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P. LIN
K. W. MORTON
E. SÜLI

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EULER CHARACTERISTIC GALERKIN SCHEME WITH RECOVERY (*)

by P. LIN [1], K. W. MORTON [1] and E. SÜLI [1]

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Abstract. — *This paper describes a general formulation of the Euler Characteristic Galerkin (ECG) scheme for scalar conservation laws, based on the theory of the Riemann-Stieltjes integral. The ECG scheme is proved to be equivalent to the projection of Brenier's transport-collapse operator. For the purpose of getting higher order accuracy, we explore two recovery procedures, namely continuous linear recovery and discontinuous linear recovery. Some estimates are obtained for proving the convergence of the ECG scheme. Finally we prove that the limit function of the approximation constructed by the ECG scheme with recovery is an admissible solution of the conservation law.*

Résumé. — *Cet article présente la formulation générale du schéma de Galerkin utilisant les caractéristiques (*(ECG) *pour les lois de conservation scalaires et dans le cadre de la théorie des intégrales de Riemann et Stieltjes. Nous montrons que ce schéma est équivalent à l'opérateur de transport-collapse de Brenier. Pour obtenir une plus grande précision, nous envisageons deux types de reconstruction, suivant que sont utilisées des fonctions linéaires continues ou des fonctions linéaires discontinues. La convergence du schéma numérique est alors monrée après avoir obtenu certaines majorations. Nous prouvons finalement que la limite de la fonction approchée, obtenue après reconstruction, est une solution admissible des équations de conservation.*

1. INTRODUCTION

In this paper we shall consider the finite element method for a purely hyperbolic conservation law in one dimension

$$u_t + f(u)_x = 0\,, \tag{1.1}$$

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[1] Oxford University Computing Laboratory, Numerical Analysis Group, 11 Keble Road, Oxford, England OX1 3QD.

where the flux function $f(\,.\,) \in C^2$. The difficulties associated with the numerical solution of (1.1) are highlighted by the Godunov [5] theorem which states that among linear finite difference schemes only first-order accurate methods have the desirable property of preserving the monotonicity of u. If a difference scheme only involves linear combinations of gridpoint values U_j^n and $f(U_j^n)$, experiments show that second and higher-order accurate approximations bring about oscillations and instabilities.

Over the past twenty years, many papers have been devoted to developing successful methods which give accurate approximations with many valuable qualitative features. One of the important approaches to constructing accurate approximations of (1.1) is based on evolving along characteristics, an idea which can be traced back to the work of Courant, Isaacson and Rees [3]. Morton [11], [9], and Morton and Stokes [12] modify the Galerkin method by using the properties of characteristics to introduce what we now call the Euler Characteristic Galerkin (ECG) method. A similar idea is also used by Lesaint [7] to deal with advection equations.

Given a family of basis functions $\{\phi_j\}$ which is, for our purpose, supposed to consist of piecewise constant or piecewise linear functions, we write

$$U^n = \sum_j U_j^n \, \phi_j \tag{1.2}$$

for the finite element approximation to the solution $u(x, t)$ of (1.1) at time t_n, where $\{t_n\}$ is a set of gridpoints in time t. We notice that the characteristics of (1.1) are straight lines, and that the solution u is constant along any characteristic. Thus at two successive times t_n and t_{n+1} the values of the solution $u(x, t_n)$ and $u(x, t_{n+1})$ satisfy

$$u(y, t_{n+1}) = u(x, t_n), \quad y = x + a(u(x, t_n))\,\Delta t_n\,, \tag{1.3}$$

where $\Delta t_n := t_{n+1} - t_n$. This relation is employed to generate a finite element approximation of the form (1.2) via Galerkin projection :

$$\langle U^{n+1}, \phi_i \rangle = \int U^n(\alpha)\, \phi_i(y(\alpha))\, dy(\alpha)\,, \tag{1.4}$$

where $y(\alpha) = x(\alpha) + a(U^n(\alpha))\,\Delta t_n$, and where $(U^n(\alpha), x(\alpha))$ is a continuous parametrisation of the curve $U^n(x)$ in $[U^n, x]$ plane. The right-hand side of (1.4) is a Riemann-Stieltjes (R-S) integral.

Using the theory of R-S integral (see Appendix) we can rewrite (1.4) as

$$\langle U^{n+1}, \phi_i \rangle = - \iint^{y(\alpha)} \phi_i(s)\, ds\, dU^n(\alpha)\,,$$

and hence obtain

$$\left\langle U^{n+1} - U^n, \phi_i \right\rangle + \iint_{x(\alpha)}^{x(\alpha)+a(U^n(\alpha))\Delta t_n} \phi_i(s)\, ds\, dU^n(\alpha) = 0 \qquad (1.5)$$

or

$$\left\langle U^{n+1} - U^n, \phi_i \right\rangle + \Delta t_n \left\langle df(U^n(\alpha)), \Phi_i^n(\alpha) \right\rangle = 0\,, \qquad (1.6)$$

where

$$\Phi_i^n(\alpha) := \frac{1}{a(U^n(\alpha))\,\Delta t_n} \int_{x(\alpha)}^{x(\alpha)+a(U^n(\alpha))\Delta t_n} \phi_i(s)\, ds\,. \qquad (1.7)$$

As shown by Childs and Morton [2], in the case when f is convex or concave and $\{\phi_j\}$ is the piecewise constant basis, (1.6) becomes the simple Engquist-Osher [4] algorithm if the characteristics from each element boundary do not cross more than one element in a time step, or in other words, if the Courant-Friedrichs-Lewy (CFL) number does not exceed one. We shall show this is true for a general flux f. We note that in this paper the mesh is supposed to be quasiregular (see Section 5). Hence a large time step can be used in practice. In other words, we do not impose any specific restriction on the CFL number.

As already noted, if we use the piecewise constant basis, the ECG scheme will, in general, give only first order accuracy. Furthermore, we see from (1.6) (*cf.* [2]) that U^{n+1} may be a good approximation to $u(\,.\,, t_{n+1})$ in (1.3), but $f(U^n)$ and $a(U^n)$ may not be very good approximations to $f(u(\,.\,, t_n))$ and $a(u(\,.\,, t_n))$. Thus we are looking for a function $\tilde{u}^n$ such that $f(\tilde{u}^n)$ and $a(\tilde{u}^n)$ model $f(u(\,.\,, t_n))$ and $a(u(\,.\,, t_n))$ better than $f(U^n)$ and $a(U^n)$ do ; for the conservation purpose we require the projection restriction $\left\langle U^n - \tilde{u}^n, \phi_i \right\rangle = 0$, $\forall i$. Then from (1.6) and (1.7) we have a more general form

$$\left\langle U^{n+1} - U^n, \phi_i \right\rangle + \Delta t_n \left\langle df(\tilde{u}^n(\alpha)), \tilde{\Phi}_i^n(\alpha) \right\rangle = 0\,, \qquad (1.8)$$

where

$$\tilde{\Phi}_i^n(\alpha) := \frac{1}{a(\tilde{u}^n(\alpha))\,\Delta t_n} \int_{x(\alpha)}^{x(\alpha)+a(\tilde{u}^n(\alpha))\Delta t_n} \phi_i(s)\, ds\,, \qquad (1.9)$$

and $(\tilde{u}^n(\alpha), x(\alpha))$ is, of course, the continuous parametrisation of the curve $\tilde{u}^n(x)$ in the $[\tilde{u}^n, x]$ plane. In this paper we examine both continuous linear recovery and discontinuous linear recovery. The ECG scheme with either of the recoveries is believed to be higher-order accurate. The first recovery is explored in detail by Morton [10], Childs and Morton [2]. For the second recovery which is motivated by the work of van Leer [18] and the minmod flux-limiter method of Roe [15] (*cf.* also Goodman and LeVeque [6], Sweby [17]), the resulting scheme is proved to be TVD. One advantage of this approach is the ease of numerical implementation, and it appears to be as good as the first approach for the shock-capturing. However, it switches to being a first-order accurate scheme at extreme points and sonic points (*cf.* Osher and Chakravarthy [14]). Under an appropriate condition (see (4.4)), the ECG scheme with continuous linear recovery is proved to be TVB, where the total variation increases slightly. This may lead to higher-order accuracy at extreme points, which could be explained partly by the fact that this approach is third order accurate for linear equations on a uniform mesh. Though it does not have the monotonicity preserving property (some remedies for that are proposed and tested successfully by Morton and Sweby [8]), we prove that it preserves the monotonicity near sharp gradients. Thus it rules out any spurious oscillations near discontinuities.

