M2AN - Modélisation mathématique et analyse numérique

Andro Mikelić
Roland Tapiéro

Mathematical derivation of the power law describing polymer flow through a thin slab

M2AN - Modélisation mathématique et analyse numérique, tome 29, nº 1 (1995), p. 3-21
<http://www.numdam.org/item?id=M2AN_1995__29_1_3_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODELISATION MATHEMATIQUE ET ANALYSE NUMERIQUE

(Vol 29, n° 1, 1995, p 3 à 22)

MATHEMATICAL DERIVATION OF THE POWER LAW DESCRIBING POLYMER FLOW THROUGH A THIN SLAB (*)

by Andro MIKELIĆ (1) and Roland TAPIÉRO (1)

Communicated by R TEMAM

Abstract — *We consider the polymer flow through a slab of thickness ε. The flow is described by 3D incompressible Navier Stokes system with a nonlinear viscosity, being a power of a norm of the shear rate (power law) We consider the limit when $\varepsilon \rightarrow 0$ and prove that the limit averaged velocity, averaged over the thickness, satisfies a nonlinear two-dimensional Poiseuille's law, with non-linear viscosity depending on the power of the length of the gradient of the pressure It is found out that the powers in the starting law and in the limit law are conjugate Furthermore, we prove a convergence theorem for velocity and pressure in appropriate functional spaces*

Résumé — *On considère l'écoulement en régime stationnaire, isotherme et incompressible, d'un polymère a l'intérieur d'un domaine de faible épaisseur ε Cet écoulement est décrit par le système tridimensionnel de Navier Stokes avec une viscosité non linéaire qui est une puissance du deuxième invariant du tenseur des taux de déformation (loi de puissance) Nous montrons un théorème de convergence quand ε tend vers zéro pour la vitesse et la pression dans des espaces fonctionnels appropriés A la limite, la moyenne dans l'épaisseur de la vitesse satisfait une loi de Poiseuille bidimensionnelle non linéaire où la viscosité est fonction d'une puissance du gradient de la pression Les puissances dans la loi de départ et dans la loi limite sont conjuguées*

1. STATEMENT OF THE PROBLEM AND THE RESULT

1.1. Introduction

We consider the stationary incompressible non-Newtonian flow through a thin slab The flow regime is assumed to be laminar and heating effects are neglected. The fluid is placed in the slab which we suppose to be a right cylinder limited at the bottom by a flat bounded surface with a characteristic

(*) Manuscript received june 10, 1992, revised July 5, 1994
(1) Laboratoire d'Analyse Numérique, C N R S U R A 740, Bât 101, Université Claude Bernard, 43 Bd du 11 Novembre 1918, F69622 Villeurbanne Cedex, France

length considerably larger than the maximum thickness of the slab. Problems of this type arise in diverse processing operations such as flat extruding (see [3] and in various lubrication problems.

Many papers are devoted to the study of boundary value problems in a thin domains. We refer to books [13], [15] for the general geometric setting applied mostly to the problems in elasticity. In [2] and [8] the simple linear Poiseuille law is derived from the Stokes or Navier-Stokes system. However this concerns essentially linear phenomena (Newtonian fluids) in which inertial term of the Navier-Stokes equations is just a compact perturbation without any impact on the convergence result.

From the other side it is well-known that for polymer (quasi-newtonian) flow through a thin slab the nonlinear Poiseuille law is used. In this paper, we study the case where viscosity is given by the non-linear power law which is widely used for melted polymers, oil, mud, ... and our goal is to give a mathematical fundation for the nonlinear averaged momentum equation.

We derive a non-linear Reynolds law when thickness of the layer tends to zero. More precisely, we prove convergence in the weak topology of appropriate Banach spaces for pressure and vertically averaged velocity given by the starting three dimensional problem, to a limit pressure and plane velocity. The limit pressure is a solution for the non-linear two-dimensional power type Reynolds equation.

It is well known [3], that in this physical situation, the non-linear power-type Poiseuille law is used as the momentum equation. This means that, for a thin slab, the quasi-newtonian Navier-Stokes system is expected to be well approximated by the non-linear Poiseuille. In this paper, we justify this important approximation from the polymer engineering. Let us note that in an analogous situation of a porous medium, homogenization of the non-linear Navier-Stokes system leads to different, more general non-linear and non-local law which is reduced to a power-type Darcy law just in the simplest model cases [4].

Our intention is to derive a new type of law giving a two-dimensional approximation of the problem. So, in order to avoid technical difficulties connected with non-homogeneous boundary conditions for velocity (or pressure in some cases), we consider a flow with no-slip condition on the boundary of the slab. The more realistic case of given non-zero injection velocity on the side boundary is considered in [5] where a compatibility condition on boundary values is supposed. The general case requires construction of boundary layers corresponding to the contact between the lateral boundary and upper and lower boundaries. Estimates in that case for a newtonian flow are given in [18].

Our starting model in its nondimensional form concerning coefficients is given by

$$- \varepsilon^{\gamma} \operatorname{div} \{ |D(v^{\varepsilon})|^{r-2} D(v^{\varepsilon}) \} + (v^{\varepsilon} \nabla) v^{\varepsilon} + \nabla p^{\varepsilon} = f \quad \text{in} \quad \Omega_{\varepsilon}, \tag{1.1}$$

$$\operatorname{div} v^{\varepsilon} = 0 \quad \text{in} \quad \Omega_{\varepsilon}, \tag{1.2}$$

$$v^{\varepsilon} = 0 \quad \text{on} \quad \partial \Omega_{\varepsilon} \tag{1.3}$$

$\Omega_{\varepsilon} \subset \mathbb{R}^3$ is a thin right cylinder of the form

$$\Omega_{\varepsilon} = \{ x \in \mathbb{R}^3, x = (x', x_3), x' = (x_1, x_2) \in \omega, 0 < x_3 < \varepsilon h(x') \}$$

where ω is an open bounded set of $\mathbb{R}^2$, representing the flat basis of the slab ; ε is a positive parameter representing the characteristic thickness of the slab and tending to zero , h is a smooth C^1 bounded function such that $1/2 \leq h \leq 1$ and $x_3 = \varepsilon h(x')$ is the equation of the top surface of Ω_{ε}.

