11}^0(p_0^1)\, s^2 + \Bigg(\frac{\partial a_{11}^0(p_0^1)}{\partial x_1} - g_{11}(x)\frac{\partial p_0^2}{\partial x_1} - g_{21}(x)\,\frac{\partial p_0^2}{\partial x_2} + + \Theta_1^*(p_0^2) + p_0^1\, l_1(x)\Bigg) s\Bigg) \quad (4.34)$$

from theorem 4.9, the coefficients of s and s^2, in (4.34), are in $L^\infty(\Omega)$. Hence it follows that when s is sufficiently large (4.34) is positive. Therefore the inequality (4.33) allows us to conclude that $(p_0^2 - p_0^1) = 0$ ◆

4.3. Transverse and longitudinal roughness

The same calculations as equation (1.1) in paragraph 3.3 give the following homogenized problems in the transverse and the longitudinal roughness.

Transverse roughness

$$\frac{\partial}{\partial x_1}\left(\frac{1}{\displaystyle\int_Y \frac{dy}{h^3(x,y_1)\,p_0(x) + \lambda h^2(x,y_1)}}\frac{\partial p_0}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left((\overline{h^3}\,p_0 + \lambda\overline{h^2})\,\frac{\partial p_0}{\partial x_2}\right) =$$

$$= \frac{\partial}{\partial x_1}\left(A_1\,\frac{\displaystyle\int_Y \frac{dy}{h^2(x,y_1)\,p_0(x) + \lambda h(x,y_1)}}{\displaystyle\int_Y \frac{dy}{h^3(x,y_1)\,p_0(x) + \lambda h^2(x,y_1)}}\, p_0\right) + \frac{\partial(A_2\,\bar{h}p_0)}{\partial x_2} \text{ on } \Omega\,. \quad (4.35)$$

Longitudinal roughness

$$\frac{\partial}{\partial x_1}\left((\overline{h^3}\,p_0 + \lambda\overline{h^2})\,\frac{\partial p_0}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{1}{\displaystyle\int_Y \frac{dy}{h^3(x,y_2)\,p_0(x) + \lambda h^2(x,y_2)}}\frac{\partial p_0}{\partial x_2}\right) =$$

$$= \frac{\partial(A_1\,\bar{h}p_0)}{\partial x_1} + \frac{\partial}{\partial x_2}\left(A_2\,\frac{\displaystyle\int_Y \frac{dy}{h^2(x,y_2)\,p_0(x) + \lambda h(x,y_2)}}{\displaystyle\int_Y \frac{dy}{h^3(x,y_2)\,p_0(x) + \lambda h^2(x,y_2)}}\, p_0\right) \text{ on } \Omega\,. \quad (4.36)$$

Note that $\lambda = 0$ gives the homogenized problems (3.28) and (3.29) of the continuum problem.

Without any proof, Mitsuya *et al.* [13] have proposed the « averaged Reynolds » equation extended to the slip-flow regime as follows

$$\frac{\partial}{\partial x_1}\left((\hat{H}^3 \bar{p} + \lambda \hat{H}^2) \frac{\partial \bar{p}}{\partial x_1} \right) + \frac{\partial}{\partial x_2}\left((\overline{H^3}\, \bar{p} + \lambda \overline{H^2}) \frac{\partial \bar{p}}{\partial x_2} \right) = \Lambda_1 \frac{\partial}{\partial x_1} (\hat{H} \bar{p}) + \Lambda_2 \frac{\partial}{\partial x_2} (\bar{H} \bar{p}) \quad (4.37)$$

for stationary transverse roughness.

Where $\bar{H}^3$, $\bar{H}^2$ and $\bar{H}$ are arithmetically averaged film thickness, and $\hat{H}^3 = \frac{1}{\overline{H^{-3}}}$, $\hat{H}^2 = \frac{1}{\overline{H^{-2}}}$ and $\hat{H} = \frac{\bar{H}^{-2}}{\bar{H}^{-3}}$ are harmonically averaged film thickness. It will be proved by using numerical experiments (*see fig.* 5.7) that this « averaged equation » doesn't give satisfactory results.

