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A SPACE-TIME VARIATIONAL FORMULATION FOR THE BOUNDARY INTEGRAL EQUATION IN A 2D ELASTIC CRACK PROBLEM (*)

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Abstract. — *This paper investigates the transient elastic wave scattering by a crack in* R^2 *by means of Boundary Integral Equation Method. The analysis of the Laplace-Fourier transform (in time) of the integral operator allows to obtain existence, uniqueness and continuity dependence of the solution with respect to the data, in a Sobolev functional framework. A regularisation of the hypersingular BIE is applied in order to remove the hypersingularity and to write the associated time-space variational formulation on a tractable form. A Galerkin-type approximation is then performed to solve this variational formulation and we finally present some numerical results.*

Résumé. — *Nous nous intéressons à la résolution par une méthode d'équations intégrales d'un problème de diffraction d'ondes élastiques transitoires par une fissure. L'analyse de la transformée de Fourier-Laplace (en temps) de l'opérateur intégral permet d'obtenir des résultats d'existence, d'unicité et de continuité de la solution par rapport aux données dans des espaces fonctionnels de type Sobolev. Le noyau de l'équation intégrale étant hypersingulier, on ne dispose pas directement d'une expression calculable de la forme bilinéaire associée. Cette difficulté est surmontée en appliquant une méthode de régularisation qui fournit finalement la formulation variationnelle espace-temps du problème. Cette formulation est enfin approchée à l'aide d'un schéma de Galerkin et nous présentons quelques résultats numériques.*

1. INTRODUCTION

This paper deals with the transient elastic scattering by a crack in R^2. An important field of application of such problem is non-destructive evaluation research (for elastic materials). It consists in deducing the

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presence and characteristics of the crack in a material from the analysis of the scattered wave (inverse problem) and therefore it necessits a good knowledge of the direct problem. The BIEM (Boundary Integral Equation Method) has been shown to give accurate numerical results in external problems and is commonly applied in scattering problems with collocation type discretization (see e.g. Beskos [7], for a survey on these methods in elastodynamics) as well as with Galerkin type discretization (see e.g. Cortey-Dumont [10]).

However, until recently, the literature concerning the time-dependent BIE was rather poor. A mathematical analysis of time-dependent integral equations has been developed these last years by Bamberger & Ha Duong [1], [2], [12], in the case of acoustic waves. We present here a generalisation of these results to the elastodynamic case. The time-dependent integral operator is studied as a spatial pseudo-differential operator, with the frequency variable (i.e. the Fourier-Laplace transform of the time variable) as a complex parameter. In the Sobolev type functional framework (linked to the elastic energy), we obtain the existence and uniqueness of the solution of the BIE and its continuous dependence with respect to the data.

A difficulty appears when we want to solve this BIE numerically, related to the double layer representation of the solution. Indeed, the BIE has a hypersingular kernel. However, a lot of regularisation technics have been investigated these last years to evaluate the hypersingular integral in the time domain (Nishimura *et al.* [21], Sladek & Sladek [24]...) as well as in the frequency domain (*cf.* Martin & Rizzo [18], Krishnasamy *et al.* [15], Bui [9], Bonnet [8]...). Another method has been proposed by Nédélec [20], for the frequency problem based on the variational formulation and leads to weakly singular kernels (after integrations by parts). We use in the present paper the regularisation technique proposed by Nishimura & Kobayashi [22], and extended by Bécache *et al.* [6], to anisotropic media, which is very general and can be used with collocation as well as Galerkin type discretization.

This paper is organised as follows. We set in the next section the transient scattering problem and the associated Boundary Integral Equation. The mathematical analysis of the integral operator is done in section 3, through the analysis of the Fourier-Laplace transformed problem. The Fourier-Laplace transformed operator is shown to be an isomorphism in some Sobolev spaces. The knowledge of the dependence of the constants with respect to the frequency ω allows to derive the properties of the time-dependent integral operator (and the time-dependent variational formulation) from the previous analysis. The main result of this section concerns the existence, the uniqueness and the continuity of the solution with respect to the data. The variational formulation has then to be explicited on a tractable form, in order to be implemented. We thus have to remove the hypersingularity of the BIE by using a regularisation method. This is the object of the section 4, where we describe the way to derive the space-time variational formulation from the regularisation method developed in [6]. Finally, this problem is solved by means of a Galerkin type approximation in section 5 for a rectilinear crack. A comparison between numerical results obtained with different formulations ([5] and [21]) is then presented. We note that the formulation proposed here yields more stable results than the one used in [5].

2. NOTATIONS AND GOVERNING EQUATIONS

Let Γ be an open curve representing a crack in a two-dimensional isotropic elastic medium and $\Omega = R^2 \backslash \Gamma$ the scene where elastic waves are propagating. We suppose that Γ is part of a simple, closed Jordan curve $\tilde{\Gamma}$, so that a normal vector $\vec{n}$ can be defined on Γ separating locally (except at the two end-points of Γ) the medium Ω into two disjoint sets, one of which is exterior to $\tilde{\Gamma}$ and pointed in by $\vec{n}$. We denote by Ω_- (Ω_+) the interior (exterior) domain delimited by $\tilde{\Gamma}$, and $\Gamma_0 = \tilde{\Gamma} \backslash \Gamma$ (see 2.1).

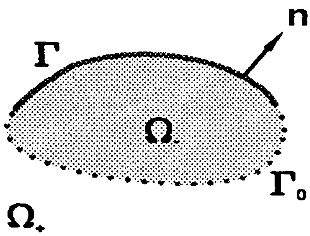


Figure 2.1.

We consider the scattered wave problem :

$$\begin{cases} \rho \dfrac{\partial^2 \vec{u}}{\partial t^2} - \operatorname{div} \sigma(\vec{u}) = 0 & \text{in} \quad \Omega \times R \qquad \text{(i)} \\ \vec{u}(x, t) = \dfrac{\partial \vec{u}}{\partial t}(x, t) = 0 & \text{in} \quad \Omega \times R_- \qquad \text{(ii)} \\ \sigma(\vec{u})\, \vec{n} = \vec{g} & \text{on} \quad \Gamma \times R \qquad \text{(iii)} \end{cases} \tag{2.1}$$

where $\vec{u}$ is the displacement field, $\sigma(\vec{u})$ the stress tensor which, in the isotropic case, is related to the deformation tensor ε with the Hooke's law

$$\begin{cases} \sigma_{ij} = \lambda\, \varepsilon_{kk}\, \delta_{ij} + 2\, \mu\, \varepsilon_{ij} \\ \varepsilon_{ij} = \dfrac{1}{2} \left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i} \right) \end{cases} \tag{2.2}$$

and λ, μ are the Lamé's constants. The given data $\vec{g}$ is due to an incident wave and we assume that it is also null for $t \leq 0$. From the Betti's reciprocal theorem, it is known that a solution of (2.1) can be represented as a double layer elastic potential

$$\vec{u}(x, t) = -\int_{\Gamma} T(\vec{n}(y))(x, y, \cdot) \overset{(t)}{*} [\vec{u}(y, \cdot)]\, d\gamma_y \tag{2.3}$$

where

(i) $[f] = f_- - f_+$ is the jump of a function over $\tilde{\Gamma}$

(ii) $\overset{(t)}{*}$ denotes the convolution in the time variable

(iii) $T(\vec{n}_y) = \Sigma \vec{n}(y)$ i.e. $T_{ij}(\vec{n}_y) = \Sigma_{ik;j}\, \vec{n}_k(y)$ where $\Sigma_{ik;j} = \sigma_{ik}(\vec{U}_j)$ is the stress tensor of the fundamental displacement tensor $\vec{U}_j$, which is defined by

$$\rho\, \frac{\partial^2 (\vec{U}_j)_i}{\partial t^2} - \sigma_{ik,k}(\vec{U}_j) = \delta(t)\, \delta(x-y)\, \delta_{ij}\,. \tag{2.4}$$

The analytic expression of the fundamental displacement tensor $U = U_{ij} = (\vec{U}_j)_i$ is given by

$$U_{ij} = \frac{1}{2\pi\rho} \times$$

$$\times \left[\left(\frac{\left(2t^2 - \frac{r^2}{C_P^2}\right) H\left(t - \frac{r}{C_P}\right)}{\sqrt{t^2 - \frac{r^2}{C_P^2}}} - \frac{\left(2t^2 - \frac{r^2}{C_S^2}\right) H\left(t - \frac{r}{C_S}\right)}{\sqrt{t^2 - \frac{r^2}{C_S^2}}} \right) \frac{r_i r_j}{r^4} - \right.$$

$$- \frac{\delta_{ij}}{r^2} \left(H\left(t - \frac{r}{C_P}\right) \sqrt{t^2 - \frac{r^2}{C_P^2}} - H\left(t - \frac{r}{C_S}\right) \sqrt{t^2 - \frac{r^2}{C_S^2}} \right)$$

$$\left. + \frac{\delta_{ij} H\left(t - \frac{r}{C_S}\right)}{C_S^2 \sqrt{t^2 - \frac{r^2}{C_S^2}}} \right].$$

To obtain (2.3), it may be useful to consider $\vec{u}$ as the solution of interior and exterior elastic equation

$$\rho\, \frac{\partial^2 \vec{u}}{\partial t^2} - \operatorname{div} \sigma(\vec{u}) = 0 \quad \text{in} \quad (\Omega_+ \times R) \cup (\Omega_- \times R) \tag{2.5}$$

with the boundary conditions

$$[\vec{u}] = 0 \quad \text{on} \quad \Gamma_0 \times R \tag{2.6}$$

$$[\sigma(\vec{u})\,\vec{n}] = 0 \quad \text{on} \quad \tilde{\Gamma} \times R\,. \tag{2.7}$$

It is easy to show that from (2.6) and (2.7), the equation (2.5) is also satisfied for $(x, t) \in \Gamma_0 \times R$, thus the first equation of (2.1) is obtained again. Now, formula (2.3) yields the solution of (2.1) as soon as the jump function $\vec{\varphi} = [\vec{u}]$ is known on $\Gamma \times R$. From the boundary condition (2.1, iii), $\vec{\varphi}$ satisfies the following boundary integral equation on $\Gamma \times R$

$$\lim_{\substack{x' = x + \varepsilon \vec{n}(x) \\ \varepsilon \to 0}} \sigma\left(- \int_{\Gamma} T(\vec{n}(y))\,(x, y, .) \overset{(t)}{*} \vec{\varphi}(y, .)\, d\gamma_y \right) = \vec{g}(x, t) \tag{2.8}$$

Since formal permutation of derivations and integration in (2.8) gives rise to singular and hypersingular integrals, we shall come back to (2.5), (2.6), (2.7) to give another definition of the LHS of (2.8). Let us denote it by $D\vec{\varphi}$, then we have

$$D\vec{\varphi} = (\sigma(\vec{u})\,\vec{n})_{/(\Gamma \times R)} \tag{2.9}$$

where $\vec{u}$ satisfies (2.5), (2.7) and

$$[\vec{u}] = \begin{cases} 0 & \text{on} \quad \Gamma_0 \times R \\ \vec{\varphi} & \text{on} \quad \Gamma \times R\,. \end{cases} \tag{2.10}$$

In the following sections, we study the operator D defined by (2.9), giving in particular a space-time variational formulation of problem (2.8) with only weakly singular integrals. A weak form of well-posedness of equation (2.8) is also obtained securing the stability of calculations by the variational formulation. Numerical experimentations will be shown confirming this.

3. MATHEMATICAL ANALYSIS

We first study the following problem

$$\left\{ \begin{array}{rlll} \rho \dfrac{\partial^2 \vec{u}}{\partial t^2} - \operatorname{div} \sigma(\vec{u}) = 0 & \text{in} & (\Omega_+ \times R) \cup (\Omega_- \times R) & (a) \\ \vec{u}(x, t) = \dfrac{\partial \vec{u}}{\partial t}(x, t) = 0 & \text{for} & t \leqslant 0 & (b) \\ [(\sigma(\vec{u})\,\vec{n})] = 0 & \text{on} & \tilde{\Gamma} \times R & (c) \\ (\sigma(\vec{u})\,\vec{n})_+ = (\sigma(\vec{u})\,\vec{n})_- = \vec{g} & \text{on} & \tilde{\Gamma} \times R & (d) \end{array} \right. \tag{3.1}$$

Thus, we don't impose, at that time, the condition $[\vec{u}] = 0$ on $\Gamma_0 \times R$. Following the ideas of [1], [2] we start by Fourier-Laplace transforming the problem with respect to the time variable ans study the transformed problem in function of the dual variable. Results for (3.1) will be obtained by the inverse transform. To fix the notations, we denote by

$$\hat{f}(\omega) = \int_R f(t)\, e^{i\omega t}\, dt \tag{3.2}$$

the Fourier transform for $f \in L^1(R)$, and extend it as usual to the complex half-plane $\mathfrak{Im}\,(\omega) > 0$ when f is causal (i.e. $f(t) = 0$ for $t < 0$). This Fourier-Laplace transform is also extended to less regular functions or distributions f, with values in a Hilbert space (see [28]). This point of view is useful here, as we consider $\hat{u}$ as a function of t with values in functional spaces in x.