The term *recovery* was introduced by Morton [11] in order to indicate the link to the field of optimal recovery. An important point in the process of recovery is that the recovery function $\tilde{u}^n$ is obtained using a priori knowledge available about the function being approximated. Usually, the recovery function $\tilde{u}^n$ is smoother than U^n. Thus higher order accuracy is achieved. The continuous piecewise linear recovery is used in Childs and Morton [2] where experiments show dramatic improvement, compared with no recovery. Indeed, in [2] this improvement of accuracy is proved for linear advection equations.

The plan of the paper is as follows. In the next section we give a general formulation of the Euler Characteristic Galerkin scheme ; then in Section 3 we establish a relation between the ECG scheme and Brenier's [1] transport-collapse operator. Then in Section 4 and Section 5 we examine continuous linear recovery and discontinuous linear recovery, respectively. Some estimates are obtained to prove convergence. Finally in Section 6 we prove that the entropy inequality is satisfied by the limit function of the approximation.

2. THE EULER CHARACTERISTIC GALERKIN SCHEME

We consider the Cauchy problem for a scalar conservation law :

$$\partial_t u + \partial_x f(u) = 0 , \tag{2.1}$$

$$u(x, 0) = u^0(x) , \tag{2.2}$$

where $f \in C^2(R)$. We know that the solution of (2.1) is constant along characteristics which are therefore straight lines.

We define an evolutionary operator $\hat{E}(t)$ by

$$y = x + a(u^0(x))\, t\,, \tag{2.3}$$

$$(\hat{E}(t)\, u)(y) = u^0(x)\,, \tag{2.4}$$

and note that, in general, after a finite time, $(\hat{E}(t)\, u)(y)$ becomes a multivalued function (*cf.* Brenier [1], for example).

We will assume that the initial datum $u^0(x)$, although possibly discontinuous, has a continuous graph $[u^0, x]$ in the (u, x) plane. Thus we can write it in the form

$$u^0 = u^0(\alpha), \quad x = x(\alpha)\,, \tag{2.5}$$

where α is a parameter and $u^0(\alpha)$, $x(\alpha)$ are continuous with respect to α. We emphasize that $x(\alpha)$ is a non-decreasing function of α. If u^0 is continuous at the point x, it follows that

$$u^0(x(\alpha)) = u^0(x) = u^0(\alpha)\,.$$

We then have, corresponding to (2.3),

$$y(\alpha) = x(\alpha) + a(u^0(\alpha))\, t\,. \tag{2.6}$$

Hence we obtain another continuous graph $[\hat{E}(t)\, u, y]$ which can be written in the form

$$y = y(\alpha), \quad (\hat{E}(t)\, u)(\alpha) = u^0(\alpha)\,. \tag{2.7}$$

Given any set of discrete times $\{t_n\}$, $t_{n+1} = t_n + \Delta t_n$, suppose that $u(t_n) = u(x, t_n)$ is approximated by the finite element expansion

$$U^n(x) = \sum_j U_j^n\, \phi_j(x) \tag{2.8}$$

in terms of the basis functions $\{\phi_j\}$, where $\{\phi_j\}$ is considered to be the basis of piecewise constant or piecewise linear functions ; henceforth we assume $U^n(x)$ is piecewise continuous. Then the Galerkin projection leads to the time-stepping algorithm

$$\langle U^{n+1}, \phi_i \rangle = \langle \hat{E}(\Delta t_n)\, U^n, \phi_i \rangle = \int U^n(\alpha)\, \phi_i(y(\alpha))\, dy(\alpha) \quad \forall i\,, \tag{2.9}$$

where the right-hand side is a R-S integral, and $y(\alpha) = x(\alpha) + a(U^n(\alpha))\, \Delta t_n$.

We also assume that the initial datum $u^0(x)$ has bounded support, i.e.,

$$u^0(x) = 0 \quad \text{if} \quad |x| \geq N\,,$$

where N is some integer. Therefore, for any $t_n \in [0, T]$, there exists $M > 0$, depending only on given $T > 0$, such that

$$U^n(x) = 0 \quad \text{if} \quad |x| \geq M .$$

Using the basic results for the R-S integral in the Appendix and noting that $y(s)$ is Lipschitz-continuous in s, we can rewrite the right-hand side of (2.9) as

$$\int U^n(\alpha)\, \phi_i(y(\alpha))\, dy(\alpha) = \int U^n(\alpha)\, d\left(\int^{y(\alpha)} \phi_i(s)\, ds\right) = \\ - \iint^{y(\alpha)} \phi_i(s)\, ds\, dU^n(\alpha), \quad (2.10)$$

where $y(\alpha) = x(\alpha) + a(U^n(\alpha))\,\Delta t_n$. This defines the successive approximations $\{U^n(x)\}$; on the other hand, we will prove below that their projections can be written as a R-S integral, i.e.,

$$\langle U^n, \phi_i \rangle = \int U^n(\alpha)\, \phi_i(x(\alpha))\, dx(\alpha) = - \iint^{x(\alpha)} \phi_i(s)\, ds\, dU^n(\alpha). \quad (2.11)$$

Hence,

$$\langle U^{n+1} - U^n, \phi_i \rangle = - \iint_{x(\alpha)}^{x(\alpha) + a(U^n(\alpha))\,\Delta t_n} \phi_i(s)\, ds\, dU^n(\alpha) = \\ - \Delta t_n \int \frac{1}{a(U^n(\alpha))\,\Delta t_n} \int_{x(\alpha)}^{x(\alpha) + a(U^n(\alpha))\,\Delta t_n} \phi_i(s)\, ds\, df(U^n(\alpha)), \quad (2.12)$$

or, equivalently,

$$\langle U^{n+1} - U^n, \phi_i \rangle + \Delta t_n \langle \Phi_i^n, df(U^n) \rangle = 0 ,$$

where Φ_i^n is given by (1.7) ; this is called the Euler Characteristic Galerkin (ECG) Scheme.