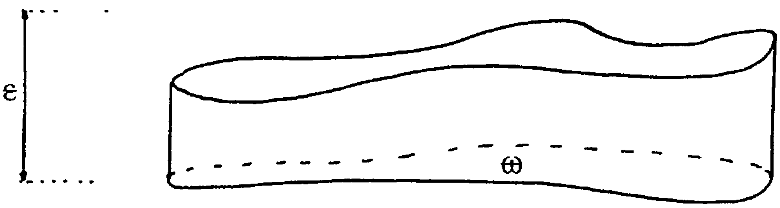

Figure 1. — Ω_{ε} .

v^{ε} is the velocity ; p^{ε} is the pressure ; $\varepsilon^{-\gamma}$ represents the behavior of the Reynolds number, f is the density of an external body force independant of the vertical coordinate x_3.

$D(v^{\varepsilon})$ is the rate of strain tensor i.e. the symmetrized gradient of the velocity

$$D(v^{\varepsilon}) = \frac{1}{2} (\nabla v^{\varepsilon} + (\nabla v^{\varepsilon})^t)$$

The matrix norm $|\ |$ is defined by

$$|\xi|^2 = T_r(\xi \xi^t), \xi \in \mathbb{R}^3 .$$

Hence, the viscosity which is, in our case given by the non-dimensional power law [1], [17] can be expressed in term of the second invariant of the strain tensor $D_{II}(v^{\varepsilon}) = D(v^{\varepsilon}) D(v^{\varepsilon})^t$ by

$$|D(v^{\varepsilon})|^{r-2} = |D_{II}(v^{\varepsilon})|^{r/2-1}$$

We suppose

$$f = f(x') \in L^{\infty}(\omega) . \tag{1.4}$$

Then, for $\varepsilon > 0$, the classical theory [7], [14] gives the existence of at least one weak solution

$$(v^{\varepsilon}, p^{\varepsilon}) \in (W_0^{1,r}(\Omega_\varepsilon))^3 \times L_0^{r'}(\Omega_\varepsilon) , \quad r' = \frac{r}{r-1}$$

for (1.1)-(1.3), under assumption $9/5 \leqslant r < +\infty$. This assumption is carried out through all sections.

1.2. Rescaled Problem, Notations and Technical Lemmas

Let us list some notations to be frequently used. A function with values in $\mathbb{R}^3$ is written in the form

$$u = (\tilde{u}, u_3) \quad \text{where} \quad \tilde{u} \in \mathbb{R}^2 .$$

Furthermore, following [13], for a sequence of functions u^{ε} defined on Ω_ε, we introduce the rescaling

$$u(\varepsilon)(x', z) = u^{\varepsilon}(x', \varepsilon z) .$$

Hence, $u(\varepsilon)$ is a sequence of functions defined on a fixed domain : $\Omega = \Omega_1$.

We introduce operators : $\nabla_{x'}$, $\operatorname{div}_{x'}$ and ∇_ε, D_ε, $\operatorname{div}_\varepsilon$ by

$$(\nabla_\varepsilon u)_{i,j} = (\nabla_{x'} u)_{i,j} = \frac{\partial u_i}{\partial x_j} \qquad \text{for} \quad i = 1, 2, 3 ; \quad j = 1, 2 .$$

$$(\nabla_\varepsilon u)_{i,3} = \frac{1}{\varepsilon} \frac{\partial u_i}{\partial z} \qquad \text{for} \quad i = 1, 2, 3 .$$

$$D_\varepsilon(u) = \frac{1}{2} (\nabla_\varepsilon u + (\nabla_\varepsilon u)^t) ,$$

$$\operatorname{div}_\varepsilon(u) = \operatorname{div}_{x'} \tilde{u} + \frac{1}{\varepsilon} \frac{\partial u_3}{\partial z} = \sum_{i=1}^{2} \frac{\partial u_i}{\partial x_i} + \frac{1}{\varepsilon} \frac{\partial u_3}{\partial z} .$$

It is easily observed that problem (1.1)-(1.3) is equivalent, after rescaling, to

$$- \varepsilon^{\gamma} \operatorname{div}_\varepsilon \left\{ |D_\varepsilon(v(\varepsilon))|^{r-2} D_\varepsilon(v(\varepsilon)) \right\} + \\ + (v(\varepsilon) \nabla_\varepsilon) v(\varepsilon) + \nabla_\varepsilon p(\varepsilon) = f \quad \text{in} \quad \Omega , \tag{1.5}$$

$$\operatorname{div}_{\varepsilon} v(\varepsilon) = 0 \quad \text{in} \quad \Omega\,, \tag{1.6}$$

$$v(\varepsilon) = 0 \quad \text{on} \quad \partial\Omega\,. \tag{1.7}$$

Also, we are going to frequently use the following three well-known results :

LEMMA 1.1 : *For* $v \in L^r(\Omega_\varepsilon)$, $1 \leqslant r < +\infty$, *the rescaled function* $v(\varepsilon)$ *verifies*

$$\| v(\varepsilon) \|_{L^r(\Omega)} = \varepsilon^{-1/r} \| v \|_{L^r(\Omega_\varepsilon)}\,. \tag{1.8}$$

□

LEMMA 1.2 : *(Poincaré inequality). For* $v \in W_0^{1,r}(\Omega_\varepsilon)$, $1 \leqslant r < +\infty$,

$$\| v \|_{L^r(\Omega_\varepsilon)} \leqslant \varepsilon \left\| \frac{\partial v}{\partial x_3} \right\|_{L^r(\Omega_\varepsilon)}\,. \tag{1.9}$$

□

LEMMA 1.3 : *(Korn inequality). For* $v \in (W_0^{1,r}(\Omega_\varepsilon))^3$, $1 < r < +\infty$,

$$\| \nabla v \|_{(L^r(\Omega_\varepsilon))^9} \leqslant C \| D(v) \|_{(L^r(\Omega_\varepsilon))^9} \tag{1.10}$$

where C *is independent of* u *and* ε. □

Proof of Lemma 1.3 : The proof for a fixed domain can be found in [12] and [17] ; by a simple extension argument, we see that, in our case, the constant C does not depend on ε. ■