3.1. The Fourier-Laplace transform problem

We consider the problem

$$\left\{\begin{array}{rlll} \operatorname{div} \sigma(\vec{u}) + \rho\,\omega^2 \vec{u} = 0 & \text{in} & \Omega_+ \cup \Omega_- & (a) \\ [\sigma(\vec{u})\,\vec{n}] = 0 & \text{on} & \tilde{\Gamma} & (b) \\ (\sigma(\vec{u})\,\vec{n})_+ = (\sigma(\vec{u})\,\vec{n})_- = \vec{g} & \text{on} & \tilde{\Gamma}\,. & (c) \end{array}\right. \tag{3.3}$$

The equation (a) in (3.3) is clearly the transformation of the equation (a) in (3.1). The equation (b) of (3.1) will be taken into account when we look for the only solutions of (3.3) which satisfy the Paley-Wiener theorem for causal functions. Usually, the elastic equation in (3.3) is treated with a real frequency ω and a radiation condition must be added to (3.3) to be well-posed. With $\omega_I = \mathfrak{Im}\,(\omega) > 0$, this radiation condition can be included in a functional framework, and we show below that (3.3) defines an isomorphism for $\vec{g} \in (H^{-1/2}(\tilde{\Gamma}))^2$ to $\vec{u}$ belonging to a closed subspace of $(H^1(\Omega_+ \cup \Omega_-))^2$. This can be done as in the work of [1], [2] for acoustic waves. However, to have a finer behavior of $\vec{u}$, we will define ω-dependent norms for these spaces. When Γ is a straight crack, these norms were introduced in [12]. For a curved crack, it is necessary to use a localisation trick. Consider a finite covering of $\tilde{\Gamma}$ by open sets $(\mathcal{O}_i)_{i \in I}$, a smooth partition of unity (α_i) subordinate to this cover and diffeomorphisms (φ_i) transporting $\mathcal{O}_i$ to the open set $Q = \{-1 < x_1, x_2 < 1\}$ in R^2, with $\varphi_i(\mathcal{O}_I \cap \tilde{\Gamma}) = \{|x_1| < 1,\, x_2 = 0\}$. Usual compatibility assumptions are done when $\mathcal{O}_i \cap \mathcal{O}_j \neq \emptyset$. Now, given a function f defined on $\tilde{\Gamma}$, we set

$$(\theta_i f)(x_1) = (\alpha_i f) \circ \varphi_i^{-1}(x_1, 0) - 1 < x_1 < 1\,. \tag{3.4}$$

Since supp $\alpha_i \subset \mathcal{O}_i$ and φ_i is a diffeomorphism, it is clear that $\theta_i f$ has its support included in $]-1, +1[$. Therefore, we identify this function with its extension by 0 out of the interval $]-1, +1[$ and can define

$$|f|_{s, \omega, \tilde{\Gamma}} = \left(\sum_I \int_R (|\omega|^2 + |\xi|^2)^s \left| \widehat{\theta_i f}(\xi) \right|^2 d\xi \right)^{1/2} \tag{3.5}$$

It is clear that for every $\omega \neq 0$, $|f|_{s, \omega, \tilde{\Gamma}}$ is a norm on $H^s(\tilde{\Gamma})$ equivalent to the usual norm. On the other hand, we define

$$\|f\|_{s, \omega} = \left(\int_{R^2} (|\omega|^2 + |\xi|^2)^s \, |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2} \tag{3.6}$$

for $\omega \neq 0$, $f \in H^s(R^2)$ and $\|f\|_{s, \omega, \mathcal{O}}$ the induce norm on $H^s(\mathcal{O})$ if $\mathcal{O}$ is an open set in R^2 :

$$\|f\|_{s, \omega, \mathcal{O}} = \inf \left\{ \|\tilde{f}\|_{s, \omega} ; \tilde{f} \in H^s(R^2), \tilde{f}_{/\mathcal{O}} = f \right\} . \tag{3.7}$$

For $s = 1$, one has

$$C_1 \|f\|_{1, \omega, \mathcal{O}} \leqslant \left(\int_{\mathcal{O}} (|\omega f|^2 + |\nabla f|^2) \, dx \right)^{1/2} \leqslant C_2 \|f\|_{1, \omega, \mathcal{O}} \tag{3.8}$$

where C_1 and C_2 are positive constants independent of ω, and we will use indifferently these two equivalent norms. Obviously, similar remarks can be made for positive integer s, but we need only the case $s = 1$ below. The first thing we have to do is to precise the dependence on ω of the usual trace theorems, when the norms (3.5) and (3.7) are used. Let $\mathcal{O}$ be a regular open set with compact boundary $\partial \mathcal{O}$. We denote by $H^1_\omega(\mathcal{O})$ and $H^{1/2}_\omega(\partial \mathcal{O})$ the spaces $H^1(\mathcal{O})$ and $H^{1/2}(\partial \mathcal{O})$ equipped with the norms (3.7) and (3.5) respectively.

LEMMA 3.1 : a) *There exists a positive constant C depending only on* $\mathcal{O}$ *and* ω_I^0 *such that*

$$|\gamma u|_{1/2, \omega, \partial \mathcal{O}} \leqslant C \left(\frac{1 + \omega_I^2}{\omega_I^2} \right)^{1/2} \|u\|_{1, \omega, \mathcal{O}} \tag{3.9}$$

for all u of $H^1(\mathcal{O})$ *and* ω *of the half-plane* $\left\{ \Im\mathrm{m}\ (\omega) = \omega_I \geqslant \omega_I^0 > 0 \right\}$, *where* γ *denotes the trace application.*

b) *On the other sense, one can construct an extension operator* Λ *from* $H^{1/2}(\partial \mathcal{O})$ to $H^1(\mathcal{O})$ *such that*

$$\|\Lambda \varphi\|_{1, \omega, \mathcal{O}} \leqslant C \left(\frac{1 + \omega_I^2}{\omega_I^2} \right)^{1/2} |\varphi|_{1, 2, \omega, \partial \mathcal{O}} \tag{3.10}$$

for all $\varphi \in H^{1/2}(\partial \mathcal{O})$ *and* $\omega \in \left\{ \Im\mathrm{m}\ (\omega) = \omega_I \geqslant \omega_I^0 > 0 \right\}$.

Proof : The proof consists in similar calculations as for the usual norms (see [11]), so we only sketch out here the steps.

(i) Using the atlas $(\mathcal{O}_i, \varphi_i)$, one reduces the problem of the trace application in the half-space $R^2_+ = \{(x_1, x_2) ; x_2 > 0\}$.

(ii) In this case, a partial Fourier transform with respect to x_1 can be carried out, leaving a one real variable problem to be dealed with. Following exactly the path of Dautray-Lions's proof, one obtains the inequalities

$$\|\gamma u\|_{1/2, \omega, \partial\mathcal{O}} \leq C \; \|u\|_{1, \omega, \mathcal{O}} \tag{3.11}$$

$$\|\Lambda\varphi\|_{1, \omega, \mathcal{O}} \leq C \; |\varphi|_{1/2, \omega, \partial\mathcal{O}} \tag{3.12}$$

with C independent od ω, instead of (3.9) and (3.10).

(iii) However, in the process of step (i), one has to evaluate $\nabla(\theta u) = u\,\nabla\theta + \theta\,\nabla u$ where θ is a regular truncature function. Thus, in the expression $|\omega(\theta u)|^2 + |\nabla(\theta u)|^2$, a non homogeneous term (with respect to ω) $u\,\nabla\theta$ is introduced. The dependence on ω of the continuity constants in (3.9) and (3.10) is now clear because of the inequality

$$\frac{C_1 + C_2\,|\omega|^2}{|\omega|^2} \leq C\,\frac{1+\omega_I^2}{\omega_I^2} \quad \text{for} \quad |\omega| \geq \omega_I > 0 \tag{3.13}$$

where C_1, C_2 are constants and $C = \max(C_1, C_2)$. □

An immediat useful consequence of this lemma concerns the analogous result with the jump operator, which appears in the following :

COROLLARY 3.1 : *(of lemma* 3.1*). Let $\mathcal{O}_-$ (resp. $\mathcal{O}_+$) a bounded regular domain (resp. its exterior domain), and let γ^- and Λ^- (resp. γ^+ and Λ^+) the corresponding trace and extension operators. One defines the jump operator from $H^1(\mathcal{O}_- \cup \mathcal{O}_+)$ to $H^{1/2}(\partial\mathcal{O}_-)$ as $\gamma = \gamma^- - \gamma^+ = [\,.\,]$.*

a) *There exists a positive constant C depending only on $\mathcal{O}_-$ and $\mathcal{O}_+$ such that*

$$|\gamma u|_{1/2, \omega, \partial\mathcal{O}} \leq C\left(\frac{1+\omega_I^2}{\omega_I^2}\right)^{1/2} \|u\|_{1, \omega, \mathcal{O}_- \cup \mathcal{O}_+} \tag{3.14}$$

for all u of $H^1(\mathcal{O}_- \cup \mathcal{O}_+)$ and ω of the half-plane $\{\Im m\,(\omega) = \omega_I \geq \omega_I^0 > 0\}$.

b) *On the other sense, one can construct an « extension » operator Λ from $H^{1/2}(\partial\mathcal{O}_-)$ to $H^1(\mathcal{O}_- \cup \mathcal{O}_+)$ such that $\gamma\Lambda\varphi = \varphi$ and*

$$\|\Lambda\varphi\|_{1, \omega, \mathcal{O}_- \cup \mathcal{O}_+} \leq C\left(\frac{1+\omega_I^2}{\omega_I^2}\right)^{1/2} |\varphi|_{1/2, \omega, \partial\mathcal{O}} \tag{3.15}$$

for all $\varphi \in H^{1/2}(\partial\mathcal{O}_-)$ and $\omega \in \{\Im m\,(\omega) = \omega_I \geq \omega_I^0 > 0\}$.

A second ω-dependence question concerns the Korn inequality. More precisely, we would like to compare the $H^1_\omega(\Omega)$ norm of the elastic displacement $\vec{u}$ with its energy

$$E_\omega(\vec{u}\,;\,\Omega) = \frac{1}{2}\rho\ \|\omega\vec{u}\|^2_{0,\,\Omega} + \frac{1}{2}\int_\Omega \sigma(\vec{u})\,.\,\bar{\varepsilon}(\vec{u})\,dx\,. \tag{3.16}$$

LEMMA 3.2 : *For regular open sets Ω with compact boundary, we have*

$$C_1\left(\frac{\omega_I^2}{1+\omega_I^2}\right)\|\vec{u}\|^2_{1,\,\omega,\,\Omega} \leqslant E_\omega(\vec{u}\,;\,\Omega) \leqslant C_2\ \|\vec{u}\|^2_{1,\,\omega,\,\Omega} \tag{3.17}$$

for all $\vec{u}\in(H^1_\omega(\Omega))^2$, where the constants C_1 and C_2 depend only on Ω and on the elastic medium and $\omega_I \geqslant \omega_I^0 > 0$.

Proof : The second inequality comes from the obvious $\|\varepsilon(\vec{u})\|^2_0 \leqslant \|\nabla\vec{u}\|^2_0$. The first one comes from the ellipticity assumption, which is always supposed for the elastic constants λ, μ : $\langle\sigma(\vec{u}),\ \varepsilon(\vec{u})\rangle \geqslant C\ \|\varepsilon(\vec{u})\|^2_0$, and the Korn inequality

$$\|\nabla\vec{u}\|^2_0 \leqslant C\,(\|\varepsilon(\vec{u})\|^2_0 + \|\vec{u}\|^2_0) \tag{3.18}$$

which is valid for the domains with regular, compact boundary (see [23] for bounded Ω and [14] for unbounded Ω). □

Now, we come back to the problem (3.3).