To improve the accuracy of approximation, many papers have explored recovery procedures (*cf.* Morton [11], [10], Childs and Morton [2], etc.). This means that we replace U^n on the right-hand side of (2.12) by a physically more acceptable function $\tilde{u}^n$. Thus the ECG scheme takes the form

$$\langle U^{n+1} - U^n, \phi_i \rangle + \Delta t_n \langle \tilde{\Phi}_i^n, df(\tilde{u}^n) \rangle = 0 , \quad (2.13)$$

where the integral appearing in the second term of the left-hand side is a R-S integral and $\tilde{\Phi}_i^n$ is given by (1.9) ; $\tilde{u}^n$ is assumed to satisfy the recovery equations

$$\langle U^n - \tilde{u}^n, \phi_i \rangle = 0 \quad \forall i . \quad (2.14)$$

We now prove (2.11). Since U^n is piecewise continuous and has bounded support, U^n has only a finite number of points of discontinuity $\{x_i\}$, $i = 1, \dots, M$. For the mapping $x = x(\alpha)$ we can construct a series of intervals $[\alpha_i, \beta_i]$, $\beta_i \leqslant \alpha_{i+1}$, such that $x(\alpha) = x_i$, $\alpha \in [\alpha_i, \beta_i]$. Thus,

$$\int_{\alpha \in [\alpha_i, \beta_i]} U^n(\alpha)\, \phi_i(x(\alpha))\, dx(\alpha) = 0 .$$

On the other hand, $U^n(x)$ is continuous on the intervals (x_i, x_{i+1}) and $(-\infty, x_1)$, $(x_M, +\infty)$, so that

$$U^n(\alpha) = U^n(x),\ x = x(\alpha),$$
$$\alpha \in \bigcup_i [\beta_i, \alpha_{i+1}] \cup (-\infty, \alpha_1] \cup [\beta_M, +\infty) .$$

We therefore have that

$$\int_{\alpha \in [\beta_i, \alpha_{i+1}]} U^n(\alpha)\, \phi_i(x(\alpha))\, dx(\alpha) = \\ = \int_{x(\beta_i)}^{x(\alpha_{i+1})} U^n(x)\, \phi_i(x)\, dx, \quad i = 1, \dots, M-1 ,$$

$$\int_{\alpha \in (-\infty, \alpha_1]} U^n(\alpha)\, \phi_i(x(\alpha))\, dx(\alpha) = \int_{-\infty}^{x(\alpha_1)} U^n(x)\, \phi_i(x)\, dx ,$$

$$\int_{\alpha \in [\beta_M, +\infty)} U^n(\alpha)\, \phi_i(x(\alpha))\, dx(\alpha) = \int_{x(\beta_M)}^{+\infty} U^n(x)\, \phi_i(x)\, dx ,$$

where all integrals on the right-hand sides of the above are Lebesgue integrals. Combining these identities we obtain (2.11).

Finally we illustrate that, under the condition $CFL \leqslant 1$, the ECG scheme using piecewise constants with no recovery is equivalent to the Engquist-Osher scheme. In fact, if we write $a(w) = a^+(w) - a^-(w)$, where $a^+ \geqslant 0$, $a^- \geqslant 0$, then under the condition $CFL \leqslant 1$ the ECG scheme becomes

$$U_i^{n+1} = U_i^n - \frac{\Delta t_n}{\Delta x_i} \left[\int_{U_{i-1}^n}^{U_i^n} a^+(w)\, dw - \int_{U_i^n}^{U_{i+1}^n} a^-(w)\, dw \right]$$

$$= U_i^n - \frac{\Delta t_n}{\Delta x_i} \left\{ \frac{1}{2} [f(U_i^n) - f(U_{i-1}^n)] + \int_{U_{i-1}^n}^{U_i^n} \left[a^+(w) - \frac{1}{2} a(w) \right] dw \right.$$
$$\left. + \frac{1}{2} [f(U_{i+1}^n) - f(U_i^n)] + \int_{U_i^n}^{U_{i+1}^n} \left[-a^-(w) - \frac{1}{2} a(w) \right] dw \right\}$$

$$= U_i^n - \frac{1}{2} \frac{\Delta t_n}{\Delta x_i} [f(U_{i+1}^n) - f(U_{i-1}^n)] + \frac{1}{2} \frac{\Delta t_n}{\Delta x_i} [Q_{i+\frac{1}{2}}^n \Delta U_{i+\frac{1}{2}}^n - Q_{i-\frac{1}{2}}^n \Delta U_{i-\frac{1}{2}}^n] ,$$

where

$$Q_{i+\frac{1}{2}}^n = \frac{1}{U_{i+1}^n - U_i^n} \int_{U_i^n}^{U_{i+1}^n} |a(w)|\, dw, \quad \Delta U_{i+\frac{1}{2}}^n = U_{i+1}^n - U_i^n .$$

This is exactly the Engquist-Osher scheme on a non-uniform mesh.

3. PROJECTION OF THE TRANSPORT-COLLAPSE OPERATOR

In this section we prove the important result that (2.10) is indeed equivalent to the projection of Brenier's transport-collapse operator (*cf.* Childs and Morton [2]). Let $T(t)$ denote the transport-collapse operator (see Brenier [1]) ; then

$$T(t)\, u(x) = \int Ju(x - a(w)\, t,\, w)\, dw ,$$

where

$$Ju(x, w) = \begin{cases} 1 & \text{if} \quad 0 < w < u(x) \\ -1 & \text{if} \quad u(x) < w < 0 \\ 0 & \text{otherwise} . \end{cases}$$

THEOREM 3.1 : *Let $u \in BV$ have bounded support. We introduce a parameter α, as in* (2.5), *such that $u(\alpha)$ and $x(\alpha)$ are continuous BV functions of α. Then for any $f \in L^\infty(R) \cap L^1(R)$,*

$$\langle T(t)\, u, f \rangle = - \iint_{-\infty}^{x(\alpha) + a(u(\alpha))\, t} f(x)\, dx\, du(\alpha) . \tag{3.1}$$

Proof : 1st step. Suppose that $u(x)$ is piecewise constant, i.e., $u(x) = U_j$, $x \in [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$, $j = 0, \pm 1, \pm 2, \ldots$; this is an important special case, as well as being a convenient step in a more general argument. We recall Lemma 1 of Brenier [1] by which in this case

$$Ju(x, w) = \sum_j \operatorname{sgn}\,(U_j - U_{j-1}) \,.\, 1\left\{w \in [U_{j-1}, U_j]\right\} \,.\, 1\left\{x \geqslant x_{j-\frac{1}{2}}\right\} ,$$

where we have used the notation

$$1\{x \in A\} = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \notin A . \end{cases}$$

Hence, for any $f \in L^{\infty}(R) \cap L^{1}(R)$,

$$\begin{aligned}\int T(t)\, u(x)\, f(x)\, dx &= \iint Ju(x, w)\, f(x + a(w)\, t)\, dx\, dw \\ &= \sum_j \int_{U_{j-1}}^{U_j} \int_{x_{j-\frac{1}{2}} + a(w)\, t}^{+\infty} f(x)\, dx\, dw \\ &= - \sum_j \int_{U_{j-1}}^{U_j} \int_{-\infty}^{x_{j-\frac{1}{2}} + a(w)\, t} f(x)\, dx\, dw\,. \end{aligned} \tag{3.2}$$

We need to prove that the right-hand side of (3.1) is equal to the right-hand side of (3.2). Since u is piecewise constant, there exist $\alpha_{j-\frac{1}{2}}$, $\beta_{j-\frac{1}{2}}$, $\alpha_{j-\frac{1}{2}} \leq \beta_{j-\frac{1}{2}}$, such that $x(\alpha) = x_{j-\frac{1}{2}}$, $\alpha \in [\alpha_{j-\frac{1}{2}}, \beta_{j-\frac{1}{2}}]$, and $u(\alpha) = U_j = u(x(\alpha))$, $\alpha \in [\beta_{j-\frac{1}{2}}, \alpha_{j+\frac{1}{2}}]$. Then

$$\begin{aligned}&\iint_{-\infty}^{x(\alpha) + a(u(\alpha))\, t} f(x)\, dx\, du(\alpha) \\ &\quad = \sum_j \int_{\alpha \in [\alpha_{j-\frac{1}{2}}, \beta_{j-\frac{1}{2}}]} \int_{-\infty}^{x(\alpha) + a(u(\alpha))\, t} f(x)\, dx\, du(\alpha) \\ &\quad = \sum_j \int_{\alpha \in [\alpha_{j-\frac{1}{2}}, \beta_{j-\frac{1}{2}}]} \int_{-\infty}^{x_{j-\frac{1}{2}} + a(u(\alpha))\, t} f(x)\, dx\, du(\alpha)\,.\end{aligned}$$

