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STOCHASTIC HOMOGENIZATION OF NONCONVEX INTEGRAL FUNCTIONALS (*)

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Abstract. — *Almost sure epiconvergence of a sequence of random integral functionals is studied without convexity assumption. We give a proof by using an Ergodic theorem and recover and make precise the result of S. Müller in the periodic case. Finally, we study the asymptotic behaviour of corresponding random primal and dual problems in the convex case.*

Résumé. — *Le problème étudié dans cet article concerne l'épiconvergence presque sûre d'une suite de fonctionnelles intégrales aléatoires non nécessairement convexes. On présente une méthode directe utilisant un théorème ergodique, retrouvant ainsi et précisant un résultat de S. Müller obtenu dans le cas périodique. Finallement, dans le cas de fonctionnelles convexes, on étudie la convergence presque sûre des problèmes aléatoires primaux et duaux associés.*

1. INTRODUCTION

In this paper, we propose a method to establish the almost sure convergence, in a sense precised below, of a process $(F_n)_{n \in \mathbf{N}}$ with state space $(\mathcal{F}, \mathcal{B}(\mathcal{F}))$ where $\mathcal{B}(\mathcal{F})$ is a suitable σ-field on the class $\mathcal{F}$ of the integral functionals G of the following form

$$G(u, A) := \int_A g(x, \nabla u(x))\, dx .$$

We denote by A a bounded regular domain in $\mathbf{R}^d$, $u : A \to \mathbf{R}^m$ is a vector valued function defined in a Sobolev space, and $g : \mathbf{R}^d \times M^{m \times d} \to \mathbf{R}$ runs in a class of equicoercive and equibounded Carathéodory functions.

Given a probability space $(\Sigma, \mathcal{T}, P)$ and a measurable map

$$F : (\Sigma, \mathcal{T}) \to (\mathcal{F}, \mathcal{B}(\mathcal{F}))$$

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with

$$F(\omega)(u, A) := \int_A f(\omega)(x, \nabla u(x))\, dx ,$$

if the law of F possesses some *ergodic* and *periodic* properties, the process $(F_n)_{n \in \mathbf{N}}$ defined by

$$F_n(\omega)(u, A) := \int_A f(\omega)\left(\frac{x}{\varepsilon_n}, \nabla u(x)\right) dx$$

epiconverges almost surely, when ε_n tends to 0, towards a constant F^{hom} in $\mathscr{F}$. More precisely, there exists a subset Σ' in $\mathcal{T}$ with $P(\Sigma') = 1$ such that, for every ω in Σ', every bounded regular domain A

$$F^{\text{hom}}(\,.\,, A) = \tau - \underset{n \to +\infty}{\text{epi lim}}\, F_n(\omega)(\,.\,, A)$$

exists in $W^{1,p}(A, \mathbf{R}^m)$ equipped with its weak topology τ or with the strong topology of $L^p(A, \mathbf{R}^m)$. The limit functional F^{hom} is given by

$$F^{\text{hom}}(u, A) := \int_A f^{\text{hom}}(\nabla u)\, dx ,$$

where, for every matrix a of $M^{m \times d}$

$$f^{\text{hom}}(a) = \inf_{n \in N^*} \times$$

$$\times E\left(\frac{1}{\text{meas }(nY)} \inf\left\{\int_{nY} f(\omega)(x, a + \nabla u(x))\, dx,\ u \in W_0^{1,p}(nY, \mathbf{R}^m)\right\}\right)$$

Y denoting the unit cube $]0, 1[^d$ and $E(\,.\,)$ the probability average operator. For the relevant notations and definitions, see part 2.

Under standard hypothesis on the subspace V of $W^{1,p}(A, \mathbf{R}^m)$ and on the map Φ from $W^{1,p}(A, \mathbf{R}^m)$ into $\mathbf{R}$, the variational properties of epiconvergence lead to the almost sure convergence of $\inf\{F_n(\omega)(u, A) + \Phi(u)\,;\, u \in V\}$ towards $\min\{F^{\text{hom}}(\omega)(u, A) + \Phi(u)\,;\, u \in V\}$.

In this way, we generalize the results obtained by G. Dal Maso & L. Modica [11], [12], K. Sab [14] in the stochastic convex case and A. Braides [7] and S. Müller [15] in the periodic non convex case. We give a new proof, establishing the lower bound and the upper bound in the epiconvergence process, by means of an ergodic theorem which seems to be firstly used in the calculus of variation by G. Dal Maso & L. Modica [12] in connection with compactness method. By showing that the infima with respect to n in the above expression of f^{hom} is actually a limit — which follows from the Ackoglu & Krengel ergodic theorem — we slighty improve the result of S. Müller in the determinist case.

This nonconvex approach finds its motivation in non linear elasticity where $f(\omega)$ is the stored energy density of a composite material with random inclusions. With additional assumptions upon the probability space $(\Sigma, \mathcal{T}, P)$ in view to overcome the lack of coerciveness, the homogenization of functionals related to an elastic material with holes and fissures distributed at random could be treated with the same technique. Nevertheless, let us point out that our method requires an equiboundedness property on $f(\omega)$ and thus, the class $\mathcal{F}$ is not a correct model to describe functionals energy in non linear elasticity (see also S. Müller [15]). Homogenization of functionals, even polyconvex, taking their values in $\mathbf{R} \cup \{+\infty\}$, seems to be an open problem.

In the convex case, we study the asymptotic behaviour of random Primal-Dual optimization problems associated with $(F_n)_{n \in \mathbf{N}}$. More precisely, in view to obtain the structural equation $\sigma \in \partial f^{\text{hom}}(e(u))$, where $e(u) := \frac{1}{2}(\nabla u + \nabla u^t)$, which links the weaks limits σ and u of the solutions of primal and dual problems associated with $F_n(\omega)$, we consider again an almost sure epiconvergence process but now upon the sequence of classical perturbation of $F_n(\omega)$, which provides the almost sure weak convergence of corresponding saddle points sequence towards saddle point of the homogenized Lagrangian problem.

Let us clarify the plan of this paper. In the next part, we give the definition and main properties about epiconvergence. In theorem 2.3, we recall the useful almost sure convergence result of M. Ackoglu & U. Krengel [1] about superadditive set function processes. Part 3 is devoted to the definition and properties of the homogenized density f^{hom}. In part 4, we prove the almost sure epiconvergence of the sequence $(F_n)_{n \in \mathbf{N}}$ by means of two lemmas : the upper bound in lemma 4.2, the lower bound in lemma 4.4. In corollary 4.5, we establish the almost sure convergence of optimization problems associated with $(F_n)_{n \in \mathbf{N}}$. Part 5 is devoted to the description of some examples of non homegenous random functions $f(\omega)$ which are a model of stored energy density for material with random inclusions and for which our results can be applied. Finally, in part 6, in a convex situation, we study the asymptotic behaviour of random dual optimization problems.

2. NOTATIONS AND PRELIMINARY RESULTS

For m, d in $\mathbf{N}^*$, $M^{m \times d}$ denotes the space of $m \times d$ matrices $a = (a_{i,j})_{i,j}$, $i = 1, \ldots, m$; $j = 1, \ldots, d$ equipped with the euclidean Hilbert Schmidt product $a : b = \text{trace}\,(a^t b)$. In what follows, we shall denote indifferently the norms in $\mathbf{R}^m$ and $M^{m \times d}$. $\mathcal{O}$ is the set of all open bounded subset of $\mathbf{R}^d$ with Lipschitz boundary. Classically, for $1 < p < +\infty$ and A in $\mathcal{O}$, we define the two Banach spaces

$$L^p(A, \mathbf{R}^m) := \{u : A \to \mathbf{R}^m ; u = (u_i)_i, u_i \in L^p(A), i = 1, \dots, m\}$$

$$W^{1,p}(A, \mathbf{R}^m) := \left\{u \in L^p(A, \mathbf{R}^m); \frac{\partial u_i}{\partial x_j} \in L^p(A), i = 1, \dots, m\right\},$$

respectively equipped with the two norms

$$|u|_{0,A} := \left(\int_A |u(x)|^p \, dx\right)^{\frac{1}{p}}, \quad |u|_{1,A} :=$$

$$:= \left(\int_A |u(x)|^p \, dx + \int_A |\nabla u(x)|^p \, dx\right)^{\frac{1}{p}},$$

where ∇u denotes the matrix valued distribution $\frac{\partial u_i}{\partial x_j}_{i,j}$.

$W_0^{1,p}(A, \mathbf{R}^m)$ is the closed subspace of $W^{1,p}(A, \mathbf{R}^m)$ of the functions with null trace on the boundary ∂A of A. $W_{\mathrm{loc}}^{1,p}(\mathbf{R}^d, \mathbf{R}^m)$ is the space of vector valued functions u, measurable in $\mathbf{R}^d$ and satisfying the following condition : every $x \in \mathbf{R}^d$ possesses a neighborhood A such that the restriction of u in A belongs to $W^{1,p}(A, \mathbf{R}^m)$.

α, β being two given positive constants, we define the subset $\mathcal{F}$ of the product $R^{W_{\mathrm{loc}}^{1,p}(\mathbf{R}^d, \mathbf{R}^m) \times \mathcal{O}}$ as follows : G belongs to $\mathcal{F}$ iff there exists a function g from $\mathbf{R}^d \times M^{m \times d}$ into $\mathbf{R}$, measurable with respect to its first variable, and a positive constant L such that, for every a, b in $M^{m \times d}$ and x a.e.

$$\alpha \, |a|^p \leq g(x, a) \leq \beta(1 + |a|^p) \tag{2.1}$$

$$|g(x, a) - g(x, b)| \leq L(1 + |a|^{p-1} + |b|^{p-1}) \, |a - b| \,, \tag{2.2}$$

with, for every A in $\mathcal{O}$ and u in $W_{\mathrm{loc}}^{1,p}(\mathbf{R}^d, \mathbf{R}^m)$

$$G(u, A) = \int_A g(x, \nabla u(x)) \, dx \,.$$

In the class $\mathcal{F}$, for every $z \in \mathbf{Z}^d$ and every $r \in \mathbf{R}^{*,+}$, we shall need to consider the two following operators τ_z and ρ_z :

$$\tau_z G(u, A) := G(\tau_z u, z + A) = \int_A g(x + z, \nabla u(x)) \, dx \,, \tag{2.3}$$

$$\rho_r G(u, A) := G\left(\rho_r u, \frac{1}{r} + A\right) = \int_A g\left(\frac{x}{r}, \nabla u(x)\right) dx\,,$$

where $\tau_z u(x) := u(x-z)$ and $\rho_r u(x) := \frac{1}{r} u(rx)$.