THEOREM 3.1 : *For every ω in the half plane $\{\mathfrak{Im}\ (\omega) = \omega_I \geqslant \omega_I^0 > 0\}$, and $\vec{g}\in(H^{-1/2}(\tilde{\Gamma}))^2$, the problem* (3.3) *is well-posed in $(H^1(\Omega_+\cup\Omega_-))^2$.*

Proof : From the Green formula

$$\int_D \operatorname{div}\sigma(\vec{u})\,.\,\bar{\vec{v}}\,dx = -\int_D \sigma(\vec{u})\,.\,\varepsilon(\bar{\vec{v}})\,dx + \int_{\partial D}\sigma(\vec{u})\,\vec{n}\,.\,\bar{\vec{v}}\,d\gamma \tag{3.19}$$

where $\vec{n}$ denotes the exterior unit normal to D, applied to Ω_+ and Ω_-, it is easily seen that (3.3) admits the following variational formulation

$$\begin{cases}\vec{u}\in(H^1(\Omega_+\cup\Omega_-))^2\\ \displaystyle\int_{\Omega_+\cup\Omega_-}(\sigma(\vec{u})\,.\,\bar{\varepsilon}(\vec{v}) - \rho\,\omega^2\,\vec{u}\,.\,\bar{\vec{v}})\,dx = \int_{\tilde{\Gamma}}\vec{g}\,.\,[\bar{\vec{v}}]\,d\gamma\\ \forall\vec{v}\in(H^1(\Omega_+\cup\Omega_-))^2\,.\end{cases} \tag{3.20}$$

Now, denoting the LHS of the equality in (3.20) by $a(\vec{u},\,\vec{v})$ — which is a continuous sesquilinear form on $(H^1(\Omega_+ \cup \Omega_-))^2$ — we have

$$\mathscr{Re}\ (a(\vec{u}), -i\omega\vec{u}) = \omega_I \int_{\Omega_+ \cup \Omega_-} (\sigma(\vec{u}) \,.\, \bar{\varepsilon}(\vec{u}) + \rho\ |\omega|^2\ |\vec{u}|^2)\,dx$$

$$= 2\ \omega_I\ E_\omega(\vec{u}, \Omega_+ \cup \Omega_-)\,. \tag{3.21}$$

From (3.17), one gets the following coerciveness inequality satisfied by $a(\,.\,,.\,)$

$$|a(\vec{u}, \vec{u})| \geq C\ \frac{\omega_I^3}{1+\omega_I^2} \frac{\|\vec{u}\|^2_{1,\,\omega,\,\Omega_+ \cup \Omega_-}}{|\omega|}\,. \tag{3.22}$$

The conclusion follows as usual, and one gets, using the trace estimate (3.9)

$$\|\vec{u}\|_{1,\,\omega,\,\Omega_+ \cup \Omega_-} \leq C\ \frac{(1+\omega_I^2)^{3/2}}{\omega_I^4}\ |\omega|\ |\vec{g}|_{-1/2,\,\omega,\,\tilde{\Gamma}} \tag{3.23}$$

for the solution $\vec{u}$ of (3.3). □

Remark 3.1 : It is known that from the Green formula (3.19) and the density of regular functions in the space $\mathscr{H} = \{\vec{u} \in (H^1(\Omega))^2\,;\ \operatorname{div}\sigma(\vec{u}) \in L^2(\Omega)\}$, one can define for every $\vec{u} \in \mathscr{H}$ its normal stress $\sigma(\vec{u})\,\vec{n}$ in $(H^{-1/2}(\partial\Omega))^2$. It follows that the space

$$\mathscr{X}_\omega = \{\vec{u} \in (H^1(\Omega_+ \cup \Omega_-))^2\,;\quad \rho\,\omega^2\vec{u} + \operatorname{div}\sigma(\vec{u}) = 0\,;\quad \text{and}\quad [\sigma(\vec{u})\,\vec{n}] = 0\}$$

is well defined as a closed subspace of $(H^1(\Omega_+ \cup \Omega_-))^2$ and for $\vec{u} \in \mathscr{X}_\omega$, its normal stress is defined as :

$$\langle \sigma(\vec{u})\,\vec{n}, \vec{\psi} \rangle_{-1/2,\,1/2,\,\tilde{\Gamma}} = a(\vec{u}, \Lambda\vec{\psi}) \quad \forall \vec{\psi} \in (H^{1/2}(\tilde{\Gamma}))^2 \tag{3.24}$$

where Λ is the operator defined in corollary 3.1. Thus, by the Cauchy-Schwartz inequality and the estimates (3.15) and (3.17), one has

$$|\sigma(\vec{u})\,\vec{n}|_{-1/2,\,\omega,\,\tilde{\Gamma}} \leq C \left(\frac{1+\omega_I^2}{\omega_I^2} \right)^{1/2} \|\vec{u}\|_{1,\,\omega,\,\Omega_+ \cup \Omega_-} \quad \forall \vec{u} \in \mathscr{X}_\omega\,. \tag{3.25}$$

We notice here that theorem 3.1 establishes an isomorphism between $\vec{g} \in (H^{-1/2}(\tilde{\Gamma}))^2$ and $\vec{u} \in \mathscr{X}_\omega$, unique solution of problem 3.3. However, in the estimate (3.23), which is the reciprocal estimate of the inequality (3.25) through this isomorphism, a factor $|\omega|$ is introduced. We will see below that this factor corresponds to a « lost of regularity » in the time variable.

Our next task is to study the Boundary Integral Equation which allows to solve the problem (3.3). It is well known (see [19]), that, for regular function $\vec{\varphi}$ *on* $\tilde{\Gamma}$, the double layer potential

$$\vec{u}_\omega(x) = -\int_{\tilde{\Gamma}} T_\omega(\vec{n}_y)(x-y)\,\vec{\varphi}(y)\,d\gamma(y) \quad x \in \Omega_+ \cup \Omega_- \tag{3.26}$$

(where $T_\omega(\vec{n}_y)$ is the normal stress tensor of the fundamental elastic displacement tensor) satisfies the equations (a) and (b) of (3.3), and the trace property

$$[\vec{u}_\omega] = \vec{\varphi}\,. \tag{3.27}$$

For $\omega_I > 0$, this potential belongs to the space $(H^1(\Omega_+ \cup \Omega_-))^2$, therefore, from the preceeding remark, it belongs to $\mathscr{X}_\omega$ (see similar results for scalar wave equations in [2]). We are going here to extend these properties to the $(H^{1/2}(\tilde{\Gamma}))^2$ functions $\vec{\varphi}$ and proving that only one such $\vec{\varphi}$ yields the potential $\vec{u}_\omega$ solution of the problem (3.3). As usual, this $\vec{\varphi}$ is the unknown function of a hypersingular equation, of which we will give a variational formulation that circumvents the hypersingularity.

LEMMA 3.3 : *The jump operator* $\vec{u} \in (H^1(\Omega_+ \cup \Omega_-))^2 \xrightarrow{\gamma} [\vec{u}] \in (H^{1/2}(\tilde{\Gamma}))^2$ *is an isomorphism between* $\mathscr{X}_\omega$ *and* $(H^{1/2}(\tilde{\Gamma}))^2$ *for* ω *such that* $\omega_I > 0$.

Proof : We have to prove that the problem

$$\vec{u} \in \mathscr{X}_\omega \quad \text{such that} \quad [\vec{u}] = \vec{\varphi} \tag{3.28}$$

is well posed when $\omega_I > 0$. If $\vec{v} \in \mathscr{X}_\omega$ then $\rho\bar{\omega}^2\bar{\vec{v}} + \operatorname{div}\overline{\sigma(\vec{v})} = 0$ in $\Omega_+ \cup \Omega_-$, and if $\vec{u}$ satisfies (3.28), the Green formula leads to

$$\begin{aligned} -a(\vec{u}, \vec{v}) &= \int_{\Omega_+ \cup \Omega_-} (\rho\bar{\omega}^2\,\vec{u}\,.\,\bar{\vec{v}} - \overline{\sigma(\vec{v})}\,.\,\varepsilon(\vec{u}))\,dx \\ &= -\int_{\tilde{\Gamma}} \overline{\sigma(\vec{v})}\,\vec{n}\,.\,\vec{\varphi}\,d\gamma\,. \end{aligned} \tag{3.29}$$

As in the proof of theorem 3.1, one sees that the LHS of (3.29) is a continuous, coercive sesquilinear form on $\mathscr{X}_\omega$ (closed subset of $(H^1(\Omega_+ \cup \Omega_-))^2$, equipped with the induced norm). Thus, there is an unique $\vec{u}$ of $\mathscr{X}_\omega$ such that (3.29) is satisfied for all $\vec{v} \in \mathscr{X}_\omega$. Applying again the Green formula, one has

$$\int_{\tilde{\Gamma}} \overline{\sigma(\vec{v})\,\vec{n}} \,.\, [\vec{u}]\, d\gamma = \int_{\tilde{\Gamma}} \overline{\sigma(\vec{v})\,\vec{n}} \,.\, \vec{\varphi}\, d\gamma$$

and since the normal stress application $\vec{v} \to \sigma(\vec{v})\,\vec{n}$ is an isomorphism between $\mathscr{X}_{\omega}$ and $(H^{-1/2}(\tilde{\Gamma}))^2$, it follows that $[\vec{u}] = \vec{\varphi}$. □

It is now natural (see the recalls preceeding lemma 3.3) to call the solution of problem (3.28) the double layer elastic potential of density $\vec{\varphi}$. This potential satisfies (3.29) and the coerciveness estimate (3.22) leads to

$$C\,\frac{\omega_I^3}{1+\omega_I^2}\,\frac{\|\vec{u}\|^2_{1,\,\omega,\,\Omega_+\cup\Omega_-}}{|\omega|} \leqslant |a(\vec{u},\vec{u})| = \int_{\tilde{\Gamma}} \sigma(\bar{\vec{u}})\,\vec{n}\,.\,\vec{\varphi}\, d\gamma$$

$$\leqslant \left|\sigma(\vec{u})\,\vec{n}\right|_{-1/2,\,\omega}\, |\vec{\varphi}|_{1/2,\,\omega}$$

and finally the estimation (3.25) yields

$$\|\vec{u}\|_{1,\,\omega,\,\Omega_+\cup\Omega_-} \leqslant C\,\frac{(1+\omega_I^2)^{3/2}}{\omega_I^4}\,|\omega|\,|\vec{\varphi}|_{1/2,\,\omega,\,\tilde{\Gamma}}\,. \tag{3.30}$$

We denote by $D_{\omega}\,\vec{\varphi}$ the normal stress vector of $\vec{u} = (\gamma)^{-1}\,\vec{\varphi}$. When $\vec{\varphi}$ is regular, by the expression (3.26) of $\vec{u}$, one gets

$$D_{\omega}\,\vec{\varphi} = \lim_{x' \to x}\, \sigma\left(-\int_{\tilde{\Gamma}} T_{\omega}(\vec{n}_y)\,(x'-y)\,\vec{\varphi}(y)\, d\gamma(y)\right)\vec{n}_x\,, \quad x \in \tilde{\Gamma}\,. \tag{3.31}$$

That explains that D_{ω} is a hypersingular integral operator on $\tilde{\Gamma}$. We give in section 4 below a variational treatment of this singularity. Here, we notice that

THEOREM 3.2 : *For all* $\omega \in \{\Im m\,(\omega) = \omega_I \geqslant \omega_I^0 > 0\}$, D_{ω} *is an isomorphism between* $(H^{1/2}(\tilde{\Gamma}))^2$ *and* $(H^{-1/2}(\tilde{\Gamma}))^2$.

Proof : This is a simple juxtaposition of lemma 3.3 and the corollary of theorem 3.1. The application D_{ω} can be factorised as $P_{\omega}^{-1} \circ \gamma^{-1}$, where P_{ω} designates the isomorphism between $\vec{g} \in (H^{-1/2}(\tilde{\Gamma}))^2$ and $\vec{u} \in \mathscr{X}_{\omega}$, unique solution of problem 3.3, and γ the jump operator defined in corollary 3.1. □

Since we will solve the integral equation $D_{\omega}\,\vec{\varphi} = \vec{g}$ by means of a variational method, the following theorem gives a more precise statement of this isomorphism.