Since $u(\alpha)$ is continuous, $u(\alpha_{j-\frac{1}{2}}) = U_{j-1}$, $u(\beta_{j-\frac{1}{2}}) = U_j$. It follows that

$$\begin{aligned}&\int_{\alpha \in [\alpha_{j-\frac{1}{2}}, \beta_{j-\frac{1}{2}}]} \int_{-\infty}^{x_{j-\frac{1}{2}} + a(u(\alpha))\, t} f(x)\, dx\, du(\alpha) = \\ &\qquad\qquad = \int_{U_{j-1}}^{U_j} \int_{-\infty}^{x_{j-\frac{1}{2}} + a(w)\, t} f(x)\, dx\, dw\,,\end{aligned}$$

where the right-hand side is a Riemann integral. Thus we have proved the required result.

2nd step. Suppose that u is piecewise linear, i.e.,

$$u(x) = V_{j-\frac{1}{2}} + \frac{x - x_{j-\frac{1}{2}}}{\Delta x_j} (U_{j+\frac{1}{2}} - V_{j-\frac{1}{2}}), \quad x \in [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})\,, \tag{3.3}$$

where $\Delta x_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$. It is easy to approximate u by a sequence of piecewise constant functions : we just set

$$u^n(x) = U^n_{j,k} := V_{j-\frac{1}{2}} + \frac{k}{n}\,(U_{j+\frac{1}{2}} - V_{j-\frac{1}{2}}),$$
$$x \in \left[x_{j-\frac{1}{2}} + \frac{k}{n}\,\Delta x_j,\ x_{j-\frac{1}{2}} + \frac{k+1}{n}\,\Delta x_j\right],$$

for $k = 0, 1, \ldots, n-1$; $j = 0, \pm 1, \ldots$ From the 1st step we have that

$$\begin{aligned}\int T(t)\,u^n(x)\,f(x)\,dx = &-\sum_j \int_{U_{j-\frac{1}{2}}}^{V_{j-\frac{1}{2}}} \int_{-\infty}^{x_{j-\frac{1}{2}} + a(w)\,t} f(x)\,dx\,dw \\ &-\sum_j \sum_{k=1}^{n-1} \int_{U^n_{j,k-1}}^{U^n_{j,k}} \int_{-\infty}^{x_{j-\frac{1}{2}} + \frac{k}{n}\Delta x_j + a(w)\,t} f(x)\,dx\,dw \\ &-\sum_j \int_{U^n_{j,n-1}}^{U_{j+\frac{1}{2}}} \int_{-\infty}^{x_{j+\frac{1}{2}} + a(w)\,t} f(x)\,dx\,dw\,. \end{aligned} \tag{3.4}$$

Since the graph of u is a straight line on each interval $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, u^n is either increasing or decreasing on each of these intervals. We can easily find a one-to-one mapping such that $u^n(\alpha)$ is a continuous function of α where α is such that $u(\alpha)$ is continuous, and moreover $u^n(\alpha)$ converges uniformly to $u(\alpha)$. Hence, (3.4) can be written as a R-S integral :

$$\begin{aligned}\int T(t)\,u^n(x)\,f(x)\,dx &= -\iint_{-\infty}^{x(\alpha) + a(u^n(\alpha))\,t} f(x)\,dx\,du^n(\alpha) \\ &\to -\iint_{-\infty}^{x(\alpha) + a(u(\alpha))\,t} f(x)\,dx\,du(\alpha)\,,\end{aligned}$$

as $n \to \infty$, where we have used Theorem 7.1. On the other hand,

$$\begin{aligned}&\left|\int T(t)\,u^n(x)\,f(x)\,dx - \int T(t)\,u(x)\,f(x)\,dx\right| \\ &\qquad \leqslant \|f\|_{L^\infty} \int |T(t)\,u^n(x) - T(t)\,u(x)|\,dx \\ &\qquad \leqslant \|f\|_{L^\infty} \int |u^n(x) - u(x)|\,dx \to 0,\ n \to \infty\,,\end{aligned}$$

where we have used a property of $T(t)$ (see Proposition 1 of Brenier [1]). Thus we finish the proof of the 2nd step.

3rd step. Suppose that $u(x)$ is a BV function. Then $u(\alpha)$ is a continuous BV function, and hence it is absolutely continuous since it has bounded support. There exists a sequence $\{u^n(\alpha)\}$ of piecewise linear continuous functions such that $u^n(\alpha)$ converges uniformly to $u(\alpha)$. On the other hand, we can construct a series of intervals $[\alpha_j, \beta_j]$, $j = 1, 2, \ldots,$ such that $x = x(\alpha)$ is a one-to-one mapping between $\{x \in R\}$ and $\{\alpha \in \cup_j [\alpha_j, \beta_j)\}$. The inverse mapping is denoted by $\alpha = \alpha^{-1}(x)$. In this way, $u^n(\alpha^{-1}(x))$ is piecewise linear and so is $u^n(x) = u^n(\alpha^{-1}(x))$. We also have that $u^n(x)$ converges uniformly to $u(x)$. For each function u^n, the 2nd step implies that

$$\langle T(t)\, u^n, f\rangle = -\iint_{-\infty}^{x(\alpha) + a(u^n(\alpha))\, t} f(x)\, dx\, du^n(\alpha)\,.$$

Passing to the limit as $n \to \infty$, we get (3.1) and hence we finish the proof.

We end this section by remarking that the ECG scheme and Theorem 3.1 can be generalized to the case where the initial datum has unbounded support, such as the Riemann initial datum. Suppose that

$$u^0(x) = u^+\,,\ u(-x) = u^-\,,\ \text{if } x \text{ is large enough}\,,$$

where u^+, u^- are some constants. Then the right-hand sides of (2.10) and (2.11) have an additional term

$$u^+ \int_{-\infty}^{+\infty} \phi_i(s)\, ds\,,$$

but (2.12) remains unchanged. We now prove that (3.1) is still true if $u \in L^\infty(R)$. Of course, we have to introduce here the following definition.

DEFINITION 3.2 : *Given that $u \in L^\infty(R)$, let*

$$u^n(x) = u(x), \quad if \quad |x| \leqslant n, \quad u^n(x) = 0 \quad if\ |x| \geqslant n\,.$$

We define

$$T(t)\, u(x) = \lim_{n \to \infty} T(t)\, u^n(x)\,.$$

We note that $T(t)\, u(x)$ is uniquely defined.