For every $a \in M^{m \times d}$, l_a denotes the linear vector valued function whose gradient is a, and we set, for every A in $\mathcal{O}$ and G in $\mathcal{F}$

$$\mathcal{M}_A(G, a) := \inf \left\{ G(u + l_a, A)\,;\, u \in W_0^{1,p}(A, \mathbf{R}^m) \right\}\,.$$

We shall use, in the sequel, the following elementary properties.

PROPOSITION 2.1 :

(i) $$\frac{\mathcal{M}_A(\rho_r G, a)}{\text{meas}\,(A)} = \frac{\mathcal{M}_{\frac{1}{r}A}(G, A)}{\text{meas}\left(\frac{1}{r}A\right)}\,, \quad \mathcal{M}_A(\tau_z G, a) = \mathcal{M}_{A+z}(G, a)\,,$$

(ii) *there exists a positive constant L' depending only on L, α, β and p, such that, for every a and b in $M^{m \times d}$*

$$\left| \frac{\mathcal{M}_A(G, a)}{\text{meas}\,(A)} - \frac{\mathcal{M}_A(G, b)}{\text{meas}\,(A)} \right| \leq L'(1 + |a|^{p-1} + |b|^{p-1})\, |a - b|\,.$$

Proof : It is straightforward to check (i). We only prove (ii). For every a in $M^{m \times d}$, let us set $m(a) := \dfrac{\mathcal{M}_A(G, a)}{\text{meas}\,(A)}$.

Let $\eta > 0$ and $u_n \in W_0^{1,p}(A, \mathbf{R}^m)$ such that $m(b) \geq \dfrac{1}{\text{meas}\,(A)}(G(u_\eta + l_b, A) - \eta)$.

Using (2.2) and Hölder's inequality, we obtain

$$\begin{aligned}
m(a) - m(b) &\leq \\
&\leq \frac{1}{\text{meas}\,(A)}(G(u_\eta + l_a, A) - G(u_\eta + l_b, A) + \eta) \\
&\leq \frac{1}{\text{meas}\,(A)} \int_A |g(x, \nabla u_\eta(x) + a) - g(x, \nabla u_\eta(x) + b)|\, dx \\
&+ \frac{\eta}{\text{meas}\,(A)} \\
&\leq |a - b| \left(\frac{1}{\text{meas}\,(A)} \int_A (1 + |a + \nabla u_\eta(x)|^{p-1} \right. \\
&\left. + |b + \nabla u_\eta(x)|^{p-1})^{\frac{p}{p-1}}\, dx \right)^{\frac{p-1}{p}} + \frac{\eta}{\text{meas}\,(A)}
\end{aligned}$$

$$\leq CL\ |a-b| \left(\frac{1}{\text{meas}(A)} \int_A (1+|a|^p+|b|^p+|b+\nabla u_\eta(x)|^p)\,dx \right)^{\frac{p-1}{p}} + \frac{\eta}{\text{meas}(A)} \tag{2.4}$$

where C is a constant depending only on p. On the other hand, by (2.1)

$$\begin{aligned}\frac{1}{\text{meas}(A)} \int_A |b+\nabla u_\eta(x)|^p\,dx &\leq \frac{1}{\alpha\ \text{meas}(A)} G(u_\eta + l_b, A) \\ &\leq \frac{1}{\alpha}\left(m(b) + \frac{\eta}{\text{meas}(A)}\right) \\ &\leq \frac{\beta}{\alpha}(1+|b|^p) + \frac{\eta}{\alpha\ \text{meas}(A)}.\end{aligned}$$

From (2.4) and after making η tends to 0, it follows

$$m(a) - m(b) \leq L'|b-a|\ (1+|a|^{p-1}+|b|^{p-1}).$$

Where L' depends only on p, α and β, which ends the proof. ■

The following notion of convergence has been studied in a more general setting, and, for overview, we refer to H. Attouch [2], G. Dal Maso & L. Modica [11] and their bibliographies. In our case the definition is

DEFINITION 2.2 : *Let* $\{(G_n)_n, G, n \to +\infty\}$ *be a sequence of functionals mapping* $W^{1,p}(A, \mathbf{R}^m)$ *into* $\mathbf{R} \cup \{+\infty\}$ *and let* τ *denote the strong topology of* $L^p(A, \mathbf{R}^m)$ *on the space* $W^{1,p}(A, \mathbf{R}^m)$. *We say that* G_n τ*-epiconverges to* G *at* v *in* $W^{1,p}(A, \mathbf{R}^m)$ *(or* Γ *converges, according to* [10] *iff the two following sentences hold :*

(i) *there exists a sequence* $(v_n)_{n\in\mathbf{N}}$ *of* $W^{1,p}(A, \mathbf{R}^m)$, τ*-converging to* v *such, that*

$$G(v) \geq \limsup_{n\to+\infty} G_n(v_n),$$

(ii) *for every sequence* $(v_n)_{n\in\mathbf{N}}$, τ*-converging to* v *in* $W^{1,p}(A, \mathbf{R}^m)$,

$$G(v) \leq \liminf_{n\to+\infty} G_n(v_n).$$

When this property holds for every v *in* $W^{1,p}(A, \mathbf{R}^m)$, G_n *is said to be* τ*-epiconvergent to* G *in* $W^{1,p}(A, \mathbf{R}^m)$. *It is straightforward to show that* G_n τ*-epiconverges to* G *iff*

$$\tau - \text{epi}\limsup_{n\to+\infty} G_n \leq G \leq \tau - \text{epi}\liminf_{n\to+\infty} G_n$$

where

$$\tau - \operatorname{epi} \limsup_{n \to +\infty} G_n(v) := \min \left\{ \limsup_{n \to +\infty} G_n(v_n) \, ; \, v = \tau - \lim_n v_n \right\} ,$$

$$\tau - \operatorname{epi} \liminf_{n \to +\infty} G_n(v) := \min \left\{ \liminf_{n \to +\infty} G_n(v_n) \, ; \, v = \tau - \lim_n v_n \right\} .$$

Under appropriate technical assumptions, epiconvergence is stable by continuous perturbation and implies the convergence of minimizers, precisely we have

PROPOSITION 2.3 : *Assume that (G_n) τ-epiconverges to G and let Φ be a τ-continuous functional from $W^{1,p}(A, \mathbf{R}^m)$ into $\mathbf{R}$. Then $G_n + \Phi$ τ-epiconverges to $G + \Phi$.*

Moreover if $\{u_n, n \in N\}$, $u_n \in W^{1,p}(A, \mathbf{R}^m)$ is τ-relatively compact in $W^{1,p}(A, \mathbf{R}^m)$ and satisfies

$$G_n(u_n) + \Phi(u_n) < \inf \left\{ G_n(u) + \Phi(u) \, ; \, u \in W^{1,p}(A, \mathbf{R}^m) \right\} + \varepsilon_n .$$

Then any τ-cluster point u of $\{u_n, n \in N\}$ is a minimizer of $(G + \phi)$ and

$$\lim_{n \to +\infty} \operatorname{Inf} \left\{ G_n(v) + \phi(v) \, ; \, v \in W^{1,p}(A, \mathbf{R}^m) \right\} = G(u) + \phi(u) .$$

For a proof, see for instance, H. Attouch [2]. It is a classical result that every τ epilimit is τ-lower semicontinuous (see also H. Attouch [2]) so that, if G_n possesses an epilimit G which is an integral functionals whose integrand g satisfies the growth condition (2.1), g is necessary quasiconvex (see for instance J. M. Ball & F. Murat [5] and C. B. Morrey Jr. [14]).

Let us give now few definitions and results abour *Ergodic Theory*. Let $(\Sigma, \mathcal{T}, P)$ be any probability space and $(\tau_z)_{z \in \mathbf{Z}^d}$ a group of P-preserving transformations on $(\Sigma, \mathcal{T})$, that is to say

(i) τ_z is $\mathcal{T}$-measurable,

(ii) $P \circ \tau_z(E) = E$, for every E in $\mathcal{T}$ and every z in $\mathbf{Z}^d$,

(iii) $\tau_z \circ \tau_t = \tau_{z+t}$, $\tau_{-z} = \tau_z^{-1}$, for every z and t in $\mathbf{Z}^d$.