THEOREM 3.3 : *The sesqui-linear form*

$$b_\omega(\vec{\varphi}, \vec{\psi}) \overset{def}{=} \left\langle D_\omega\, \vec{\varphi}, \vec{\psi} \right\rangle \quad for \quad \vec{\varphi}, \vec{\psi} \in (H^{1/2}(\tilde{\Gamma}))^2 \tag{3.32}$$

where the brackets stands for the duality pairing between $(H^{-1/2}(\tilde{\Gamma}))^2$ *and* $(H^{1/2}(\tilde{\Gamma}))^2$, *satisfies the following continuity and coerciveness inequalities*

$$\left|b_\omega(\vec{\varphi}, \vec{\psi})\right| \leq C\, \frac{(1+\omega_I^2)^2}{\omega_I^5}\, |\omega|\, |\vec{\varphi}|_{1/2,\,\omega,\,\tilde{\Gamma}} |\vec{\psi}|_{1/2,\,\omega,\,\tilde{\Gamma}} \tag{3.33}$$

$$\left|b_\omega(\vec{\varphi}, \vec{\varphi})\right| \geq C\, \frac{\omega_I^5}{(1+\omega_I^2)^2}\, \frac{1}{|\omega|}\, |\vec{\varphi}|^2_{1/2,\,\omega,\,\tilde{\Gamma}}\,. \tag{3.34}$$

Proof : The estimate (3.33) results from (3.25) and (3.30). To obtain (3.34), we write

$$b_\omega(\vec{\varphi}, \vec{\psi}) = \int_{\tilde{\Gamma}} \sigma(\vec{v})\, \vec{n}\,.\, \overline{[\vec{v}]}\, d\gamma$$

where $\vec{u}$ and $\vec{v}$ are respectively $(\gamma)^{-1}\, \vec{\varphi}$ and $(\gamma)^{-1}\, \vec{\psi}$. Then, by applying the Green's formula, one gets :

$$b_\omega(\vec{\varphi}, \vec{\psi}) = -\int_{\Omega_+ \cup \Omega_-} (\rho\,\omega^2\, \vec{u}\,.\, \vec{v} - \sigma(\vec{u})\,.\, \overline{\varepsilon(\vec{v})})\, dx\,.$$

Therefore

$$|\omega|\, \left|b_\omega(\vec{\varphi}, \vec{\varphi})\right| = \left|b_\omega(\vec{\varphi}, -i\,\omega\, \vec{\varphi})\right| \geq \mathcal{R}e\,(b_\omega(\vec{\varphi}, -i\,\omega\,\vec{\varphi}))$$

and (3.34) follows from (3.22) and (3.14). □

Before ending this section, we have to come back to the crack problem

$$\begin{cases} \operatorname{div} \sigma(\vec{u}) + \rho\,\omega^2\, \vec{u} & = 0 \quad \text{in} \quad \Omega = R^2 \backslash \Gamma \quad (a) \\ [\sigma(\vec{u})\, \vec{n}] & = 0 \quad \text{on} \quad \Gamma \quad (b) \\ (\sigma(\vec{u})\,\vec{n})_+ = (\sigma(\vec{u})\,\vec{n})_- & = \vec{g} \quad \text{on} \quad \Gamma\,. \quad (c) \end{cases} \tag{3.35}$$

We have signalled that (3.35*a*) is equivalent to (3.1*a*) plus the conditions

$$[\vec{u}] = 0 \quad \text{on} \quad \Gamma_0 \tag{3.36}$$

$$[\sigma(\vec{u})\,\vec{n}] = 0 \quad \text{on} \quad \Gamma_0\,. \tag{3.37}$$

Thus, to solve (3.35), we can go through (3.3), with a condition however : to find an extension $\tilde{\vec{g}}$ of $\vec{g}$ on $\tilde{\Gamma} = \Gamma \cup \Gamma_0$ such that the solution of the problem (3.3) with data $\vec{\tilde{g}}$ satisfies (3.36). We will proceed differently. We notice that, if $\vec{u}$ is solution of (3.35) and

$$\vec{\varphi} = \begin{cases} [\vec{u}] & \text{on} \quad \Gamma \\ 0 & \text{on} \quad \Gamma_0 \end{cases}$$

then, if an extension $\vec{\tilde{g}}$ of $\vec{g}$ exists such that the corresponding problem (3.3) admits $\vec{u}_{/\Omega_+ \cup \Omega_-}$ as solution, it must be equal to $D_\omega \vec{\varphi}$ on $\tilde{\Gamma}$. Therefore, one gets

$$\int_{\tilde{\Gamma}} D_\omega \vec{\varphi} \cdot \vec{\bar{\psi}} \, d\gamma = \int_{\tilde{\Gamma}} \vec{\tilde{g}} \cdot \vec{\bar{\psi}} \, d\gamma \quad \forall \vec{\psi} . \tag{3.38}$$

However, since $\sigma(\vec{u}) \vec{n}_{/\Gamma} = \vec{g}$ do not depend on the extension $\tilde{\Gamma}$ of Γ and $\vec{\tilde{g}}$ of $\vec{g}$, equation (3.38) should define completely $\vec{\varphi}_{/\Gamma}$ when we restrict the test functions to those with support in Γ.

We denote by $H^s_\Gamma(\tilde{\Gamma})$ the closed subspace of functions of $H^s(\tilde{\Gamma})$ with support in Γ, equipped with the induced norm. We suppose $\mathfrak{Im}(\omega) > 0$.

THEOREM 3.4 : *For $\vec{g} \in (H^{-1/2}(\Gamma))^2$, the problem* (3.35) *is well posed in $(H^1(\Omega))^2$. Its solution is the double layer potential with the density $\vec{\varphi}$ solution of the variational problem*

$$\begin{cases} \vec{\varphi} \in (H^{1/2}_\Gamma(\tilde{\Gamma}))^2 \text{ such that} \\ b_\omega(\vec{\varphi}, \vec{\psi}) = \langle \vec{\tilde{g}}, \vec{\psi} \rangle \quad \forall \vec{\psi} \in (H^{1/2}_\Gamma(\tilde{\Gamma}))^2 \end{cases} \tag{3.39}$$

where $\vec{\tilde{g}}$ is any extension of $\vec{g}$ in $(H^{-1/2}(\tilde{\Gamma}))^2$.

Proof : Let $\vec{u}$ in $(H^1(\Omega))^2$ be a solution of (3.35) with $\vec{g} = 0$. Then one can apply the Green formula to $\vec{u}$ in $\Omega_+ \cup \Omega_-$ and find :

$$\int_{\Omega_+ \cup \Omega_-} (\rho \omega^2 \vec{u} \cdot \vec{\bar{u}} - \sigma(\vec{u}) \cdot \bar{\varepsilon}(\vec{u})) \, dx = \int_{\tilde{\Gamma}} \sigma(\vec{u}) \vec{n} \cdot [\vec{\bar{u}}] \, d\gamma .$$

But in the RHS, $\sigma(\vec{u}) \vec{n}$ is null on Γ and $[\vec{u}]$ is null on Γ_0, thus the two sides of this equality are null. It follows that $\vec{u} \equiv 0$ in Ω. This proves the uniqueness of the solution of (3.35).

Now, the continuity and coerciveness properties of the form b_ω on $(H^{1/2}(\tilde{\Gamma}))^2$ remain clearly true on the closed subspace $(H^{1/2}_\Gamma(\tilde{\Gamma}))^2$ and the problem (3.39) is well-posed on this subspace. On the other hand, since the pairing $\langle \vec{\tilde{g}}, \vec{\psi} \rangle$ do not depend on the extension of $\vec{g}$ out of Γ, the solution of (3.39) do not depend either on this extension. The double layer potential of density $\vec{\varphi}$, defined by (3.28), satisfies the conditions (3.36) — by the support of $\vec{\varphi}$ — and (3.37) — by the appartenance of $\vec{u}$ to $\mathscr{X}_{\omega}$. Thus, it is solution of (3.35). Finally, the jump condition (3.36) implies that the norms $H^1(\Omega)$ of $\vec{u}$ are equal to the corresponding norms in $(H^1(\Omega_+ \cup \Omega_-))^2$. $\square$

Remark 3.2 : The space $H_{\Gamma}^{1/2}(\tilde{\Gamma})$ depends only on Γ. It is the same space denoted by $H_{00}^{1/2}(\Gamma)$ in Lions-Magenes [16]. The idea of treating the crack as a part of a closed curved has appeared in Stephan [26].

One final remark on the regularity question. It is known that the operator D_{ω} on $\tilde{\Gamma}$ is a pseudo-differential operator of order 1, and its inverse of order -1. Consequently, if $\vec{\tilde{g}}$ is given in $(H^s(\tilde{\Gamma}))^2$ with $s \geqslant -1/2$, and $\vec{\varphi}$ the solution of the variational problem

$$\left\langle D_{\omega} \vec{\tilde{\varphi}}, \vec{\psi} \right\rangle = \left\langle \vec{\tilde{g}}, \vec{\psi} \right\rangle \quad \forall \vec{\psi} \in (H^{1/2}(\tilde{\Gamma}))^2 \tag{3.40}$$

then $\vec{\varphi} \in (H^{s+1}(\tilde{\Gamma}))^2$. Now, if $\vec{g}$ is given in $(H^s(\Gamma))^2$ with $s \geqslant -1/2$, can we conclude that the solution $\vec{\varphi}$ of the variational problem (3.39) is of regularity $(H^{s+1}(\Gamma))^2$? There are two difficulties for a positive answer to the question.

First, even if we choose $\vec{\tilde{g}}$ as an extension of $\vec{g}$ by

$$\vec{\tilde{g}}_{/\Gamma_0} = (D_{\omega} \vec{\tilde{\varphi}})_{/\Gamma_0} \in (C^{\infty}(\Gamma_0))^2$$

this is not sure that $\vec{\tilde{g}} \in (H^s(\tilde{\Gamma}))^2$! (we can only say that it is the case when $s < 1/2$).

Secondly, the variational problem (3.39) posed in a closed subspace of $(H^{1/2}(\tilde{\Gamma}))^2$, is not extensible to the whole space as in (3.40). In fact, one can only prove the regularity property for $-1/2 \leqslant s < 0$. In a 3D situation, we refer to Stephan [27] for a proof, using pseudo-differential operator technique, and an explicit expression of the singularity of $\vec{\varphi}$ at the vicinity of the edges of Γ when $\vec{g}$ is in $(H^2(\Gamma))^2$.

3.2. The time problem

The results in this section are merely the translations of those on the preceeding section when we apply the inverse Fourier-Laplace transform to the problem (3.35).

We recall that a function $\hat{f}(\omega)$ is the transform of a causal temperate distribution f if and only if $\hat{f}$ is holomorphic in a half-plane $\{\mathfrak{Im}\ (\omega) > a\}$ and bounded by a polynome in a closed half-plane included in the preceeding. This remains valid if f is of values in a Hilbert space. Moreover, in this case the « weak = strong » property is true for the holomorphicity, so that in most cases it is very simple to verify this, and generally we shall omit this proof.

On the other hand, the results in section 3.1 are all formulated in the framework of L^2 Sobolev spaces, with norms $\| \cdot \|_{s,\omega}$ and the factor $|\omega|^\tau$. Then, we have to precise the spatio-temporal norms which will result from that in the process of the inverse transform. A convenient framework was introduced in [12]. Let ω_I^0 be a positive real. We will denote by $H_+^{s,\tau,\omega_I^0}(R^n \times R)$ the space of distributions u in $R^n \times R$ which Fourier transforms satisfy the following properties :

(i) For almost all $\xi \in R^n$, $u(\xi, \omega)$ is analytic in $\{\Im m\, \omega > \omega_I^0\}$.

(ii) There exists $C > 0$ such that

$$\sup_{\omega_I > \omega_I^0} \int_{R^n \times \{R + i\omega_I\}} |\omega|^{2\tau}(|\omega|^2 + |\xi|^2)^s |\hat{u}(\xi, \omega)|^2 \, d\xi \, d\omega \leqslant C .$$

It is equivalent to say that u is a causal distribution satisfying the inequality

$$\int_{R^n \times \{R + i\omega_I^0\}} |\omega|^{2\tau}(|\omega|^2 + |\xi|^2)^s |\hat{u}(\xi, \omega)|^2 \, d\xi \, d\omega =$$

$$= \int_{R + i\omega_I^0} |\omega|^{2\tau} \|\hat{u}(\,.\,, \omega)\|_{s,\omega}^2 \, d\omega < \infty . \tag{3.41}$$

The square root of the two members of (3.41) will be the norm of $H_+^{s,\tau,\omega_I^0}(R^n \times R)$, which is clearly a Hilbertian space. For a regular domain Ω of R^n, the space $H_+^{s,\tau,\omega_I^0}(\Omega \times R)$ is, as usual, the space of distributions u in $\Omega \times R$ extensible to an element of $H_+^{s,\tau,\omega_I^0}(R^n \times R)$, with the induced norm

$$\|u\|_{H_+^{s,\tau,\omega_I^0}(\Omega \times R)} = \inf_{\tilde{u}\,;\,\tilde{u}_{/\Omega \times R} = u} \|u\|_{H_+^{s,\tau,\omega_I^0}(R^n \times R)}$$

$$= \left(\int_{R + i\omega_I^0} |\omega|^{2\tau} \|\hat{u}(\,.\,, \omega)\|_{s,\omega,\Omega}^2 \, d\omega \right)^{1/2} .$$

For $\Gamma = \partial\Omega$, one defines $H_+^{s,\tau,\omega_I^0}(\Gamma \times R)$ with the aid of an atlas as usual. We define also $\tilde{H}_+^{1/2,\tau,\omega_I^0}(\Gamma \times R)$ as the space of $\varphi \in H_+^{1/2,\tau,\omega_I^0}(\Gamma \times R)$ such that its extension by 0 in $\tilde{\Gamma} \times R$ belongs to $H_+^{1/2,\tau,\omega_I^0}(\tilde{\Gamma} \times R)$. Now, we can sum up in the following theorem our results for problem (2.1).

