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TIME-DISCRETE FINITE ELEMENT SCHEMES FOR MAXWELL'S EQUATIONS (*)

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Abstract — *We analyze a family of fully discrete finite element methods applied to Maxwell's equations To discretize in space we use the edge elements of Nedelec which are particularly suitable for discretizing electro magnetic problems To discretize in time we use a family of methods based on rational approximations of the exponential We prove error estimates for this scheme*

Résumé — *Nous étudions une famille de discrétisation complète des équations de Maxwell L'approximation en espace utilise les éléments d'arêtes de Nédelec, bien adaptés aux problemes d'electromagnetisme, la discrétisation en temps est basée sur une famille d'approximations rationnelles de l'exponentielle Nous montrons des estimations d'erreur pour le schéma obtenu*

1. INTRODUCTION

We shall analyze the use of fully discrete finite element schemes to approximate the time dependent Maxwell equations on a bounded domain The finite element scheme used to discretize in space is the standard Nédélec family of edge elements for the electric field and the corresponding Raviart-Thomas-Nédélec elements for the magnetic flux The advantages of these elements for electromagnetic computations are summarized for example in [3]. To discretize in time, we use a family of implicit schemes based on rational approximations of the exponential of order 2, 3 or 4. Among others, a reason of analyzing the implicit schemes is that these schemes may be preferred in situations in which the mesh has some irregular tetrahedra (this is often the case with meshes that are generated automatically, even though most tetrahedra may be quite regular) We note however that our analysis can be

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extended to include explicit schemes also, under of course, a restriction on the stability region of our method. For other implicit and explicit schemes for the time discretization of the Maxwell equation we refer to [1, 17].

To state Maxwell's equations, let $\Omega \subset \mathbb{R}^3$ be a bounded, polyhedral, and convex domain with boundary denoted by Γ and unit outward normal is denoted by $\mathbf{n}$. We remark that the assumption of convexity is necessary for analysis, but not for the successful use of the method in practice. The extension of the method to curved domains is possible but significantly complicates the presentation of the method (see [5] for details of how to define edge elements on a smooth domain). In the case of a smooth domain the convexity assumption is not needed. We shall also assume that the material contained in Ω is non-conducting. Some parts of the paper (in particular the semi-discrete error analysis) can be extended trivially to include conducting media, but the fully discrete time stepping scheme must be modified in that case.

We suppose that Ω is filled with a dielectric material have permitivity (or dielectric « constant ») ε and permeability μ both of which may be three by three matrix functions of position. We denote by $\mathbf{E} \equiv \mathbf{E}(\mathbf{x}, t)$ and $\mathbf{H} \equiv \mathbf{H}(\mathbf{x}, t)$ the electric and magnetic fields respectively. These fields satisfy the Maxwell equations in Ω :

$$\varepsilon \mathbf{E}_t - \nabla \times \mathbf{H} = \mathbf{J} \quad \text{in} \quad \Omega , \tag{1a}$$

$$\mu \mathbf{H}_t + \nabla \times \mathbf{E} = 0 \quad \text{in} \quad \Omega , \tag{1b}$$

where

$$\mathbf{E}_t = \frac{\partial}{\partial t} \mathbf{E} \quad \text{and} \quad \mathbf{H}_t = \frac{\partial}{\partial t} \mathbf{H} .$$

In addition, for simplicity we shall assume that the boundary of Ω is perfectly conducting so that

$$\mathbf{n} \times \mathbf{E} = 0 \quad \text{on} \quad \Gamma . \tag{2}$$

Finally, we need to specify initial data so we suppose that functions $\mathbf{E}_0$ and $\mathbf{H}_0$ are known and require that

$$\mathbf{E}(0) = \mathbf{E}_0 \quad \text{and} \quad \mathbf{H}(0) = \mathbf{H}_0 \quad \text{in} \quad \Omega \tag{3}$$

(where $\mathbf{E}(0) \equiv \mathbf{E}(\,\cdot\,, 0)$ and $\mathbf{H}(0) \equiv \mathbf{H}(\,\cdot\,, 0)$). On physical grounds, we can assume that

$$\nabla \cdot (\mu \mathbf{H}_0) = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad (\mu \mathbf{H}_0) \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma . \tag{4}$$

In order to handle the case where μ is not a constant function of $\mathbf{x}$ it is more convenient to use as dependent variables the electric field strength $\mathbf{E}$ and the magnetic flux $\mathbf{B}$. The magnetic flux is given in terms of the magnetic field $\mathbf{H}$ by the constitutive relation

$$\mathbf{B} = \mu \mathbf{H} . \tag{5}$$

Using (5) in (1) to eliminate $\mathbf{H}$, we arrive at the following equivalent equations for $\mathbf{E}$ and $\mathbf{B}$:

$$\varepsilon \mathbf{E}_t - \nabla \times (\mu^{-1} \mathbf{B}) = \mathbf{J} \quad \text{in} \quad \Omega , \tag{6a}$$

$$\mathbf{B}_t + \nabla \times \mathbf{E} = 0 \quad \text{in} \quad \Omega . \tag{6b}$$

Our goal for this paper is to develop and analyze fully discrete finite element approximations to the fields $(\mathbf{E}, \mathbf{B})$ satisfying (2)-(6) for $0 \leqslant t \leqslant T$ where $T > 0$.

In order to write down an appropriate variational formulation for the Maxwell system we will need suitable spaces for the field and flux variables. Thus we recall the standard spaces :

$$H(\text{curl} ; \Omega) = \{\mathbf{u} \in (L^2(\Omega))^3 | \nabla \times \mathbf{u} \in (L^2(\Omega))^3\} , \tag{7a}$$

$$H_0(\text{curl} ; \Omega) = \{\mathbf{u} \in H(\text{curl} ; \Omega) | \mathbf{u} \times \mathbf{n} = 0 \text{ on } \Gamma\} , \tag{7b}$$

$$H(\text{div} ; \Omega) = \{\mathbf{u} \in (L^2(\Omega))^3 | \nabla \cdot \mathbf{u} \in L^2(\Omega)\} . \tag{7c}$$

Let us now be more precise about the requirements on the functions ε, μ and $\mathbf{J}$. We assume that $\varepsilon \in (C^1(\bar{\Omega}))^9$, $\mu^{-1} \in (C^1(\bar{\Omega}))^9$ and $\mathbf{J} \in C(0, T : (L^2(\Omega))^3)$ (the continuity assumptions are made to allow analysis and can be weakened considerably in practice). The functions ε and μ^{-1} are assumed to be positive definite and uniformly bounded (above and below) on $\bar{\Omega}$.

A weak formulation of the system (2)-(6) is

$$(\varepsilon \mathbf{E}_t , \psi) - (\mu^{-1} \mathbf{B}, \nabla \times \psi) = (\mathbf{J}, \psi) \quad \forall \psi \in H_0(\text{curl} ; \Omega) \tag{8a}$$

$$(\mu^{-1} \mathbf{B}_t , \phi) + (\nabla \times \mathbf{E}, \mu^{-1} \phi) = 0 \quad \forall \phi \in H(\text{div} ; \Omega) . \tag{8b}$$

Here $\mathbf{B}$ and $\mathbf{E}$ are still subject to the initial conditions (3). We will assume that the system (8) has a unique solution smooth enough for the purposes of our analysis. For existence uniqueness of the Maxwell equations see for example [10].

To approximate (8) we use Nédélec's edge elements and the Raviart-Thomas-Nédélec divergence conforming elements [15] to construct finite element subspaces of $H_0(\mathrm{curl}\,;\Omega)$ and $H(\mathrm{div}\,;\Omega)$. We detail this construction, and the assumptions required on the mesh, in § 2. Here it suffices to say that if we are given the spaces $\mathbf{U}_h^{r,0} \subset H_0(\mathrm{curl}\,;\Omega)$ and $\mathbf{V}_h^r \subset H(\mathrm{div}\,;\Omega)$ we can discretize (8) in space in the obvious way to obtain the semi-discrete problem of finding $(\mathbf{E}_h(t), \mathbf{B}_h(t)) \in \mathbf{U}_h^{r,0} \times \mathbf{V}_h^r$ such that, for $0 < t \leqslant T$,

$$(\varepsilon \mathbf{E}_{h,t}, \psi_h) - (\mu^{-1}\mathbf{B}_h, \nabla \times \psi_h) = (\mathbf{J}, \psi_h) \quad \forall \psi_h \in \mathbf{U}_h^{r,0} \tag{9a}$$

$$(\mu^{-1}\mathbf{B}_{h,t}, \phi_h) + (\nabla \times \mathbf{E}_h, \mu^{-1}\phi_h) = 0 \quad \forall \phi_h \in \mathbf{V}_h^r . \tag{9b}$$

In addition $(\mathbf{E}_h, \mathbf{B}_h)$ satisfy a discrete version of (3) so that

$$\mathbf{E}_h(0) \approx \mathbf{E}_0 \quad \text{and} \quad \mathbf{B}_h(0) \approx \mathbf{B}_0 .$$

The first part of our paper § 2-§ 3 is devoted to analyzing the error in this semi-discrete scheme. We remark that in a previous paper on this type of scheme [14] error estimates for the corresponding semi-discrete scheme using $\mathbf{H}$ in place of $\mathbf{B}$ and constant coefficients ε and μ were proved. The novelty of the analysis here is that it allows variable matrix coefficients (but not yet discontinuous coefficients).

Having analyzed the semi-discrete scheme, we formulate a family of fully discrete time stepping schemes in § 4. We analyze the convergence of the fully discrete schemes in § 5, where we prove optimal order error estimates. These schemes belong to the class of schemes introduced in [2] for the discretization of second order hyperbolic equations. Their construction is based on a choice of a rational approximation of the exponential with certain accuracy and stability properties. Related work includes [13] where similar time stepping schemes were used for the construction of mixed finite element fully discrete schemes for the equations of elasticity, and [1] where an implicit scheme is investigated for the two dimensional analogue of the Maxwell system considered here. For some computational results with edge/face elements of the type discussed here see [12, 11].

2. PRELIMINARIES

In this section we shall summarize the construction and properties of Nédélec's first family of edge finite elements on a tetrahedral mesh [15]. We note that other families of edge elements can also be constructed, and could be used for Maxwell's equations including Nédélec's second family on tetrahedra and the first family on cuboid meshes [15, 16].