5. NUMERICAL RESULTS

The equation governing the pressure distribution in a thin film slider bearing of infinite breadth is the Reynolds equation

$$\begin{cases} \frac{d}{dx}\left((H_\varepsilon^3 p_\varepsilon + \lambda H_\varepsilon^2) \frac{dp_\varepsilon}{dx} \right) = \Lambda \frac{d}{dx} (H_\varepsilon p_\varepsilon) \text{ on }]0, 1[\\ p_\varepsilon(0) = p_\varepsilon(1) = 1 \,. \end{cases} \quad (5.1)$$

For the case of stationary transverse, sine wave roughness considered here, the bearing is given by

$$H_\varepsilon(x) = H_1 + (1 - H_1)\, x + e \sin\left(2\pi \frac{x}{\varepsilon} \right). \quad (5.2)$$

H_1 will be considered fixed $(H_1 = 2)$ throughout this numerical results and e is the roughness amplitude.

Numerical calculations of equation (5.1) was performed by two methods. The first one consists of applying the Newton-Raphson method to obtain a sequence of linear problem. Then we solve these linear problems by the LPDEM method described in appendix B (see for more details [12]). In the second scheme we linearise the problem by the fixed point method and then we apply the classical finite difference method to the linear problems.

First, a sequence of calculations are displayed in figures 5.1 and 5.2 for the continuum problem $(\lambda = 0)$ with $e = 0.6$ and $\Lambda = 300$. The dimensionless exact, homogenized and discretized pressure profiles are given on figure 5.1

for $\varepsilon = 0.02$ and on figure 5.2 for $\varepsilon = 0.003$. One finds, as ε decreases, that the pressure oscillations decrease to zero in amplitude and so it converges to the homogenized solution as predicted by the theory. We remark that a great numerical advantage is achieved for the LPDEM approximate solution, due to a very much coarser mesh ($dx = 1/11$) used for the LPDEM scheme. The finite difference approximate solution (DF) is very far from it.

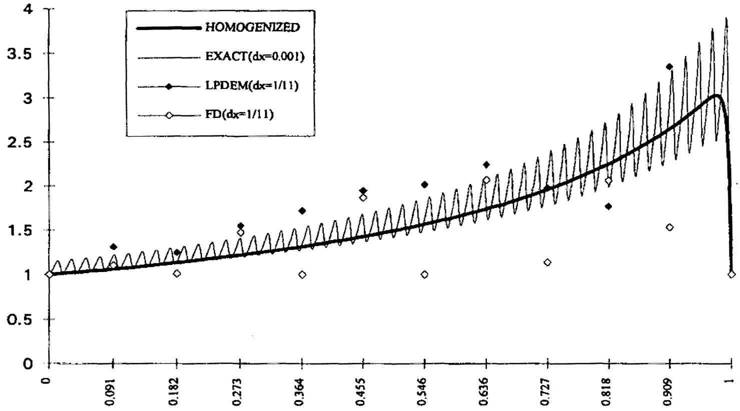

Figure 5.1. — Pressure profiles of the continuum problem ($e = 0.6$, $\Lambda = 300$, $\lambda = 0$) $\varepsilon = 0.02$.

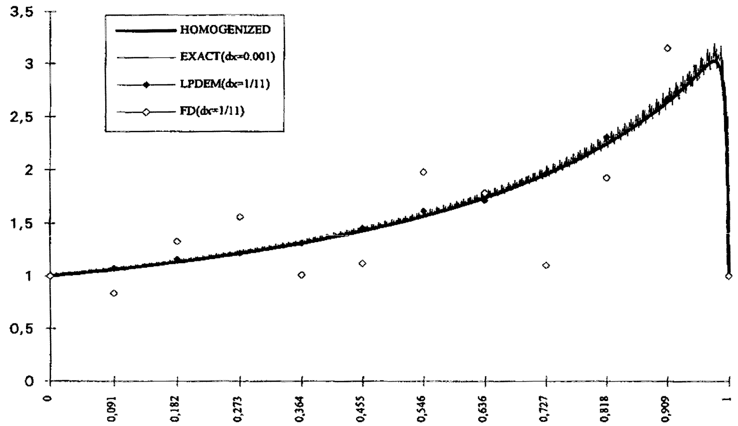

Figure 5.2. — Pressure profiles of the continuum problem ($e = 0.6$, $\Lambda = 300$, $\lambda = 0$) $\varepsilon = 0.003$.