1.3. Statement of the Result

Before stating our convergence result, we introduce the limit problem. Find $p \in W^{1,r'}(\omega) \cap L_0^{r'}(\omega)$, such that

$$\begin{aligned} &\operatorname{div}_{x'} (h^{r'+1}(x')\, |\tilde f - \nabla_{x'} p|^{r'-2} (\tilde f - \nabla_{x'} p)) = 0 \quad \text{in} \quad \omega\,, && (1.11)\\ &\nu \cdot h^{r'+1}(x')\, |\tilde f - \nabla_{x'} p|^{r'-2} (\tilde f - \nabla_{x'} p) = 0 \quad \text{on} \quad \partial\omega\,, && (1.12) \end{aligned}$$

where ν is the unit outer normal to $\partial\omega$. Moreover, we define the limit scaled projection velocity

$$\tilde u(x', z) = \frac{2^{r'/2}}{r'} \left(\left(\frac{h(x')}{2} \right)^{r'} - \left| \frac{h(x')}{2} - z \right|^{r'} \right) |\tilde f - \nabla_{x'} p|^{r'-2} (\tilde f - \nabla_{x'} p) \tag{1.13}$$

and Darcy velocity

$$U(x') = \int_0^{h(x')} \tilde{u}(x', z)\, dz = \frac{h^{r'+1}(x')}{2^{r'/2}(r'+1)} |\tilde{f} - \nabla_{x'} p|^{r'-2} (\tilde{f} - \nabla_{x'} p) . \quad (1.14)$$

We use functional spaces

$$W = \left\{ \varphi \in L^r(\Omega) ; \frac{\partial \varphi}{\partial z} \in L^r(\Omega) \right\}$$

and

$$W^2 = \left\{ \varphi \in L^r(\Omega)^2 ; \frac{\partial \varphi}{\partial z} \in L^r(\Omega)^2 \right\} .$$

We also have to define the constant $C(r)$, depending on the values of r and γ as

$$C(r) = \begin{cases} r' \text{ for } 9/4 \leqslant r \text{ and } \gamma < (r+1)/2 \\ r' \text{ for } 9/5 \leqslant r < 9/4 \text{ and } \gamma \leqslant 5\, r/2 - 6 + 9/2\, r \\ \dfrac{3\, r(r-1)}{2\, r(\gamma - r) + 6(r-1)} \\ \qquad \text{for } \begin{cases} 2 < r < 9/4 \text{ and } 5\, r/2 - 6 + 9/2\, r < \gamma < (r+1)/2 \\ \text{or} \\ 9/5 \leqslant r \leqslant 2 \text{ and } 5\, r/2 - 6 + 9/2\, r < \gamma < 2\, r - 4 + 3/r \end{cases} \\ \text{any } l' < 3/2 \textit{ for } 9/5 \leqslant r < 2 \text{ and } \gamma = 2\, r - 4 + 3/r . \end{cases} \quad (1.15)$$

Now, we are able to state our result.

Convergence theorem. *Let* $\gamma < (r+1)/2$, $r \geqslant 9/5$ *and* $(v(\varepsilon), p(\varepsilon)) \in W^{1,r}(\Omega)^3 \times L_0^{r'}(\Omega)$ *be a weak solution of* (1.5)-(1.7). *Then problem* (1.11)-(1.12) *has a unique solution* $p \in W^{1,r'}(\omega) \cap L_0^{r'}(\omega)$ *and we have*

$$\begin{aligned} \varepsilon^{-\frac{r-\gamma}{r-1}} \tilde{v}(\varepsilon) &\rightharpoonup \tilde{u} && \textit{weakly} \quad \textit{in } W^2 , \\ \varepsilon^{-\frac{r-\gamma}{r-1}} \tilde{v}(\varepsilon) &\rightharpoonup 0 && \textit{weakly} \quad \textit{in } W , \\ \varepsilon^{-\frac{r-\gamma}{r-1}} \int_0^{h(x')} \tilde{v}(\varepsilon)\,(x', z)\, dz &\rightharpoonup U(x') && \textit{weakly} \quad \textit{in } (L^r(\omega))^2 , \\ p(\varepsilon) &\rightharpoonup p && \textit{weakly} \quad \textit{in } L_0^{C(r)}(\Omega) , \end{aligned}$$

$\tilde{u}$ and U are given by (1.13), (1.14) *and $C(r)$ is given by* (1.15). $\square$

2. UNIFORM *A PRIORI* ESTIMATES

In this section, we derive uniform *a priori* estimates for solution of the rescaled problem (1.5)-(1.7).

A priori estimates for components of the velocity are to be compared to those obtained in the simpler case of Newtonian flow ($r = 2$) and with Reynolds number not depending on ε (see [2] and [8]). In our physically general model for obtaining *a priori* estimates for the pressure we have to derive very precise interpolation estimates having the correct order in ε. Also, we need them for passing to the limit in the inertial term. We use the same approach, with adaptations, as in [16] and get the convergence of the pressure in the optimal Banach spaces.

2.1. Estimates for Velocity

PROPOSITION 2.1 : *Let v^{ε} be a weak solution of* (1.1)-(1.3) *and $v(\varepsilon)$, the corresponding rescaled solution of* (1.5)-(1.7). *Then we have*

$$\|v(\varepsilon)\|_{(L^r(\Omega))^3} \leqslant C\varepsilon^{\frac{r-\gamma}{r-1}} \tag{2.1}$$

$$\left\|\frac{\partial v(\varepsilon)}{\partial z}\right\|_{(L^r(\Omega))^3} \leqslant C\varepsilon^{\frac{r-\gamma}{r-1}} \tag{2.2}$$

$$\|\nabla_{x'} v(\varepsilon)\|_{(L^r(\Omega))^{3\times 2}} \leqslant C\varepsilon^{\frac{r-\gamma}{r-1}}. \tag{2.3}$$

$\square$

Proof of Proposition 2.1 : We use the three technical lemmas of Section 1.2 ; energy equality corresponding to the momentum equation (1.1) gives

$$\varepsilon^{\gamma}\|D(v^{\varepsilon})\|^{r}_{(L^r(\Omega_\varepsilon))^9} \leqslant C\|f\|_{(L^{r'}(\Omega_\varepsilon))^3}\|v^{\varepsilon}\|_{(L^r(\Omega_\varepsilon))^3}.$$

With hypothesis (1.4) and (1.9), we have

$$\varepsilon^{\gamma}\|D(v^{\varepsilon})\|^{r}_{(L^r(\Omega_\varepsilon))^9} \leqslant C\varepsilon^{1+\frac{1}{r'}}\left\|\frac{\partial v^{\varepsilon}}{\partial x_3}\right\|_{(L^r(\Omega_\varepsilon))^3}$$

and, by (1.10),

$$\|\nabla v^{\varepsilon}\|_{(L^r(\Omega_\varepsilon))^9} \leqslant C\varepsilon^{\left(1+\frac{1}{r'}-\gamma\right)/(r-1)}.$$

Rescalıng and (1 8) gıve (2-1)-(2 3) ∎

We prefer to work wıth once more scaled velocıty $u(\varepsilon)$ defined by $u(\varepsilon)=\varepsilon^{(\gamma-r)/(r-1)}v(\varepsilon)$ where $v(\varepsilon)$ ıs a weak solutıon of (1 5)-(1 7)