First let us define some notation. $W^{1,s}(\Omega)$ will denote the standard Sobolev space of functions in $L^s(\Omega)$ having derivatives in $L^s(\Omega)$. Similarly $H^p(\Omega)$ is the standard Sobolev space of functions with p derivatives in $L^2(\Omega)$. In general we shall denote the norm on a metric space X by $\|\cdot\|_X$ where X can be a space of vector functions. In the case of $H(\text{curl}\,;\Omega)$ (see (7)) the norm is defined by

$$\|\mathbf{u}\|_{H(\text{curl}\,;\,\Omega)} = \sqrt{\|\mathbf{u}\|^2_{(L^2(\Omega))^3} + \|\nabla\times\mathbf{u}\|^2_{(L^2(\Omega))^3}}\ .$$

In our analysis, we are going to use some weighted L^2 spaces. We define $L^2_\varepsilon(\Omega)$ and $L^2_{\mu^{-1}}(\Omega)$ to be the standard L^2 spaces with weights ε and μ^{-1} respectively. We will denote their inner products by

$$(\mathbf{u},\mathbf{v})_\varepsilon = \int_\Omega \varepsilon\mathbf{u}\cdot\mathbf{v}\,dV \quad\text{and}\quad (\mathbf{u},\mathbf{v})_{\mu^{-1}} = \int_\Omega \mu^{-1}\,\mathbf{u}\cdot\mathbf{v}\,dV$$

and the corresponding norms by

$$\|\mathbf{u}\|_{L^2_\varepsilon(\Omega)} = (\mathbf{u},\mathbf{u})_\varepsilon^{1/2} \quad\text{and}\quad \|\mathbf{u}\|_{L^2_{\mu^{-1}}(\Omega)} = (\mathbf{u},\mathbf{u})^{1/2}_{\mu^{-1}}\ .$$

Note that a consequence of the properties of ε and μ^{-1} is that these norms are equivalent with the standard L^2 norm. We have already defined some spaces of vector functions in the introduction and so will not repeat them here.

Let $\{\tau_h\}_{h>0}$ be a family of tetrahedral meshes of Ω where h is the maximum diameter of the tetrahedra in τ_h. We assume the meshes are regular and quasi-uniform [4].

In order to define the curl conforming space of Nédélec, we let P_r denote the standard space of polynomials of total degree less than or equal to r, and let $\tilde{P}_r$ denote the space of homogeneous polynomials of order r. Now we define $S_r \subset (P_r)^3$ and $R_r \subset (P_r)^3$ by

$$S_r = \{\mathbf{p}\in(\tilde{P}_r)^3 \,|\, \mathbf{p}(\mathbf{x})\cdot\mathbf{x} = 0\}\,,$$

$$R_r = (P_{r-1})^3 \oplus S_r\,.$$

For example, if $r=1$ then a polynomial $p\in R_1$ has the form $\mathbf{p}(\mathbf{x}) = \boldsymbol{\alpha} + \boldsymbol{\beta}\times\mathbf{x}$ where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are constant vectors [16]. Following [15], for positive integer r, we define

$$\mathbf{U}_h^r = \{\mathbf{u}_h \in H(\text{curl}\,;\Omega) \,|\, \mathbf{u}_h|_K \in R_r \quad \forall K\in\tau_h\}\,.$$

To define the degrees of freedom in $\mathbf{U}_h^r$ we define the following moments. If $K \in \tau_h$ with general edge e and face f and if $\mathbf{t}$ is a unit vector parallel to e :

$$M_e(\mathbf{u}) = \left\{ \int_e \mathbf{u} \cdot \mathbf{t} q \, ds \quad \forall q \in P_{r-1}(e) \quad \text{for the six edges } e \text{ of } K \right\}, \tag{10}$$

$$M_f(\mathbf{u}) = \left\{ \int_f \mathbf{u} \times \mathbf{n} \cdot \mathbf{q} \, dA \quad \forall \mathbf{q} \in (P_{r-2}(f))^2 \quad \text{for all four faces } f \text{ of } K \right\}, \tag{11}$$

$$M_K(\mathbf{u}) = \left\{ \int_K \mathbf{u} \cdot \mathbf{q} \, d\mathbf{x} \quad \forall \mathbf{q} \in (P_{r-3}(K))^3 \right\}. \tag{12}$$

These moments are defined if $\mathbf{u} \in (W^{1,s}(K))^3$ for some $s > 2$. Nédélec [15] shows that the above three sets of degrees of freedom are R_r-unisolvent and curl conforming. Using these degrees of freedom we can define an interpolant denoted $\mathbf{r}_h \mathbf{u} \in \mathbf{U}_h^r$ for any function $\mathbf{u}$ for which (10)-(12) are defined. On each $K \in \tau_h$ we pick $\mathbf{r}_h \mathbf{u}|_K \in R_r$ such that

$$M_e(\mathbf{u} - \mathbf{r}_h \mathbf{u}) = M_f(\mathbf{u} - \mathbf{r}_h \mathbf{u}) = M_K(\mathbf{u} - \mathbf{r}_h \mathbf{u}) = \{0\}.$$

To approximate functions in $H_0(\mathrm{curl}\,; \Omega)$ we define

$$\mathbf{U}_h^{r,0} = \{\mathbf{u}_h \in \mathbf{U}_h^r | \mathbf{n} \times \mathbf{u}_h = 0 \quad \text{on} \quad \Gamma\}. \tag{13}$$

The constraint $\mathbf{n} \times \mathbf{u}_h = 0$ on Γ is easily implemented by taking the degrees of freedom associated with edges or faces on Γ to be zero [8].

The following estimate is known for $\mathbf{r}_h$ [15] (for other estimates including estimates using weaker norms see [8, 5, 14]) : if $\mathbf{u} \in (H^{r+1}(\Omega))^3$ then

$$\|\mathbf{u} - \mathbf{r}_h \mathbf{u}\|_{H(\mathrm{curl}\,; \Omega)} \leqslant C h^r \|\mathbf{u}\|_{(H^{r+1}(\Omega))^3}. \tag{14}$$

Of crucial importance in our analysis is the fact $\mathbf{U}_h^{r,0}$ admits a discrete Helmholtz decomposition. Let

$$S_h^{r,0} = \{\phi_h \in H^1(\Omega) | \phi_h|_K \in P_r \quad \forall K \in \tau_h, \phi_h|_\Gamma = 0\}. \tag{15}$$

Then $\nabla S_h^{r,0} \subset \mathbf{U}_h^{r,0}$, and

$$\mathbf{U}_h^{r,0} = (\nabla S_h^{r,0}) \oplus (\nabla S_h^{r,0})^\perp \tag{16}$$

Next we define a space of divergence conforming finite elements. These facial elements are the Raviart-Thomas-Nédélec spaces [15]. Let

$$D_r = (P_{r-1})^3 \oplus \{p(\mathbf{x})\,\mathbf{x} \mid p \in \tilde{P}_{r-1}\},$$

then we define

$$\mathbf{V}_h^r = \{\mathbf{v} \in H(\operatorname{div}; \Omega) \mid \mathbf{v}|_K \in D_r \quad \forall K \in \tau_h\}.$$

An appropriate set of degrees of freedom for this space are the following which are defined for any function $\mathbf{v} \in (H^1(K))^3$ where $K \in \tau_h$:

$$M_f^D(\mathbf{u}) = \left\{ \int_f \mathbf{u} \cdot \mathbf{n} q \, ds \quad \forall q \in P_{r-1}(f) \quad \text{for all four faces } f \text{ of } K \right\}, \tag{17}$$

$$M_K^D(\mathbf{u}) = \left\{ \int_K \mathbf{u} \cdot \mathbf{q} \, dx \quad \forall \mathbf{q} \in (P_{r-2}(K))^3 \right\}. \tag{18}$$

These degrees of freedom are $H(\operatorname{div}; \Omega)$ conforming and unisolvent. Thus we can define an interpolation operator $\mathbf{w}_h : (H^1(\Omega))^3 \to \mathbf{V}_h^r$ by requiring that on each $K \in \tau_h$

$$M_f^D(\mathbf{u} - \mathbf{w}_h \mathbf{u}) = M_K^D(\mathbf{u} - \mathbf{w}_h \mathbf{u}) = \{0\}.$$

This interpolant satisfies the error estimate that if $\mathbf{u} \in (H^r(\Omega))^3$ then (see [15])

$$\|\mathbf{u} - \mathbf{w}_h \mathbf{u}\|_{L^2_\varepsilon(\Omega)} \leq C h^r \|\mathbf{u}\|_{(H^r(\Omega))^3}. \tag{19}$$

Furthermore, if $\mathbf{u}$ is smooth enough that both interpolants are defined, than $\mathbf{w}_h \nabla \times \mathbf{u} = \nabla \times \mathbf{r}_h \mathbf{u}$.

An important property of the space $\mathbf{U}_h^{r,0}$ and $\mathbf{V}_h^r$ is that

$$\nabla \times \mathbf{U}_h^{r,0} \subset V_h^r.$$

We define $\tilde{\mathbf{V}}_h^r = \nabla \times \mathbf{U}_h^{r,0}$ and then can define the space $\tilde{\mathbf{V}}_h^{r,\perp}$ by the orthogonal decomposition

$$\mathbf{V}_0^r = \tilde{\mathbf{V}}_h^r \oplus_{\mu^{-1}} \tilde{\mathbf{V}}_h^{r,\perp}. \tag{20}$$

where the orthogonality is with respect to the $L^2_{\mu^{-1}}(\Omega)$ inner product. This is another discrete Helmholtz decomposition. Note that by virtue of the connection between $\mathbf{w}_h$ and $\mathbf{r}_h$, if $\mathbf{u}$ is divergence free then $\mathbf{w}_h \mathbf{u}$ is divergence free.

We remark that $\tilde{\mathbf{V}}_h^r$ is a good space for approximating the magnetic flux $\mathbf{B}$ but it is usually more convenient to compute with the larger space $\mathbf{V}_h^r$. Nevertheless, we shall carry out some of our analysis in $\tilde{\mathbf{V}}_h^r$. This is possible since by (20) we can write

$$\mathbf{B}_h = \tilde{\mathbf{B}}_h + \tilde{\mathbf{B}}_h^{\perp}$$

where $\tilde{\mathbf{B}}_h \in \tilde{\mathbf{V}}_h^r$ and $\tilde{\mathbf{B}}_h^{\perp} \in \tilde{\mathbf{V}}_h^{r,\perp}$. Substituting this expansion in (9*a*) and (9*b*), and using the orthogonality properties of $\tilde{\mathbf{B}}_h$, we can see that $(\mathbf{E}_h, \tilde{\mathbf{B}}_h) \in \mathbf{U}_h^{r,0} \times \tilde{\mathbf{V}}_h^r$ satisfies

$$(\varepsilon \mathbf{E}_{h,t}, \psi_h) - (\mu^{-1} \tilde{\mathbf{B}}_h, \nabla \times \psi_h) = (\mathbf{J}, \psi_h) \quad \forall \psi_h \in \mathbf{U}_h^{r,0} \tag{21a}$$

$$(\mu^{-1} \tilde{\mathbf{B}}_{h,t}, \phi_h) + (\nabla \times \mathbf{E}_h, \mu^{-1} \phi_h) = 0 \quad \forall \phi_h \in \tilde{\mathbf{V}}_h^r . \tag{21b}$$

In the same way, we can see that

$$\tilde{\mathbf{B}}_{h,t}^{\perp} = 0 \quad \text{so that} \quad \tilde{\mathbf{B}}_h^{\perp}(t) = \tilde{\mathbf{B}}_h^{\perp}(0) \quad \text{for all} \quad t . \tag{22}$$

In view of (22) we will see that it suffices to carry out the error analysis for the approximation (21) to (8).