THEOREM 3.5 : *Let* $\vec{g} \in (H_+^{-1/2, \tau, \omega_I^0}(\Gamma \times R))^2$ *with* $\omega_I^0 > 0$. *Then*

a) *The problem* (2.1) *admits a unique solution* $\vec{u}$ *in* $(H_+^{1, \tau - 2, \omega_I^0}(\Omega \times R))^2$ *which is the double layer potential of density* $\vec{\varphi} \in (\tilde{H}_+^{1/2, \tau - 1, \omega_I^0}(\Gamma \times R))^2$.

b) *If* $\tau \geq 1$, $\vec{\varphi}$ *is solution of the following variational problem*

$$\left\{\begin{array}{l} \vec{\varphi} \in (\tilde{H}_+^{1/2, 0, \omega_I^0}(\Gamma \times R))^2 \text{ such that} \\ \displaystyle\int_0^\infty e^{-2\omega_I^0 t} \int_\Gamma D\vec{\varphi}(x, t) \frac{\partial \vec{\psi}}{\partial t}(x, t)\, d\gamma(x)\, dt = \\ \displaystyle = \int_0^\infty e^{-2\omega_I^0 t} \int_\Gamma \vec{g}(x, t) \frac{\partial \vec{\psi}}{\partial t}(x, t)\, d\gamma(x)\, dt \\ \forall \vec{\psi} \in (\tilde{H}_+^{1/2, 1, \omega_I^0}(\Gamma \times R))^2 . \end{array}\right. \tag{3.42}$$

c) *Denote the bilinear form of* (3.42) *as* $b(\vec{\varphi}, \vec{\psi})$. *Then if* $\vec{\varphi} \in (\tilde{H}_+^{1/2, 1, \omega_I^0}(\Gamma \times R))^2$ *one has the following weak coerciveness inequality*

$$b(\vec{\varphi}, \vec{\varphi}) \geq C(\omega_I^0) \| \vec{\varphi} \|^2_{\tilde{H}_+^{1/2, 0, \omega_I^0}(\Gamma \times R)} . \tag{3.43}$$

Proof : If $\vec{u}$ in $(H_+^{1, \tau - 2, \omega_I^0}(\Omega \times R))^2$ is solution of (2.1) with $\vec{g} = 0$, then its Fourier-transform $\hat{\vec{u}}$ is solution of (3.35) for every $\omega \in \{\Im m\ \omega > \omega_I^0\}$ with null data $\hat{\vec{g}}$. Thus $\vec{u} = 0$.

Inversely, let $\vec{g} \in (H_+^{-1/2, \tau, \omega_I^0}(\Gamma \times R))^2$ and consider the solution $\hat{\vec{u}}_\omega$ of problem (3.35) constructed in theorem 3.4 : $\hat{\vec{u}}_\omega$ is the double layer potential with the density $\hat{\vec{\varphi}}_\omega$ solution of 3.39. Since the form b_ω is analytic in $\mathbb{C}$ and coercive in $\{\Im m\ \omega > 0\}$, the analyticity of $\hat{\vec{g}}_\omega$ in $\{\Im m\ \omega > \omega_I^0\}$ implies that $\hat{\vec{\varphi}}_\omega$ is also analytic in this half-plane, and then the same is true for $\hat{\vec{u}}_\omega$. On the other hand, the estimates (3.34) and (3.30) give

$$|\hat{\vec{\varphi}}_\omega| \leq C(\omega_I^0) |\omega|\, |\hat{\vec{g}}|_{-1/2, \omega} \tag{3.44}$$

$$|\hat{\vec{u}}_\omega| \leq C(\omega_I^0) |\omega|^2 |\hat{\vec{g}}|_{-1/2, \omega} \tag{3.45}$$

for all $\omega \in \{\Im m\ \omega > \omega_I^0\}$ (one has used the monoticity of the constants depending on ω_I in (3.34) and (3.30)).

Thus, part (a) is proved by the inverse Fourier transform. By using the Plancherel formula, one obtains easily part (b), from the variational problem (3.39) and part (c) from the estimates (3.14) and (3.22). □

Finally, we notice that the conditions of part (c) is satisfied as soon as $\vec{g} \in (H_+^{-1/2,\, 2,\, \omega_I^0}(\Gamma \times R))^2$ and the LHS of (3.43) is then linked to the elastic energy of the solution u of (2.1) by

$$b(\vec{\varphi}, \vec{\varphi}) = 2\,\omega_I^0 \int_0^\infty e^{-2\,\omega_I^0 t}\, E(\vec{u}, t)\, dt \tag{3.46}$$

where

$$E(\vec{u}, t) = \frac{1}{2} \int_\Gamma \left(\rho \left| \frac{\partial \vec{u}}{\partial t} \right|^2 + \sigma(\vec{u})\, \varepsilon(\vec{u})\right) dx\,. \tag{3.47}$$

In the following section, we will give a more tractable expression of the form b, with which the numerical experiments will be done, giving good results.

4. OBTENTION OF THE VARIATIONAL FORMULATION

This part deals with the problem of the kernel hypersingularity. By using a regularisation method developed in [6], the bilinear form can be rewritten as the sum of two parts containing weakly singular kernels. The application of this regularisation method to the obtention of the bilinear form is explained in details, in the 2D case. The 3D case is very similar and is treated in details in [4]. The bilinear form in part 3 is obtained by integrating the frequency bilinear form with respect to the frequencies

$$b(\vec{\varphi}, \vec{\psi}) = \frac{1}{2\,\pi} \int_{R + i\omega_I} b_\omega(\hat{\vec{\varphi}}, -i\,\omega\,\hat{\vec{\psi}})\, d\omega\,. \tag{4.1}$$

We therefore first deal with the frequency bilinear form b_ω and the time domain formulation is given in the second part.

4.1. The frequency bilinear form

The bilinear form is expressed as

$$b_\omega(\vec{\varphi}, \vec{\psi}) = \int_\Gamma \sigma(\vec{u})\, \vec{n}\, \bar{\vec{\psi}}\, d\gamma \tag{4.2}$$

where $\vec{u}$ is solution of the problem $(P_{\vec{\varphi}}^{\omega})$

$$\begin{cases} \operatorname{div} \sigma(\vec{u}) + \rho\,\omega^2\,\vec{u} = 0 & \text{in} \quad \Omega \\ [\sigma(\vec{u})\,\vec{n}] = 0 & \\ [\vec{u}] = \vec{\varphi}\,. & \end{cases} \qquad (P_{\vec{\varphi}}^{\omega})$$

We consider the derivatives in $\varepsilon(\vec{u})$ in the distributional sense and we denote by $\tilde{\varepsilon}(\vec{u})$ the function part, defined as

$$\tilde{\varepsilon}(\vec{u}) = \varepsilon(\vec{u})_{/\Omega} . \tag{4.3}$$

The relation between the distribution ε and its function part is given in the

LEMMA 4.1 :

$$\varepsilon(\vec{u}) = \tilde{\varepsilon}(\vec{u}) - t\delta_{\Gamma} \tag{4.4}$$

where $t\delta_{\Gamma}$ is a distribution with support on the crack and t is defined as

$$t_{ij} = \frac{1}{2}(\varphi_i n_j + \varphi_j n_i) . \tag{4.5}$$

The proof can be found in [20]. The problem (P_{φ}^{ω}) can now be rewritten in the distributional sense

$$\begin{cases} \operatorname{div} \sigma(\vec{u}) + \rho\omega^2 \vec{u} = 0 & \text{in} \quad \Omega \\ \tilde{\varepsilon}(\vec{u}) - \varepsilon(\vec{u}) = t\delta_{\Gamma} \end{cases} \tag{4.6}$$

and in Ω the Hooke's law is valid, thus we have

$$\tilde{\varepsilon}(\vec{u}) = A\sigma(\vec{u}) . \tag{4.7}$$

The problem (4.6) is transformed into a stress problem in the whole plane R^2, in which only σ remains as unknown :

$$\varepsilon(\operatorname{div} \sigma(\vec{u})) + \rho\omega^2 A\sigma = \rho\omega^2 t\delta_{\Gamma} . \tag{4.8}$$

If we denote by G the fundamental tensor associated to (4.8), the stress tensor can be expressed as the convolution of this fundamental tensor with the right hand side :

$$\sigma = G * t\delta_{\Gamma} . \tag{4.9}$$

This representation of the stress tensor is used in (4.2) to rewrite the bilinear form :

$$\begin{cases} b_{\omega}(\vec{\varphi}, \vec{\psi}) = \langle \sigma(\vec{u})\vec{n}, \vec{\psi}\delta_{\Gamma} \rangle \\ \qquad\qquad = \langle G_{ij}^{kl} * t_{kl}\, \delta_{\Gamma} n_j, \psi_i\, \delta_{\Gamma} \rangle . \end{cases} \tag{4.10}$$

In order to reduce the singularity, we use the decomposition of G given in [6] in the general anisotropic case. However, in the isotropic case, it can be directly computed and we have the

THEOREM 4.1 : *The fundamental tensor G can be decomposed as*

$$\hat{G}_{ij}^{kl} = \hat{D}_{ij}^{kl} + \rho\,\omega^2 \hat{R}_{ij}^{kl} \tag{4.11}$$

where denotes the spatial Fourier transform. D represents the static part, which contains the hypersingularity and R the dynamic part, which is weakly singular. Moreover, the static part D can be expressed as a fourth order differentiating operator of a regular kernel ϕ called the stress function. The expressions of $\hat{R}$, $\hat{D}$ and $\hat{\phi}$ are given in (4.16)-(4.19) *below.*

Proof : see [6]. □

Notations 4.1 :

- $$\varepsilon_{ij} = \begin{cases} 1 & \text{if } (i,j) = (1,2) \\ -1 & \text{if } (i,j) = (2,1) \\ 0 & \text{otherwise}. \end{cases} \tag{4.12}$$

- $$\begin{cases} D_P = \rho\,\omega^2 - (\lambda + 2\,\mu)|\xi|^2 \\ D_S = \rho\,\omega^2 - \mu\,|\xi|^2 \end{cases} \tag{4.13}$$

where $\xi = (\xi_1, \xi_2)$ is the dual variable of x in R^2, and

$$|\xi|^2 = \xi_1^2 + \xi_2^2 \tag{4.14}$$

$$\operatorname{rot} \vec{\alpha} = \partial_1 \alpha_2 - \partial_2 \alpha_1 = \varepsilon_{ij}\, \partial_i \alpha_j\,. \tag{4.15}$$

Expressions of $\hat{R}$, $\hat{D}$ and $\hat{\phi}$

$$\begin{aligned} \hat{R}_{ij}^{kl} = \frac{1}{D_P\,D_S} [&- \mu\,(\lambda + 3\,\mu)(\delta_{ki}\,\delta_{lj} + \delta_{kj}\,\delta_{li})|\xi|^2 + \mu^2\,\hat{r}_{ij}^{kl} \\ &- 3\,\lambda\,\mu\,|\xi|^2\,\delta_{kl}\,\delta_{ij} + 2\,\lambda\,\mu\,(\delta_{kl}\,\xi_i\,\xi_j + \delta_{ij}\,\xi_k\,\xi_l)] \\ &+ \frac{\rho\,\omega^2}{D_P\,D_S} [\mu\,(\delta_{ki}\,\delta_{lj} + \delta_{kj}\,\delta_{li}) + \lambda\,\delta_{kl}\,\delta_{ij}] \end{aligned} \tag{4.16}$$

$$\hat{r}_{ij}^{kl} = \delta_{kj}\,\xi_i\,\xi_l + \delta_{lj}\,\xi_k\,\xi_i + \delta_{ki}\,\xi_l\,\xi_j + \delta_{li}\,\xi_k\,\xi_j \tag{4.17}$$

$$\hat{D}_{ij}^{kl} = \varepsilon_{km}\,\varepsilon_{jn}\,\varepsilon_{lq}\,\varepsilon_{ip}\,\xi_m\,\xi_n\,\xi_p\,\xi_q\,\hat{\phi} \tag{4.18}$$

$$\hat{\phi} = \frac{4\,\mu^2(\lambda + \mu)}{D_P\,D_S}\,. \tag{4.19}$$

Remark 4.1 : 1. One can easily check, on the form of (4.18) that D represents a fourth order derivative of ϕ.

2. The equations (4.16) and (4.19) give the asymptotic behaviours of $\hat{R}$ and $\hat{\phi}$ when $|\xi|$ tends to infinity

$$\hat{R} = O\left(\frac{1}{|\xi|^2}\right) \quad \text{and} \quad \hat{\phi} = O\left(\frac{1}{|\xi|^4}\right).$$

This shows that R and any second derivative of ϕ are locally integrable. The decomposition (4.11) is now used in (4.10) and we finally have.