It is sufficient that we prove (3.1) under the condition that u is piecewise constant, i.e.,

$$u(x) = U_j,\ x \in [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}), \quad j = 0, \pm 1, \ldots, \pm M\,.$$

$$u(x) = u^+,\ x \in [x_{M+\frac{1}{2}}, +\infty)\,; \quad u(x) = u^-,\ x \in [-\infty, x_{-M-\frac{1}{2}})\,.$$

Let

$$u^n(x) = \begin{cases} 0 & x > x_{M+\frac{1}{2}} + n \\ 0 & x < x_{-M-\frac{1}{2}} - n \\ u(x) & \text{otherwise}. \end{cases}$$

$u^n(x)$ has bounded support, so that from (3.1) for $f \in L^\infty(R) \cap L^1(R)$, $\langle T(t)\, u^n, f\rangle$

$$= -\int_0^{u^-} \int_{-\infty}^{x_{-M-\frac{1}{2}} - n + a(w)t} f(x)\, dx\, dw - \int_{u^-}^{U_{-M}} \int_{-\infty}^{x_{-M-\frac{1}{2}} + a(w)t} f(x)\, dx\, dw$$

$$- \sum_{j=-M+1}^{M} \int_{U_{j-1}}^{U_j} \int_{-\infty}^{x_{j-\frac{1}{2}} + a(w)t} f(x)\, dx\, dw$$

$$- \int_{U_M}^{u^+} \int_{-\infty}^{x_{M+\frac{1}{2}} + a(w)t} f(x)\, dx\, dw - \int_{u^+}^{0} \int_{-\infty}^{x_{M+\frac{1}{2}} + n + a(w)t} f(x)\, dx\, dw$$

$$\to - \int_{u^-}^{U_{-M}} \int_{-\infty}^{x_{-M-\frac{1}{2}} + a(w)t} f(x)\, dx\, dw - \sum_{j=-M+1}^{M} \int_{U_j-1}^{U_j} \int_{-\infty}^{x_{j-\frac{1}{2}} + a(w)t} f(x)\, dx\, dw$$

$$- \int_{U_M}^{u^+} \int_{-\infty}^{x_{M+\frac{1}{2}} + a(w)t} f(x)\, dx\, dw + u^+ \int_{-\infty}^{+\infty} f(x)\, dx$$

$$= - \iint_{-\infty}^{x(\alpha) + a(u(\alpha))t} f(x)\, dx\, du(\alpha) + u^+ \int_{-\infty}^{+\infty} f(x)\, dx$$

which is exactly the right-hand side of (3.1) plus $u^+ \int_{-\infty}^{+\infty} f(x)\, dx$.

On the other hand, we have from Definition 3.1 that

$$\langle T(t)\, u^n, f\rangle \to \langle T(t)\, u, f\rangle\,,$$

which completes the proof of (3.1) in this more general case.

4. CONTINUOUS LINEAR RECOVERY

From now on we are only concerned with quasiregular meshes. Following Childs and Morton [2], a mesh $\{t_n\}$, $\left\{x_{k+\frac{1}{2}}\right\}$ is called quasiregular if there exists a constant D such that as $h \to 0$

$$\frac{1}{D} h < \Delta x_i < h, \quad \forall i,\ \Delta x_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}\,, \tag{4.1}$$

$$\frac{1}{D} h < \Delta t_n < h, \quad t_n \leqslant T,\ \Delta t_n = t_{n+1} - t_n\,. \tag{4.2}$$

We also consider the piecewise constant basis $\{\phi_i\}$:

$$\phi_i = \begin{cases} 1 & x \in [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \\ 0 & \text{otherwise} . \end{cases}$$

The ECG scheme with continuous linear recovery technique, in the form (2.13), was first used by Morton [10], Morton and Sweby [8] to improve the accuracy of the algorithm. As shown in Childs and Morton [2], the linear recovery function $\tilde{u}^n$ must, from assumption (2.14), satisfy the relation

$$\frac{1}{4}\,\theta^n_{i-\frac{1}{2}}\,\frac{\Delta x_i}{\Delta x_{i-1}+\Delta x_i}\,(\tilde{u}^n_{i-1}-\tilde{u}^n_i)\,+$$
$$+\,\frac{1}{4}\,\theta^n_{i+\frac{1}{2}}\,\frac{\Delta x_i}{\Delta x_i+\Delta x_{i+1}}\,(\tilde{u}^n_{i+1}-\tilde{u}^n_i)+\tilde{u}^n_i = U^n_i\,, \quad (4.3)$$

where $\theta^n_{i+\frac{1}{2}}$, $0 \leq \theta^n_{i+\frac{1}{2}} \leq 1$, is the parameter that at zero corresponds to no spreading of the discontinuity at $x_{j+\frac{1}{2}}$, and at unity to linear variation between $\tilde{u}^n_j$ at x_j and $\tilde{u}^n_{j+1}$ at x_{j+1}, where $x_j = \frac{1}{2}\,(x_{j+\frac{1}{2}} + x_{j-\frac{1}{2}})$.

THEOREM 4.1 : *Assume that the mesh is quasiregular, that* $\{U^n\}$ *is generated by the ECG scheme* (2.13) *with recovery by* (4.3), *and that there exists a constant* $C > 0$ *such that*

$$\theta^n_{i+\frac{1}{2}}\left|U^n_{i+1}-U^n_i\right| \leq Ch\ \forall i,\ \forall n\,. \quad (4.4)$$

Then there exists a constant K, depending on the given $t_N > 0$, *such that*

$$V(U^n(\,.\,,t_n)) \leq K, \quad t_n \leq t_N\,, \quad (4.5)$$
$$\sum_j \left|U^n_j - U^m_j\right|\,\Delta x_j \leq K(t_n - t_m)\,, \quad (4.6)$$

where $0 < t_m < t_n \leq t_N$.

We shall use the following lemma to prove Theorem 4.1.

LEMMA 4.2 : *Under the condition (4.4), there exists a constant* $M > 0$, *independent of h and N, such that*

$$\|U^n\|_{L^\infty} \leq \|U^0\|_{L^\infty} + Mt_n\,, \quad t_n \leq t_N\,. \quad (4.7)$$

((4.7) holds even when the mesh is not quasiregular.)

Proof : We first prove that

$$\theta^n_{i-\frac{1}{2}}\left|\tilde{u}^n_i - \tilde{u}^n_{i-1}\right| \leq 4\,Ch \quad \forall i\,. \quad (4.8)$$

Indeed, if (4.8) were not true, there would exist i_0 such that

$$\theta^n_{i_0-\frac{1}{2}}\left|\tilde{u}^n_{i_0}-\tilde{u}^n_{i_0-1}\right| \geqslant C_0 h, \quad C_0 > 4C,$$
$$\theta^n_{i-\frac{1}{2}}\left|\tilde{u}^n_i-\tilde{u}^n_{i-1}\right| \leqslant C_0 h \quad \forall i, \quad i \neq i_0.$$

Since from (4.3)

$$U^n_i - U^n_{i-1} = \left(1-\frac{1}{4}\theta^n_{i-\frac{1}{2}}\right)(\tilde{u}^n_i-\tilde{u}^n_{i-1}) + \frac{1}{4}\theta^n_{i+\frac{1}{2}}\frac{\Delta x_i}{\Delta x_{i+1}+\Delta x_i}(\tilde{u}^n_{i+1}-\tilde{u}^n_i)$$
$$+\frac{1}{4}\theta^n_{i-\frac{3}{2}}\frac{\Delta x_{i-1}}{\Delta x_{i-2}+\Delta x_{i-1}}(\tilde{u}^n_{i-2}-\tilde{u}^n_{i-1}), \tag{4.9}$$

we then have

$$\theta^n_{i_0-\frac{1}{2}}\left|U^n_{i_0}-U^n_{i_0-1}\right| \geqslant$$
$$\geqslant \left(1-\frac{1}{4}\theta^n_{i_0-\frac{1}{2}}\right)C_0 h - \frac{1}{4}\frac{\Delta x_{i_0}}{\Delta x_{i_0+1}+\Delta x_{i_0}}C_0 h - \frac{1}{4}\frac{\Delta x_{i_0-1}}{\Delta x_{i_0-2}+\Delta x_{i_0-1}}C_0 h$$
$$\geqslant \frac{3}{4}C_0 h - \frac{2}{4}C_0 h = \frac{1}{4}C_0 h > Ch$$

which contradicts (4.4). Hence (4.8) holds.