For $u(\varepsilon)$, we obtaın

PROPOSITION 2 2 *a) $\tilde{u}(\varepsilon)$ verıfies*

$$\operatorname{div}_x \int_0^{h(x')} \tilde{u}(\varepsilon)(x', z)\, d = 0 \quad \text{in} \quad \omega, \tag{2 4}$$

$$\left\| \int_0^{h(x)} \tilde{u}(\varepsilon)(x', z)\, dz \right\|_{L^r(\omega)^2} \leq C, \tag{2 5}$$

$$\left\| \nu \int_0^{h(x')} \tilde{u}(\varepsilon)(x', z)\, dz \right\|_{W^{-1/r, r}(\partial\omega)} \leq C \tag{2 6}$$

b) $u_3(\varepsilon)$ verıfies

$$\left\| \frac{\partial u_3(\varepsilon)}{\partial z} \right\|_{W^{-1, r}(\Omega)} \leq C\varepsilon \tag{2 7}$$

□

Proof of Propositıon 2 2 (2 4) ıs a dırect consequence of (1 6) and (1 7) The Holder ınequalıty and (2 2) gıve (2 5)

Furthermore, (2 4) and (2 5) ımply (2 6), by sımple modıficatıon of the proof for the case $r = 2$ ın [10], pages 27-28 (see also [11] and [20]) Fınally, for every $\varphi \in W_0^{1, r'}(\Omega)$

$$\int_\Omega \frac{\partial u_3(\varepsilon)}{\partial z} \varphi\, dx = -\varepsilon \int_\Omega \operatorname{div}_x \tilde{u}(\varepsilon)\, \varphi\, dx = \varepsilon \int_\Omega \tilde{u}(\varepsilon)\, \nabla_{x'} \varphi\, dx$$

and, by (2 1), we have

$$\left| \int_\Omega \frac{\partial u_3(\varepsilon)}{\partial z} \varphi\, dx \right| \leq C\varepsilon \|\varphi\|_{W_0^{1, r'}(\Omega)} \tag{2 8}$$

(2 8) gıves (2 7) ∎

2.2. Estimates for pressure

In thıs subsectıon we show that, for the values of γ and r satısfyıng condıtıons specıfied ın the definıtıon (1 15), of $C(r)$, ınertıal term $(v(\varepsilon)\nabla_\varepsilon)v(\varepsilon)$ weakly converges to zero ın $(W^{-1, C(r)}(\Omega))^3$ and we obtaın an estımate for the pressure ın $L_0^{C(r)}$

PROPOSITION 2.3 : *With* $C(r)$ *defined by* (1.15), *any weak solution* $p(\varepsilon)$ *of* (1.5)-(1.7) *satisfies inequalities*

$$\| p(\varepsilon) \|_{L_0^{C(r)}(\Omega)} \leqslant C, \tag{2.9}$$

$$\| \nabla_{x'} p(\varepsilon) \|_{(W^{-1, C(r)}(\Omega))^2} \leqslant C, \tag{2.10}$$

$$\left\| \frac{\partial p(\varepsilon)}{\partial z} \right\|_{W^{-1, C(r)}(\Omega)} \leqslant C\varepsilon . \tag{2.11}$$

□

Proof of Proposition 2.3 : Rescaled momentum equation (1.5) gives

$$\begin{aligned} \langle \nabla_\varepsilon p(\varepsilon), \varphi \rangle_\Omega = \langle \nabla_{x'} p(\varepsilon), \tilde{\varphi} \rangle_\Omega &+ \frac{1}{\varepsilon} \left\langle \frac{\partial p(\varepsilon)}{\partial z}, \varphi_3 \right\rangle_\Omega \\ &= - \varepsilon^\gamma \int_\Omega |D_\varepsilon(v(\varepsilon))|^{r-2} D_\varepsilon(v(\varepsilon)) \, D_\varepsilon(\varphi) \, dx \\ &+ \int_\Omega f\varphi \, dx - \int_\Omega (v(\varepsilon) \, \nabla_\varepsilon) \, v(\varepsilon) \, \varphi \, dx \\ &\forall \varphi \in (W_0^{1,q}(\Omega))^3, q \geqslant r . \end{aligned} \tag{2.12}$$

By Proposition 2.1, we have

$$\begin{aligned} \varepsilon^\gamma \left| \int_\Omega |D_\varepsilon(v(\varepsilon))|^{r-2} D_\varepsilon(v(\varepsilon)) \, D_\varepsilon(\varphi) \, dx \right| &\leqslant \\ \leqslant C\varepsilon^{\gamma - 1} \; \| \, |D_\varepsilon(v(\varepsilon))|^{r-2} D_\varepsilon(v(\varepsilon)) \|_{(L^{r'}(\Omega))^9} \, \| \varphi \|_{(W_0^{1, r}(\Omega))^3} \\ \leqslant C \| \varphi \|_{(W_0^{1, r}(\Omega))^3} \end{aligned}$$

and

$$\left| \int_\Omega f\varphi \, dx \right| \leqslant C \| \varphi \|_{W_0^{1, r}(\Omega)^3} .$$

Hence, to derive (2.9)-(2.11) from (2.12), we just need to consider the inertial term. It can be written

$$\begin{aligned} \int_\Omega (v(\varepsilon) \, \nabla_\varepsilon) \, v(\varepsilon) \, \varphi \, dx &= - \int_\Omega v(\varepsilon) \, \tilde{\otimes} \, v(\varepsilon) \, \nabla_{x'} \varphi \, dx + \\ &+ \frac{1}{\varepsilon} \left\{ \int_\Omega \frac{\partial v_3(\varepsilon)}{\partial z} v(\varepsilon) \, \varphi \, dx + \int_\Omega v_3(\varepsilon) \frac{\partial v(\varepsilon)}{\partial z} \varphi \, dx \right\} \end{aligned} \tag{2.13}$$

where

$$(u \tilde{\otimes} v)_{ij} = u_i v_j, \quad i = 1, 2, \quad j = 1, 2, 3$$

We consider separately the two terms in the right hand side of (2 13)