THEOREM 4.2 : *The frequency bilinear form has the following expression*

$$b_\omega(\vec{\varphi}, \vec{\psi}) = b_\omega^1(\vec{\varphi}, \vec{\psi}) + b_\omega^2(\vec{\varphi}, \vec{\psi}) \tag{4.20}$$

with

$$b_\omega^1(\vec{\varphi}, \vec{\psi}) = \int_\Gamma \int_\Gamma F_i^k(x-y) \frac{d\varphi_k}{ds}(x) \frac{d\bar{\psi}_i}{ds}(y)\, d\gamma_x\, d\gamma_y \tag{4.21}$$

$$b_\omega^2(\vec{\varphi}, \vec{\psi}) = \int_\Gamma \int_\Gamma \rho\,\omega^2 R_{ij}^{kl}(x-y)\, \varphi_k(x)\, n_l(x)\, \bar{\psi}_i(y)\, n_j(y)\, d\gamma_x\, d\gamma_y \tag{4.22}$$

where the kernels F and R are weakly singular kernels expressed as some second derivatives of the kernel N :

$$N(r, \omega) = \frac{i}{4}\left[H_0^{(1)}\left(\frac{\omega r}{C_S}\right) - H_0^{(1)}\left(\frac{\omega r}{C_P}\right)\right] \tag{4.23}$$

$$F_i^k = \frac{-4\mu^2}{\rho\,\omega^2}\, \varepsilon_{km}\, \varepsilon_{ip}\, \partial_{mp}^2 N(r, \omega) \tag{4.24}$$

$$\begin{aligned} R_{ij}^{kl} = {} & \frac{\mu}{\rho\,\omega^2(\lambda+\mu)}\left[((\lambda + 3\mu)(\delta_{ki}\,\delta_{lj} + \delta_{kj}\,\delta_{li}) + 3\lambda\,\delta_{kl}\,\delta_{ij})\,\Delta\right. \\ & \left. - 2\lambda(\delta_{kl}\,\partial_{ij}^2 + \delta_{ij}\,\partial_{kl}^2) - \mu(\delta_{kj}\,\partial_{il}^2 + \delta_{lj}\,\partial_{ik}^2 + \delta_{ki}\,\partial_{jl}^2 + \delta_{li}\,\partial_{jk}^2)\right] N(r, \omega) \\ & + \frac{1}{(\lambda+\mu)}\left[\mu(\delta_{ki}\,\delta_{lj} + \delta_{kj}\,\delta_{li}) + \lambda\,\delta_{kl}\,\delta_{ij}\right] N(r, \omega). \end{aligned} \tag{4.25}$$

Proof : The proof contains 3 steps

1. Use of the decomposition of $\hat{G}$.
2. Integration by parts of the static part, in Fourier variables.
3. Back to the space variables.

1. Use of the decomposition of $\hat{G}$.

The expression (4.10) of the bilinear form is transformed by means of Parseval identity into

$$b_\omega(\vec{\varphi}, \vec{\psi}) = \left\langle \hat{G}_{ij}^{kl}\, \widehat{t_{kl}\,\delta_\Gamma}, \widehat{\psi_i\, n_j\, \delta_\Gamma}\right\rangle$$

or by using (4.5) and the symmetries of G :

$$b_\omega(\vec{\varphi}, \vec{\psi}) = \left\langle \hat{G}_{ij}^{kl}\, \widehat{\varphi_k\, n_l\, \delta_\Gamma}, \widehat{\psi_i\, n_j\, \delta_\Gamma} \right\rangle .$$

The decomposition (4.11) of $\hat{G}$ leads to

$$b_\omega(\vec{\varphi}, \vec{\psi}) = b_\omega^1(\vec{\varphi}, \vec{\psi}) + b_\omega^2(\vec{\varphi}, \vec{\psi})$$

with

$$b_\omega^1(\vec{\varphi}, \vec{\psi}) = \left\langle \hat{D}_{ij}^{kl}\, \widehat{\varphi_k\, n_l\, \delta_\Gamma}, \widehat{\psi_i\, n_j\, \delta_\Gamma} \right\rangle \tag{4.26}$$

$$b_\omega^2(\vec{\varphi}, \vec{\psi}) = \left\langle \rho\, \omega^2 \hat{R}_{ij}^{kl}\, \widehat{\varphi_k\, n_l\, \delta_\Gamma}, \widehat{\psi_i\, n_j\, \delta_\Gamma} \right\rangle .$$

The hypersingularity is now contained in the static part $b_\omega^1(\vec{\varphi}, \vec{\psi})$.

2. Integration by parts of the static part.

In order to reduce the singularity, we use the remark that D coincids with a fourth-order differentiate operator of the stress function ϕ. This remark allows to integrate by parts and to transform this singularity into some derivatives of the unknown $\vec{\varphi}$ and of the test function $\vec{\psi}$. Let us replace in (4.26) $\hat{D}$ by its expression :

$$\begin{cases} b_\omega^1(\vec{\varphi}, \vec{\psi}) = \left\langle \varepsilon_{km}\, \varepsilon_{jn}\, \varepsilon_{lq}\, \varepsilon_{ip}\, \xi_m\, \xi_n\, \xi_p\, \xi_q\, \hat{\phi}\, \widehat{\varphi_k\, n_l\, \delta_\Gamma}, \widehat{\psi_i\, n_j\, \delta_\Gamma} \right\rangle \\ \qquad = - \left\langle (\varepsilon_{km}\, \varepsilon_{ip}\, i\xi_m\, i\xi_p\, \hat{\phi})\, \mathrm{rot}\, (\widehat{\varphi_k\, \vec{n}\delta_\Gamma}), \mathrm{rot}\, (\widehat{\psi_i\, \vec{n}\delta_\Gamma}) \right\rangle . \end{cases}$$

Before concluding, we have to introduce the following identity, proved in [20]

$$\mathrm{rot}\, (\alpha \vec{n} \delta_\Gamma) = \frac{d\alpha}{ds}\, \delta_\Gamma \tag{4.27}$$

valid for any scalar function α. Apply again Parseval identity, to come back in the space domain, and use identity (4.27) to get

$$b_\omega^1(\vec{\varphi}, \vec{\psi}) = - \left\langle \varepsilon_{km}\, \varepsilon_{ip}\, \partial_{mp}^2\, \phi * \left(\frac{d\varphi_k}{ds}\, \delta_\Gamma \right), \frac{d\psi_i}{ds}\, \delta_\Gamma \right\rangle .$$

We thus obtain (4.21) with

$$F_i^k = - \varepsilon_{km}\, \varepsilon_{ip}\, \partial_{mp}^2\, \phi . \tag{4.28}$$

3. Back to the space variables.

We now need the expressions of R and F. We thus have to compute the inverse Fourier transforms of $\hat{R}$ and of $\hat{\phi}$, which can be expressed in terms of the following quantities

$$\left\{\begin{aligned} \frac{1}{D_S} &= \mathscr{F}\left[-\frac{i}{4\mu} H_0^{(1)}\left(\frac{\omega r}{C_S}\right)\right] \\ \frac{1}{D_S D_P} &= \mathscr{F}\left[\frac{1}{\rho\,\omega^2(\lambda+\mu)}\frac{i}{4}\left(H_0^{(1)}\left(\frac{\omega r}{C_S}\right) - H_0^{(1)}\left(\frac{\omega r}{C_P}\right)\right)\right] \\ &= \mathscr{F}\left[\frac{1}{\rho\,\omega^2(\lambda+\mu)} N(r,\omega)\right]. \end{aligned}\right. \tag{4.29}$$

Relation (4.29) together with the expressions (4.16)-(4.19) give finally the expressions (4.24) and (4.25) of F and R. □

Remark 4.2 : The result of theorem 4.1 is interesting by itself, if we want to solve the scattering problem in the frequency domain with a variational method. In most cases, regularisation technics have been developed for collocation methods. However, in the frequency domain, several similar variational formulations are available (see [3], [20]) and some numerical results can be found in [10]. There are two main advantages of the method presented here. First, the procedure to get the bilinear form, based on the decomposition of the fundamental tensor G is very general and can be extended to anisotropic media, as far as the third step of the proof is realisable, i.e., as far as one knows how to come back from the Fourier domain to the space domain. If this is not the case, one can always compute the inverse Fourier Transform numerically. Secondly, and this is our main purpose here, this formulation is well suited to the time domain, thanks to the causality principle satisfied by the kernels F and R, as we will see in the following.

4.2. The time domain bilinear form

The time domain bilinear form is obtained by integrating the frequency bilinear form (see equation 4.1). We apply again Parseval identity, written for the Fourier-Laplace transform to get its expression. It essentially consists in computing the inverse Fourier-Laplace transform of the kernels F and R. We finally have the

THEOREM 4.3 : *The space-time bilinear form has the following expression*

$$b(\vec{\varphi}, \dot{\vec{\psi}}) = b^1(\vec{\varphi}, \dot{\vec{\psi}}) + b^2(\vec{\varphi}, \dot{\vec{\psi}}) \tag{4.30}$$

with

$$\left\{\begin{aligned} b^1(\vec{\varphi}, \dot{\vec{\psi}}) = \int_{R^+} e^{-2\omega_I t} \int_\Gamma \int_\Gamma \left(\mathscr{F}_{(t)}^{-1}(F_i^k(x-y, .)) \overset{(t)}{*} \frac{d\varphi_k}{ds}(x, .)\right)(t) \\ \frac{d\dot{\psi}_i}{ds}(y, t)\, d\gamma_x\, d\gamma_y\, dt \end{aligned}\right. \tag{4.31}$$

and

$$\left\{\begin{aligned} b^2(\vec{\varphi}, \vec{\psi}) &= \int_{R^+} e^{-2\omega_I t} \int_\Gamma \int_\Gamma \rho n_l(x)\, n_j(y) \\ &\quad (\mathscr{F}_{(t)}^{-1}(R_{ij}^{kl}(x-y, .)) \overset{(t)}{*} \ddot{\varphi}_k(x, .))(t)\, \bar{\dot{\psi}}_i(y, t)\, d\gamma_x\, d\gamma_y\, dt\,. \end{aligned}\right. \tag{4.32}$$

The inverse Fourier-Laplace of F and R are given by (4.33), (4.34) *below.*

Expressions of $\mathscr{F}_{(t)}^{-1}$ and $\mathscr{F}_{(t)}^{-1}(R)$

$$\mathscr{F}_{(t)}^{-1}(F_i^k)(r, t) = -4\mu^2 \varepsilon_{kn}\varepsilon_{iq} A_{nq}(r, t) \tag{4.33}$$

$$\mathscr{F}_{(t)}^{-1}(R_{ij}^{kl})(r, t) = P(A)_{ij}^{kl}(r, t) + \zeta_{ij}^{kl} N(r, t) \tag{4.34}$$

where we still denote by $N(r, t)$ the inverse Fourier-Laplace of $N(r, \omega)$. $N(r, t)$, A, $P(A)$ and ζ are defined by

$$\left\{\begin{aligned} N(r, t) &= N_{C_S}(r, t) - N_{C_P}(r, t) \\ N_C(r, t) &= \frac{1}{2\pi} \frac{H\left(t - \frac{r}{C}\right)}{\left(t^2 - \frac{r^2}{C^2}\right)^{1/2}} \end{aligned}\right. \tag{4.35}$$

$$\left\{\begin{aligned} A_{nq} &= \mathscr{F}_{(t)}^{-1}\left(\frac{1}{\rho\omega^2}\partial_{nq}^2 N\right) \\ &= \frac{1}{2\pi\rho} H\left(t - \frac{r}{C_P}\right)\left(\frac{r_{,n} r_{,q}}{r^2} \frac{t^2}{\left(t^2 - \frac{r^2}{C_P^2}\right)^{1/2}} - \frac{r_{,nq}}{r}\left(t^2 - \frac{r^2}{C_P^2}\right)^{1/2}\right) \\ &\quad - \frac{1}{2\pi\rho} H\left(t - \frac{r}{C_S}\right)\left(\frac{r_{,n} r_{,q}}{r^2} \frac{t^2}{\left(t^2 - \frac{r^2}{C_S^2}\right)^{1/2}} - \frac{r_{,nq}}{r}\left(t^2 - \frac{r^2}{C_S^2}\right)^{1/2}\right) \end{aligned}\right. \tag{4.36}$$

$$\begin{aligned} P(A)_{ij}^{kl} = \frac{\mu}{\lambda + \mu} [&((\lambda + 3\mu)(\delta_{ki}\delta_{lj} + \delta_{kj}\delta_{li}) + 3\lambda\delta_{kl}\delta_{ij})\, tr(A) \\ &- 2\lambda(\delta_{kl}A_{ij} + \delta_{ij}A_{kl}) - \mu(\delta_{kj}A_{il} + \delta_{lj}A_{ik} + \delta_{ki}A_{jl} + \delta_{li}A_{jk})] \end{aligned} \tag{4.37}$$

$$\zeta_{ij}^{kl} = \frac{1}{(\lambda + \mu)} [\mu(\delta_{ki}\delta_{lj} + \delta_{kj}\delta_{li}) + \lambda\delta_{kl}\delta_{ij}]\,. \tag{4.38}$$

Proof : Relation (4.1) together with (4.20)-(4.22) yields

$$b(\vec{\varphi}, \vec{\psi}) =$$

$$= \frac{1}{2\pi} \int_{R + i\omega_I} (b^1_\omega (\hat{\vec{\varphi}} - i\omega \hat{\vec{\psi}}) + b^2_\omega (\hat{\vec{\varphi}}, - i\omega \hat{\vec{\psi}}))\, d\omega$$

$$= \int_\Gamma \int_\Gamma \frac{1}{2\pi} \int_{R + i\omega_I} \Big(F^k_i (x - y, \omega) \frac{d\hat{\varphi}_k}{ds} (x, \omega)\, i\bar{\omega} \frac{d\bar{\hat{\psi}}_i}{ds} (y, \omega)$$

$$+ \rho\omega^2 R^{kl}_{ij} (x - y, \omega)\, \hat{\varphi}_k(x, \omega)\, n_l(x)\, i\bar{\omega} \bar{\hat{\psi}}_i (y, \omega)\, n_j(y)\big)\, d\gamma_x\, d\gamma_y\, d\omega\ .$$

$$(4.39)$$

The Parseval identity, applied to the Fourier-Laplace transform is the following

$$\frac{1}{2\pi} \int_{R + i\omega_I} \hat{f}(\omega)\, \bar{\hat{g}}(\omega)\, d\omega = \int_R e^{-2\omega_I t} f(t)\, \bar{g}(t)\, dt\ . \qquad (4.40)$$