The error analysis of the semi-discrete and fully discrete problems rests on the use of a suitable pair of projections in to the spaces $\mathbf{U}_h^{r,0}$ and $\tilde{\mathbf{V}}_h$. Following [14] we define first $\Pi_h : H(\text{curl}\,;\Omega) \to \mathbf{U}_h^{r,0}$ by requiring that if $\mathbf{u} \in H(\text{curl}\,;\Omega)$, then $\Pi_h \mathbf{u} \in \mathbf{U}_h^{r,0}$ is the unique solution of

$$(\nabla \times \Pi_h \mathbf{u}, \psi_h)_{\mu^{-1}} = (\nabla \times \mathbf{u}, \psi_h)_{\mu^{-1}} \quad \forall \psi_h \in \tilde{\mathbf{V}}_h \tag{23a}$$

$$(\Pi_h \mathbf{u}, \nabla \phi_h) = (\mathbf{u}, \nabla \phi_h)_{\varepsilon} \quad \forall \phi_h \in S_h^{r,0} . \tag{23b}$$

(Here Π_h corresponds to the solution operator for an appropriate discrete static problem.) Existence and error estimates for projections of this type (with constant coefficients) have been studied in [15, 6, 7, 14]. Existence and uniqueness are proved by using the Babuška-Brezzi theory of mixed problems where we must use the Freidrichs type inequality proved in [9] for the continuous problem. This inequality can also be proved for the discrete problem following Nédélec's analysis [15] provided the domain and coefficients are such that the following problem of computing $\mathbf{w} \in H(\text{curl}\,;\Omega)$ such that

$$\nabla \times \mathbf{w} = \nabla \times \mathbf{f} \quad \text{in} \quad \Omega \tag{24a}$$

$$\nabla \cdot (\varepsilon \mathbf{w}) = 0 \quad \text{in} \quad \Omega \tag{24b}$$

$$\mathbf{w} \times \mathbf{n} = 0 \quad \text{on} \quad \Gamma \tag{24c}$$

has a unique solution obeying the *a priori* estimate

$$\|\mathbf{w}\|_{W^{1,s}(\Omega)} \leqslant C \|\nabla \times \mathbf{f}\|_{(L^s(\Omega))^3} \tag{25}$$

for $2 \leqslant s \leqslant s_0$ where $s_0 > 2$. In this case the constant coefficient results also hold for the variable coefficient problem. Our assumptions on ε and μ (continuous differentiability) are sufficient for (25) to hold, but are probably not minimal. The result of this theory is that the following estimates hold :

$$\|\mathbf{u} - \Pi_h \mathbf{u}\|_{L^2_\varepsilon(\Omega)} \leqslant C h^r \|\mathbf{u}\|_{(H^{r+1}(\Omega))^3} . \tag{26}$$

In proving this estimate we use the estimate for the interpolant in (14). We remark that an almost optimal estimate of this type, but with an improved norm on the right hand side can be proved (see [14]) but we will not examine that case here.

We also need a projection into the space of magnetic fluxes. So we define $\tilde{P}_h : L^2_{\mu^{-1}}(\Omega) \to \tilde{\mathbf{V}}^r_h$ to be the $L^2_{\mu^{-1}}(\Omega)$ projection onto $\tilde{\mathbf{V}}^r_h$ so that for any $\mathbf{v} \in L^2_{\mu^{-1}}(\Omega)$, the function $\tilde{P}_h \mathbf{v} \in \tilde{\mathbf{V}}^r_h$ satisfies

$$(\mu^{-1} \tilde{P}_h \mathbf{v}, \boldsymbol{\psi}_h) = (\mu^{-1} \mathbf{v}, \boldsymbol{\psi}_h) \quad \forall \boldsymbol{\psi}_h \in \tilde{\mathbf{V}}^r_h . \tag{27}$$

We are only interested in the approximation properties of $\tilde{P}_h$ on divergence free vector fields so we shall only analyze this case. Let $\tilde{\mathbf{V}} = \nabla \times H_0(\text{curl} ; \Omega)$ then if $\mathbf{v} \in \tilde{\mathbf{V}}$ there is a function $\mathbf{u} \in H_0(\text{curl} ; \Omega)$ such that $\mathbf{v} = \nabla \times \mathbf{u}$. In view of the fact that $\tilde{P}_h \mathbf{v} \in \tilde{\mathbf{V}}^r_h$, we know that there is a function $\mathbf{u}_h \in \mathbf{U}^{r,0}_h$ such that $\tilde{P}_h \mathbf{v} = \nabla \times \mathbf{u}_h$, we conclude that (27) may be rewritten as

$$(\nabla \times \mathbf{u}_h, \boldsymbol{\psi}_h)_{\mu^{-1}} = (\nabla \times \mathbf{u}, \boldsymbol{\psi}_h)_{\mu^{-1}} \quad \forall \boldsymbol{\psi}_h \in \tilde{\mathbf{V}}^r_h$$

which is exactly (23a). Thus if we make the arbitrary choice that $\mathbf{u}_h$ will also satisfy the divergence condition (23b), we may conclude that if $\mathbf{v} \in \tilde{\mathbf{V}}$, then

$$\tilde{P}_h \mathbf{v} = \nabla \times \Pi_h \mathbf{u} .$$

By virtue of the fact that $\mathbf{w}_h$ maps $\tilde{\mathbf{V}} \cap (H^r(\Omega))^3$ into $\tilde{\mathbf{V}}_h^r$, we may estimate the error in the projection $\tilde{P}_h$ using (19) as follows

$$\|\mathbf{v} - \tilde{P}_h \mathbf{v}\|_{(L^2(\Omega))^3} \leq Ch^r \|\mathbf{v}\|_{(H^r(\Omega))^3} . \tag{28}$$

3. ERROR ESTIMATES FOR THE SEMI-DISCRETE PROBLEM

It will prove convenient (when we come to analyze the fully discrete scheme) to convert the variational equations (9) to operator form. For this we introduce two discrete operators C and $\tilde{C}$ as follows :
Let $C : H(\text{curl}\,;\Omega) \to V_h^r$ be defined by

$$(C\mathbf{u}, \boldsymbol{\phi}_h)_{\mu^{-1}} = (\nabla \times \mathbf{u}, \boldsymbol{\phi}_h)_{\mu^{-1}} \quad \forall \boldsymbol{\phi}_h \in \mathbf{V}_h^r . \tag{29}$$

Since $\nabla \times \mathbf{U}_0^{r,0} \subset \mathbf{V}_h^r$, we have that

$$C\mathbf{u}_h = \nabla \times \mathbf{u}_h \quad \text{if} \quad \mathbf{u}_h \in \mathbf{U}_h^{r,0} , \tag{30}$$

and so

$$C\boldsymbol{\psi}_h \in \tilde{\mathbf{V}}_h^r, \quad \forall \boldsymbol{\psi}_h \in \mathbf{U}_h^{r,0} . \tag{31}$$

Let $\tilde{C} : L^2_{\mu^{-1}}(\Omega) \to \mathbf{U}_h^{r,0}$ be the operator defined as follows : for $\mathbf{v} \in L^2_{\mu^{-1}}(\Omega)$, the function $\tilde{C}\mathbf{v} \in \mathbf{U}_h^{r,0}$ is the solution of the variational problem

$$(\tilde{C}\mathbf{v}, \boldsymbol{\phi}_h) = (\mathbf{v}, \nabla \times \boldsymbol{\phi}_h)_{\mu^{-1}} \quad \forall \boldsymbol{\phi}_h \in \mathbf{U}_h^{r,0} . \tag{32}$$

Note that we have used the standard $(L^2(\Omega))^3$ inner product on the left hand side here. Using the weighted inner-product notation as above we may write (9*b*) as

$$(\mathbf{B}_{h,t}, \boldsymbol{\psi}_h)_{\mu^{-1}} + (\nabla \times \mathbf{E}_h, \boldsymbol{\psi}_h)_{\mu^{-1}} = 0 \quad \forall \boldsymbol{\psi}_h \in \mathbf{V}_h^r .$$

Hence using (29) and (30), we obtain the operator equation

$$\mathbf{B}_{h,t} + C\mathbf{E}_h = 0 . \tag{33}$$

To obtain an operator form for (9*a*) we need two more projection operators. Let P_h be the standard $(L^2(\Omega))^3$ orthogonal projection onto $\mathbf{U}_h^{r,0}$. We define $\mathbf{A}_h : \mathbf{U}_h^{r,0} \to \mathbf{U}_h^{r,0}$ by

$$\mathbf{A}_h\, \mathbf{u}_h = P_h(\varepsilon \mathbf{u}_h) , \quad \text{where} \quad \mathbf{u}_h \in \mathbf{U}_h^{r,0} .$$

(Note here that we can extend $\mathbf{A}_h : (L^2_\varepsilon(\Omega))^3 \to \mathbf{U}_h^{r,0}$ as $\mathbf{A}_h\ \mathbf{u} = P_h(\varepsilon \mathbf{u})$ for $\mathbf{u} \in (L^2_\varepsilon(\Omega))^3$.) Using $\mathbf{A}_h$ and the curl operator $\tilde{C}$, equation (9*a*) can be written as

$$\mathbf{A}_h\ \mathbf{E}_{h,t} - \tilde{C}\mathbf{B}_h = P_h\,\mathbf{J} \tag{34}$$

and for simplicity we define $\mathbf{J}_h = P_h\,\mathbf{J}$.

From the above analysis, we conclude that the semidiscrete problem (9) can be written as :

$$\mathbf{A}_h\ \mathbf{E}_{h,t} - \tilde{C}\mathbf{B}_h = \mathbf{J}_h\ , \tag{35a}$$

$$\mathbf{B}_{h,t} + C\mathbf{E}_h = 0\ , \tag{35b}$$

where $\mathbf{E}_h(t) \in \mathbf{U}_h^{r,0}$ and $\mathbf{B}_h(t) \in \mathbf{V}_h^r$ for each t.

Analogously, using the fact that $C\mathbf{E}_h \in \tilde{\mathbf{V}}_h^r$ and the fact that $\tilde{C}\tilde{\mathbf{B}}^{\perp}_{k=1}$ we obtain the operator form of (21) : find $(\mathbf{E}_h, \tilde{\mathbf{B}}_h) : [0, T] \to \mathbf{U}_h^{r,0} \times \tilde{\mathbf{V}}_h^r$ such that

$$\mathbf{A}_h\ \mathbf{E}_{h,t} - \tilde{C}\tilde{\mathbf{B}}_h = \mathbf{J}_h\ , \tag{36a}$$

$$\tilde{\mathbf{B}}_{h,t} + C\mathbf{E}_h = 0\ . \tag{36b}$$

To write the above equations more compactly, let the operator matrix $\mathscr{C}_h$

$$\mathscr{C}_h = \begin{pmatrix} 0 & \mathbf{A}_h^{-1}\,\tilde{C} \\ -C & 0 \end{pmatrix}. \tag{37}$$

The operator $\mathscr{C}_h$ maps $\mathbf{U}_h^{r,0} \times \mathbf{V}_h^r$ into itself, and because of the orthogonality in the definition of $\tilde{\mathbf{V}}_h^r$ (see (31)), we can conclude that $\mathscr{C}_h : \mathbf{U}_h^{r,0} \times \mathbf{V}_h^r \to \mathbf{U}_h^{r,0} \times \tilde{\mathbf{V}}_h^r$.