(a) Let us first estimate the term $\int_\Omega v(\varepsilon) \tilde{\otimes} v(\varepsilon) \nabla_x \varphi \, dx$ For $\varphi \in W_0^{1,q}(\Omega)^3$, it follows from Holder inequality that

$$\int_\Omega v(\varepsilon) \tilde{\otimes} v(\varepsilon) \nabla_{x'} \varphi \, dx \leqslant C\left(\int_\Omega |v(\varepsilon)|^{q'}\right)^{2/q'} \left(\int_\Omega |\nabla_{x'} \varphi|^q\right)^{1/q}$$

with

$$\frac{2}{q'} + \frac{1}{q} = 1$$

Using (2 1) and (2 3), the Sobolev embedding theorem implies that

$$\| v(\varepsilon) \|_{(L^{r^*}(\Omega))^3} \leqslant C \varepsilon^{\frac{1-\gamma}{r-1}},$$

where $r^* = 3r/(3-r)$ if $r < 3$, $r^* \in [r, +\infty[$ if $r = 3$ and $r^* = +\infty$ if $r > 3$ Hence, for all $\theta \in [0, 1]$, if q' is such that

$$\frac{1}{q'} = \frac{\theta}{r} + \frac{1-\theta}{r^*},$$

we have by interpolation

$$\| v(\varepsilon) \|_{(L^{q'}(\Omega))^3} \leqslant \| v(\varepsilon) \|^{\theta}_{(L^r(\Omega))^3} \, \| v(\varepsilon) \|^{1-\theta}_{(L^{r^*}(\Omega))^3} \leqslant \varepsilon^{\theta \frac{r-\gamma}{r-1} + (1-\theta)\frac{1-\gamma}{r-1}}$$

Therefore, we may conclude that $\int_\Omega v(\varepsilon) \tilde{\otimes} v(\varepsilon) \nabla_x \varphi \, dx \to 0$ for all $\varphi \in W_0^{1,q}(\Omega)^3$ if we can find $\theta \in [0, 1]$ such that

$$\theta \frac{r-\gamma}{r-1} + (1-\theta)\frac{1-\gamma}{r-1} > 0$$

that is

$$1 \geqslant \theta > \theta_0 = \max\left\{0, \frac{\gamma-1}{r-1}\right\}$$

Our goal in the ensuing discussion is finding an optimal θ such as the corresponding q is as close as possible to r

Recall that

$$\frac{1}{q} = 1 - \frac{2\,\theta}{r} - \frac{2(1-\theta)}{r^*}$$

and let us examine the different possible values of γ and r

(a1) Let $\gamma \geqslant r$, then $\theta_0 \geqslant 1$ and we are not able to find a bound for the pressure except for $\gamma = r$ However then

$$\int_\Omega v(\varepsilon) \tilde{\otimes} v(\varepsilon) \nabla_x \varphi \, dx$$

is just bounded and may have an influence in the limit We do not consider this case and limit us to $\gamma < r$

(a2) Let $\gamma \leqslant 1$, then $\theta_0 = 0$, we set $q = r$ and see that $q' = 2\,r/(r-1) \leqslant 3\,r/(3-r)$ for $r \geqslant 9/5$ So, in this case

$$\left| \int_\Omega v(\varepsilon) \tilde{\otimes} v(\varepsilon) \nabla_x \varphi \, dx \right| \leqslant C\varepsilon^{2\left(\theta + \frac{1-\gamma}{r-1}\right)} \|\varphi\|_{(W_0^{1,r}(\Omega))^3}$$

(a3) Let $1 < \gamma < r$, then $\theta_0 = (\gamma - 1)/(r-1)$ and

(a3 1) $r > 3$, $1/q \leqslant 1 - 2(\gamma-1)/r(r-1)$ which clearly allows the choice $q = r$ because of $r > 3$ and $\gamma < r$

(a3 2) $r = 3$, $1/q \leqslant 1 - (\gamma - 1)/3 - \delta(3-\gamma)$ and $q = 3$ can be chosen because of $\gamma < 3$

(a3 3) $r < 3$, $1/q \leqslant 1 + 2(r-\gamma)/3(r-1) - 2/r$ which is positive for $\gamma < 5\,r/2 - 9/2 + 3/r$ The former inequality always holds only if $r \geqslant 2$ For $9/5 \leqslant r < 2$, we have to restrict γ further (see (2 15)) Moreover, $q = r$ can be chosen for $\gamma \leqslant 5\,r/2 - 6 + 9/2\,r$ Otherwise, we have to choose $q > r$ using formula (2 15)

Let us summarize all cases of part (a)

$$\left| \int_\Omega v(\varepsilon) \tilde{\otimes} v(\varepsilon) \nabla_x \varphi \, dx \right| \leqslant C\varepsilon^{\alpha} \|\varphi\|_{(W_0^{1,q}(\Omega))^3} \tag{2 14}$$

with $\alpha > 0$ and

$$q = \begin{cases} r \text{ for } r \geqslant 3 \text{ and } \gamma < r \\ r \text{ for } 9/5 \leqslant r < 3 \text{ and } \gamma \leqslant 5\,r/2 - 6 + 9/2\,r \\ \dfrac{3\,r(r-1)}{5\,r^2 - 9\,r + 6 - 2\,\gamma r} \\ \qquad \text{for} \begin{cases} 2 \leqslant r < 3 \text{ and } 5\,r/2 - 6 + 9/2\,r < \gamma < r \\ \text{or} \\ 9/5 \leqslant r < 2 \text{ and } 5\,r/2 - 6 + 9/2\,r < \gamma < 5\,r/2 - 9/2 + 3/r \end{cases} \end{cases} \tag{2 15}$$

(b) Estimate of the second part of the right hand side of (2 13) has the form

$$C\varepsilon^{-1}\left\|\frac{\partial v(\varepsilon)}{\partial z}\right\|_{(L^{r}(\Omega))^3}\|v(\varepsilon)\|_{(L^{q}(\Omega))^3}\|\varphi\|_{(L^{q_1}(\Omega))^3}$$

As in part (a), the goal is again to have the best interpolation estimate for $v(\varepsilon)$ in L^q with φ in $W_0^{1,q}$ where $q \geqslant r$ is as close as possible to r

Hence, $q_1 = 1/\delta$, where

$$\delta = \begin{cases} 0 \quad \text{for} \quad q > 3 \\ \text{any finite positive number for} \quad q = 3 \\ 1/q - 1/3 \quad \text{for} \quad q < 3 \end{cases}$$

and

$$1/q' \leqslant 1 - 1/r - \delta \tag{2 16}$$

Interpolation between L^r and $W^{1,r}$ gives

$$1/q' = \theta/r + \delta'(1-\theta) \tag{2 17}$$

with

$$\delta' = \begin{cases} 0 \quad \text{for} \quad r > 3 \\ \text{any finite positive number for} \quad r = 3 \\ 1/r - 1/3 \quad \text{for} \quad r < 3 \end{cases}$$