Apply this identity to relation (4.39) to get the expression of the bilinear form given in (4.30)-(4.32). □

As mentioned in the remark 4.1, the preceeding bilinear form is well suited to the time domain. In fact, the kernel F and R satisfy the causality principle, which is an essential property of the wave propagation, since it traduces the fact that the wave is travelling with a finite velocity. Of course, the fundamental solution G satisfies this causality, and this can be seen on its Fourier transform expression :

$$\hat{G} = \frac{P(\omega, \xi)}{D_P\, D_S} \qquad (4.41)$$

where $P(\omega, \xi)$ is a polynomial in ω and ξ. The fraction $L = \dfrac{1}{D_P\, D_S}$ is linked to the wave velocities and gives the causal property. From (4.41), one deduces that G is given in terms of some derivatives of the inverse Fourier transform of L, which is proportional to the kernel N, and therefore possesses the causality property. Since F and R have the same form than G — i.e., they coincid with some derivatives of N — they also possess the causality property. In [20], Nédélec proposed another decomposition of G on the following form

$$\hat{G} = \frac{P_1(\omega, \xi)}{|\xi|^4} + \frac{P_2(\omega, \xi)}{|\xi|^4} \qquad (4.42)$$

where the first term contains the hypersingularity and the second term is weakly singular. He also performs an integration by parts on the first term in order to reduce the singularity. But this time, the kernels appearing in (4.42) are given in terms of some derivatives of $\frac{1}{|\xi|^4}$ — which corresponds to the Laplace operator and not to the wave equation — and violate the causality principle. Therefore this method works in the frequency domain, but is not suited to the time domain.

5. NUMERICAL RESULTS

In the numerical application, we have considered the case of a rectilinear crack of length ℓ, located on $[0, \ell]$. In this case, two problem have been solved : the scalar antiplane problem and the vectorial plane problem. We don't enter into details in the discretisation since it has been done, for the antiplane crack, in [5]. Since we present simultaneously both problems (scalar and vectorial) we will « forget » the sign $\vec{\ }$ in the following notations, unless in sections specific to the plane problem.

In the particular case of a rectilinear (or plane) crack, the bilinear form $b_\omega(\varphi, \psi)$ satisfies a coerciveness inequality without multipliying by the factor $-i\omega$, even if $\omega_I = 0$, as Ha Duong showed in [13]. We will call the resulting formulation the « first » formulation. This is the formulation directly obtained from the Boundary Integral Equation by multiplying with a test function and by integrating on the space-time domain. The formulation presented previously (3.42), with $\omega_I = 0$, will be called the « second » formulation.

First formulation

$$\int_0^\infty \int_\Gamma D\varphi(x, t)\, \psi(x, t)\, d\gamma(x)\, dt = \int_0^\infty \int_\Gamma g(x, t)\, \psi(x, t)\, d\gamma(x)\, dt\,.$$

Second formulation

$$\int_0^\infty \int_\Gamma D\varphi(x, t) \frac{\partial \psi}{\partial t}(x, t)\, d\gamma(x)\, dt = \int_0^\infty \int_\Gamma g(x, t) \frac{\partial \psi}{\partial t}(x, t)\, d\gamma(x)\, dt\,.$$

The crack $[0, \ell]$ is divided into subintervals $\Gamma_j = [x_j, x_{j+1}]$ of length $\Delta x_j = x_{j+1} - x_j$, with $0 = x_0 < x_1 < \dots < x_j < \dots < x_N = \ell$, the time interval is divided into intervals of length Δt and we define $t_k = k\,\Delta t$. In both cases, the approximate problem is on the classical following form : Find φ_h in the finite dimensional space V_h such that

$$b(\varphi_h, \psi_h) = L(\psi_h) \quad \forall \psi_h \in V_h\,. \tag{5.1}$$

For both formulations, the space V_h is the space of piecewise linear functions in space and in time and the test functions are denoted by χ_i^k where i denotes the space index and k the time index

$$\chi_i^k(x, t) = \varphi_i(x)\, \gamma^k(t)\; i = 1, \ldots, N_h, \quad k \geq 0 \tag{5.2}$$

where $N_h = N - 2$. The unknown is decomposed on the basis of test functions :

$$\varphi_h(x, t) = \sum_{j, m} \alpha_j^m \chi_j^m(x, t)\,. \tag{5.3}$$

It can be shown (see [4]) that the Galerkin problem (5.1) can be written as a marching-in-time scheme, which is the discrete equivalent of the convolution character of the continuous integral operator

$$M^0 \alpha^k = b^k - (M^1 \alpha^{k-1} + \cdots + M^k \alpha^0) \quad \text{for} \quad k \geq 0 \tag{5.4}$$

where the matrices M^k and the vectors b^k are defined by

$$\begin{cases} M_{ij}^k = b(\chi_j^0, \chi_i^k) \\ b_i^k = L(\chi_i^k)\,. \end{cases} \tag{5.5}$$

Numerical Experiments

The incident wave is a plane wave propagating in the direction $\vec{k}$ with an amplitude A

$$u^I(x, t) = Af\left(t - \frac{\vec{k}\,.\,\vec{x}}{C}\right) d\,. \tag{5.6}$$

The function f is choosen to be linear (unless otherwise mentioned) :

$$f(t) = tH(t) \tag{5.7}$$

and we will call the corresponding wave a linear incident wave. In (5.6) d is a scalar equal to 1 in the antiplane problem and an unit vector corresponding to the direction of motion in the plane problem.

5.1. The Antiplane Problem

For this scalar problem, we know the asymptotic behaviour of the solution, as t tends to infinity. The solution of the antiplane problem with a linear incident wave tends to the solution of the static corresponding problem and this solution can be computed explicitly (see [25]). If θ denotes the incident angle, we have :

$$\varphi(x, t) \xrightarrow{t \to +\infty} \varphi_{\theta}^{\infty}(x) = 2A \sin \theta \sqrt{x(\ell - x)} . \tag{5.8}$$

This case has been dealed with in [5] by using the « first » formulation. In this paper, we pointed out numerically the existence of a CFL type stability condition. We now compare this result with the « second » formulation. We use 20 boundary elements of constant length and we will denote by CFL the quantity $\dfrac{\Delta t}{\Delta x}$.

Figures (5.1) show the dynamical crack opening displacement (φ) at some fixed instant compared with the static one. On figures (5.2), we have represented the evolution in time of the COD, at a fixed point on the crack ($x = 0.1$). The first axis represents t/ℓ — i.e. the unit represents the travel time necessary for the wave to propagate from one tip to the other of the crack.

The comparison in figures (5.1 — Left) and (5.2 — Left) show that the « second » formulation is stable for a CFL = 0.4 while the first one was unstable (the first one is stable for a CFL $\leqslant$ 0.3, see [5]). The first

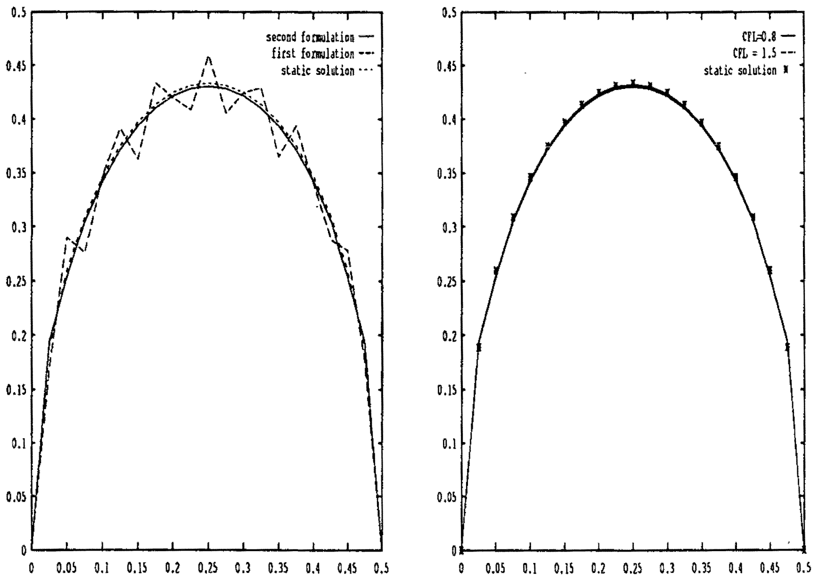


Figure 5.1. — Axis : $(x, \varphi(x))$ (t being fixed). Left : Comparison between the first, the second formulation at some fixed instant and the static solution — $\theta = 60°$ — CFL = 0.4. Right : Second formulation and static solution — $\theta = 60°$ — CFL = 0.8 and CFL = 1.5.

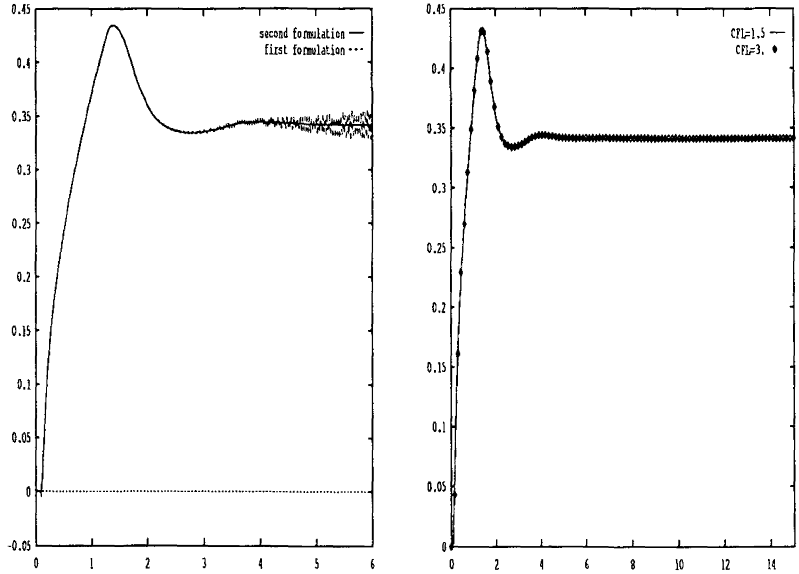


Figure 5.2. — Axis : $(t/\ell, \varphi(0.1, t))$ Left : Comparison between the first and the second formulation — evolution in time — $\theta = 60°$ — CFL = 0.4. Right : Time evolution of φ computed with the second formulation — $\theta = 60°$ — CFL = 1.5 and CFL = 3.

approximate solution oscillates around the static solution and the second coincids perfectly with the static solution (we have choosen enough large times). On figure (5.1 — Right), we see that the agreement of the second solution with the static solution is still very good even for large values of CFL.

Figure (5.2 — Left) show that the oscillations of the first solution appear around $t/\ell = 2$, i.e. after one round trip of the wave along the crack. On the right, the time evolution of the second solution has been observed for large CFL (CFL = 1.5 and CFL = 3.) up to times equal to 15 times the travelling time along the crack. We have also carried out some other computations with different choices of CFL and all this experiments showed that the « second » formulation is unconditionally stable. This result is numerically very interesting since it allows to compute the solution at large time without too much step size. Thus, this method is much less expensive. However, if we choose a too large CFL, the accuracy is not so good. This is not obvious to see it with a linear incident wave but it can be seen on the figure (5.3), where the solution corresponding to an harmonique incident wave is presented (i.e. the function f is choosen harmonique).