We may now rewrite (35) as

$$\frac{\partial}{\partial t}\begin{pmatrix} \mathbf{E}_h \\ \mathbf{B}_h \end{pmatrix} - \mathscr{C}_h \begin{pmatrix} \mathbf{E}_h \\ \mathbf{B}_h \end{pmatrix} = \begin{pmatrix} \mathbf{A}_h^{-1}\ \mathbf{J}_h \\ 0 \end{pmatrix}. \tag{38}$$

and rewrite (36) as

$$\frac{\partial}{\partial t}\begin{pmatrix} \mathbf{E}_h \\ \tilde{\mathbf{B}}_h \end{pmatrix} - \mathscr{C}_h \begin{pmatrix} \mathbf{E}_h \\ \tilde{\mathbf{B}}_h \end{pmatrix} = \begin{pmatrix} \mathbf{A}_h^{-1}\ \mathbf{J}_h \\ 0 \end{pmatrix}. \tag{39}$$

The operator $\mathscr{C}_h$ has useful properties that we describe next. In order to do this, we define the following inner product on vector functions $(\phi_i, \psi_i)^T \in L^2_\varepsilon(\Omega) \times L^2_{\mu^{-1}}(\Omega)$, $i = 1, 2$ by

$$\left(\left(\begin{pmatrix} \phi_1 \\ \psi_1 \end{pmatrix}, \begin{pmatrix} \phi_2 \\ \psi_2 \end{pmatrix}\right)\right) = (\phi_1, \phi_2)_\varepsilon + (\psi_1, \psi_2)_{\mu^{-1}}$$

with the associated norm

$$\left|\left|\left|\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right|\right|\right| = \left(\left(\left(\begin{pmatrix}\phi\\ \psi\end{pmatrix},\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right)\right)\right)^{1/2}. \tag{40}$$

In the above inner-product, $\mathscr{C}_h$ is anti-symmetric since

$$\begin{aligned}\left(\left(\mathscr{C}_h\begin{pmatrix}\phi_1\\ \psi_1\end{pmatrix},\begin{pmatrix}\phi_2\\ \psi_2\end{pmatrix}\right)\right) &= (\mathbf{A}_h^{-1}\ \tilde{C}\psi_1, \phi_2)_\varepsilon - (C\phi_1, \psi_2)_{\mu^{-1}}\\ &= (\mathbf{A}_h^{-1}\ \tilde{C}\psi_1, \varepsilon\phi_2) - (C\phi_1, \psi_2)_{\mu^{-1}}\\ &= (\mathbf{A}_h^{-1}\ \tilde{C}\psi_1, \mathbf{A}_h\ \phi_2) - (C\phi_1, \psi_2)_{\mu^{-1}}\\ &= (\tilde{C}\psi_1, \phi_2) - (C\phi_1, \psi_2)_{\mu^{-1}}.\end{aligned}$$

But from the definitions of C and $\tilde{C}$, if $\phi_h \in \mathbf{U}_h^{r,0}$ and $\psi_h \in \mathbf{V}_h^r$ we have

$$(C\phi_h, \psi_h)_{\mu^{-1}} = (\nabla\times\phi_h, \psi_h)_{\mu^{-1}} = (\phi_h, \tilde{C}\psi_h).$$

Putting the above two equations together proves the antisymmetry of $\mathscr{C}_h$:

$$\left(\left(\mathscr{C}_h\begin{pmatrix}\phi_1\\ \psi_1\end{pmatrix},\begin{pmatrix}\phi_2\\ \psi_2\end{pmatrix}\right)\right) = -\left(\left(\begin{pmatrix}\phi_1\\ \psi_1\end{pmatrix},\mathscr{C}_h\begin{pmatrix}\phi_2\\ \psi_2\end{pmatrix}\right)\right)$$

$$\forall\begin{pmatrix}\phi_1\\ \psi_1\end{pmatrix},\begin{pmatrix}\phi_2\\ \psi_2\end{pmatrix} \in \mathbf{U}_h^{r,0}\times\mathbf{V}_h^r. \tag{41}$$

It folllows that if $(\phi_h, \psi_h) \in \mathbf{U}_h^{r,0}\times\mathbf{V}_h^r$ then

$$\left(\left(\mathscr{C}_h\begin{pmatrix}\phi_h\\ \psi_h\end{pmatrix},\begin{pmatrix}\phi_h\\ \psi_h\end{pmatrix}\right)\right) = 0. \tag{42}$$

Now we shall state and prove our theorem concerning the approximation error in the semidiscrete problem (9).

THEOREM 3.1 : *Let* $(\mathbf{E}, \mathbf{B})$ *a solution of* (8) *such that*

$$\mathbf{E} \in C(0, T : (H^{r+1}(\Omega))^3) \quad \textit{and} \quad \mathbf{B} \in C(0, T : L^2_{\mu^{-1}}(\Omega))$$

and let $(\mathbf{E}_h(t), \mathbf{B}_h(t)) \in \mathbf{U}_h^{r,0} \times \mathbf{V}_h^r$ *be a solution of* (9) *then the following error estimate holds with constant C independent of h, T and* $\mathbf{J}$ *for* $0 \leqslant t \leqslant T$:

$$\|(\mathbf{E}-\mathbf{E}_h)(t)\|_{L^2_\varepsilon(\Omega)} + \|(\mathbf{B}-\mathbf{B}_h)(t)\|_{L^2_{\mu^{-1}}(\Omega)}$$

$$\leqslant C(\|(\mathbf{E}-\Pi_h\mathbf{E})(t)\|_{L^2_\varepsilon(\Omega)} + \|(\mathbf{B}-\tilde{P}_h\mathbf{B})(t)\|_{L^2_{\mu^{-1}}(\Omega)}$$

$$+ \|(\mathbf{E}_h - \Pi_h\mathbf{E})(0)\|_{L^2_\varepsilon(\Omega)} + \|(\mathbf{B}_h - \tilde{P}_h\mathbf{B})(0)\|_{L^2_{\mu^{-1}}(\Omega)} +$$

$$+ h^r \int_0^t \|\mathbf{E}_t(s)\|_{(H^{r+1}(\Omega))^3}\, ds) . \quad (43)$$

Remark : This theorem also holds for the more general problem of approximating the solutions $\mathbf{E}$ and $\mathbf{B}$ of

$$\varepsilon\mathbf{E}_t + \sigma\mathbf{E} - \nabla\times(\mu^{-1}\mathbf{B}) = \mathbf{J} ,$$

$$\mathbf{B}_t + \nabla\times\mathbf{E} = 0 ,$$

using an obvious extension of (9) provided the necessary smoothness of $\mathbf{E}$ is present (here the conductivity σ is a non-negative function of $\mathbf{x}$).

To prove this theorem, we first prove that (43) holds for (21) and then show that this implies (43) in general. To derive this intermediate result we shall compare the solution of (21) with the couple (ξ, ζ) defined by

$$\begin{pmatrix} \xi(t) \\ \zeta(t) \end{pmatrix} = \begin{pmatrix} \Pi_h\mathbf{E}(t) \\ \tilde{P}_h\mathbf{B}(t) \end{pmatrix} \quad (44)$$

where Π_h and $\tilde{P}_h$ are defined in (23) and (27) respectively. Our first lemma derives the equation satisfied by (ξ, ζ).

LEMMA 3.2 : *Under the hypotheses of Theorem* 3.1 *the pair* (ξ, ζ) *satisfies*

$$\frac{\partial}{\partial t}\begin{pmatrix} \xi \\ \zeta \end{pmatrix} - \mathscr{C}_h\begin{pmatrix} \xi \\ \zeta \end{pmatrix} = \begin{pmatrix} \mathbf{A}_h^{-1}\mathbf{J}_h \\ 0 \end{pmatrix} - \begin{pmatrix} P_{h,\varepsilon}\mathbf{E}_t - \xi_t \\ 0 \end{pmatrix} \quad (45)$$

where $P_{h,\varepsilon}$ *is the* $L^2_\varepsilon(\Omega)$ *orthogonal projection onto* $\mathbf{U}_h^{r,0}$.

Proof: We prove (45) component by component, starting with the lower equation. Using (8*b*), (23), (27) and (29) we have that for any $\boldsymbol{\psi}_h \in \tilde{\mathbf{V}}_h^r$

$$\begin{aligned} 0 &= (\mathbf{B}_t, \boldsymbol{\psi}_h)_{\mu^{-1}} + (\nabla \times \mathbf{E}, \boldsymbol{\psi}_h)_{\mu^{-1}} \\ &= (\tilde{P}_h \mathbf{B}_t, \boldsymbol{\psi}_h)_{\mu^{-1}} + (\nabla \times \Pi_h \mathbf{E}, \boldsymbol{\psi}_h)_{\mu^{-1}} \\ &= (\tilde{P}_h \mathbf{B}_t, \boldsymbol{\psi}_h)_{\mu^{-1}} + (C\Pi_h \mathbf{E}, \boldsymbol{\psi}_h)_{\mu^{-1}} \\ &= (\tilde{P}_h \mathbf{B}_t, \boldsymbol{\psi}_h)_{\mu^{-1}} + (C\xi, \boldsymbol{\psi}_h)_{\mu^{-1}} . \end{aligned}$$

Now since $\tilde{P}_h$ commutes with time differentiation $\tilde{P}_h \mathbf{B}_t = \zeta_t$ and from (30) we see that $C\xi \in \tilde{\mathbf{V}}_h^r$. Thus we have proved that

$$\zeta_t + C\xi = 0 . \tag{46}$$

Next we analyze the first equation in (45). For all $\boldsymbol{\phi}_h \in \mathbf{U}_h^{r,0}$ we have (using the fact that $\nabla \times \boldsymbol{\phi}_h \in \tilde{\mathbf{V}}_h^r$) that

$$(\tilde{C}\zeta, \boldsymbol{\phi}_h) = (\zeta, \nabla \times \boldsymbol{\phi}_h)_{\mu^{-1}} = (P_h \mathbf{B}, \nabla \times \boldsymbol{\phi}_h)_{\mu^{-1}} = (\mathbf{B}, \nabla \times \boldsymbol{\phi}_h)_{\mu^{-1}} = \\ = (\tilde{C}\mathbf{B}, \boldsymbol{\phi}_h) ,$$

so that

$$\tilde{C}\zeta = \tilde{C}\mathbf{B} . \tag{47}$$

Furthermore, using $\boldsymbol{\phi} = \boldsymbol{\phi}_h \in \mathbf{U}_h^{r,0}$ in (8*a*) shows that $P_h(\varepsilon \mathbf{E}_t) - \tilde{C}\mathbf{B} = P_h \mathbf{J}$.

Using the above results,

$$\begin{aligned} \mathbf{A}_h \xi_t - \tilde{C}\zeta &= \mathbf{A}_h \xi_t - \tilde{C}\mathbf{B} = \mathbf{A}_h \xi_t + P_h \mathbf{J} - P_h(\varepsilon \mathbf{E}_t) \\ &= P_h \mathbf{J} + \mathbf{A}_h (\xi_t - \mathbf{E}_t) , \end{aligned}$$

where in the last equality we have used the definition of $\mathbf{A}_h$. We will show below that

$$\mathbf{A}_h^{-1} \mathbf{A}_h = P_{h,\varepsilon} \tag{48}$$

where $P_{h,\varepsilon}$ is the $L^2_\varepsilon(\Omega)$ projection onto $\mathbf{U}_h^{r,0}$. Using (48) we have

$$\begin{aligned} \xi_t - \mathbf{A}_h^{-1} \tilde{C}\zeta &= \mathbf{A}_h^{-1} P_h \mathbf{J} + \mathbf{A}_h^{-1} A_h (\xi_t - \mathbf{E}_t) \\ &= \mathbf{A}_h \mathbf{J}_h + (\xi_t - P_{h,\varepsilon} \mathbf{E}_t) . \end{aligned} \tag{49}$$

Taken together equations (46) and (49) give (45).