In addition, if every set E in $\mathcal{T}$ satisfying for every $z \in \mathbf{Z}$, $\tau_z(E) = E$, has a probability 0 or 1, $(\tau_z)_{z \in \mathbf{Z}^d}$ is said to be *Ergodic*. A sufficient condition to ensure Ergodicity of $(\tau_z)_{z \in \mathbf{Z}^d}$ is the following *mixing property* : for every E and F in $\mathcal{T}$

$$\lim_{|z| \to +\infty} P(\tau_z E \cap F) = P(E) P(F) .$$

We denote by $\mathfrak{J}$ the set of intervals $[x, y[$ where x and y belong to

$\mathbf{Z}^d$ and consider a set function $\mathscr{S}$ from $\mathfrak{I}$ into $L^1(\Sigma, \mathcal{T}, P)$ verifying the three conditions :

(i) $\mathscr{S}$ is superadditive, that is, for every $A \in \mathfrak{I}$ such that there exists a finite family $(A_i)_{i \in I}$ of disjoint sets in $\mathfrak{I}$ whose union A belongs to $\mathfrak{I}$, then

$$\mathscr{S}_A(\,.\,) \geqslant \sum_{i \in I} \mathscr{S}_{A_i}(\,.\,),$$

(ii) $\mathscr{S}$ is covariant, that is, for every $A \in \mathfrak{I}$, every $z \in \mathbf{Z}^d$,

$$\mathscr{S}_{A+z} = \mathscr{S}_A \circ \tau_z ,$$

(iii) $\sup \left\{ \dfrac{1}{\text{meas }(A)} \displaystyle\int_{\Sigma} \mathscr{S}_A \, dP, \, A \in \mathfrak{I}, \text{ meas } (A) \neq \emptyset \right\} < +\infty.$

Following M. A. Ackoglu & U. Krengel [1], $\mathscr{S}$ is called a discrete superadditive process. If $-\mathscr{S}$ is superadditive, $\mathscr{S}$ is said subadditive. The following useful almost sure convergence result holds (see M. A. Ackoglu & U. Krengel [1] theorem (2.4), Lemma (3.4) and Remark p. 59) :

THEOREM 2.4 : *When n tends to $+\infty$, $\dfrac{1}{n^d} \mathscr{S}_{[0, n[^d}$ converges almost surely. Moreover, if $(\tau_z)_{z \in k\mathbf{Z}^d}$ is Ergodic, we have, almost surely :*

$$\lim_{n \to +\infty} \frac{1}{n^d} \mathscr{S}_{[0, n[^d}(\omega) = \sup_{n \in \mathbf{N}^*} \frac{1}{n^d} E(\mathscr{S}_{[0, n[^d}(\,.\,))$$

where $E(\,.\,)$ denotes the probability average operator.

3. DEFINITION OF THE PROCESS $\{F_n, F^{\text{hom}};\ n \to +\infty\}$

We denote by $\mathscr{B}(\mathscr{F})$ the trace on $\mathscr{F}$ of the product σ-field of $\mathbf{R}^{W^{1,p}_{\text{loc}}(\mathbf{R}^d, \mathbf{R}^m) \times \mathcal{O}}$, that is, the smallest σ-field on $\mathscr{F}$ such that all the evaluation maps

$$G \mapsto G(u, A), \; u \in W^{1,p}_{\text{loc}}(\mathbf{R}^d, \mathbf{R}^m), \quad A \in \mathcal{O}$$

are $(\mathscr{B}(\mathscr{F}), B(\mathbf{R}))$ measurable, $\mathscr{B}(\mathbf{R})$ denoting the Borel σ-field of $\mathbf{R}$.

The following property is a direct consequence of the definition of $\mathscr{B}(\mathscr{F})$.

PROPOSITION 3.1 : *For every z in $\mathbf{Z}$ and r in $\mathbf{R}^{*,+}$, τ_z and ρ_r are measurable from $(\mathscr{F}, \mathscr{B}(\mathscr{F}))$ into itself.*

We define now the process $\{F_n, n \to +\infty\}$. $(\Sigma, \mathcal{T}, P)$ is a given probability space and F a given measurable map

$$F : (\Sigma, \mathcal{T}) \to (\mathcal{F}, \mathcal{B}(\mathcal{F}))$$
$$\omega \mapsto F(\omega)$$

where

$$F(\omega)(u, A) = \int_A f(\omega)(x, \nabla u(x))\, dx .$$

Conditions on f under which the map F is measurable are well known and will be examined later in section 5.

We assume that $(\tau_z)_{z \in \mathbf{Z}^d}$ defined in (2.3) is a group of μ preserving transformations on the probability space $(\mathcal{F}, \mathcal{B}(\mathcal{F}), \mu)$ where μ is the probability image $P \circ F^{-1}$ of P in $\mathcal{F}$ (or the law of F). Following G. Dal Maso & L. Modica [12], we shall summarize these properties upon F by saying that F is a *random integral functional, periodic in law and ergodic*.

Finally, $\{\varepsilon_n, n \to +\infty\}$ being a given sequence in $\mathbf{R}^{*,+}$ which tends to 0, we define the process $\{F_n, n \to +\infty\}$ as follows

$$F_{\varepsilon_n} : (\Sigma, \mathcal{T}) \to (\mathcal{F}, \mathcal{B}(\mathcal{F}))$$
$$\omega \mapsto F_{\varepsilon_n}(\omega)$$

where

$$F_{\varepsilon_n}(\omega)(u, A) = \int_A f(\omega)\left(\frac{x}{\varepsilon_n}, \nabla u(x)\right) dx$$
$$= \rho_{\varepsilon_n} F(\omega) .$$

Note that the measurability of F_{ε_n} comes from proposition 3.1. In the next part, we shall study, in the sense precised in introduction, the asymptotic behaviour of $\{F_n, n \to +\infty\}$. The main tool, to define the limit denoted by F^{hom}, is the superadditive ergodic theorem 2.4 applied to the map $A \mapsto -\mathcal{M}_A(\cdot\,, a)$ where A belongs to the set $\mathfrak{I}$ of all open bounded intervals $]x, y[$ in $\mathbf{Z}^d$ (or equivalently to the set $\mathfrak{I}$). Let us give some properties of this map.

PROPOSITION 3.2 : *For every a in $M^{m \times d}$, the map $\mathcal{M}(\cdot\,, a)$ defined by*

$$\mathfrak{I} \to L^1(\mathcal{F}, \mathcal{B}(\mathcal{F}), \mu)$$
$$A \mapsto \mathcal{M}_A(\cdot\,, a)$$

is a discrete subadditive ergodic process in $(\mathcal{F}, \mathcal{B}(\mathcal{F}), \mu)$. Moreover $\mathcal{M}_A(\cdot\,, a)$ satisfies the following equiboundedness property on

$L^1(\mathcal{F}, \mathcal{B}(\mathcal{F}), \mu)$

$$|\mathcal{M}_A(\,.\,, a)|_{L^1(\mathcal{F}, \mathcal{B}(\mathcal{F}), \mu)} \leq \beta(1 + |a|^p)\,\text{meas}\,(A)\,.$$

Proof : We begin to establish the $(\mathcal{B}(\mathcal{F}), \mathcal{B}(\mathbf{R}))$ measurability of the map $G \mapsto \mathcal{M}_A(G, a)$ for every A in $\mathfrak{J}$. Noticing that $u \mapsto G(u + l_a, A)$ is a continuous map from the separable space $W^{1,p}(A, \mathbf{R}^m)$ equipped with its strong topology, into $\mathbf{R}$, there exists a dense countable subset $\{u_k,\ k \in \mathbf{N}\}$ of $W^{1,p}(A, \mathbf{R}^m)$ such that

$$\begin{aligned}\mathcal{M}_A(G, a) &= \inf_{k \in \mathbf{N}} \{G(u_k + l_a, A)\} \\ &= \inf_{k \in \mathbf{N}} \psi_k(G)\end{aligned}$$

where, from the definition of the σ-field $\mathcal{B}(\mathcal{F})$, the maps ψ_k : $G \mapsto G(u_k + l_a, A)$ are measurable.

The equiboundedness inequality is a straightforward consequence of the upper growth condition (2.1). So $\mathcal{M}_A(\,.\,, a)$ belongs to $L^1(\mathcal{F}, \mathcal{B}(\mathcal{F}), \mu)$.

Let us prove the subadditivity. Let $A \in \mathfrak{J}$ such that there exists a finite family $(A_i)_{i \in I}$ of disjoint sets in $\mathfrak{J}$ with $A_i \subset A$, $i \in I$, and meas $(A \setminus \cup_{i \in I} A_i) = 0$. For $\eta > 0$, $i \in I$ and $G \in \mathcal{F}$, let $u_{i,\eta} \in W_0^{1,p}(A_i, \mathbf{R}^m)$ such that

$$G(u_{i,\eta} + l_a, A_i) \leq \mathcal{M}_{A_i}(G, a) + \eta$$

and define u_η in $W_0^{1,p}(A, \mathbf{R}^m)$ by setting $u_\eta := u_{i,\eta}$ on A_i. We get

$$\begin{aligned}\mathcal{M}_A(G, a) &\leq G(u_\eta, A) \\ &= \sum_{i \in I} G(u_{i,\eta}, A_i) \\ &\leq \sum_{i \in I} \mathcal{M}_{A_i}(G, a) + \eta\ \text{Card}\ (I)\end{aligned}$$

and subadditivity is obtained as η tends to 0. Covariance property has been already proved in proposition 2.1 (i). ■

We are now in position to define an integral functional F^{hom} in $\mathcal{F}$ which will be the expected limit.

COROLLARY 3.3 : *There exists a subset Σ' of Σ in $\mathcal{C}$ with $P(\Sigma') = 1$ and a function $f^{\text{hom}} : M^{m \times d} \to \mathbf{R}$ such that, for all ω in Σ', all cube Q in $\mathbf{R}^d$ and all a in $M^{m \times d}$*

$$\begin{aligned}f^{\text{hom}}(a) &:= \lim_{t \to +\infty} \frac{\mathcal{M}_{tQ}(F(\omega), a)}{\text{meas}\ (tQ)} \\ &= \inf_{n \in N^*} \left\{E\left(\frac{\mathcal{M}_{nY}(F(\,.\,), a)}{\text{meas}\ (nY)}\right)\right\}\end{aligned}$$

where $E(\,.\,)$ denotes the average operator with respect to the probability *measure P. Moreover* f^{hom} *satisfies* (2.1) *and* (2.2) *with* L' *defined in proposition* 2.1.

Proof: *First step.* We assume that a belongs to the subset $M'^{m \times d}$ of matrices in $M^{m \times d}$ with rational entries.