(2 16) and (2 17) imply that

$$\theta(1/r - \delta') \leqslant 1 - 1/r - \delta - \delta'$$

We define θ by

$$\theta = \min\left\{1, \frac{1 - 1/r - \delta - \delta'}{1/r - \delta'}\right\} \tag{2 18}$$

and check that $\theta \in [0, 1]$ for every values of $q \geqslant r \geqslant 9/5$

Estimate of the second part of (2 13) and proposition 2 1 imply that θ has to verify

$$\frac{1-\gamma}{r-1} + \theta\frac{r-\gamma}{r-1} + (1-\theta)\frac{1-\gamma}{r-1} > 0$$

or, equivalently

$$1 \geq \theta > \theta_0 = \max\left\{0, 2\frac{\gamma - 1}{r - 1}\right\}. \tag{2.19}$$

Therefore, the whole problem reduces to the study of compatibility of (2.18) and (2.19). We have, according to the different possible values of γ and r, the following situation :

(b1) Let $\gamma \geq (r+1)/2$, then $\theta_0 \geq 1$ and as in case (a1), we are not able to find a bound for the inertial term proportional to a positive power of ε. We eliminate this case.

(b2) Let $\gamma \leq 1$, then $\theta_0 = 0$ and (2.18) and (2.19) are obviously compatible.

(b3) Let $1 < \gamma < (r+1)/2$. We have $\theta_0 = 2(\gamma - 1)/(r-1)$ and there are three different subcases :

(b3.1) $r > 3, \delta' = \delta = 0$ and $\theta = \min\{1, r-1\} = 1 > \theta_0$. Obvious choice is $q = r$.

(b3.2) $r = 3$. Then, after choosing very small $\delta'$and δ, we get $\theta = 1$ and again, $q = r$.

(b3.3) $9/5 \leq r < 3. \delta' = (3-r)/3r$ and we study the inequality

$$\theta = \min\{1{,}4 - 6/r - 3\delta\} \geq 2\frac{\gamma - 1}{r - 1}. \tag{2.20}$$

It is natural to try with $q = r$. Then, we have

$$\theta = \min\{1, 5 - 9/r\}.$$

For $9/4 \leq r < 3$ it is obvious that our choice is correct. For $9/5 \leq r < 9/4$, (2.20) reduces to an inequality for γ

$$\gamma \leq 5r/2 - 6 + 9/2r.$$

Note that the right hand side of this inequality is always less than $(r+1)/2$ for $r < 9/4$.

Our next step is to study the case $q > r$ and $9/5 \leq r < 9/4$. The first subcase is $r < q < 3$. Then

$$\theta = \min\{1, 5 - 6/r - 3/q\}.$$

For $2 < r < 9/4$, we have $3r/(2(2r-3)) < 3$ and choice

$$3 > \frac{3r}{2(2r-3)} \geq q > r$$

leads to $\theta = 5 - 6/r - 3/q$ and q is given by (2 21)

Finally, we consider the case $9/5 \leqslant r \leqslant 2$ In this case $3\,r/(2(2\,r-3)) \geqslant 3$ and upper bound for q is simply 3 Once again $\theta = 5 - 6/r - 3/q$ and q is given by (2 21) However, the bound $q < 3$ gives rise to the new bound for γ

$$\gamma < 2\,r - 4 + 3/r$$

which is less or equal $(r+1)/2$ for $9/5 \leqslant r \leqslant 2$ In order to get some estimate for the situation $2\,r - 4 + 3/r \leqslant \gamma < (r+1)/2$, we investigate the possibility $q \geqslant 3$ Simple calculation shows that the only compatible case is

$$q = l \quad \text{for} \quad \gamma = 2\,r - 4 + 3/r\,, \quad 9/5 \leqslant r < 2$$

where l is an arbitrary number greater than 3

Let us summarize (b) the second part of the right hand side of (2 13) is satisfying the estimate analogous to (2 14) for q given by

$$q = \begin{cases} r \text{ for } 9/4 \leqslant r \text{ and } \gamma < (r+1)/2 \\ r \text{ for } 9/5 \leqslant r < 9/4 \text{ and } \gamma \leqslant 5\,r/2 - 6 + 9/2\,r \\ \dfrac{3\,r(r-1)}{5\,r^2 - 9\,r + 6 - 2\,\gamma r} \\ \qquad \text{for} \begin{cases} 2 < r < 9/4 \text{ and } 5\,r/2 - 6 + 9/2\,r < \gamma < (r+1)/2 \\ \text{or} \\ 9/5 \leqslant r \leqslant 2 \text{ and } 5\,r/2 - 6 + 9/2\,r < \gamma < 2\,r - 4 + 3/r \end{cases} \\ l > 3 \textit{ for } 9/5 \leqslant r < 2 \text{ and } \gamma = 2\,r - 4 + 3/r \end{cases} \tag{2 21}$$

Putting together (2 15) and (2 21) and choosing $C(r) = q/(q-1)$, we obtain (1 15) and estimates (2 9)-(2 11) ∎

3. CONVERGENCE THEOREM

After getting the *a priori* estimates in the previous section, we prove a convergence of our singular perturbation process when characteristic thickness ε of the slab tends to zero

We start with a weak compactness type result, being a direct consequence of estimates obtained in Section 2.

PROPOSITION 3.1 : *For the choice of* γ *and* r *and the value* $C(r)$ *given by* (1.15), *there is a subsequence* $(u(\varepsilon), p(\varepsilon))$ *chosen from a sequence of solutions* $(v(\varepsilon), p(\varepsilon)) \in (W_0^{1,r}(\Omega))^3 \times L_0^{r'}(\Omega)$ *of* (1.5)-(1.7) *such that*

$$
\begin{aligned}
\tilde{u}(\varepsilon) &\rightharpoonup \tilde{u} && \textit{weakly} \quad \textit{in } W^2 ,\\
u_3(\varepsilon) &\rightharpoonup 0 && \textit{weakly} \quad \textit{in } W ,\\
\int_0^{h(x')} \tilde{u}(\varepsilon)(x', z)\, dz &\rightharpoonup \int_0^{h(x')} \tilde{u}(x', z)\, dz && \textit{weakly} \quad \textit{in } (L^r(\omega))^2 ,\\
p(\varepsilon) &\rightharpoonup p && \textit{weakly} \quad \textit{in } L_0^{C(r)}(\Omega) .
\end{aligned}
$$