It remains to prove (48). Note first that $\mathbf{A}_h|_{\mathbf{U}_h^{r,0}}$ is symmetric and positive definite and therefore its inverse in $\mathbf{U}_h^{r,0}$, $\mathbf{A}_h^{-1}$, exists and is symmetric and positive definite too. If $\mathbf{u} \in (L^2(\Omega))^3$ and $\boldsymbol{\phi}_h \in \mathbf{U}_h^{r,0}$, the properties of the various projections show that

$$\begin{aligned}(\mathbf{A}_h^{-1}\ \mathbf{A}_h\ \mathbf{u}, \boldsymbol{\phi}_h) &= (\mathbf{A}_h\ \mathbf{u}, \mathbf{A}_h^{-1}\ \boldsymbol{\phi}_h) = (P_h(\varepsilon \mathbf{u}), \mathbf{A}_h^{-1}\ \phi_h)\\ &= (\varepsilon \mathbf{u}, \mathbf{A}_h^{-1}\ \boldsymbol{\phi}_h) = (\varepsilon P_{h,\varepsilon}\mathbf{u}, \mathbf{A}_h^{-1}\ \boldsymbol{\phi}_h)\\ &= (P_h(\varepsilon P_{h,\varepsilon}\mathbf{u}), \mathbf{A}_h^{-1}\ \boldsymbol{\phi}_h) = (\mathbf{A}_h\ P_{h,\varepsilon}\mathbf{u}, \mathbf{A}_h^{-1}\ \boldsymbol{\phi}_h)\\ &= (P_{h,\varepsilon}\mathbf{u}, \boldsymbol{\phi}_h)\,.\end{aligned}$$

Thus (48) is proved. □

LEMMA 3.3 : *Under the hypotheses of Theorem* 3.1, *let* $(\mathbf{E}, \mathbf{B})$ *be a sufficiently smooth solution of* (8) *and let* $(\mathbf{E}_h, \tilde{\mathbf{B}}_h) \in \mathbf{U}_h^{r,0} \times \tilde{\mathbf{V}}_h^r$ *be a solution of* (21) *then the following error estimate holds with constant* C *independent of* h, T *and* $\mathbf{J}$:

$$\begin{aligned}&\|(\mathbf{E} - \mathbf{E}_h)(t)\|_{L^2_\varepsilon(\Omega)} + \|(\mathbf{B} - \tilde{\mathbf{B}}_h)(t)\|_{L^2_{\mu^{-1}}(\Omega)}\\ &\leqslant C\big(\|(\mathbf{E} - \Pi_h\mathbf{E})(t)\|_{L^2_\varepsilon(\Omega)} + \|(\mathbf{B} - \tilde{P}_h\mathbf{B})(t)\|_{L^2_{\mu^{-1}}(\Omega)}\\ &\quad + \|(\mathbf{E}_h - \Pi_h\mathbf{E})(0)\|_{L^2_\varepsilon(\Omega)} + \|(\tilde{\mathbf{B}}_h - \tilde{P}_h\mathbf{B})(0)\|_{L^2_{\mu^{-1}}(\Omega)} +\\ &\quad + h^r\int_0^t \|\mathbf{E}_t(s)\|_{(H^{r+1}(\Omega))^3}\,ds\big)\,.\end{aligned} \tag{50}$$

Proof: Subtracting (45) from (39) we have that if

$$\tilde{\boldsymbol{\theta}} = \begin{pmatrix}\mathbf{E}_h - \xi\\ \tilde{\mathbf{B}}_h - \zeta\end{pmatrix}$$

then

$$\tilde{\boldsymbol{\theta}}_t - \mathscr{C}_h\,\tilde{\boldsymbol{\theta}} = \begin{pmatrix}P_{h,\varepsilon}\mathbf{E}_t - \xi_t\\ 0\end{pmatrix}. \tag{51}$$

Now let $\mathscr{E}_t$ denote the solution operator for the problem of computing $\mathbf{W}_h(t) \in \mathbf{U}_h^{r,0} \times \tilde{\mathbf{V}}_h^r$ such that

$$\frac{\partial}{\partial t}\mathbf{W}_h - \mathscr{C}_h\,\mathbf{W}_h = 0\,, \quad \text{and} \quad \mathbf{W}_h(0) = \mathbf{W}_h^0 \quad \text{with} \quad \mathbf{W}_h^0 \in \mathbf{U}_h^{r,0} \times \tilde{\mathbf{V}}_h^r\,.$$

Thus $\mathbf{W}_h(t) = \mathscr{E}_t \mathbf{W}_h^0$. By virtue of (42) and a standard energy argument we have that

$$|||\mathbf{W}_t(t)||| = |||\mathscr{E}_t \mathbf{W}_h^0||| = |||\mathbf{W}_h^0|||.$$

Duhamel's principle applied to (51) gives us that

$$\tilde{\boldsymbol{\theta}}(t) = \mathscr{E}_t \tilde{\boldsymbol{\theta}}(0) + \int_0^t \mathscr{E}_{t-s} \begin{pmatrix} P_{h,\varepsilon} \mathbf{E}_t - \xi_t \\ 0 \end{pmatrix} ds,$$

and hence

$$|||\tilde{\boldsymbol{\theta}}(t)||| \leqslant |||\tilde{\boldsymbol{\theta}}(0)||| + \int_0^t \left|\left|\left| \begin{pmatrix} P_{h,\varepsilon} \mathbf{E}_t - \xi_t \\ 0 \end{pmatrix} \right|\right|\right| ds. \tag{52}$$

But by (26)

$$\begin{aligned} \left|\left|\left| \begin{pmatrix} P_{h,\varepsilon} \mathbf{E}_t - \xi_t \\ 0 \end{pmatrix} \right|\right|\right| &= \| P_{h,\varepsilon} \mathbf{E}_t - \xi_t \|_{L^2_\varepsilon(\Omega)} \\ &\leqslant \| \mathbf{E}_t - \Pi_h \mathbf{E}_t \|_{L^2_\varepsilon(\Omega)} \\ &\leqslant C \| \mathbf{E}_t - \Pi_h \mathbf{E}_t \|_0 \leqslant C h^r \| \mathbf{E}_t \|_{(H^{r+1}(\Omega))^3}. \end{aligned}$$

Thus

$$|||\tilde{\boldsymbol{\theta}}(t)||| \leqslant |||\tilde{\boldsymbol{\theta}}(0)||| + C h^r \int_0^t \| \mathbf{E}_t(s) \|_{(H^{r+1}(\Omega))^3} ds \tag{53}$$

and after standard manipulations we have completed the proof of this lemma. □

Proof [of Theorem 3.1] : We recall that $\mathbf{B}_h(t) = \tilde{\mathbf{B}}_h(t) + \tilde{\mathbf{B}}_h^\perp(t)$ where $\tilde{\mathbf{B}}_h(t) \in \tilde{\mathbf{V}}_h^r$ and $\tilde{\mathbf{B}}_h^\perp(t) \in \tilde{\mathbf{V}}_h^{r,\perp}$. Further we have shown that $(\partial/\partial t)\, \tilde{\mathbf{B}}_h^\perp(t) = 0$. But since $\tilde{\mathbf{B}}_h^\perp(t) \in \tilde{\mathbf{V}}_h^{r,\perp}$ we have

$$\mathscr{C}_h \begin{pmatrix} 0 \\ \tilde{\mathbf{B}}_h^\perp \end{pmatrix} = \begin{pmatrix} \mathbf{A}_h^{-1} \tilde{C} \tilde{\mathbf{B}}_h^\perp \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore we have that

$$\frac{\partial}{\partial t} \begin{pmatrix} 0 \\ \tilde{\mathbf{B}}_h^\perp \end{pmatrix} + \mathscr{C}_h \begin{pmatrix} 0 \\ \tilde{\mathbf{B}}_h^\perp \end{pmatrix} = 0. \tag{54}$$

Now let

$$\boldsymbol{\theta} = \begin{pmatrix} \mathbf{E}_h - \xi \\ \mathbf{B}_h - \zeta \end{pmatrix} \quad \text{so that} \quad \boldsymbol{\theta} = \tilde{\boldsymbol{\theta}} + \begin{pmatrix} 0 \\ \tilde{\mathbf{B}}_h^\perp \end{pmatrix}.$$

Equations (51) and (54) give

$$\frac{\partial}{\partial t}\boldsymbol{\theta} + \mathscr{C}_h \boldsymbol{\theta} = \begin{pmatrix} P_{h,\varepsilon}\mathbf{E}_t - \xi_t \\ 0 \end{pmatrix}.$$

The same argument as in the proof of the last lemma, *via* Duhammel's principle, shows that $\boldsymbol{\theta}$ satisfies (52) so that (53) holds for $\boldsymbol{\theta}$. Standard manipulations now prove the desired estimate. □

4. FULLY DISCRETE SCHEMES

In this section we shall describe how to discretize the semi-discrete problem (9) in time. It will prove convenient to work from the semi-discrete equations in operator form given by (38).

To discretize (38) we use a rational approximation of the exponential as in [13]. Let

$$r(z) = \frac{p(z)}{q(z)} \quad \text{where} \quad p(z) = 1 + p_1 z + p_2 z^2 \quad \text{and} \quad q(z) = 1 + q_1 z + q_2 z^2$$

where p_1, p_2, q_1 and q_2 are constants chosen so that firstly there exists a ν with $1 \leqslant \nu \leqslant 4$ with

$$|r(z) - e^z| \leqslant c|z|^{\nu+1}$$

for all z in a neighborhood of zero and $z \in \mathbb{C}$ and secondly such that

$$|r(z)| \leqslant 1 \quad \text{for all} \quad z \in i\mathbb{R}. \tag{55}$$

By the correct choice of p and q we can obtain a number of interesting schemes [13]

Method	ν	q_1	q_2	p_1	p_2
Backward Euler	1	− 1	0	0	0
Crank-Nicolson	2	− 1/2	0	1/2	0
Padé	3	− 2/3	1/6	1/3	0
Padé	4	− 1/2	1/12	1/2	1/12

As a result of these assumptions, we know that if $y(t)$ is $\nu+1$ times differentiable and $k>0$ then

$$y(t+k)+q_1\,ky'(t+k)+q_2\,k^2\,y''(t+k)$$
$$=y(t)+p_1\,ky'(t)+p_2\,k^2\,y''(t)+O(k^{\nu+1}\,y^{(\nu+1)})\,. \quad (56)$$