Combining proposition 3.2, theorem 2.4 with the probability space $(\mathcal{F}, \mathcal{B}(\mathcal{F}), \mu)$, for every a in $M'^{m \times d}$, we obtain the existence of a set E_a in $\mathcal{B}(\mathcal{F})$ with $\mu(E_a) = 1$ and a real $f^{\text{hom}}(a)$ such that, for all G in E_a

$$f^{\text{hom}}(a) := \lim_{n \to +\infty} \frac{\mathcal{M}_{nY}(G, a)}{\text{meas }(nY)}$$

$$= \inf_{n \in \mathbf{N}^*} \left\{ \int_{\mathcal{F}} \frac{\mathcal{M}_{nY}(H, a)}{\text{meas }(nY)} \, d\mu(H) \right\}.$$

Setting $\Sigma' = F^{-1}\left(\bigcap_{a \in M'^{m \times d}} E_a \right)$, we obtain, from above

$$\begin{aligned} f^{\text{hom}}(a) &:= \lim_{n \to +\infty} \frac{\mathcal{M}_{nY}(F(\omega), a)}{\text{meas }(nY)} \\ &= \inf_{n \in N^*} \left\{ E\left(\frac{\mathcal{M}_{nY}(F(\,.\,), a)}{\text{meas }(nY)} \right) \right\} \end{aligned} \tag{3.1}$$

for every a in $M'^{m \times d}$ and ω in Σ'.

Let Q be any open cube in $\mathbf{R}^d$ with side η and, for every t in $\mathbf{R}^{*,+}$, set $k^- = [t\eta] - 1$, $k^+ = [t\eta] + 1$, and consider $Q^- = k^-(Y + z)$, $Q^+ = k^+(Y + z')$ the two cubes such that $z, z' \in Z^d$, $Q^- \subset tQ \subset Q^+$. Thanks to the inequality

$$\mathcal{M}_A(F(\omega), a) \leq \mathcal{M}_B(F(\omega), a) + \beta(1 + |a|^p) \text{ meas } (A \setminus B)$$

whenever $B \subset A$ in $\mathcal{O}$ and noticing that meas (tQ) is equivalent to meas $(k^+ Y)$ and meas $(k^- Y)$ whenever t tends to $+\infty$, we get, from (3.1) and the covariance property,

$$\begin{aligned} f^{\text{hom}}(a) = \lim_{t \to +\infty} \frac{\mathcal{M}_{k^+Y}(F(\tau_z \omega), a)}{\text{meas }(tQ)} &\leq \liminf_{t \to +\infty} \frac{\mathcal{M}_{tQ}(F(\omega), a)}{\text{meas }(tQ)} \\ &\leq \limsup_{t \to +\infty} \frac{\mathcal{M}_{tQ}(F(\omega), a)}{\text{meas }(tQ)} \\ &\leq \lim_{t \to +\infty} \frac{\mathcal{M}_{k^-Y}(F(\tau_{z'} \omega), a)}{\text{meas }(tQ)} \\ &= f^{\text{hom}}(a) \end{aligned}$$

for every a in $M'^{m \times d}$ and ω in Σ', which concludes this first step.

Second step. We extend the result of previous step to every a in $M^{m \times d}$. In that follows, ω will be a fixed element of Σ'. Using Proposition 2.1 and above step, it is clear that f^{hom} satisfies the locally Lipschitz condition (2.2) with the new constant L' for every a and b in $M'^{m \times d}$. So, by a classical argument, one can extend f^{hom} to $M^{m \times d}$ by setting, for every r in $M^{m \times d}$, $f^{\text{hom}}(r) := \lim_{n \to +\infty} f^{\text{hom}}(a_n)$ where $\{a_n, n \to +\infty\}$ is any sequence in $M'^{m \times d}$ converging towards r. It is straightforward to check that this extension verifies the same condition (2.2).

On the other hand, from

$$\left| f^{\text{hom}}(r) - \frac{\mathcal{M}_{tQ}(F(\omega), r)}{\text{meas}(tQ)} \right| \leq \left| f^{\text{hom}}(r) - f^{\text{hom}}(a_n) \right| + \left| f^{\text{hom}}(a_n) - \frac{\mathcal{M}_{tQ}(F(\omega), a_n)}{\text{meas}(tQ)} \right| + \left| \frac{\mathcal{M}_{tQ}(F(\omega), a_n)}{\text{meas}(tQ)} - \frac{\mathcal{M}_{tQ}(F(\omega), r)}{\text{meas}(tQ)} \right|,$$

using proposition 2.1 (ii) for the last term and going to the limit in a_n towards r, in t towards $+\infty$, we get

$$f^{\text{hom}}(r) = \lim_{t \to +\infty} \frac{\mathcal{M}_{tQ}(F(\omega), r)}{\text{meas}(tQ)}$$

which concludes this step.

Third step. It remains to prove that f^{hom} satisfies the growth condition (2.1). The upper bound is just a consequence of the proposition 3.2. For the lower bound, using (2.1) and the convexity of $r \mapsto |r|^p$, we get

$$\frac{\mathcal{M}_{tQ}(F(\omega), r)}{\text{meas}(tQ)} \geq$$

$$\geq \alpha \inf \left\{ \frac{1}{\text{meas}(tQ)} \int_{tQ} |a + \nabla u(x)|^p \, dx, \; u \in W_0^{1,p}(tQ, \mathbf{R}^m) \right\}$$

$$= \alpha |a|^p$$

which gives our result after going to the limit in t. ■

We can now define the integral functional F^{hom} in the class $\mathcal{F}$ by

$$F^{\text{hom}}(u, A) := \int_A f^{\text{hom}}(\nabla u(x)) \, dx .$$

4. ALMOST SURE EPICONVERGENCE OF THE PROCESS $\{F_n, F^{\mathrm{hom}}; n \to +\infty\}$

THEOREM 4.1 : *Let Σ' be the subset of Σ with $P(\Sigma') = 1$ defined in the previous section. For all ω in Σ' and all A in $\mathcal{O}$, we have*

$$F^{\mathrm{hom}}(\,.\,, A) = \tau - \operatorname*{epi\,lim}_{n \to +\infty} F_n(\omega)(\,.\,, A)$$

in $W^{1,p}(A, \mathbf{R}^m)$ equipped with its weak topology τ, or the strong topology of $L^p(A, \mathbf{R}^m)$.

We shall give the proof with τ denoting the strong topology of $L^p(A, \mathbf{R}^m)$. From the lower growth condition (2.1) and the compact embedding from $W^{1,p}(A, \mathbf{R}^m)$ into $L^p(A, \mathbf{R}^m)$, we can easily conclude in the other case.

The proof of theorem 4.1 will be established by means of two lemmas : the upper bound in the definition of epiconvergence is proved in lemma 4.2, the lower bound in lemma 4.4, lemma 4.3 being just a simple technical lemma. In all what follows, ω denotes a fixed element of Σ'.

LEMMA 4.2 : *For every A in $\mathcal{O}$, every u in $W^{1,p}(A, \mathbf{R}^m)$*

$$F^{\mathrm{hom}}(u, A) \leq \operatorname*{epi\,lim\,inf}_{n \to +\infty} F_n(\omega)(u, A)\,,$$

that is to say, for every sequence $\{u_n, n \to +\infty\}$, $\tau -$ converging towards u,

$$F^{\mathrm{hom}}(u, A) \leq \liminf_{n \to +\infty} F_n(\omega)(u_n, A).$$

Proof of lemma 4.2.

First step. A is an open cube Q and u is defined by $u(x) = l_a(x)$, $a \in M^{m \times d}$. It is convenient and involves no loss of generality, to assume that $u_n - l_a$ belongs to $W_0^{1,p}(Q, \mathbf{R}^m)$ (see for instance S. Müller [15] or G. Dal Maso & L. Modica [10], [11]). In this case, by definition of F^{hom}, by corollary 3.3 and proposition 2.1 (i), we get

$$\begin{aligned} F^{\mathrm{hom}}(u, Q) &= \operatorname{meas}(Q)\, f^{\mathrm{hom}}(a) \\ &= \operatorname{meas}(Q) \lim_{n \to +\infty} \frac{\mathcal{M}_{\frac{1}{\varepsilon_n} Q}(F(\omega), a)}{\operatorname{meas}\left(\dfrac{1}{\varepsilon_n} Q\right)} \\ &= \operatorname{meas}(Q) \lim_{n \to +\infty} \frac{\mathcal{M}_Q(F_n(\omega), a)}{\operatorname{meas}(Q)} \\ &\leq \liminf_{n \to +\infty} F_n(\omega)(u_n, Q) \end{aligned}$$

which ends the first step.

Second step. We assume that $A \in \mathcal{O}$ and $u = l_a$.

For $\eta > 0$, there exists a finite family $(Q_i)_{i \in I(\eta)}$ of disjoint open cubes include in A, such that meas $\left(A \backslash \bigcup_{i \in I(\eta)} Q_i\right) \leq \eta$. Since f^{hom} satisfies the growth condition (2.1), we get

$$F^{\text{hom}}(u, A) \leq \sum_{i \in I(\eta)} F^{\text{hom}}(u, Q_i) + \eta \beta (1 + |a|^p).$$

Using previous step, superadditivity and non decreasing properties of the set function $B \mapsto \tau - \text{epi} \liminf_{n \to +\infty} F_n(\omega)(\,.\,, B)$ (see H. Attouch [2], pp. 156-157), we obtain

$$\begin{aligned} F^{\text{hom}}(u, A) &\leq \sum_{i \in I(\eta)} \tau - \text{epi} \liminf_{n \to +\infty} F_n(\omega)(u, Q_i) + \eta \beta (1 + |a|^p) \\ &\leq \tau - \text{epi} \liminf_{n \to +\infty} F_n(\omega)(u, A) + \eta \beta (1 + |a|^p) \end{aligned}$$

and we conclude by letting η tends to 0.

Third step. $A \in \mathcal{O}$ and $u \in W^{1,p}(A, \mathbf{R}^m)$. We use the density of piecewise affine continuous function in $W^{1,p}(A, \mathbf{R}^m)$ (see I. Ekeland & R. Temam [13]) and the previous step.