Using (4.3) and (4.8) we get that

$$\left|\tilde{u}^n_i - U^n_i\right| \leqslant 2Ch \quad \forall i. \tag{4.10}$$

On the other hand, since Theorem 3.1 gives

$$\left\langle U^{n+1}, \phi_i\right\rangle = \left\langle T(\Delta t_n)\,\tilde{u}^n, \phi_i\right\rangle,$$

we have that

$$\|U^{n+1}\|_{L^\infty} \leqslant \left\|T(\Delta t_n)\,\tilde{u}^n\right\|_{L^\infty} \leqslant \left\|\tilde{u}^n\right\|_{L^\infty}, \tag{4.11}$$

where we have used the properties of $T(\,.\,)$ (*cf.* Brenier [1]).

Combining (4.10) and (4.11) we get (4.7).

Remark 1 : (4.4) implies that when $\left|U^n_{i+1}-U^n_i\right|$ is large, $\theta^n_{i+\frac{1}{2}}$ has to be small. In other words, we do little recovery near shocks and contact discontinuities. On the other hand, we see from (4.8), (4.9) that

$$\operatorname{sgn}(U^n_{i+1}-U^n_i) = \operatorname{sgn}(\tilde{u}^n_{i+1}-\tilde{u}^n_i) \quad \text{if} \quad \left|U^n_{i+1}-U^n_i\right| > Ch,$$

which means that the recovery keeps the monotonicity near sharp gradients, and thus prevents the generation of spurious oscillations near discontinuities.

It is worth noticing that this condition on sgn $(\tilde{u}_{i+1}^n - \tilde{u}_i^n)$ is imposed everywhere in the recovery algorithms used in [2] and [8].

Remark 2 : The good accuracy of the continuous linear recovery scheme stems in part from the recovery near peaks and troughs. This means that the total variation is increased at the recovery stage, making the following proof rather long and delicate.

Proof of Theorem 4.1 : From the continuous linear recovery procedure we see that : (a) if $\theta_{j+\frac{1}{2}}^n = 0$, $\tilde{u}^n(\,.\,)$ has a jump at $x_{j+\frac{1}{2}}$; (b) if $\theta_{j+\frac{1}{2}}^n \neq 0$, $\tilde{u}^n(\,.\,)$ is continuous at $x_{j+\frac{1}{2}}$, and there exists an interval $[a_{j+\frac{1}{2}}, b_{j+\frac{1}{2}}] \subset [x_{j-\frac{1}{2}}, x_{j+\frac{3}{2}}]$, $a_{j+\frac{1}{2}} < x_{j+\frac{1}{2}} < b_{j+\frac{1}{2}}$, such that

$$\frac{d}{dx}\tilde{u}^n(x) = (\tilde{u}_{j+1}^n - \tilde{u}_j^n)/(b_{j+\frac{1}{2}} - a_{j+\frac{1}{2}}), \qquad x \in [a_{j+\frac{1}{2}}, b_{j+\frac{1}{2}}],$$

$$\frac{d}{dx}\tilde{u}^n(x) = 0, \qquad x \in [b_{j+\frac{1}{2}}, a_{j+\frac{3}{2}}).$$

Therefore, (2.13) is equivalent to

$$\begin{aligned}\langle U^{n+1} - U^n, \phi_i\rangle &= -\iint_{x(\alpha)}^{x(\alpha)+a(\tilde{u}^n(\alpha))\Delta t_n} \phi_i(s)\, ds\, d\tilde{u}^n(\alpha) \\ &= -\sum_{j,\ \left(\theta_{j+\frac{1}{2}}^n \neq 0\right)} \int_{\tilde{u}_j^n}^{\tilde{u}_{j+1}^n} \int_x^{x+a(\tilde{u}^n(x))\Delta t_n} \phi_i(s)\, ds\, d\tilde{u}^n(x) \\ &\quad - \sum_{j,\ \left(\theta_{j+\frac{1}{2}}^n = 0\right)} \int_{\tilde{u}_j^n}^{\tilde{u}_{j+1}^n} \int_{x_{j+\frac{1}{2}}}^{x_{j+\frac{1}{2}}+a(\tilde{u}^n(x))\Delta t_n} \phi_i(s)\, ds\, d\tilde{u}^n(x). \end{aligned} \tag{4.12}$$

Since

$$\tilde{u}^n(x) = \tilde{u}_j^n + \frac{x - a_{j+\frac{1}{2}}}{b_{j+\frac{1}{2}} - a_{j+\frac{1}{2}}}(\tilde{u}_{j+1}^n - \tilde{u}_j^n), \quad x \in [a_{j+\frac{1}{2}}, b_{j+\frac{1}{2}}],$$

(4.12) can be written as

$$\begin{aligned}&\langle U^{n+1} - U^n, \phi_i\rangle = \\ &= -\sum_{j,\ \left(\theta_{j+\frac{1}{2}}^n \neq 0\right)} (\tilde{u}_{j+1}^n - \tilde{u}_j^n)\int_0^1 d\lambda\ \times \\ &\qquad \times \int_{(b_{j+\frac{1}{2}} - a_{j+\frac{1}{2}})\lambda + a_{j+\frac{1}{2}}}^{(b_{j+\frac{1}{2}} - a_{j+\frac{1}{2}})\lambda + a_{j+\frac{1}{2}} + a(\tilde{u}_j^n + \lambda(\tilde{u}_{j+1}^n - \tilde{u}_j^n))\Delta t_n} \phi_i(s)\, ds\end{aligned}$$

$$- \sum_{j,\ \left(\theta^n_{j+\frac{1}{2}}=0\right)} (\tilde{u}^n_{j+1} - \tilde{u}^n_j) \int_0^1 d\lambda \int_{x_{j+\frac{1}{2}}}^{x_{j+\frac{1}{2}} + a(\tilde{u}^n_j + \lambda(\tilde{u}^n_{j+1} - \tilde{u}^n_j))\Delta t_n} \phi_i(s)\, ds$$

$$= - \sum_j (\tilde{u}^n_{j+1} - \tilde{u}^n_j) \int_0^1 d\lambda \int_{x_{j+\frac{1}{2}}(\lambda)}^{x_{j+\frac{1}{2}}(\lambda) + a(\tilde{u}^n_j + \lambda(\tilde{u}^n_{j+1} - \tilde{u}^n_j))\Delta t_n} \phi_i(s)\, ds\,, \tag{4.13}$$

where we have used the notation

$$x_{j+\frac{1}{2}}(\lambda) = \begin{cases} x_{j+\frac{1}{2}} & \text{if} \quad \theta^n_{j+\frac{1}{2}} = 0 \\ (b^n_{j+\frac{1}{2}} - a^n_{j+\frac{1}{2}})\,\lambda + a^n_{j+\frac{1}{2}} & \text{if} \quad \theta^n_{j+\frac{1}{2}} \neq 0\,. \end{cases}$$

Similarly, we have that

$$\langle T(\Delta t_n)\, U^n - U^n,\ \phi_i \rangle =$$

$$= - \iint_{x(\alpha)}^{x(\alpha) + a(U^n(\alpha))\Delta t_n} \phi_i(s)\, ds\, dU^n(\alpha)$$

$$= - \sum_j \int_{U^n_j}^{U^n_{j+1}} \int_{x_{j+\frac{1}{2}}}^{x_{j+\frac{1}{2}} + a(w)\Delta t_n} \phi_i(s)\, ds\, dw$$

$$= - \sum_j (U^n_{j+1} - U^n_j) \int_0^1 d\lambda \int_{x_{j+\frac{1}{2}}}^{x_{j+\frac{1}{2}} + a(U^n_j + \lambda(U^n_{j+1} - U^n_j))\Delta t_n} \phi_i(s)\, ds\,. \tag{4.14}$$