Next, the same bearing geometry was investigated but with the slip flow equation ($\lambda = 0.4$). The results are presented in figures 5.3 and 5.4. One can conclude similar remarks to the ones for the continuum problem.

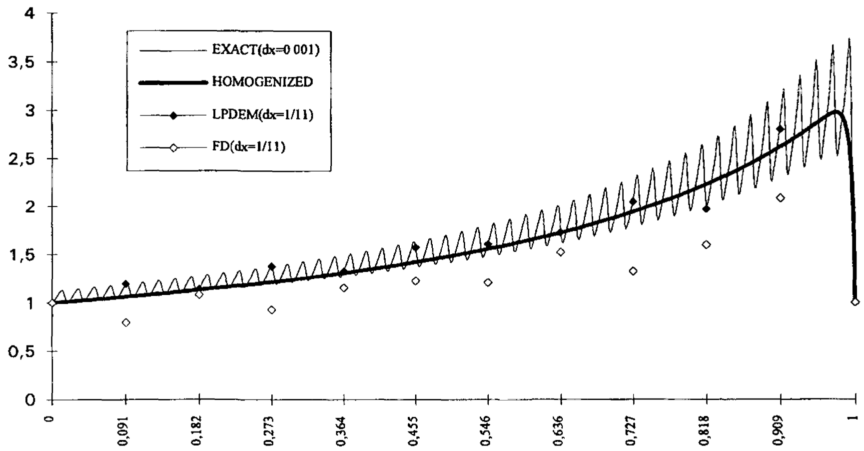

Figure 5.3. — Pressure profiles of the slip flow problem ($e = 0.6$, $\Lambda = 300$, $\lambda = 0.4$) $\varepsilon = 0.02$.

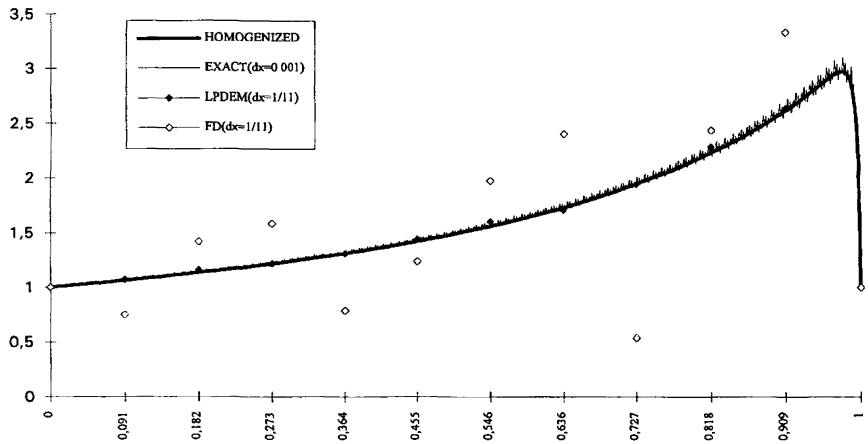

Figure 5.4. — Pressure profiles of the slip flow problem ($e = 0.6$, $\Lambda = 300$, $\lambda = 0.4$) $\varepsilon = 0.003$.

In figure 5.5, we compare the Mitsuya averaged proposed pressure, given by equation (4.37) and used in tribology, and the homogenized solution $p0$ (equation (4.35)). The exact solution does in no way seems to tend to the average solution but it converges to the homogenized solution.

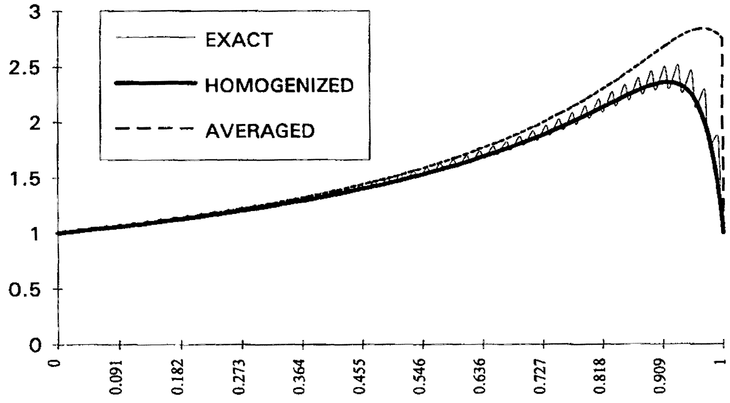

Figure 5.5. — Comparison of the Mitsuya averaged and homogenized solutions($\lambda = 10$, $\varepsilon = 0.02$, $\Lambda = 300$).