Let u, u_n in $W^{1,p}(A, \mathbf{R}^m)$ such that $u = \tau - \lim_{n \to +\infty} u_n$. For $\eta > 0$, there exist a finite partition $(A_i)_{i \in I}$ of A with $A_i \in \mathcal{O}$, and u_η in $W^{1,p}(A_i, \mathbf{R}^m)$ such that $|u - u_\eta|_{1,A} \leq \eta$ and such that its restriction $u_{\eta,i}$ is affine on A_i.

Set $v_{n,\eta} := u_\eta + u_n - u$ and denote by $v_{n,\eta,i}$ its restriction to A_i. By using the second step, we get, for every $i \in I$

$$\begin{cases} u_{\eta,i} = \tau - \lim_{n \to +\infty} v_{n,\eta,i} \quad \text{in} \quad L^p(A_i, \mathbf{R}^m) \\ F^{\text{hom}}(u_{\eta,i}, A_i) \leq \liminf_{n \to +\infty} F_n(\omega)(v_{n,\eta,i}, A_i). \end{cases}$$

After summation over i, with superadditivity of lim inf, we obtain

$$F^{\text{hom}}(u_\eta, A) \leq \liminf_{n \to +\infty} F_n(\omega)(v_{n,\eta}, A). \tag{4.1}$$

On the other hand, by (2.2)

$$\begin{aligned} F_n(\omega)(v_{n,\eta}, A) &\leq F_n(\omega)(u_n, A) + \\ &+ L \int_A (1 + |\nabla u_n(x)|^{p-1} + |\nabla v_{n,\eta}(x)|^{p-1}) |\nabla u_n(x) - \nabla v_{n,\eta}(x)| \, dx \end{aligned}$$

and after using Hölder inequality, we get, up to a further subsequence with respect to n

$$F_n(\omega)(v_{n,\eta}, A) \le F_n(\omega)(u_n, A) + C\,|u_\eta - u|_{1,A}$$
$$\le F_n(\omega)(u_{n,A}) + C\eta\ , \qquad (4.2)$$

where C will denote any constant that does not depends on η and n. Note that we have assumed $\liminf_{n\to+\infty} F_n(\omega)(u_n, A) < +\infty$ and so, thanks to the growth condition (2.1), up to a further subsequence with respect to n, u_n and $v_{n,\eta}$ bounded in $W^{1,p}(A, \mathbf{R}^m)$.

Finally, by continuity property of $F^{\hom}$,

$$F^{\hom}(u_\eta, A) \ge F^{\hom}(u, A) -$$
$$- L' \int_A (1 + |\nabla u(x)|^{p-1} + |\nabla u_\eta(x)|^{p-1})|\nabla u(x) - \nabla u_\eta(x)|\, dx$$
$$\ge F^{\hom}(u, A) - C\eta\ . \qquad (4.3)$$

From (4.1), (4.2) and (4.3), after letting η tends to 0, we get

$$F^{\hom}(u, A) \le \liminf_{n\to+\infty} F_n(\omega)(u_n, A)$$

which ends the proof of lemma (4.2). ■

Before proving the lower bound in the definition of epiconvergence, we shall need the following estimation for any η-approximate minimizer of $\mathcal{M}_Q(F_n(\omega), a)$.

LEMMA 4.3 : *Let $\eta > 0$, Q an open cube in $\mathbf{R}^d$ with side η of the lattice in $\mathbf{R}^d$ spanned by $]0, \eta[$, and $v_{n,\eta}(\omega)$ in $W_0^{1,p}(Q, \mathbf{R}^m)$ such that*

$$F_n(\omega)(v_{n,\eta}(\omega) + l_a, Q) \le \mathcal{M}_Q(F_n(\omega), a) + \eta\ .$$

Then

$$|(v_{n,\eta}(\omega))|_{0,Q}^p \le C\eta^p(\text{meas}\ (Q) + \eta)$$

where the constant C depends only on α, β, and a.

Proof : In that follows, C will denote different constants depending only on α, β, and a. By the growth condition (2.1), omitting the variable ω, we get

$$\begin{aligned} |\nabla v_{n,\eta} + a|_{0,Q}^p &\le \frac{1}{\alpha} F_n(\omega)(v_{n,\eta} + l_a, Q) \\ &\le \frac{1}{\alpha}\mathcal{M}_Q(F_n(\omega), a) + \frac{\eta}{\alpha} \\ &\le \frac{\beta}{\alpha}(1 + |a|^p)\ \text{meas}\ (Q) + \frac{\eta}{\alpha} \\ &\le C\ \text{meas}\ (Q) + \frac{\eta}{\alpha}\ . \end{aligned} \qquad (4.4)$$

On the other hand

$$|v_{n,\eta}|^p_{0,Q} \leq C\,\eta^p |\nabla v_{n,\eta}|^p_{0,Q},$$

where C is the Poincaré's constant in $W_0^{1,p}(Y, \mathbf{R}^m)$. Recalling (4.4), we obtain

$$|v_{n,\eta}(\omega)|^p_{0,Q} \leq C\,\eta^p(\text{meas}\,(Q) + \eta)$$

which closes the proof of lemma 4.3. ■

LEMMA 4.4 : *For every A in $\mathcal{O}$ and every u in $W^{1,p}(A, \mathbf{R}^m)$, there exists a sequence $\{u_n(\omega)\,;\, n \to +\infty\}$ in $W^{1,p}(A, \mathbf{R}^m)$ such that*

$$\begin{cases} u = \tau - \lim\limits_{n \to +\infty} u_n(\omega), \\ F^{\text{hom}}(u, A) \geq \limsup\limits_{n \to +\infty} F_n(\omega)(u_n(\omega), A). \end{cases}$$

Proof :

First step. We prove the previous lemma when $u = l_a$, $a \in M^{m \times d}$.

Let $\eta > 0$ and $(Q_i)_{i \in I(\eta)}$, $(Q_j)_{j \in J(\eta)}$ two finite family of open disjoint cubes with side η of the lattice in $\mathbf{R}^d$ spanned by $]0, \eta[$, such that

$$\bigcup_{i \in I(\eta)} Q_i \subset A \subset \bigcup_{i \in I(\eta) \cup J(\eta)} Q_i$$

and meas $\left(\bigcup_{j \in J(\eta)} Q_j\right) = \delta(\eta)$ with $\lim\limits_{\eta \to 0} \delta(\eta) = 0$.

From the definition of F^{hom}, corollary 3.3 and proposition 2.1, we get

$$\begin{aligned} F^{\text{hom}}(u, A) &\geq F^{\text{hom}}\left(u, \bigcup_{i \in I(\eta)} Q_i\right) \\ &= \sum_{i \in I(\eta)} \text{meas}\,(Q_i)\, f^{\text{hom}}(a) \\ &= \lim_{n \to +\infty} \sum_{i \in I(\eta)} \mathcal{M}_{Q_i}(F_n(\omega), a). \end{aligned} \tag{4.5}$$

The suitable sequence $\{u_n(\omega), n \to +\infty\}$ will be deduced from the approximate minimizers of $\mathcal{M}_{Q_i}(F_n(\omega), a)$. Precisely, let $v_{n,\eta,i}(\omega)$ in $W_0^{1,p}(A, \mathbf{R}^m)$ such that

$$F_n(\omega)(v_{n,\eta,i}(\omega) + l_a, Q_i) \leq \mathcal{M}_{Q_i}(F_n(\omega), a) + \frac{\eta}{\text{card}\,(I(\eta) \cup J(\eta))},$$

and define $v_{n,\eta}$, $u_{n,\eta}$ in $W^{1,p}_{\text{loc}}(\mathbf{R}^d, \mathbf{R}^m)$ by $v_{n,\eta} = v_{n,\eta,i}$ in Q_i and $u_{n,\eta} = v_{n,\eta} + u$. Recalling (4.5), we get

$$F^{\text{hom}}(u, A) \geq \limsup_{n \to +\infty} F_n(\omega)\left(u_{n,\eta}, \bigcup_{i \in I(\eta)} Q_i\right) - \eta$$
$$\geq \limsup_{n \to +\infty} F_n(\omega)(u_{n,\eta}, A) - \beta(1 + |a|^p)\,\delta(\eta) - 2\eta .$$

Therefore

$$F^{\text{hom}}(u, A) \geq \limsup_{\eta \to 0} \limsup_{n \to +\infty} F_n(\omega)(u_{n,\eta}, A) . \tag{4.6}$$

On the over hand

$$|u_{n,\eta} - u|^p_{0,A} = |v_{n,\eta}|^p_{0,A}$$

and thanks to the lemma 4.3

$$|u_{n,\eta} - u|^p_{0,A} \leq C \sum_{i \in I(\eta) \cup J(\eta)} \left(\eta^p\left(\text{meas}\,(Q_i) + \frac{\eta}{\text{card}\,(I(\eta) \cup J(\eta))}\right)\right)$$
$$\leq C\,\eta^p(\text{meas}\,(B) + \eta) \tag{4.7}$$

where C is a constant that depends only on p, α, β, a, and B is any bounded set containing A. From (4.6), (4.7) and using a diagonalization argument (see H. Attouch [2], cororollary 1.16), there exists a map $n \mapsto \eta(n)$ such that $\eta(n)$ tends to 0 when n tends to $+\infty$ and such that

$$\begin{cases} u = \tau - \lim_{n \to +\infty} u_{n,\eta(n)} \\ F^{\text{hom}}(u, A) \geq \limsup_{n \to +\infty} F_n(\omega)(u_{n,\eta(n)}, A) \end{cases}$$

and it suffices to set $u_n := u_{n,\eta(n)}$.

Second step. We prove lemma 4.4 for any u in $W^{1,p}(A, \mathbf{R}^m)$.