Now (38) implies that

$$\frac{\partial^2}{\partial t^2}\begin{pmatrix}\mathbf{E}_h\\ \mathbf{B}_h\end{pmatrix}=\mathscr{C}_h\begin{pmatrix}\mathbf{E}_{h,t}\\ \mathbf{B}_{h,t}\end{pmatrix}+\begin{pmatrix}\mathbf{A}_h^{-1}\,\mathbf{J}_{h,t}\\ 0\end{pmatrix}$$
$$=\mathscr{C}_h^2\begin{pmatrix}\mathbf{E}_h\\ \mathbf{B}_h\end{pmatrix}+\mathscr{C}_h\begin{pmatrix}\mathbf{A}_h^{-1}\,\mathbf{J}_h\\ 0\end{pmatrix}+\begin{pmatrix}\mathbf{A}_h^{-1}\,\mathbf{J}_{h,t}\\ 0\end{pmatrix}$$
$$=\begin{pmatrix}-\mathbf{A}_h^{-1}\,\tilde{C}C & 0\\ 0 & -C\mathbf{A}_h^{-1}\,\tilde{C}\end{pmatrix}\begin{pmatrix}\mathbf{E}_h\\ \mathbf{B}_h\end{pmatrix}+\begin{pmatrix}\mathbf{A}_h^{-1}\,\mathbf{J}_{h,t}\\ -C\mathbf{A}_h^{-1}\,\mathbf{J}_h\end{pmatrix}. \quad (57)$$

Motivated by (56) and using (57) to replace second derivative terms, we arrive at the a fully discrete approximation to (8). Let $(\mathbf{E}_h^n,\mathbf{B}_h^n)\in\mathbf{U}_h^{r,0}\times\mathbf{V}_h^r$ be a given approximation to $(\mathbf{E}(t_n),\mathbf{B}(t_n))$ where $t_n=nk$. We define $(\mathbf{E}_h^{n+1},\mathbf{B}_h^{n+1})$ as the solution of

$$\mathscr{R}\begin{pmatrix}\mathbf{E}_h^{n+1}\\ \mathbf{B}_h^{n+1}\end{pmatrix}=\mathscr{S}\begin{pmatrix}\mathbf{E}_h^n\\ \mathbf{B}_h^n\end{pmatrix}+\mathscr{F}^n \quad (58)$$

where

$$\mathscr{R}=q(k\mathscr{C}_h)=\begin{pmatrix}I-k^2\,q_2\,\mathbf{A}_h^{-1}\,\tilde{C}C & kq_1\,\mathbf{A}_h^{-1}\,\tilde{C}\\ -kq_1\,C & I-k^2\,q_2\,C\mathbf{A}_h^{-1}\,\tilde{C}\end{pmatrix} \quad (59a)$$

$$\mathscr{S}=p(k\mathscr{C}_h)=\begin{pmatrix}I-k^2\,p_2\,\mathbf{A}_h^{-1}\,\tilde{C}C & kp_1\,\mathbf{A}_h^{-1}\,\tilde{C}\\ -kp_1\,C & I-k^2\,p_2\,C\mathbf{A}_h^{-1}\,\tilde{C}\end{pmatrix} \quad (59b)$$

and

$$\mathscr{F}^n=-\begin{pmatrix}k\mathbf{A}_h^{-1}\,(q_1\,\mathbf{J}_h^{n+1}-p_1\,\mathbf{J}_h^n)+k^2\,\mathbf{A}_h^{-1}\,(q_2\,\mathbf{J}_{h,t}^{n+1}-p_2\,\mathbf{J}_{h,t}^n)\\ -k^2\,C\mathbf{A}_h^{-1}\,(q_2\,\mathbf{J}_h^{n+1}-p_2\,\mathbf{J}_h^n)\end{pmatrix} \quad (60)$$

Remark : Our error analysis covers explicit schemes too. In particular, the scheme (58) is explicit if $q_1 = q_2 = 0$. In this case (55) should be replaced by $|r(z)| \leq 1$, $z \in (-i\alpha, i\alpha), \alpha > 0$. Then the stability of the scheme in this case will be ensured provided that kh^{-1} remains small.

5. ERROR ANALYSIS OF FULLY DISCRETE SCHEMES

This section is devoted to proving the following theorem which gives convergence results for the fully discrete method. We also will state and prove a corollary of this theorem that covers the practically useful case.

THEOREM 5.1 : *Suppose that* **E** *and* **B** *are solutions of* (8) *with the regularity :*

$$\mathbf{E} \in C^1(0, T : (H^{r+1}(\Omega))^3) \cap C^{\nu+1}(0, T : L^2_\varepsilon(\Omega))$$

$$\mathbf{B} \in C(0, T : H(\operatorname{div}; \Omega)) \cap C^{\nu+1}(0, T : L^2_{\mu^{-1}}(\Omega)) .$$

Then if

$$\mathbf{\Theta}^n = \begin{pmatrix} \mathbf{E}_h^n - \Pi_h \mathbf{E}(t_n) \\ \mathbf{B}_h^n - \tilde{P}_h \mathbf{B}(t_n) \end{pmatrix}$$

there is a constant C independent of k and h such that for $0 \leq t_n \leq T$ *the following estimate holds where* $|||\cdot|||$ *is the norm defined in* (40) :

$$|||\mathbf{\Theta}^n||| \leq C \Bigg\{ |||\mathbf{\Theta}^0||| + k|||\mathscr{C}_h \mathbf{\Theta}^0||| + k^2 |||\mathscr{C}_h^2 \mathbf{\Theta}^0||| +$$

$$+ t_n \Bigg((h^r + kh^{r-1}) \sup_{0 \leq s \leq t_n} \|\mathbf{E}_t(s)\|_{(H^{r+1}(\Omega))^3} +$$

$$+ t_n k^\nu \Bigg(\sup_{s \in [0, t_n]} \left\| \frac{\partial^{\nu+1} \mathbf{E}}{\partial t^{\nu+1}}(s) \right\|_{L^2_\varepsilon(\Omega)} + \sup_{s \in [0, t_n]} \left\| \frac{\partial^{\nu+1} \mathbf{B}}{\partial t^{\nu+1}}(s) \right\|_{L^2_{\mu^{-1}}(\Omega)} \Bigg) \Bigg) \Bigg\} . \tag{61}$$

Remark : If ε is constant, the term kh^{r-1} does not appear in the above estimate. For general ε, if we take $k = O(h)$, the bound in the theorem will be of order $O(h^r + k^\nu)$.

The next corollary summarizes the results in a particularly practical case :

COROLLARY 5.2 : *Suppose the conditions of Theorem* 5.1 *hold. In addition suppose that* $\mathbf{E}_h^0 = \mathbf{r}_h \mathbf{E}(0)$ *and* $\mathbf{B}_h^0 = \mathbf{w}_h \mathbf{B}(0)$. *Then* $\mathbf{B}_h^n \in \tilde{\mathbf{V}}_h^r$ *for each n. In addition the following error estimate holds for* $0 \leq t_n \leq T$ *provided* **E** *and* **B** *are smooth enough :*

$$\|\mathbf{E}(t_n) - \mathbf{E}_h^n\|_{L^2_\varepsilon(\Omega)} + \|\mathbf{B}(t_n) - \mathbf{B}_h^n\|_{L^2_{\mu^{-1}}(\Omega)} \leqslant$$

$$\leqslant C\Big\{ (h^r + kh^{r-1} + k^2 h^{r-2})(\|\mathbf{E}(0)\|_{(H^{r+1}(\Omega))^3} + \|\mathbf{B}(0)\|_{(H^r(\Omega))^3}) +$$

$$+ h^r(\|\mathbf{E}(t_n)\|_{(H^{r+1}(\Omega))^3} + \|\mathbf{B}(t_n)\|_{(H^r(\Omega))^3}) +$$

$$+ t_n\Big((h^r + kh^{r-1}) \sup_{0 \leqslant s \leqslant t_n} \|\mathbf{E}_t(s)\|_{(H^{r+1}(\Omega))^3} +$$

$$+ t_n k^\nu\Big(\sup_{s \in [0, t_n]} \Big\|\frac{\partial^{\nu+1}\mathbf{E}}{\partial t^{\nu+1}}(s)\Big\|_{L^2_\varepsilon(\Omega)} + \sup_{s \in [0, t_n]} \Big\|\frac{\partial^{\nu+1}\mathbf{B}}{\partial t^{\nu+1}}(s)\Big\|_{L^2_{\mu^{-1}}(\Omega)}\Big)\Big)\Big\} .$$

Remark : This case, in which we simply interpolate the initial data, is easy to use in practice. Furthermore, the magnetic flux is exactly divergence free at each time step.

To avoid the term $(h^r + kh^{r-1} + k^2 h^{r-2})$ in the error estimate we could choose $\mathbf{B}_h^0 = \tilde{P}_h \mathbf{B}(0)$ and $\mathbf{E}_h^0 = \Pi_h \mathbf{E}(0)$ but this choice is very costly to implement.

Proof [of Corollary 5.2] : Since $\mathbf{B}(0) \in \tilde{\mathbf{V}}$, we know that $\mathbf{w}_h\ \mathbf{B}(0) \in \tilde{\mathbf{V}}_h^r$ and so is divergence free. Now we show that since $\mathbf{B}_h^0 \in \tilde{\mathbf{V}}_h^r$ then $\mathbf{B}_h^n \in \tilde{\mathbf{V}}_h^r$ for each n. As we have seen if $\mathbf{B}_h = \tilde{\mathbf{B}}_h + \tilde{\mathbf{B}}_h^\perp$ with $\tilde{\mathbf{B}}_h \in \tilde{\mathbf{V}}_h^{r,\perp}$ then $\mathscr{C}_h(0, \tilde{\mathbf{B}}_h^\perp)^T = 0$ or

$$\mathscr{C}_h\begin{pmatrix} 0 \\ \mathbf{B}_h \end{pmatrix} = \mathscr{C}_h\begin{pmatrix} 0 \\ \tilde{\mathbf{B}}_h \end{pmatrix} . \tag{62}$$

Therefore, if we write

$$\begin{pmatrix} \mathbf{E}_h^n \\ \mathbf{B}_h^n \end{pmatrix} = \begin{pmatrix} \mathbf{E}_h^n \\ \tilde{\mathbf{B}}_h^n \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \tilde{\mathbf{B}}_h^{n,\perp} \end{pmatrix}$$

for each n where $\tilde{\mathbf{B}}_h^n \in \tilde{\mathbf{V}}_h^r$ and $\tilde{\mathbf{B}}_h^{n,\perp} \in \tilde{\mathbf{V}}_h^{r,\perp}$ then we have the following equations :

$$\mathscr{C}_h\begin{pmatrix} \mathbf{E}_h^n \\ \mathbf{B}_h^n \end{pmatrix} = \mathscr{C}_h\begin{pmatrix} \mathbf{E}_h^n \\ \tilde{\mathbf{B}}_h^n \end{pmatrix} \in \mathbf{U}_h^{r,0} \times \tilde{\mathbf{V}}_h^r$$

$$\mathscr{C}_h^2\begin{pmatrix} \mathbf{E}_h^n \\ \mathbf{B}_h^n \end{pmatrix} = \mathscr{C}_h^2\begin{pmatrix} \mathbf{E}_h^n \\ \tilde{\mathbf{B}}_h^n \end{pmatrix} \in \mathbf{U}_h^{r,0} \times \tilde{\mathbf{V}}_h^r . \tag{63}$$