Combining (4.13) and (4.14) we get that

$$|\langle U^{n+1} - T(\Delta t_n)\, U^n,\ \phi_i \rangle| =$$

$$= |\langle U^{n+1} - U^n,\ \phi_i \rangle - \langle T(\Delta t_n)\, U^n - U^n,\ \phi_i \rangle|$$

$$\leqslant \sum_j \Big\{ \big|(\tilde{u}^n_{j+1} - \tilde{u}^n_j) - (U^n_{j+1} - U^n_j)\big| \times$$

$$\times \int_0^1 d\lambda \left| \int_{x_{j+\frac{1}{2}}(\lambda)}^{x_{j+\frac{1}{2}}(\lambda) + a(\tilde{u}^n_j + \lambda(\tilde{u}^n_{j+1} - \tilde{u}^n_j))\Delta t_n} \phi_i(s)\, ds \right|$$

$$+ \big|U^n_{j+1} - U^n_j\big| \int_0^1 d\lambda \left| \int_{x_{j+\frac{1}{2}}(\lambda) + a(U^n_j + \lambda(U^n_{j+1} - U^n_j))\Delta t_n}^{x_{j+\frac{1}{2}}(\lambda) + a(\tilde{u}^n_j + \lambda(\tilde{u}^n_{j+1} - \tilde{u}^n_j))\Delta t_n} \phi_i(s)\, ds \right| \Big\}$$

$$+ \sum_j \big|U^n_{j+1} - U^n_j\big| \int_0^1 d\lambda$$

$$\times \left| \int_{x_{j+\frac{1}{2}}(\lambda)}^{x_{j+\frac{1}{2}}(\lambda) + a^+ (U_j^n + \lambda (U_{j+1}^n - U_j^n)) \Delta t_n} \phi_i(s)\, ds \right.$$

$$\left. - \int_{x_{j+\frac{1}{2}}}^{x_{j+\frac{1}{2}} + a^+ (U_j^n + \lambda (U_{j+1}^n - U_j^n)) \Delta t_n} \phi_i(s)\, ds \right|$$

$$+ \sum_j \left| U_{j+1}^n - U_j^n \right| \int_0^1 d\lambda$$

$$\times \left| \int_{x_{j+\frac{1}{2}}(\lambda)}^{x_{j+\frac{1}{2}}(\lambda) - a^- (U_j^n + \lambda (U_{j+1}^n - U_j^n)) \Delta t_n} \phi_i(s)\, ds \right.$$

$$\left. - \int_{x_{j+\frac{1}{2}}}^{x_{j+\frac{1}{2}} - a^- (U_j^n + \lambda (U_{j+1}^n - U_j^n)) \Delta t_n} \phi_i(s)\, ds \right|$$

$$:= I_1^i + I_2^i + I_3^i ,$$

where I_l^i, $l = 1, 2, 3$, denote respectively the l-th term of the right-hand side of the above inequality, and we have used that $a(w) = a^+(w) - a^-(w)$, $a^+ \geqslant 0$, $a^- \geqslant 0$.

We first notice that

$$\left| a(U^n)\, \Delta t_n \right| \leqslant Ah \leqslant AD \left(\frac{1}{D} h \right), \quad t_n \leqslant t_N ,$$

$$\left| a(\tilde{u}^n)\, \Delta t_n \right| \leqslant Ah \leqslant AD \left(\frac{1}{D} h \right), \quad t_n \leqslant t_{N-1} ,$$

where

$$A = \max_{|u| \leqslant \| U^0 \|_\infty + M t_N} |a(u)| ,$$

because of (4.7), (4.10) and the quasiregularity of the mesh. This means that for each fixed i, all the integrands in I_l^i, $l = 1, 2, 3$, are zero if $|j - i| > k$, where $k := [AD] + 1$ with $[AD]$ being the integer part of AD. Thus the summation of j in each I_l^i has at most $2k + 1$ non-zero terms. Furthermore, since $U^n(\,.\,)$, $\tilde{u}^n(\,.\,)$, $t_n \leqslant t_N$, have bounded supports, there are only a finite number of non-zero terms I_l^i, say, when $i = 0, \pm 1, \pm 2, \pm 3, \ldots, \pm M$.

Having the above preparation, we now turn to estimate I_1^i. Using (4.10) we get that

$$\sum_{i=-M}^{M} I_1^i =$$

$$= \sum_{i=-M}^{M} \sum_{j=i-k}^{i+k} \left\{ \left| (\tilde{u}_{j+1}^n - \tilde{u}_j^n) - (U_{j+1}^n - U_j^n) \right| \times \right.$$

$$\times \int_0^1 d\lambda \left| \int_{x_{j+\frac{1}{2}}}^{x_{j+\frac{1}{2}}(\lambda) + a(\tilde{u}_j^n + \lambda(\tilde{u}_{j+1}^n - \tilde{u}_j^n))\Delta t_n} \phi_i(s)\, ds \right|$$

$$\left. + \left| U_{j+1}^n - U_j^n \right| \int_0^1 d\lambda \left| \int_{x_{j+\frac{1}{2}}(\lambda) + a(U_j^n + \lambda(U_{j+1}^n - U_j^n))\Delta t_n}^{x_{j+\frac{1}{2}}(\lambda) + a(\tilde{u}_j^n + \lambda(\tilde{u}_{j+1}^n - \tilde{u}_j^n))\Delta t_n} \phi_i(s)\, ds \right| \right\}$$

$$= \sum_{j=-k}^{k} \sum_{i=-M+j}^{j+M} \left\{ \left| (\tilde{u}_{j+1}^n - \tilde{u}_j^n) - (U_{j+1}^n - U_j^n) \right| \times \right.$$

$$\times \int_0^1 d\lambda \left| \int_{x_{j+\frac{1}{2}}(\lambda)}^{x_{j+\frac{1}{2}}(\lambda) + a(\tilde{u}_j^n + \lambda(\tilde{u}_{j+1}^n - \tilde{u}_j^n))\Delta t_n} \phi_i(s)\, ds \right|$$

$$\left. + \left| U_{j+1}^n - U_j^n \right| \int_0^1 d\lambda \left| \int_{x_{j+\frac{1}{2}}(\lambda) + a(U_j^n + \lambda(U_{j+1}^n - U_j^n))\Delta t_n}^{x_{j+\frac{1}{2}}(\lambda) + a(\tilde{u}_j^n + \lambda(\tilde{u}_{j+1}^n - \tilde{u}_j^n))\Delta t_n} \phi_i(s)\, ds \right| \right\}$$

$$\leqslant \sum_{j=-k}^{k} \left\{ Ch \int_0^1 d\lambda \int_{x_{j+\frac{1}{2}}(\lambda) - A\,\Delta t_n}^{x_{j+\frac{1}{2}}(\lambda) + A\,\Delta t_n} \sum_{i=-M+j}^{j+M} \phi_i(s)\, ds + Ch \int_0^1 d\lambda \, \times \right.$$

$$\left. \times \int_{x_{j+\frac{1}{2}}(\lambda) + \min\left\{a(\tilde{u}_j^n + \lambda(\tilde{u}_{j+1}^n - \tilde{u}_j^n))\Delta t_n,\, a(U_j^n + \lambda(U_{j+1}^n - U_j^n))\Delta t_n\right\}}^{x_{j+\frac{1}{2}}(\lambda) + \max\left\{a(\tilde{u}_j^n + \lambda(\tilde{u}_{j+1}^n - \tilde{u}_j^n))\Delta t_n,\, a(U_j^n + \lambda(U_{j+1}^n - U_j^n))\Delta t_n\right\}} \sum_{i=-M+j}^{j+M} \phi_i(s)\, ds \right\}$$

$$\leqslant Ch\,\Delta t_n\,, \tag{4.15}$$

where C is independent of h.