By continuity of F^{hom} in $W^{1,p}(A, \mathbf{R}^m)$, it suffices to prove lemma 4.4 when u is assumed to be a piecewise affine continuous function by applying previous step, and to conclude again by a diagonalization argument. More precisely, there exists a finite partition $(A_i)_{i \in I}$ of A, $A_i \in \mathcal{O}$ such that $u = l_{a_i} + b_i$ in A_i, with $a_i \in M^{m \times d}$, $b_i \in \mathbf{R}^m$. Using the first step, there exists $v_{n,i}$ in $W^{1,p}(A_i, \mathbf{R}^m)$, possibly depending on ω, such that

$$\begin{cases} u = \tau_i - \lim_{n \to +\infty} v_{n,i} \\ F^{\text{hom}}(u, A_i) \geq \limsup_{n \to +\infty} F_n(\omega)(v_{n,i}, A_i) \end{cases}$$

where τ_i denotes the strong topology of $L^p(A_i, \mathbf{R}^m)$. By an argument proved in G. Dal Maso & L. Modica [10], [11], modifying $v_{n,i}$ in a neighbourhood of the boundary ∂A_i of A_i, we can construct a sequence of functions $u_{n,i}$ in $W^{1,p}(A_i, \mathbf{R}^m)$, such that

$$\begin{cases} u = \tau_i - \lim\limits_{n \to +\infty} u_{n,i},\ u_{n,i} = u \quad \text{on} \quad \partial A_i \\ F^{\text{hom}}(u, A_i) \geq \limsup\limits_{n \to +\infty} F_n(\omega)(u_{n,i}, A_i)\,. \end{cases}$$

The sequence $(u_n)_{n \in N}$ of $W^{1,p}(A, \mathbf{R}^m)$ defined by $u_n := u_{n,i}$ in A_i, satisfies, after summing over i

$$\begin{cases} u = \tau - \lim\limits_{n \to +\infty} u_n \\ F^{\text{hom}}(u, A) \geq \limsup\limits_{n \to +\infty} F_n(\omega)(u_n, A)\,. \end{cases}$$

When u belongs to $W^{1,p}(A, \mathbf{R}^m)$, we conclude, like in the last step in the proof of lemma 4.2, by a density and diagonalization argument (see also S. Müller [15]), which ends the proof of lemma 4.4 and theorem 4.1. ■

We give now the following consequence of theorem 4.1.

COROLLARY 4.5 : *Let Ω be a set of $\mathcal{O}$, Γ_0 a subset of the boundary $\partial\Omega$ of Ω with strictly positive surface measure and V the subset*

$$\{u \in W^{1,p}(\Omega, \mathbf{R}^m),\ u = u_0 \ \textit{on}\ \Gamma_0\}$$

where u_0 is a given element of $W^{1,p}(\Omega, \mathbf{R}^m)$. If F is a random integral functional, periodic in law and ergodic, Φ a continuous map from V into $\mathbf{R}$, V being equipped with the weak topology of $W^{1,p}(\Omega, \mathbf{R}^m)$, then $F^{\text{hom}}(\,.\,, \omega)$ is lower semi continuous for the weak topology of $W^{1,p}(\Omega, \mathbf{R}^m)$, f^{hom} is quasiconvex, and

$$\inf \{F(\omega)(u, \Omega) + \Phi(u),\ u \in V\}$$

converges almost surely towards

$$\min \{F^{\text{hom}}(u, \Omega) + \Phi(u),\ u \in V\}\,.$$

Proof : ω is a fixed element of Σ'. Every τ-epilimit being τ-lower semi continuous, it follows, from theorem 4.1 that $F^{\text{hom}}(\,.\,, \omega)$ is lower semi continuous for the weak topology of $W^{1,p}(\Omega, \mathbf{R}^m)$ and that f^{hom} is quasiconvex (see J. M. Ball & F. Murat [5], C. B. Morrey Jr. [14]).

For the last statement, using variational properties of epiconvergence recalled in the second section, it remains to prove that

$$F^{\hom}(\,.\,,\Omega)+\Phi=\tau-\operatorname{epi}\lim_{n\to+\infty}F_n(\omega)(\,.\,,\Omega)+\Phi$$

in V equipped with the weak topology of $W^{1,p}(\Omega,\mathbf{R}^m)$. But Φ being a τ-continuous perturbation of $F_n(\Omega)(\,.\,,\Omega)$, it suffices to prove that :

$$F^{\hom}(\,.\,,\Omega)=\tau-\operatorname{epi}\lim_{n\to+\infty}F_n(\omega)(\,.\,,\Omega)$$

takes place in V, and thus, that, for every u in V, there exists a sequence $\{u_n(\omega)\,;\,n\to+\infty\}$ in V satisfying

$$\begin{cases} u=\tau-\lim\limits_{n\to+\infty}u_n(\omega),\\ F^{\hom}(u,\Omega)\geqslant\limsup\limits_{n\to+\infty}F_n(\omega)(u_n(\omega)\,.\,\Omega)\,.\end{cases}$$

For this, it suffices to modify, in a neighbourhood of $\partial\Omega$, the sequence of functions $u_n(\omega)$ obtained in lemma 4.4, in such a way to preserve above condition, with, in addition, $u=u_n(\omega)$ in $\partial\Omega$ (see again G. Dal Maso & L. Modica [10], [11]). ■

5. SOME EXAMPLES OF RANDOM INTEGRAL FUNCTIONALS

We would like to give in this section, some examples of non homogeneous random functions $f(\omega)$ which will be a model of stored energy density for material with inclusions distributed at random and for which, the corresponding integral functional is *a random integral functional, periodic in law and ergodic.*

Let us denote by $\mathscr{A}$ the set of functions g defined in part 2, equipped with the trace σ-field $\sigma(\mathscr{A})$ of the product σ-field of $\mathbf{R}^{\mathbf{R}^d\times M^{m\times d}}$ and define the group of transformation $(\tau_z)_{z\in\mathbf{Z}^d}$ in $\mathscr{A}$, by

$$\tau_z\,g(x,a)=g(x+z,a)\,.$$

Consider a map f from $\Sigma\times\mathbf{R}^d\times M^{m\times d}$ into $\mathbf{R}$, which is $(\mathscr{T}\otimes\mathscr{B}(\mathbf{R}^d)\otimes\mathscr{B}(M^{m\times d}),\mathscr{B}(\mathbf{R}))$ measurable and such that, for every ω in Σ, $f(\omega,\,.\,,\,.\,)$ belongs to $\mathscr{A}$. It is clear that the maps $\tau_z f$ from Σ into $\mathscr{A}$ are $(\mathscr{T},\sigma(\mathscr{A}))$ measurable.

Following G. Dal Maso & L. Modica, we say that f is *periodic in law* if, for every z in $\mathbf{Z}^d$, $P\circ f^{-1}=P\circ(\tau_z f)^{-1}$, and that f is *ergodic,* if, for every E in $\sigma(\mathscr{A})$ such that, for every z in $\mathbf{Z}^d$ $\tau_z(E)=E$, we have $P\circ f^{-1}(E)\in\{0,1\}$.

With some slight modifications of the proof of G. Dal Maso & L. Modica [12], one can easily show that corresponding random integral functional $\omega \mapsto F(\omega)$ from Σ into $\mathcal{F}$ defined by

$$F(\omega)(u, A) := \int_A f(\omega)(x, \nabla u(x))\, dx$$

is *periodic in law and ergodic* in the sense of the part 3. (Note that no convexity asumption is required to obtain this result in the proof of [12].)

So, we have, by definition of the σ-field $\sigma(\mathscr{A})$, the two following sufficient conditions to obtain the *periodicity in law and ergodicity* of F.

PROPOSITION 5.1 : *If, for all finite family* $(x_i, a_i)_{i \in I}$ *of* $\mathbf{R}^d \times M^{m \times d}$, *the random vectors* $(f(\,.\,, x_i, a_i))_{i \in I}$ *and* $(f(\,.\,, x_i + z, a_i))_{i \in I}$ *have the same law for every* z *in* $\mathbf{Z}^d$, *then* F *is periodic in law.*

If, for all finite family $(x_i, a_i, r_i)_{i \in I}$ *and* $(y_j, b_j, s_j)_{j \in J}$ *of* $\mathbf{R}^d \times M^{m \times d} \times \mathbf{R}$

$$\lim_{|z| \to +\infty,\, z \in \mathbf{Z}^d} P([f(\,.\,, x_i + z, a_i) > r_i] \cap [f(\,.\,, y_j, b_j) > s_j]) =$$
$$= P([f(\,.\,, x_i, a_i) > r_i])\, P([f(\,.\,, y_j, b_j) > s_j])$$

then F *is ergodic.*

Example 1 : Let $D = \{g_i \,;\, i \in I\}$ be a given finite set of homogeneous stored energy density g_i from $M^{m \times d}$ into $\mathbf{R}$ which satisfies the conditions (2.1) and (2.2) of part 2. Define the set Σ by $\Sigma := \{\omega = (\omega_z)_{z \in \mathbf{Z}^d}\,;\, \omega_z \in D\}$ equipped with the σ-field generated by the cylinders $E_{z,i} := \{\omega \,;\, \omega_z = g_i\}$, $i \in I$, $z \in \mathbf{Z}^d$, and let P be the probability product, construct from the probability presence of g_i in D. We define a non homogeneous random stored energy density f by :

$$f(\omega, x, a) := \omega_z(a) \quad \text{if} \quad x \in Y + z\,.$$

f is then a model for a stored energy density of a composite material in $\mathbf{R}^d$ with a random presence of inclusions in a rescaled periodic structure.