Also (31) implies that the second component of $\mathcal{F}^n$ is in $\tilde{\mathbf{V}}_h^r$ so

$$k^2\, C\mathbf{A}_h^{-1}\,(q_2\,\mathbf{J}_h^{n+1} - p_2\,\mathbf{J}_h^n) \in \tilde{\mathbf{V}}_h^r\,. \tag{64}$$

Using (58) and (62)-(64) we get :

$$\begin{pmatrix}\mathbf{E}_h^{n+1}\\ \tilde{\mathbf{B}}_h^{n+1}\end{pmatrix} + \begin{pmatrix}0\\ \tilde{\mathbf{B}}_h^{n+1,\perp}\end{pmatrix} + kq_1\,\mathcal{C}_h\begin{pmatrix}\mathbf{E}_h^{n+1}\\ \tilde{\mathbf{B}}_h^{n+1}\end{pmatrix} + k^2\,q_2\,\mathcal{C}_h^2\begin{pmatrix}\mathbf{E}_h^{n+1}\\ \tilde{\mathbf{B}}_h^{n+1}\end{pmatrix} =$$

$$= \begin{pmatrix}\mathbf{E}_h^{n}\\ \tilde{\mathbf{B}}_h^{n}\end{pmatrix} + \begin{pmatrix}0\\ \tilde{\mathbf{B}}_h^{n,\perp}\end{pmatrix} + kp_1\,\mathcal{C}_h\begin{pmatrix}\mathbf{E}_h^{n}\\ \tilde{\mathbf{B}}_h^{n+1}\end{pmatrix} + k^2\,p_2\,\mathcal{C}_h^2\begin{pmatrix}\mathbf{E}_h^{n}\\ \tilde{\mathbf{B}}_h^{n+1}\end{pmatrix} - \mathcal{F}^n\,. \tag{65}$$

This equation implies that

$$\tilde{\mathbf{B}}_h^{n+1,\perp} = \tilde{\mathbf{B}}_h^{n,\perp} \tag{66}$$

and proves the result. Note also that

$$\mathcal{R}\begin{pmatrix}\mathbf{E}_h^{n+1}\\ \tilde{\mathbf{B}}_h^{n+1}\end{pmatrix} = \mathcal{S}\begin{pmatrix}\mathbf{E}_h^{n}\\ \tilde{\mathbf{B}}_h^{n}\end{pmatrix} + \mathcal{F}^n \tag{67}$$

where $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{F}^n$ are given by (59)-(60).

The remainder of the corollary is proved by using (61) and estimating the errors at $t = 0$ and $t = t_n$ using (14), (19), (28) and (26) together with the fact that since the mesh is quasi-uniform we may estimate

$$|||\mathcal{C}_h\,\boldsymbol{\psi}||| \leq Ch^{-1}|||\boldsymbol{\psi}|||\,,\quad \forall \boldsymbol{\psi} \in \mathbf{U}_h^{r,0} \times \mathbf{V}_h^r\,. \qquad \square$$

We start the proof of Theorem 5.1 by proving a consistency estimate for the fully discrete scheme. To do this we let Π_h and $\tilde{P}_h$ be the projections defined in (23) and (27) and let

$$\begin{pmatrix}\xi^n\\ \zeta^n\end{pmatrix} = \begin{pmatrix}\Pi_h\,\mathbf{E}(t_n)\\ \tilde{P}_h\,\mathbf{B}(t_n)\end{pmatrix} \tag{68}$$

where **E** and **B** are the exact solutions of (8). Let us define γ^n and α^n by

$$\begin{pmatrix}\gamma^n\\ \alpha^n\end{pmatrix} = \mathcal{R}\begin{pmatrix}\xi^{n+1}\\ \zeta^{n+1}\end{pmatrix} - \mathcal{S}\begin{pmatrix}\xi^{n}\\ \zeta^{n}\end{pmatrix}. \tag{69}$$

LEMMA 5.3 : *Suppose all the conditions of Theorem* 5.1 *are satisfied. Let* (ξ^n, ζ^n) *be as defined in* (68) *and* (γ^n, α^n) *be as defined in* (69) *then we can write*

$$\begin{pmatrix}\gamma^n\\ \alpha^n\end{pmatrix} = \begin{pmatrix}\xi^{n+1} - \xi^n\\ \zeta^{n+1} - \zeta^n\end{pmatrix} + k\begin{pmatrix}\gamma_1^n\\ \alpha_1^n\end{pmatrix} + k^2\begin{pmatrix}\gamma_2^n\\ \alpha_2^n\end{pmatrix} \tag{70}$$

where

$$\begin{pmatrix} \gamma_1^n \\ \alpha_1^n \end{pmatrix} = \begin{pmatrix} P_{h,\varepsilon}(q_1 \mathbf{E}_t(t_{n+1}) - p_1 \mathbf{E}_t(t_n)) \\ (q_1 \zeta_t^{n+1} - p_1 \zeta_t^n) \end{pmatrix} - \begin{pmatrix} \mathbf{A}_h^{-1}(q_1 \mathbf{J}_h^{n+1} - p_1 \mathbf{J}_h^n) \\ 0 \end{pmatrix}. \tag{71}$$

and

$$\begin{pmatrix} \gamma_2^n \\ \alpha_2^n \end{pmatrix} = \begin{pmatrix} P_{h,\varepsilon}(q_2 \mathbf{E}_{tt}(t_{n+1}) - p_2 \mathbf{E}_{tt}(t_n)) \\ q_2 \zeta_{tt}^{n+1} - p_2 \zeta_{tt}^n \end{pmatrix} -$$

$$- \begin{pmatrix} A_h^{-1}(q_2 \mathbf{J}_t^{n+1} - p_2 \mathbf{J}_t^n) \\ C\mathbf{A}_h^{-1}(-q_2 \mathbf{J}_h^{n+1} + p_2 \mathbf{J}_h^n) \end{pmatrix} - \begin{pmatrix} 0 \\ q_2 \mathbf{W}^{n+1} - p_2 W^n \end{pmatrix}, \tag{72}$$

where $\mathbf{W}^{n+1}$ *and* $\mathbf{W}^n$ *satisfy the estimate* (80).

Proof: By expanding (69) using the definitions of $\mathcal{R}$ and $\mathcal{S}$ we may write

$$\begin{pmatrix} \gamma^n \\ \alpha^n \end{pmatrix} = \begin{pmatrix} \xi^{n+1} - \xi^n \\ \zeta^{n+1} - \zeta^n \end{pmatrix} + k \begin{pmatrix} \mathbf{A}_h^{-1} \tilde{C}(q_1 \zeta^{n+1} - p_1 \zeta^n) \\ -C(q_1 \xi^{n+1} - p_1 \xi^n) \end{pmatrix} -$$

$$- k^2 \begin{pmatrix} \mathbf{A}_h^{-1} \tilde{C}C(q_2 \xi^{n+1} - p_2 \xi^n) \\ C\mathbf{A}_h^{-1} \tilde{C}(q_2 \zeta^{n+1} - p_2 \zeta^n) \end{pmatrix}. \tag{73}$$

We study each term on the right hand side of this expansion. First we study the term multiplying k on the right hand side. So we define

$$\begin{pmatrix} \gamma_1^n \\ \alpha_1^n \end{pmatrix} = \begin{pmatrix} \mathbf{A}_h^{-1} \tilde{C}(q_1 \zeta^{n+1} - p_1 \zeta^n) \\ -C(q_1 \xi^{n+1} - p_1 \xi) \end{pmatrix}. \tag{74}$$

Using (46) we have :

$$C\xi^n = -\zeta_t^n . \tag{75}$$

Now using (47) we have that $\tilde{C}\zeta^n = \tilde{C}\mathbf{B}(t_n)$, and hence from the second equation of (8) we have that for $\psi_h \in \mathbf{U}_h^{r,0}$

$$\begin{aligned} (\tilde{C}\zeta^n, \psi_h) &= (\tilde{C}\mathbf{B}(t_n), \psi_h) = (\mathbf{B}(t_n), \nabla \times \psi_h)_{\mu^{-1}} \\ &= (\varepsilon \mathbf{E}_t(t_n), \psi_h) - (\mathbf{J}^n, \psi) \\ &= (\varepsilon P_{h,\varepsilon} \mathbf{E}_t(t_n), \psi_h) - (P_h \mathbf{J}^n, \psi) \\ &= (P_h \varepsilon P_{h,\varepsilon} \mathbf{E}_t(t_n), \psi_h) - (\mathbf{J}_h^n, \psi) \\ &= (\mathbf{A}_h P_{h,\varepsilon} \mathbf{E}_t(t_n), \psi_h) - (\mathbf{J}_h^n, \psi) \end{aligned}$$

where P_h is the $(L^2(\Omega))^3$ projection on $\mathbf{U}_h^{r,0}$ and $P_{h,\varepsilon}$ is the $L^2_\varepsilon(\Omega)$ projection on $\mathbf{U}_h^{r,0}$. Now since $P_{h,\varepsilon}\mathbf{E}_t \in \mathbf{U}_h^{r,0}$ we have

$$\mathbf{A}_h^{-1}\tilde{C}\zeta^n = P_{h,\varepsilon}\mathbf{E}_t(t_n) - \mathbf{A}_h^{-1}\mathbf{J}_h^n . \tag{76}$$

The identities (75) and (76) in (74) give (71).

Now we estimate the term multiplying k^2 in (73). To this end we define

$$\begin{pmatrix} \gamma_2^n \\ \alpha_2^n \end{pmatrix} = - \begin{pmatrix} \mathbf{A}_h^{-1}\tilde{C}C(q_2\xi^{n+1} - p_2\xi^n) \\ C\mathbf{A}_h^{-1}\tilde{C}(q_2\zeta^{n+1} - p_2\zeta) \end{pmatrix} . \tag{77}$$

By using (75) and differentiating (76) with respect to t we have

$$\mathbf{A}_h^{-1}\tilde{C}C\xi^n = -\mathbf{A}_h^{-1}\tilde{C}\zeta_t^n = -P_{h,\varepsilon}\mathbf{E}_{tt}(t_n) + \mathbf{A}_h^{-1}\mathbf{J}_{h,t}^n . \tag{78}$$

Also by (76), (29) and the definition of ζ we have

$$\begin{aligned} C\mathbf{A}_h^{-1}\tilde{C}\zeta^n &= CP_{h,\varepsilon}\mathbf{E}_t(t_n) - C\mathbf{A}_h^{-1}\mathbf{J}_h^n \\ &= C(P_{h,\varepsilon} - I)\mathbf{E}_t(t_n) + C\mathbf{E}_t(t_n) - C\mathbf{A}_h^{-1}\mathbf{J}_h^n \\ &= C(P_{h,\varepsilon} - I)\mathbf{E}_t(t_n) - \tilde{P}_h\mathbf{B}_{tt}(t_n) - C\mathbf{A}_h^{-1}\mathbf{J}_h^n \\ &= C(P_{h,\varepsilon} - I)\mathbf{E}_t(t_n) - \zeta_{tt}^n - C\mathbf{A}_h^{-1}\mathbf{J}_h^n \\ &= \mathbf{W}^n - \zeta_{tt}^n - C\mathbf{A}_h^{-1}\mathbf{J}_h^n \end{aligned} \tag{79}$$

where $\mathbf{W}^n = C(P_{h,\varepsilon} - I)\mathbf{E}_t(t_n)$ and $\mathbf{W}^n$ can be estimated as follows. Using the definition of C we have

$$\begin{aligned} \|\mathbf{W}^n\|^2_{L^2_{\mu^{-1}}(\Omega)} &= (C(P_{h,\varepsilon} - I)\mathbf{E}_t(t_n), \mathbf{W}^n)_{\mu^{-1}} \\ &= (\nabla \times (P_{h,\varepsilon} - I)\mathbf{E}_t(t_n), \mathbf{W}^n)_{\mu^{-1}} \\ &\leq \|\nabla \times (P_{h,\varepsilon} - I)\mathbf{E}_t(t_n)\|_{L^2_{\mu^{-1}}(\Omega)} \|\mathbf{W}^n\|_{L^2_{\mu^{-1}}(\Omega)} \\ &\leq C\|(P_{h,\varepsilon} - I)\mathbf{E}_t(t_n)\|_{H(\mathrm{curl}\,;\,\Omega)} \|\mathbf{W}^n\|_{L^2_{\mu^{-1}}(\Omega)} . \end{aligned}$$

So that for some positive constant C,

$$\|\mathbf{W}^n\|_{L^2_{\mu^{-1}}(\Omega)} \leq C\|(P_{h,\varepsilon} - I)\mathbf{E}_t(t_n)\|_{H(\mathrm{curl}\,;\,\Omega)} . \tag{80}$$

The equalities (78) and (79) can be combined in (77) to give (72). □

Our next lemma shows that the time stepping scheme is stable.