Moreover,

$$p = p(x') \in L_0^{C(r)}(\omega) ,$$

$$\operatorname{div}_{x'} \int_0^{h(x')} \tilde{u}(x', z)\, dz = 0 \quad \text{in} \quad \omega , \tag{3.1}$$

$$\nu \cdot \int_0^{h(x')} \tilde{u}(x', z)\, dz = 0 \quad \text{on} \quad \partial\omega , \tag{3.2}$$

$$\tilde{u}(x', z) = \tilde{u}(x', h(x')) = 0 \quad \text{in} \quad \omega . \tag{3.3}$$

□

PROPOSITION 3.2 : *Any cluster point* $(\tilde{u}, p) \in W^2 \times L_0^{C(r)}(\omega)$ *as in Proposition 3.1, verifies*

$$-\frac{\partial}{\partial z}\left\{ \left|\frac{\partial \tilde{u}}{\partial z}\right|^{r-2} \frac{\partial \tilde{u}}{\partial z} \right\} = 2^{r/2}(\tilde{f} - \nabla_{x'} p) \quad \textit{in} \quad \Omega . \tag{3.4}$$

□

Proof of Proposition 3.2 : Due to Minty's Lemma [9] and monotonicity, (1.5)-(1.7) is equivalent to

$$
\begin{aligned}
\varepsilon^{\gamma} \int_{\Omega} |D_\varepsilon(\varphi)|^{r-2} D_\varepsilon(\varphi)\, D_\varepsilon(\varphi - v(\varepsilon))\, dx &\geqslant \int_{\Omega} f(\varphi - v(\varepsilon))\, dx \\
- \int_{\Omega} (v(\varepsilon)\, \nabla_\varepsilon)\, v(\varepsilon)\, \varphi\, dx - \langle \nabla_\varepsilon p(\varepsilon), \varphi \rangle_{\Omega} &\quad \forall \varphi \in (C_0^\infty(\Omega))^3 .
\end{aligned}
$$

Replacing φ by $\varepsilon^{(r-\gamma)/(r-1)}\varphi$ and dividing by $\varepsilon^{(r-\gamma)/(r-1)}$ gives

$$\int_\Omega |\varepsilon D_\varepsilon(\varphi)|^{r-2}\varepsilon^2 D_\varepsilon(\varphi)\, D_\varepsilon(\varphi - u(\varepsilon))\, dx \geqslant \int_\Omega f(\varphi - u(\varepsilon))\, dx$$
$$- \int_\Omega (v(\varepsilon)\nabla_\varepsilon)\, v(\varepsilon)\,\varphi\, dx + \int_\Omega p(\varepsilon)\operatorname{div}_\varepsilon \varphi\, dx .$$

Because of the choice of γ and r in (1.15), inertial term tends to zero and we obtain

$$\int_\Omega \left|\left(\frac{\partial\varphi_3}{\partial z}\right)^2 + \frac{1}{2}\sum_{j=1}^{2}\left(\frac{\partial\varphi_j}{\partial z}\right)^2\right|^{\frac{r}{2}-1}\left\{\frac{\partial\varphi_3}{\partial z}\frac{\partial}{\partial z}(\varphi_3 - u_3) + \right.$$
$$\left. + \frac{1}{2}\sum_{j=1}^{2}\frac{\partial\varphi_j}{\partial z}\frac{\partial}{\partial z}(\varphi_j - u_j)\right\} dx \geqslant$$
$$\geqslant \int_\Omega f(\varphi - u)\, dx + \lim_{\varepsilon\to 0}\int_\Omega p(\varepsilon)\operatorname{div}_\varepsilon \varphi\, dx .$$

But, as $u_3 = 0$, we take $\varphi_3 = 0$ and $\varphi = \tilde{\varphi}$. Then, by Proposition 3.1.

$$\lim_{\varepsilon\to 0}\int_\Omega p(\varepsilon)\operatorname{div}_{x'}\tilde{\varphi}\, dx = \int_\Omega p(x')\operatorname{div}_{x'}\tilde{\varphi}\, dx = -\langle\nabla_{x'}p, \tilde{\varphi}\rangle_\Omega . \tag{3.5}$$

Furthermore, (3.1) implies

$$\int_\Omega p\operatorname{div}_{x'}\tilde{u}\, dx = 0 .$$

Finally we get

$$2^{-r/2}\int_\Omega \left|\frac{\partial\tilde{\varphi}}{\partial z}\right|^{r-2}\frac{\partial\tilde{\varphi}}{\partial z}\frac{\partial}{\partial z}(\tilde{\varphi} - \tilde{u})\, dx \geqslant \int_\Omega \tilde{f}(\tilde{\varphi} - \tilde{u})\, dx$$
$$- \langle\nabla_{x'}p, \tilde{\varphi} - \tilde{u}\rangle_\Omega \quad \forall\tilde{\varphi} \in (C_0^\infty(\Omega))^2 \tag{3.6}$$

which is equivalent to (3.4) because of Minty's Lemma. ■

PROPOSITION 3.3 : *Any cluster point $(\tilde{u}, p)$ from Proposition 3.1, satisfies*

$$p \in W^{1,r'}(\omega) \cap L_0^{r'}(\omega) .$$

□

Proof of Proposition 3.3 : As $\tilde{u} \in W^2$, we have

$$w = \left|\frac{\partial \tilde{u}}{\partial z}\right|^{r-2} \frac{\partial \tilde{u}}{\partial z} \in (L^{r'}(\Omega))^2 .$$

Let $\tilde{\varphi} = \phi(z)\,\zeta(x')$, where $\zeta \in (C_0^\infty(\omega))^2$ is arbitrary and $\phi \in C_0^\infty\left(0, \frac{1}{2}\right)$ is a fixed nonnegative function satisfying $\int_0^{\frac{1}{2}} \phi(z)\,dz = 1$. By choosing $\tilde{\varphi}$ as a test function for (3.6) we get

$$-\int_\omega p(x')\,\mathrm{div}_{x'}\,\zeta\,dx' = \int_\omega \tilde{f}(x')\,\zeta(x')\,dx' - \int_\Omega \zeta(x')\,w(x', z)\,\frac{d\phi(z)}{dz}\,dz\,dx' ,$$

for every $\zeta \in (C_0^\infty(\omega))^2$. As $\tilde{f}$ is regular, $\nabla_{x'} p \in (L^{r'}(\omega))^2$ and as $p \in L_0^{C(r)}(\omega)$, $1 < C(r) \leqslant r'$, Poincaré-Wirtinger inequality [6] implies that $p \in L^{r'}(\omega)$. ■