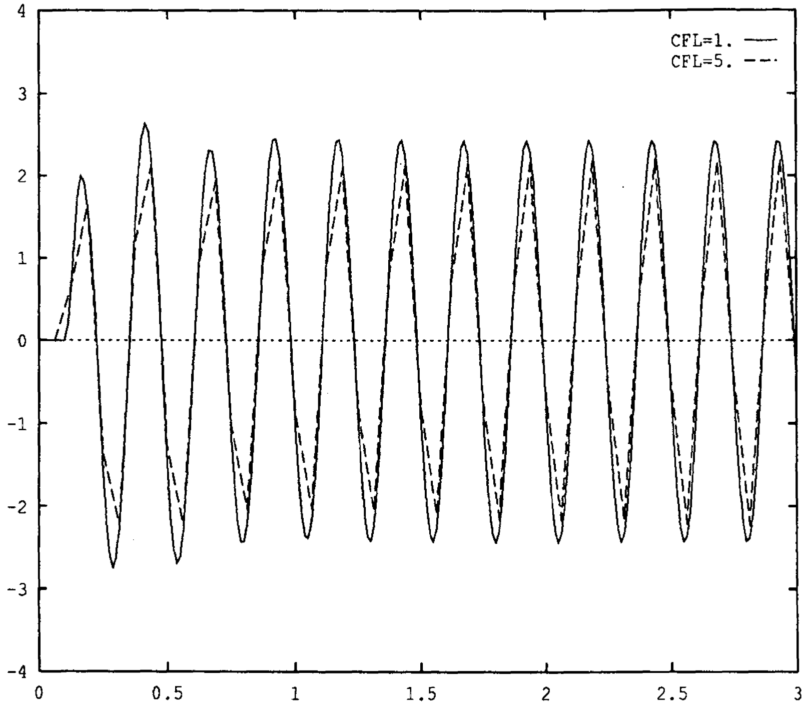


Figure 5.3. — Axis : $(t, \varphi(0.2, t))$ — Second formulation — Comparison between CFL = 1 and CFL = 5 for an harmonique incident wave — $\theta = 60°$.

5.2. The Plane Problem

In this case, there are two kinds of waves propagating

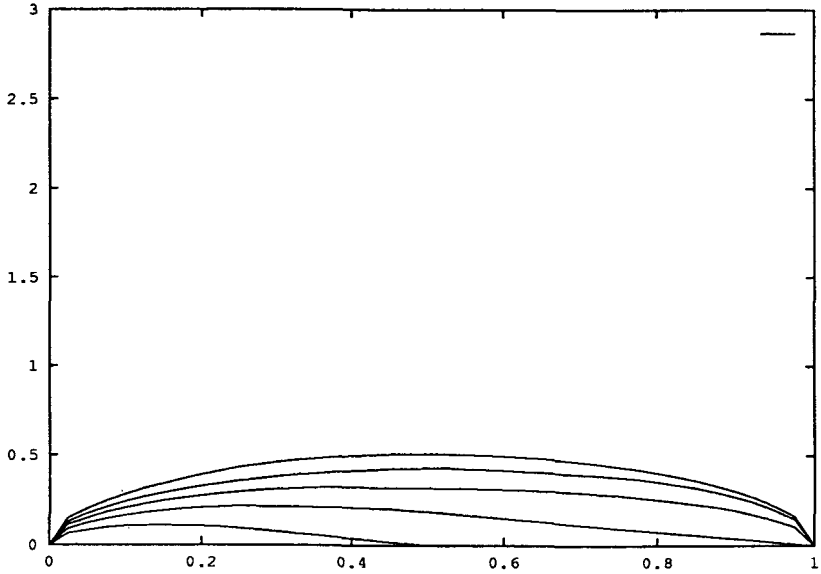
1. **The Pressure waves *P***

$$\begin{cases} C = C_P \\ \vec{d} = \vec{k}\,. \end{cases} \tag{5.9}$$

2. **The Shear waves S**

$$\begin{cases} C = C_S \\ \vec{d} = \vec{k}^{\perp}\,. \end{cases} \tag{5.10}$$

A comparison is done with the results of N. Nishimura, Q. C. Guo and S. Kobayashi (see [21]). In this paper, they solve this problem by using a collocation method. In order to compare with their result we multiply our



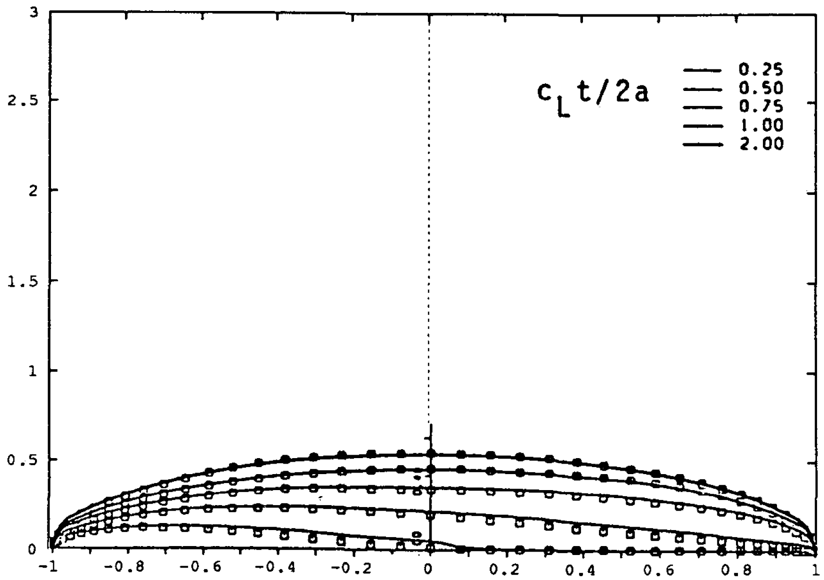


Figure 5.4. — Axis : (x_1/ℓ, cte × φ_1). Incident *P* wave $\theta = 60°$ — top : variational method — buttom : collocation method.

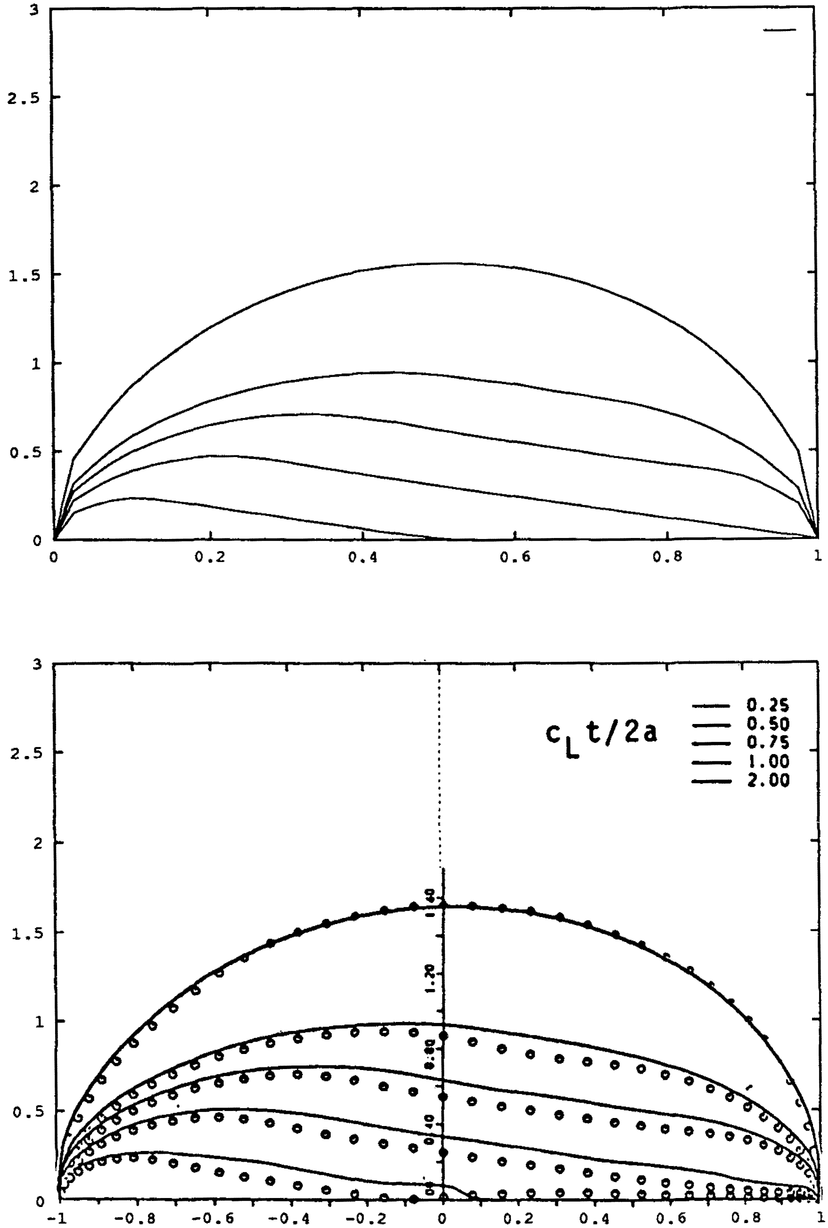


Figure 5.5. — Axis : $(x_1/\ell, \text{cte} \times \varphi_2)$. Incident P wave $\theta = 60°$.

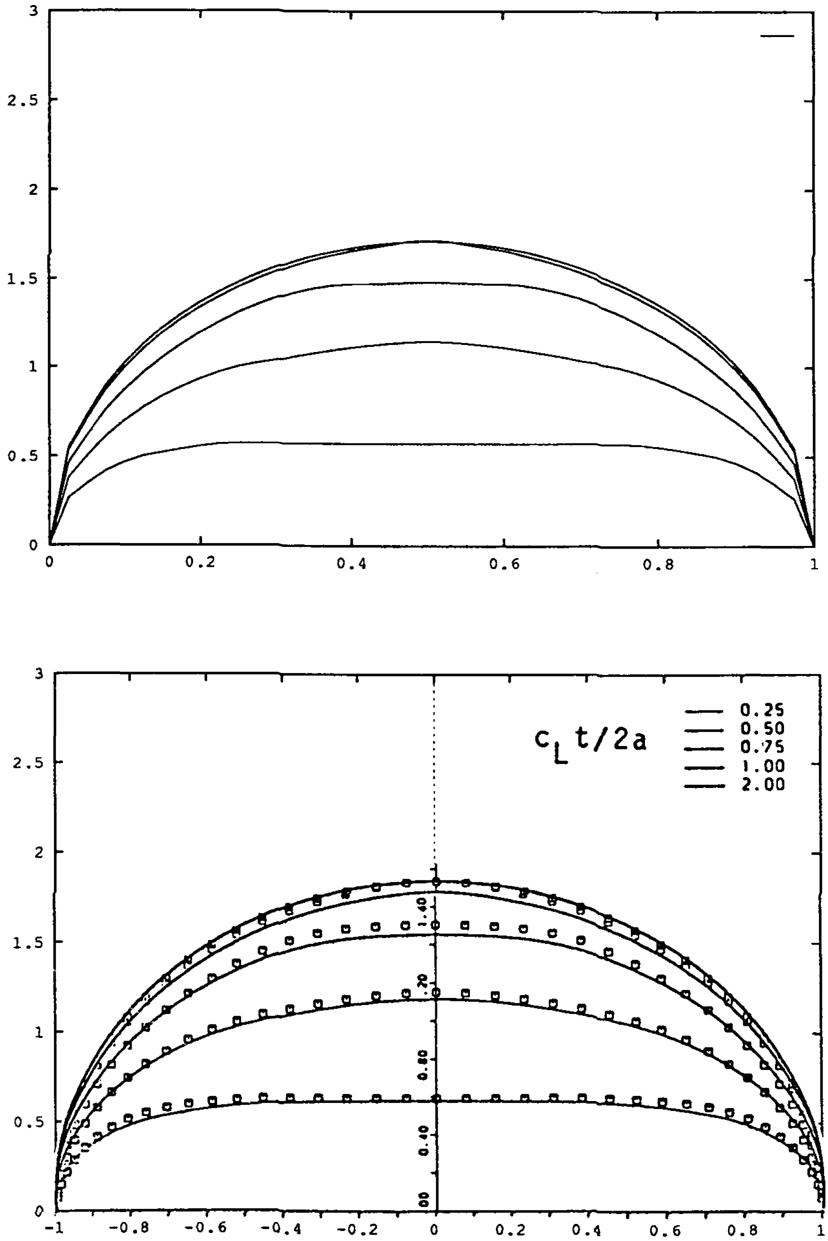


Figure 5.6. — Axis : (x_1/ℓ, cte × φ_1). Incident S wave $\theta = 90°$ — top : variational method — buttom : collocation method.

resulting COD by a constant : cte $= \dfrac{2\,C_S^2}{C_P\,A\ell}$ for a P incident wave and Cte $= \dfrac{2\,C_S}{A\ell}$ for an S incident wave. Poisson's ratio is set equal to 1/4. We used 40 boundary elements of constant length and we will denote by CFL the quantity $\dfrac{C_P\,\Delta t}{\Delta x}$.

The agreement is satisfactory. The conclusion are the same as for the antiplane problem. The solution has a singularity near the typ of the crack as $\sqrt{r}$ where r represents the distance to the tip. In [21], they used shape functions that have this behaviour near the tip, it explains the difference between their and our results in the vicinity of the tips. Of course, the same shape functions could be used in the variational approach, but we choose piecewise linear shape function for the sake of simplicity.

6. CONCLUSION

We have presented a space-time variational formulation of the BIE for an elastic crack problem. The numerical experiments show its accuracy and stability. The numerical analysis of our method of discretization can be done following the classical procedure, as done in [1], [2] for the acoustic problem. Also, it would be interesting to compare the framework of Galerkin discretization used here with the framework of multistep time discretization proposed in a recent work of Lubich [17] for time-dependent boundary integral equations.

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