LEMMA 5.4 : *For any* $\mathbf{\Psi} \in \mathbf{U}_h^{r,0} \times \mathbf{V}_h^r$,

$$|||\mathcal{S}\mathbf{\Psi}||| \leq |||\mathcal{R}\mathbf{\Psi}||| . \tag{81}$$

Proof : From (42) we have that $\mathcal{C}_h$ has pure imaginary eigenvalues and since the rational function r satisfies (55) we have the desired result. □

Proof [of Theorem 5.1] : Let us recall the definition

$$\mathbf{\Theta}^n = \begin{pmatrix} \mathbf{E}_h^n \\ \mathbf{B}_h^n \end{pmatrix} - \begin{pmatrix} \xi^n \\ \zeta^n \end{pmatrix}.$$

Using (69)-(73), (74), (71), (72) and (67) we have the estimate :

$$\mathcal{R}\mathbf{\Theta}^{n+1} - \mathcal{S}\mathbf{\Theta}^n = \begin{pmatrix} (P_{h,\varepsilon} - \Pi_h)(\mathbf{E}(t_{n+1}) - \mathbf{E}(t_n)) \\ 0 \end{pmatrix} -$$

$$- \begin{pmatrix} P_\varepsilon & 0 \\ 0 & \tilde{P}_h \end{pmatrix} \left[\left(I + q_1 k \frac{\partial}{\partial t} + q_2 k^2 \frac{\partial^2}{\partial t^2} \right) \begin{pmatrix} \mathbf{E}(t_{n+1}) \\ \mathbf{B}(t_{n+1}) \end{pmatrix} - \right.$$

$$\left. - \left(I + p_1 k \frac{\partial}{\partial t} + p_2 k^2 \frac{\partial^2}{\partial t^2} \right) \begin{pmatrix} \mathbf{E}(t_n) \\ \mathbf{B}(t_n) \end{pmatrix} \right] +$$

$$+ k^2 \begin{pmatrix} 0 \\ q_2 \mathbf{W}^{n+1} - p_2 \mathbf{W}^n \end{pmatrix}. \tag{82}$$

where $\mathbf{W}^n$ is defined after (79). We define $\mathbf{D}^{n,1}$, $\mathbf{D}^{n,2}$ and $\mathbf{D}^{n,3}$ to be respectively the first second and third terms on the right hand side of (82). Each of these is estimated separately.

We estimate $\mathbf{D}^{n,1}$ first using the mean value theorem and the error estimate for Π_h in (26) :

$$\begin{aligned} |||\mathbf{D}^{n,1}||| &= \| (P_{h,\varepsilon} - \Pi_h)(\mathbf{E}(t_{n+1}) - \mathbf{E}(t_n)) \|_{L^2_\varepsilon(\Omega)} \\ &= \| P_{h,\varepsilon}(I - \Pi_h)(\mathbf{E}(t_{n+1}) - E(t_n)) \|_{L^2_\varepsilon(\Omega)} \\ &\leq \| (I - \Pi_h)(\mathbf{E}(t_{n+1}) - E(t_n)) \|_{L^2_\varepsilon(\Omega)} \\ &\leq Ck \sup_{s \in [t_n, t_{n+1}]} \| (I - \Pi_h) \mathbf{E}_t(s) \|_{L^2_\varepsilon(\Omega)} \\ &\leq Ckh^r \sup_{s \in [t_n, t_{n+1}]} \| \mathbf{E}_t(s) \|_{H^{r+1}(\Omega)} . \end{aligned} \tag{83}$$

The relation (56) gives

$$|||\mathbf{D}^{n,2}||| \leqslant Ck^{\nu+1}\left(\sup_{s\in[t_n,t_{n+1}]}\left\|\frac{\partial^{\nu+1}\mathbf{E}}{\partial t^{\nu+1}}(s)\right\|_{L^2_\varepsilon(\Omega)} + \right.$$

$$\left. + \sup_{s\in[t_n,t_{n+1}]}\left\|\frac{\partial^{\nu+1}\mathbf{B}}{\partial t^{\nu+1}}(s)\right\|_{L^2_{\mu^{-1}}(\Omega)}\right). \quad (84)$$

The estimation of $\mathbf{D}^{n,3}$ is more troublesome. Since the grid is quasi-uniform, we can easily show that

$$\|(P_{h,\varepsilon}-I)\,\psi\|_{H(\mathrm{curl}\,;\,\Omega)} \leqslant Ch^{r-1}\|\psi\|_{H^{r+1}(\Omega)}\,. \quad (85)$$

In view of the fact that the finite element interpolant approximates a function to $O(h^r)$ in the $H(\mathrm{curl}\,;\Omega)$ norm, one might hope to improve (85) with h^{r-1} replace by h^r. Using (85) we have

$$|||\mathbf{D}^{n,3}||| \leqslant Ck^2h^{r-1}\sup_{s\in[t_n,t_{n+1}]}\|\mathbf{E}_t(s)\|_{H^{r+1}(\Omega)}\,. \quad (86)$$

Now (82) and (81) and the estimates (83), (84) and (86) give :

$$|||\mathcal{R}\mathbf{\Theta}^{n+1}||| \leqslant |||\mathcal{S}\mathbf{\Theta}^n||| + Ck\left((h^r+kh^{r-1})\sup_{s\in[t_n,t_{n+1}]}\|\mathbf{E}_t(s)\|_{H^{r+1}(\Omega)} + \right.$$

$$\left. + k^{\nu+1}\left(\sup_{s\in[t_n,t_{n+1}]}\left\|\frac{\partial^{\nu+1}\mathbf{E}}{\partial t^{\nu+1}}(s)\right\|_{L^2_\varepsilon(\Omega)} + \sup_{s\in[t_n,t_{n+1}]}\left\|\frac{\partial^{\nu+1}\mathbf{B}}{\partial t^{\nu+1}}(s)\right\|_{L^2_{\mu^{-1}}(\Omega)}\right)\right)$$

$$\leqslant |||\mathcal{R}\mathbf{\Theta}^n||| + Ck\left((h^r+kh^{r-1})\sup_{s\in[t_n,t_{n+1}]}\|\mathbf{E}_t(s)\|_{H^{r+1}(\Omega)} + \right.$$

$$\left. + k^{\nu+1}\left(\sup_{s\in[t_n,t_{n+1}]}\left\|\frac{\partial^{\nu+1}\mathbf{E}}{\partial t^{\nu+1}}(s)\right\|_{L^2_\varepsilon(\Omega)} + \sup_{s\in[t_n,t_{n+1}]}\left\|\frac{\partial^{\nu+1}\mathbf{B}}{\partial t^{\nu+1}}(s)\right\|_{L^2_{\mu^{-1}}(\Omega)}\right)\right).$$

Iterating this estimate we prove that

$$|||\mathcal{R}\Theta^n||| \leqslant |||\mathcal{R}\mathbf{\Theta}^0||| + Cnk\left((h^r+kh^{r-1})\sup_{s\in[0,t_n]}\|\mathbf{E}_t(s)\|_{H^{r+1}(\Omega)} + \right.$$

$$\left. + nkk^{\nu}\left(\sup_{s\in[0,t_n]}\left\|\frac{\partial^{\nu+1}\mathbf{E}}{\partial t^{\nu+1}}(s)\right\|_{L^2_\varepsilon(\Omega)} + \sup_{s\in[0,t_n]}\left\|\frac{\partial^{\nu+1}\mathbf{B}}{\partial t^{\nu+1}}(s)\right\|_{L^2_{\mu^{-1}}(\Omega)}\right)\right)$$

$$\leqslant |||\mathcal{R}\Theta^0||| + Ct_n\left((h^r+kh^{r-1})\sup_{s\in[0,t_n]}\|\mathbf{E}_t(s)\|_{H^{r+1}(\Omega)} + \right.$$

$$\left. + t_n k^{\nu}\left(\sup_{s\in[0,t_n]}\left\|\frac{\partial^{\nu+1}\mathbf{E}}{\partial t^{\nu+1}}(s)\right\|_{L^2_\varepsilon(\Omega)} + \sup_{s\in[0,t_n]}\left\|\frac{\partial^{\nu+1}\mathbf{B}}{\partial t^{\nu+1}}(s)\right\|_{L^2_{\mu^{-1}}(\Omega)}\right)\right).$$

But using the antisymmetry of $\mathscr{C}_h$ in the $((\cdot\,,\cdot))$ inner product [13] we obtain

$$|||\mathscr{R}\Theta^n|||^2 = ((\Theta^n, \Theta^n)) + k^2(q_1^2 - 2\,q_2)((\mathscr{C}_h\,\Theta^n, \mathscr{C}_h\,\Theta^n)) + + k^4\,q_2^4((\mathscr{C}_h\,\Theta^n, \mathscr{C}_h^2\,\Theta^n))\,. \quad (87)$$

It is easy to see that (55) implies that $q_1^2 - 2\,q_2 \geqslant 0$, [13]. Therefore we have proved that

$$|||\Theta^n||| \leqslant |||\mathscr{R}\Theta^0||| + Ct_n\Big((h^r + kh^{r-1})\sup_{s\,\in\,[0,t_n]}\|\mathbf{E}_t(s)\|_{H^{r+1}(\Omega)} + t_n\,k^{\nu} \left(\sup_{s\,\in\,[0,t_n]}\left\|\frac{\partial^{\nu+1}\mathbf{E}}{\partial t^{\nu+1}}(s)\right\|_{L^2_\varepsilon(\Omega)} + \sup_{s\,\in\,[0,t_n]}\left\|\frac{\partial^{\nu+1}\mathbf{B}}{\partial t^{\nu+1}}(s)\right\|_{L^2_{\mu^{-1}}(\Omega)}\right)\Big)\,. \quad (88)$$

Use of (87) in (88) completes the estimate. □

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