It is straightforward to check that f satisfies the hypothesis of proposition 5.1 and so defines a *random integral functional* F, *periodic in law and ergodic.*

Example 2 : Let g, h be two homogeneous stored energy density which satisfy the conditions (2.1) and (2.2). On the other hand, consider a ponctual *Poisson process* $\omega \mapsto \mathcal{N}(\omega, .)$ from a probability space $(\Sigma, \mathcal{T}, P)$ into $\mathbf{N}^{\mathcal{B}(\mathbf{R}^d)}$ which satisfies (see for instance N. Bouleau [6]) :

(i) For every bounded Borel set A in $\mathbf{R}^d$,

$$\mathcal{N}(\omega, A) = \sum_{y \in D(\omega)} \delta_y(A)$$

where $\delta_y(A)$ denotes the Dirac measure with support $\{y\}$ and $D(\omega)$ is a given countable subset of $\mathbf{R}^d$ without cluster point,

(ii) for every finite family $(A_i)_{i \in I}$ of bounded Borel sets in $\mathbf{R}^d$, two by two disjoint, $(\mathcal{N}(\,.\,, A_i))_{i \in I}$ are independent *random variables,*

(iii) for every bounded Borel set A and every $k \in \mathbf{N}$

$$P([\mathcal{N}(\,.\,, A) = k]) = \mu^k \text{meas}\,(A)^k \frac{\exp(-\mu \text{ meas}\,(A))}{k!}\,.$$

(Note that $\mathcal{N}(\omega, A) = \text{card}\,(A \cap D(\omega))$ and that $E(\mathcal{N}(\,.\,, A)) = \mu \text{ meas}\,(A)$.)

For a given $r > 0$, we define the random non homogeneous stored energy density by

$$f(\omega, x, a) := g(a) + (h(a) - g(a)) \min\,(1, \mathcal{N}(\,.\,, B(x, r)))$$

$$\text{that is} \quad f(\omega, x, a) = \begin{cases} h(a) & \text{if} \quad x \in \bigcup_{y \in D(\omega)} B(y, r)\,, \\ g(a) & \text{if not}\,. \end{cases}$$

f is then a model for a stored energy density of a composite material in $\mathbf{R}^d$, $(B(y, r))_{y \in D(\omega)}$ being the rescaled random inclusions with a probability expectation μ meas (A) in every bounded Borel set A. One can see that f satisfies the hypothesis of proposition 5.1 and so defines a *random integral functional F, periodic in law and ergodic.*

6. STOCHASTIC HOMOGENIZATION AND DUALITY IN THE CONVEX CASE

In this section, we study the asymptotic behaviour of the classical perturbed optimization problem when $f(\omega, x, .)$ is convex, leading to the limit of its dual formulation. We get in this way, the structural equation $\sigma \in \partial f^{\text{hom}}(e(u))$ where $e(u) := \dfrac{\nabla u + {}^t\nabla u}{2}$, which links the weak limits u and σ of the solutions of *Primal and Dual* problems corresponding to $F_n(\omega)$. We adopt again an epiconvergence process on the sequence of the perturbed functionals, which provides the almost sure weak convergence of the saddle points sequence towards the saddle point of the *Lagrangian* of the homogenized problem.

The situation and notations are those of section 2 but here $d = m$ and more specifically, we study the asymptotic behaviour of the dual formulation of the problem

$$(\mathcal{P}_n) : \inf\,\{F_n(\omega)(u, \Omega) + \Phi(u),\, u \in V\}$$

and the asymptotic behaviour of corresponding saddle points, in linearized elasticity, with

$$F(\omega)(u, A) := \int_A f(\omega)(x, e(u))\, dx ,$$

where $f(\omega)$ is measurable on x, convex with respect to the matrix variable and satisfies almost surely the following condition, for every a in the subspace $M_s^{d \times d}$ of symetric elements of $M^{d \times d}$:

$$\alpha |a|^p \leq f(\omega)(x, a) \leq \beta (1 + |a|^p) . \tag{6.1}$$

It is easy to see that (2.2) of section 2 is automatically satisfied. Indeed, every σ which belongs to the subdifferential $\partial f(\omega)(x, a)$, satisfies $|\sigma| \leq C(1 + |a|^{p-1})$ where C is a constant depending only on β (see H. Attouch [2], p. 52 for $p = 2$ or B. Dacorogna [9] in a more general setting) and with this bound, the convexity inequality leads to (2.2).

V will be the space $W_0^{1,p}(\Omega, \mathbf{R}^d)$ and Φ, the following functional

$$\Phi(u) := \int_\Omega \phi(x) . u(x)\, dx ,$$

where ϕ is any element of $L^{p'}(\Omega, \mathbf{R}^d)$, $p' := \dfrac{p}{p-1}$.

Thanks to Korn's inequality,

$$\left(\int_A |u(x)|^p\, dx + \int_A |e(u)(x)|^p\, dx \right)^{\frac{1}{p}}$$

defines an equivalent norm in $W^{1,p}(A, \mathbf{R}^d)$ still denoted by $|u|_{1,A}$.

With these new hypothesis, one could obtain similar results of previous sections for functionals of the form

$$F(\omega)(u, A) := \int_A f(\omega)(x, e(u))\, dx ,$$

and infimum become minimum.

A classical way to perturb our optimization problem, is to define, for every A in $\mathcal{O}$, the following bivariate functional $\Psi_n(\omega)(\,.\,, A)$ from $W^{1,p}(A, \mathbf{R}^d) \times \Sigma(A)$ into $\mathbf{R}$

$$\Psi_n(\omega)((u, \sigma), A) := \int_A f(\omega)\left(\frac{x}{\varepsilon_n}, e(u) + \sigma(x) \right) dx + \int_A \phi(x) . u(x)\, dx ,$$

where

$$\Sigma(A) := \left\{ \sigma : A \mapsto M_s^{d \times d},\ \sigma = (\sigma_{i,j})_{i,j},\ \sigma_{i,j} = \sigma_{j,i},\ \sigma_{i,j} \in L^{p'}(A) \right\} .$$

The primal $(\mathscr{P}_n)$ and dual $(\mathscr{P}_n)^*$ problems take the form :

$(\mathscr{P}_n)$ min $\{\Psi_n(\omega)((u, 0), \Omega), u \in W_0^{1,p}(\Omega, \mathbf{R}^d)\}$,

$(\mathscr{P}_n)^*$ sup $\{- \Psi_n^*(\omega)((0, \sigma), \Omega), \sigma \in \Sigma(\Omega)\}$ =

$$= \min \left\{ \int_\Omega f^*(\omega)\left(\frac{x}{\varepsilon_n}, \sigma(x)\right) dx, \operatorname{div} \sigma = \phi, \sigma \in \Sigma(\Omega) \right\}$$

where $\Psi_n^*(\omega)(\,.\,, \Omega)$ and $f^*(\omega)$ denote respectively the *Fenchel conjugates* of $\Psi_n(\omega)(\,.\,, \Omega)$ and $f(\omega)$.

Similarly the following perturbation of the homogenized limit problem defined in section 4

$$\Psi^{\mathrm{hom}}((u, \sigma), A) := \int_A f^{\mathrm{hom}}(e(u) + \sigma(x))\, dx + \int_A \phi(x) \,.\, u(x)\, dx ,$$

leads to the primal $(\mathscr{P}^{\mathrm{hom}})$ and dual $(\mathscr{P}^{\mathrm{hom}})^*$ problems

$(\mathscr{P}^{\mathrm{hom}})$ min $\{\Psi^{\mathrm{hom}}((u, 0), \Omega), u \in W_0^{1,p}(\Omega, \mathbf{R}^d)\}$,

$(\mathscr{P}^{\mathrm{hom}})^*$ sup $\{- \Psi^{\mathrm{hom}*}((0, \sigma), \Omega), \sigma \in \Sigma(\Omega)\}$ =

$$= \min \left\{ \int_\Omega f^{\mathrm{hom}*}(\sigma(x))\, dx, \operatorname{div} \sigma = \phi, \sigma \in \Sigma(\Omega) \right\} .$$

$u_n(\omega)$ and $\sigma_n(\omega)$ being respectively a solution of $(\mathscr{P}_n)$ and $(\mathscr{P}_n)^*$, $(u_n(\omega), \sigma_n(\omega))$ is a saddle point of the associated *Lagrangian* defined from $W^{1,p}(\Omega, \mathbf{R}^d) \times \Sigma(\Omega)$ into $\mathbf{R}$ by

$$L_n(\omega)((u, \sigma)) := - \Psi_n^{\overset{*}{\bar{\sigma}}}(\omega)((u, \sigma), \Omega) =$$

$$= \int_\Omega \sigma(x) : e(u)(x)\, dx - \int_\Omega \phi(x) \,.\, u(x)\, dx - \int_\Omega f^*(\omega)\left(\frac{x}{\varepsilon_n}, \sigma(x)\right) dx$$

where $\Psi_n^{\overset{*}{\bar{\sigma}}}(\omega)(\,.\,, \Omega)$ denotes the Frenchel conjugate of $\Psi_n(\omega)(\,.\,, \Omega)$ with respect to its second variable.

Finally, if u and σ are respectively solution of $(\mathscr{P}^{\mathrm{hom}})$ and $(\mathscr{P}^{\mathrm{hom}})^*$, (u, σ) is a saddle point of the associated Lagrangian

$$\mathrm{L}^{\mathrm{hom}}(u, \sigma) =$$

$$= \int_\Omega \sigma(x) : e(u)(x)\, dx - \int_\Omega \phi(x) \,.\, u(x)\, dx - \int_\Omega f^{\mathrm{hom}*}(\sigma(x))\, dx .$$

For further details about these notions, we refer to I. Ekeland & R. Temam [12].

Let ω be a fixed element of Σ, we have the following result.

PROPOSITION 6.1 : *Every saddle point* $((u_n(\omega), \sigma_n(\omega)))$ *of the Lagrangian* $L_n(\omega)$, *is bounded in* $W_0^{1,p}(\Omega, \mathbf{R}^d) \times \Sigma(\Omega)$. *Therefore, there exists* $(u(\omega), \sigma(\omega))$ *in* $W_0^{1,p}(\Omega, \mathbf{R}^d) \times \Sigma(\Omega)$ *such that, up to a further subsequence,* $(u_n(\omega), \sigma_n(\omega))$ *tends towards* $(u(\omega), \sigma(\omega))$ *in* $W_0^{1,p}(\Omega, \mathbf{R}^d) \times \Sigma(\Omega)$ *equipped with the product of the weak topology of* $W_0^{1,p}(\Omega, \mathbf{R}^d)$ *and* $L^{p'}(\Omega, M_s^{d \times d})$.