APPENDIX A.

Some regularity results

Let Ω be an open and bounded set of R^n and consider the homogeneous Dirichlet problem

$$u \in H_0^1(\Omega) : \int_\Omega a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_j} dx = \int_\Omega f_j \frac{\partial \varphi}{\partial x_j} dx , \quad \forall \varphi \in H_0^1(\Omega) \qquad \text{(A.1)}$$

for $f_j \in L^\infty(\Omega), j = 1, 2, \ldots, n$.
Then we have the following theorems (see Rodrigues [16]).

THEOREM A.1 : *Let* $u \in H_0^1(\Omega)$ *be the unique solution to* (A.1) *with* $a_{ij} \in L^\infty(\Omega)$.

(i) *If* $f_j \in L^p(\Omega), p > n$ *and* $\Omega \in C^{0,1}$, *then* $u \in C^{0,\alpha}(\bar{\Omega})$, *for some* $0 < \alpha < 1$;

(ii) *If* $\partial\Omega \in C^1$, *there exists* $p^* > 2$, *such that,* $u \in W_0^{1,p}(\Omega)$ *for* $2 < p < p^*$, *provided* $f_j \in L^p(\Omega)$.

(iii) *If, in addition to* $\partial\Omega \in C^1, a_{ij} \in C^0(\bar{\Omega})$ *then* $u \in W_0^{1,p}(\Omega)$ *for each* $2 < p < \infty$, *whenever* $f_j \in L^p(\Omega)$.

THEOREM A.2 : *Let* $u \in H_0^1(\Omega)$ *be the unique solution to* (A.1).

(i) *If* $a_{ij} \in C^{m,\alpha}(\bar{\Omega}), f_j \in C^{m,\alpha}(\bar{\Omega})$ *and* $\partial\Omega_j \in C^{m+1,\alpha}$, *then*

$$u \in C^{m+1,\alpha}(\bar{\Omega}), \quad \forall m \geq 0, 0 < \alpha < 1 .$$

(ii) *If* $a_{ij} \in C^{m,1}(\bar{\Omega}), f_j \in W^{m+1,p}(\Omega)$ *and* $\partial\Omega_j \in C^{m+1,1}$, *then*

$$u \in W^{m+2,p}(\Omega), \quad \forall m \geq 0, 1 < p < \infty .$$

APPENDIX B.

The LPDEM method

By introducing the new variable

$$u_\varepsilon = \frac{1}{2} p_\varepsilon^2 + \frac{\lambda p_\varepsilon}{h_\varepsilon} \tag{B.1}$$

in equation (5.1) we obtain the problem

$$\begin{cases} \dfrac{d}{dx}\left(h_\varepsilon^3 \dfrac{d}{dx} u_\varepsilon\right) = \dfrac{d}{dx}\left(\beta(x, u_\varepsilon)\left(\Lambda - \lambda h'_\varepsilon\right)\right) \quad 0 < x < 1 \\ u_\varepsilon(0) = \dfrac{p_0^2}{2} + \dfrac{p_0}{h_\varepsilon(0)} = u_0 \, ; \, u_\varepsilon(1) = \dfrac{p_0^2}{2} + \dfrac{p_0}{h_\varepsilon(1)} = u_1 \, . \end{cases} \tag{B.2}$$