PROPOSITION 3.4 : *The boundary value problem* (3.3), (3.4), *where* $p \in W^{1,r'}(\omega)$ *is given and* $x' \in \omega$ *is a parameter, is equivalent to* (1.13) *i.e.*

$$\tilde{u}(x', z) = \frac{2^{r'/2}}{r'}\left(\left(\frac{h(x')}{2}\right)^{r'} - \left|\frac{h(x')}{2} - z\right|^{r'}\right) |\tilde{f} - \nabla_{x'} p|^{r'-2} (\tilde{f} - \nabla_{x'} p)$$

□

Proof of Proposition 3.4 : Let

$$g(x') = 2^{r'/2}(\tilde{f}(x') - \nabla_{x'} p(x')) ,$$

then, by integrating the ordinary differential equation (3.4), we get that existence of a function $C(x') \in (L^{r'}(\omega))^2$, such that

$$\left|\frac{\partial \tilde{u}}{\partial z}\right|^{r-2} \frac{\partial \tilde{u}}{\partial z} = C(x') - z g(x') = \zeta(x', z)$$

and then

$$\tilde{u}(x', z) = \int_0^z |\zeta(x', \xi)|^{r'-2}\,\zeta(x', \xi)\,d\xi . \tag{3.7}$$

Note that x' is acting as a parameter. The boundary condition (3.3) implies that

$$\psi(C) = \int_0^h |C - \xi g|^{r'-2}\,(C - \xi g)\,d\xi = 0 . \tag{3.8}$$

$\psi(C)$ is the gradient with respect to C of a strictly convex function

$$\Phi(C) = \frac{1}{r'} \int_0^h |C - \xi g|^{r'} d\xi$$

and, consequently, C is uniquely determined.

The (unique) trivial solution for the Euler equation (3.8) is

$$C(x') = h(x')\, g(x')/2 .$$

Hence $\tilde{u}(x', z)$ is uniquely determined and (3.7) can be written as

$$\tilde{u}(x', z) = \beta(x', z)\, |g(x')|^{r' - 2} g(x')$$

where

$$\beta(x', z) = \int_0^z \left| \frac{h(x')}{2} - \xi \right|^{r' - 2} \left(\frac{h(x')}{2} - \xi \right) d\xi . \tag{3.9}$$

A simple calculation shows that (3.9) gives (1.13). ■

The standard theory of monotone operators [14] gives :

PROPOSITION 3.5 : *The system* (1.14), (3.1), (3.2) *has a unique solution*

$$(U(x'), p(x')) \in (L^{r'}(\omega))^2 \times W^{1,r'}(\omega) \cap L_0^{r'}(\omega) .$$

□

Proof of the convergence theorem : It is a direct consequence of the preceeding propositions. ■

ACKNOWLEDGEMENT

The authors would like to thank Prof. A. Bourgeat for suggesting them the problem and pointing its physical significance. Furthermore, we would like to thank the anonymous referee for his very constructive remarks.

REFERENCES

[1] J. BARANGER, K. NAJIB, 1990, Analyse numérique des écoulements quasi newtoniens dont la viscosité obéit à la loi de puissance ou à la loi de Carreau, *Numer. Math.* **58,** 35-49.

[2] G. BAYADA, M. CHAMBAT, 1986,The transition between the Stokes equations and the Reynolds equation, a mathematical proof, *Appl. Math. and Opt.,* **14,** 73-93.

[3] R. B. BIRD, R. C. ARMSTRONG, O. HASSAGER, 1987, *Dynamics of polymeric liquids, vol. 1 : fluid mechanics,* John Wiley and Sons, New York.

[4] A. BOURGEAT, A. MIKELIĆ, 1992, Homogenization of a polymer flow through a porous medium, preprint no. 132, Équipe d'Analyse Numérique Lyon-Saint-Étienne (to appear in *Nonlinear Anal. T.M.A.).*

[5] A. BOURGEAT, A. MIKELIĆ, R. TAPIÉRO, 1993, Dérivation des équations moyennées décrivant un écoulement non newtonien dans un domaine de faible épaisseur, *C. R. Acad. Sci,* **316,** Série I, 965-970.

[6] H. BREZIS, 1983, *Analyse fonctionnelle. Théorie et applications,* Masson, Paris.

[7] L. CATTABRIGA, 1961, Su un problema di contorno relativo ai sistemi di equazioni di Stokes, *Rend. Mat. Sem. Univ. Padova,* **31,** 308-340.

[8] H. DRIDI, 1982, Comportement asymptotique des équations de Navier-Stokes dans des domaines applatis, *Bull. Sc. Math.,* **106,** 369-385.

[9] I. EKELAND, R. TEMAM, 1974, *Analyse convexe et problèmes variationnels,* Dunod, Paris.

[10] V. GIRAULT, P. A. RAVIART, 1986, *Finite element methods for Navier-Stokes equations,* Springer Verlag, Berlin.

[11] P. GRISVARD, 1985, *Elliptic problems in non-smooth domains,* Pitman, Boston.

[12] V. A. KONDRATIEV, O. A. OLEINIK, 1988, Boundary value problems for the system of elasticity theory in unbounded domains. Korn's inequality, *Russian Math Surveys,* **43,** 5, 65-119.

[13] H. LE DRET, 1991, *Problèmes variationnels dans les multidomaines,* R.M.A. 19, Masson, Paris.

[14] J.-L. LIONS, 1969, *Quelques méthodes de résolution de problèmes aux limites non linéaires,* Dunod, Paris.

[15] J.-L. LIONS, 1973, *Perturbations singulières dans les problèmes aux limites et en contrôle optimal,* Springer Verlag, Berlin.

[16] A. MIKELIĆ, 1991, Homogenization of nonstationary Navier-Stokes equations in a domain with a grained boundary, *Annali di Mat. pura ed appl.,* **158,** 167-179.

[17] P. P. MOSOLOV, V. P. MJASNIKOV, 1971, A proof of Korn's inequality, *Soviet Math. Doklady.,* **12,** No. 6, 1618-1622.

[18] S. A. NAZAROV, 1990, Asymptotic Solution of the Navier-Stokes Problem on the flow of a thin layer of fluid, translated from *Siberian Math. J.,* **31,** 2, 131-144.

[19] D. SANDRI, 1993, Sur l'approximation numérique des écoulements quasi-newtoniens dont la viscosité suit la loi de puissance ou le modèle de Carreau, $M^2 AN$, **27,** 131-155.

[20] R. TEMAM, 1977, *Navier Stokes equations,* North Holland, Amsterdam.