Proof: It is easy to show, thanks to the growth condition (6.1), that $u_n(\omega)$ is bounded in $W_0^{1,p}(\Omega, \mathbf{R}^d)$. On the other hand, again by (6.1) and convexity asumption, one can prove that every element σ that belongs to $\partial f(\omega)\left(\frac{x}{\varepsilon_n}, e(u_n(\omega))\right)$ satisfies

$$|\sigma| \leqslant C(1 + |e(u_n(\omega))|^{p-1})$$

which, with the property

$$\sigma_n(\omega) \in \partial f(\omega)\left(\frac{x}{\varepsilon_n}, e(u_n(\omega))\right),$$

leads to the conclusion. ■

In the sequel, we show that almost surely, every cluster point $(u(\omega), \sigma(\omega))$ of a saddle point $(u_n(\omega), \sigma_n(\omega))$ is a saddle point of the Lagrangian $L^{\hom}$ and so does not depends on ω and satisfies: $\sigma \in \partial f^{\hom}(e(u))$, u and σ are respectively solution of $(\mathcal{P}^{\hom})$ and $(\mathcal{P}^{\hom})^*$. For this, the main tool is the following proposition, direct consequence of theorems 2.4 and 3.2 of H. Attouch, D. Aze & R. Wets [3].

PROPOSITION 6.2 : *If* ω *is a fixed element of* Σ *such that*

$$\Psi^{\hom}(\,.\,, \Omega) := \tau \times s - \operatorname{epi}\lim_{n \to +\infty} \Psi_n(\omega)(\,.\,, \Omega)$$

where $\tau \times s$ *denotes the product topology of the weak topology of* $W_0^{1,p}(\Omega, \mathbf{R}^d)$ *and the strong topology of* $\Sigma(\Omega)$, *then every cluster point* $(u(\omega), \sigma(\omega))$ *of proposition* 6.1 *is a saddle point of* $L^{\hom}$.

We are now in position to prove the main result of this section. Let Σ' the subset of probability one defined in section 3. We have

THEOREM 6.3 : *For every* ω *in* Σ',

$$\Psi^{\hom}(\,.\,, \Omega) := \tau \times s - \operatorname{epi}\lim_{n \to +\infty} \Psi_n(\omega)(\,.\,, \Omega).$$

Moreover every cluster point $(u(\omega), \sigma(\omega))$, *in the sense of proposition* 6.1, *of a sequence of saddle point* $(u_n(\omega), \sigma_n(\omega))$ *of* $L_n(\omega)$, *is a saddle point of*

L^{hom} and so does not depends on ω. σ is then a solution of the dual problem $(\mathscr{P}^{\text{hom}})^*$ where

$$f^{\text{hom}*}(a) = \sup_{n \in \mathbf{N}^*} \frac{1}{n^d} \times$$

$$\times \left(\text{epi} \int_\Sigma \min \left\{ \int_{nY} f^*(\omega)(x, \sigma + .\,)\, dx,\ \sigma \in K(nY) \right\} dP(\omega) \right)(a),$$

where epi $\int_\Sigma$ *denotes the continuous infimal convolution defined by*

$$\left(\text{epi} \int_\Sigma g(\omega)(\,.\,)\, dP(\omega) \right)(a) :=$$

$$:= \inf \left\{ \int_\Sigma g(\omega)(a(\omega))\, dP(\omega),\ \int_\Sigma a(\omega)\, dP(\omega) = a \right\}$$

and where

$$K(nY) := \left\{ \sigma \in V(nY)\,;\ \int_{nY} \sigma(y)\, dy = 0,\ \text{div}\, \sigma = 0 \right\}.$$

Proof : Above expression of $f^{\text{hom}*}$ is a straightforward consequence of the definition of the Fenchel conjugate, permutation of two sup, property of the continuous infimal convolution which is, in our case, the Fenchel conjugate of

$$\int_\Sigma \min \left\{ \int_{nY} f(\omega)(x, e(u)(x) + .\,)\, dx,\ u \in W_0^{1,p}(nY, \mathbf{R}^d) \right\} dP(\omega),$$

and finally, classical expression of the Fenchel conjugate of

$$\frac{1}{n^d} \min \left\{ \int_{nY} f(\omega)(x, e(u)(x) + .\,)\, dx,\ u \in W_0^{1,p}(nY, \mathbf{R}^d) \right\},$$

which is

$$\frac{1}{n^d} \min \left\{ \int_{nY} f(\omega)^*\,(x, \sigma(x) + .\,)\, dx,\ \sigma \in K(nY) \right\}.$$

We refer to H. Attouch [2] for this last result and to C. Castaing & Valadier [8] for more about continuous infimal convolution.

It remains to prove that $\Psi^{\text{hom}}(\,.\,, \Omega) := \tau \times s - \text{epi}\lim_{n \to +\infty} \Psi_n(\omega)(\,.\,, \Omega)$.

Noticing that for every A in $\mathcal{O}$, $u \mapsto \int_A \phi(x) \,.\, u(x)\, dx$ is a $\tau - s$ continuous perturbation of $\Psi_n(\omega)(\,.\,,\Omega)$, we can neglect the presence of this term in the expression of $\Psi_n(\omega)(\,.\,,A)$ and $\Psi^{\text{hom}}(\,.\,,A)$ (see section 2).

On the other hand, with this convention, we get, when σ is constant

$$\Psi_n(\omega)((u,\sigma),A)=F_n(\omega)(u+l_\sigma,A)$$

and

$$\Psi^{\text{hom}}((u,\sigma),A)=F^{\text{hom}}(u+l_\sigma,A).$$

These remarks lead to the two following steps :

First step : We prove $\Psi^{\text{hom}}(\,.\,,\Omega):=\tau\times s-\operatorname{epi}\lim_{n\to+\infty}\Psi_n(\omega)(\,.\,,\Omega)$ in $W_0^{1,p}(\Omega,\mathbf{R}^d)\times\mathscr{E}(\Omega)$ where $\mathscr{E}(\Omega)$ denotes the subspace of piecewise constant functions of $\Sigma(\Omega)$.

(i) *Upper bound.* Let $u=\tau-\lim_{n\to+\infty}u_n$ and $\sigma=s-\lim_{n\to+\infty}\sigma_n$ with $(u_n,\sigma_n)\in W_0^{1,p}(\Omega,\mathbf{R}^d)\times\mathscr{E}(\Omega)$. We have $\sigma=\sum_{i\in I}a_i\,\chi_{\Omega_i}$ where $(\Omega_i)_{i\in I}$ is a finite partition of Ω, $\Omega_i\in\mathcal{O}$, and $u+l_{a_i}=\tau-\lim_{n\to+\infty}u_n+l_{a_i}$. So, by theorem 4.1,

$$F^{\text{hom}}(u+l_{a_i},\Omega_i)\leq\liminf_{n\to+\infty}F_n(\omega)(u_n+l_{a_i},\Omega_i)$$

that is

$$\Psi^{\text{hom}}(((u,\sigma)),\Omega_i)\leq\liminf_{n\to+\infty}\Psi_n(\omega)((u_n,\sigma),\Omega_i).\tag{6.2}$$

But, by convexity

$$\Psi_n(\omega)((u_n,\sigma_n),\Omega_i)\geq\Psi_n(\omega)((u_n,\sigma),\Omega_i)+$$
$$+\int_{\Omega_i}q(\omega)(x,e(u_n)(x)+\sigma(x)):(\sigma_n-\sigma)(x)\,dx\tag{6.3}$$

where $x\mapsto q(\omega)(x,e(u_n)(x)+\sigma(x))$ is an integrable selection of the closed set multivalued function $x\mapsto\partial f(\omega)(x,e(u_n)(x)+\sigma(x))$ (for more about integral of set valued maps and existence of integrable selections, we refer to J. P. Aubin & H. Frankovska [4] or C. Castaing & M. Valadier [8]).

So, (6.2) and (6.3), after summing over i, lead to

$$\Psi^{\text{hom}}((u,\sigma),\Omega)\leq\liminf_{n\to+\infty}\Psi_n(\omega)((u_n,\sigma_n),\Omega)$$

where we have use Hölder's inequality and the estimation

$$\left|q(\omega)(x, e(u_n)(x) + \sigma(x))\right| \leqslant C\left(1 + \left|e(u_n)(x) + \sigma(x)\right|^{p-1}\right)$$

in the last term of (6.3).

Lower bound. Let (u, σ) in $W_0^{1,p}(\Omega, \mathbf{R}^d) \times \mathscr{E}(\Omega)$. By theorem 4.1, there exists $v_{i,n}(\omega) \in W^{1,p}(\Omega_i, \mathbf{R}^d)$ such that $u + l_{a_i} = \tau - \lim_{n \to +\infty} v_{i,n}(\omega)$ and $v_{i,n}(\omega) = l_{a_i}$ on $\partial \Omega_i$. Setting $u_n(\omega) := v_{i,n}(\omega) - l_{a_i}$ in every Ω_i, we get

$$F^{\hom}(u + l_{a_i}, \Omega_i) \geqslant \limsup_{n \to +\infty} F_n(\omega)(u_n(\omega) + l_{a_i}, \Omega_i)$$

and, after summing over i

$$\begin{cases} u = \tau - \lim\limits_{n \to +\infty} u_n(\omega) \quad \text{and} \quad \sigma = \sigma_n \\ \Psi^{\hom}(((u, \sigma)), \Omega) \geqslant \limsup\limits_{n \to +\infty} \Psi_n(\omega)((u_n(\omega), \sigma_n), \Omega)\,. \end{cases}$$

Second step : We end the proof by using the s-density of $\mathscr{E}(\Omega)$ in $\Sigma(\Omega)$, a continuity and a diagonalization argument like in the proof of theorem 4.1. ■

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