Applying the Newton-Raphson method to linearise the problem (B.2) we obtain the sequence

$$u_\varepsilon^0 \text{ is given with } u_\varepsilon^0(0) = u_0 \text{ and } u_\varepsilon^0(1) = u_1$$

u_ε^{m+1} is solution of

$$\begin{cases} \dfrac{d}{dx}\left(h_\varepsilon^3 \dfrac{d}{dx} u_\varepsilon^{m+1}\right) - \dfrac{d}{dx}\left(\dfrac{h_\varepsilon^2(\Lambda - \lambda h'_\varepsilon)}{\sqrt{\lambda^2 + 2 h_\varepsilon^2 u_\varepsilon^m}} u_\varepsilon^{m+1}\right) = \\ \quad = \dfrac{d}{dx}\left(\dfrac{(\Lambda - \lambda h'_\varepsilon)\left(\lambda^2 + h_\varepsilon^2 u_\varepsilon^m - \lambda\sqrt{\lambda^2 + 2 h_\varepsilon^2 u_\varepsilon^m}\right)}{\sqrt{\lambda^2 + 2 h_\varepsilon^2 u_\varepsilon^m}}\right) \\ u_\varepsilon^{m+1}(0) = u_0 \, ; \, u_\varepsilon^{m+1}(1) = u_1 \, . \end{cases} \tag{B.3}$$

Let N be a positive integer and $dx = \frac{1}{N+1}$ and $x_i = i\, dx$ ($i = 0, 1, \ldots, N+1$) the grid points. Introducing the notations :

$$F_{x_i}(s, u_\varepsilon^m) = \exp\left(\int_{x_i}^{s} \frac{(\Lambda - \lambda h'_\varepsilon(t))}{h_\varepsilon(t)\sqrt{\lambda^2 + 2\,h_\varepsilon^2\, u_\varepsilon^m(t)}}\, dt\right)$$

$$a^+(x_i, u_\varepsilon^m) = \int_{x_i}^{x_{i+1}} \frac{ds}{h_\varepsilon^3(s)\, F_{x_i}(s, u_\varepsilon^m)} \quad a^-(x_i, u_\varepsilon^m) = \int_{x_{i-1}}^{x_i} \frac{ds}{h_\varepsilon^3(s)\, F_{x_i}(s, u_\varepsilon^m)}$$

$$w^+(x_i, u_\varepsilon^m) = -\int_{x_i}^{x_{i+1}} \frac{(\Lambda - \lambda h'_\varepsilon)(\lambda^2 + h_\varepsilon^2 u_\varepsilon^m - \lambda\sqrt{\lambda^2 + 2\,h_\varepsilon^2\, u_\varepsilon^m})}{h_\varepsilon^3(s)\sqrt{\lambda^2 + 2\,h_\varepsilon^2\, u_\varepsilon^m}\; F_{x_i}(s, u_\varepsilon^m)}\, ds$$

$$w^-(x_i, u_\varepsilon^m) = -\int_{x_{i-1}}^{x_i} \frac{(\Lambda - \lambda h'_\varepsilon)(\lambda^2 + h_\varepsilon^2 u_\varepsilon^m - \lambda\sqrt{\lambda^2 + 2\,h_\varepsilon^2\, u_\varepsilon^m})}{h_\varepsilon^3(s)\sqrt{\lambda^2 + 2\,h_\varepsilon^2\, u_\varepsilon^m}\; F_{x_i}(s, u_\varepsilon^m)}\, ds$$

$$c(x_i, u_\varepsilon^m) = -\frac{a^+(x_i, u_\varepsilon^m)}{F_{x_i}(x_{i-1}, u_\varepsilon^m)}\,; \quad d(x_i, u_\varepsilon^m) = (a^+ + a^-)(x_i, u_\varepsilon^m)\,;$$

$$e(x_i, u_\varepsilon^m) = -\frac{a^+(x_i, u_\varepsilon^m)}{F_{x_i}(x_{i+1}, u_\varepsilon^m)}\,.$$

The LPDEM discretized scheme is

$$\begin{cases} c(x_i, u_\varepsilon^m)\, u_{\varepsilon, i-1}^{m+1} + d(x_i, u_\varepsilon^m)\, u_{\varepsilon, i}^{m+1} + e(x_i, u_\varepsilon^m)\, u_{\varepsilon, i+1}^{m+1} = \\ \quad = (a^+\, w^- - a^-\, w^+)(x_i, u_\varepsilon^m)\ 1 \leqslant i \leqslant N \\ u_{\varepsilon, 0}^{m+1} = u_0\,;\ u_{\varepsilon, N+1}^{m+1} = u_1\,. \end{cases} \tag{B.4}$$

The convergence of the LPDEM scheme (B.4) is given in [12].

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