We now deal with I_2^i. By the definition of $x_{j+\frac{1}{2}}(\lambda)$, there is $\lambda_0 \in (0, 1)$ such that

$$x_{j+\frac{1}{2}}(\lambda) \leqslant x_{j+\frac{1}{2}}\,, \quad \text{if} \quad \lambda \in [0, \lambda_0]\,,$$

$$x_{j+\frac{1}{2}}(\lambda) \geqslant x_{j+\frac{1}{2}}\,, \quad \text{if} \quad \lambda \in [\lambda_0, 1]\,.$$

Then it is clear that

$$\int_{x_{j+\frac{1}{2}}(\lambda)}^{x_{j+\frac{1}{2}}(\lambda) + a^+(U_j^n + \lambda(U_{j+1}^n - U_j^n))\Delta t_n} \phi_i(s)\, ds$$

$$\begin{cases} \geqslant \displaystyle\int_{x_{j+\frac{1}{2}}}^{x_{j+\frac{1}{2}} + a^+(U_j^n + \lambda(U_{j+1}^n - U_j^n))\Delta t_n} \phi_i(s)\, ds & \text{if} \quad \lambda \in [\lambda_0, 1] \\ \leqslant \displaystyle\int_{x_{j+\frac{1}{2}}}^{x_{j+\frac{1}{2}} + a^+(U_j^n + \lambda(U_{j+1}^n - U_j^n))\Delta t_n} \phi_i(s)\, ds & \text{if} \quad \lambda \in [0, \lambda_0] \end{cases} \quad \forall i \geqslant j+1\,.$$

Thus

$$\sum_{i=-M}^{M} I_2^i = \sum_{j=-k}^{k} \sum_{i=-M+j}^{M+j} \left|U_{j+1}^n - U_j^n\right| \int_{\lambda_0}^{1} d\lambda$$

$$\times \left| \int_{x_{j+\frac{1}{2}}(\lambda)}^{x_{j+\frac{1}{2}}(\lambda) + a^+ (U_j^n + \lambda (U_{j+1}^n - U_j^n)) \Delta t_n} \phi_i(s)\, ds \right.$$

$$\left. - \int_{x_{j+\frac{1}{2}}}^{x_{j+\frac{1}{2}} + a^+ (U_j^n + \lambda (U_{j+1}^n - U_j^n)) \Delta t_n} \phi_i(s)\, ds \right|$$

$$+ \sum_{j=-k}^{k} \sum_{i=-M+j}^{M+j} \left|U_{j+1}^n - U_j^n\right| \int_{0}^{\lambda_0} d\lambda$$

$$\times \left| \int_{x_{j+\frac{1}{2}}(\lambda)}^{x_{j+\frac{1}{2}}(\lambda) + a^+ (U_j^n + \lambda (U_{j+1}^n - U_j^n)) \Delta t_n} \phi_i(s)\, ds \right.$$

$$\left. - \int_{x_{j+\frac{1}{2}}}^{x_{j+\frac{1}{2}} + a^+ (U_j^n + \lambda (U_{j+1}^n - U_j^n)) \Delta t_n} \phi_i(s)\, ds \right|$$

$$:= J_1 + J_2 .$$

We notice that

$$J_1 = \sum_{j=-k}^{k} \int_{\lambda_0}^{1} d\lambda \sum_{i \geq j+1}^{M+j} \left|U_{j+1}^n - U_j^n\right| \times$$

$$\times \left| \int_{x_{j+\frac{1}{2}}(\lambda)}^{x_{j+\frac{1}{2}}(\lambda) + a^+ (U_j^n + \lambda (U_{j+1}^n - U_j^n)) \Delta t_n} \phi_i(s)\, ds \right.$$

$$\left. - \int_{x_{j+\frac{1}{2}}}^{x_{j+\frac{1}{2}} + a^+ (U_j^n + \lambda (U_{j+1}^n - U_j^n)) \Delta t_n} \phi_i(s)\, ds \right|$$

$$\leq \sum_{j=-k}^{k} \int_{\lambda_0}^{1} d\lambda \left\{ \left|U_{j+1}^n - U_j^n\right| \right.$$

$$\times \left| \int_{x_{j+\frac{1}{2}}}^{x_{j+\frac{1}{2}}(\lambda) + a^+ (U_j^n + \lambda (U_{j+1}^n - U_j^n)) \Delta t_n} \sum_{i=j+1}^{M+j} \phi_i(s)\, ds \right.$$

$$\left. - \int_{x_{j+\frac{1}{2}}}^{x_{j+\frac{1}{2}} + a^+ (U_j^n + \lambda (U_{j+1}^n - U_j^n)) \Delta t_n} \sum_{i=j+1}^{M+j} \phi_i(s)\, ds \right|$$

$$+ \left| U_{j+1}^n - U_j^n \right| \int_{x_{j+\frac{1}{2}}}^{x_{j+\frac{1}{2}}(\lambda)} \phi_{j+1}(s)\, ds \Bigg\}$$

$$\leqslant 2 \sum_{j=-k}^{k} \left| U_{j+1}^n - U_j^n \right| \int_{\lambda_0}^{1} \left| x_{j+\frac{1}{2}}(\lambda) - x_{j+\frac{1}{2}} \right| d\lambda\ ,$$

and, similarly,

$$J_2 = \sum_{j=-k}^{k} \int_0^{\lambda_0} d\lambda \left\{ \sum_{i=j+1}^{M+j} \left| U_{j+1}^n - U_j^n \right| \right.$$

$$\times \left| \int_{x_{j+\frac{1}{2}}(\lambda)}^{x_{j+\frac{1}{2}}(\lambda) + a^+ (U_j^n + \lambda (U_{j+1}^n - U_j^n))\Delta t_n} \phi_i(s)\, ds \right.$$

$$\left. - \int_{x_{j+\frac{1}{2}}}^{x_{j+\frac{1}{2}} + a^+ (U_j^n + \lambda (U_{j+1}^n - U_j^n))\Delta t_n} \phi_i(s)\, ds \right|$$

$$\left. + \left| U_{j+1}^n - U_j^n \right| \int_{x_{j+\frac{1}{2}}(\lambda)}^{x_{j+\frac{1}{2}}(\lambda) + a^+ (U_j^n + \lambda (U_{j+1}^n - U_j^n))\Delta t_n} \phi_j(s)\, ds \right\}$$

$$\leqslant 2 \sum_{j=-k}^{k} \left| U_{j+1}^n - U_j^n \right| \int_0^{\lambda_0} \left| x_{j+\frac{1}{2}}(\lambda) - x_{j+\frac{1}{2}} \right| d\lambda\ .$$

Recall that when $\theta_{j+\frac{1}{2}}^n = 0$, $x_{j+\frac{1}{2}}(\lambda) = x_{j+\frac{1}{2}}$, and we then have that

$$\int_0^1 \left| x_{j+\frac{1}{2}}(\lambda) - x_{j+\frac{1}{2}} \right| d\lambda = 0\,;$$

if $\theta_{j+\frac{1}{2}}^n \neq 0$, then

$$\int_0^1 \left| x_{j+\frac{1}{2}}(\lambda) - x_{j+\frac{1}{2}} \right| d\lambda \leqslant \theta_{j+\frac{1}{2}}^n (\Delta x_j + \Delta x_{j+1}) \leqslant C h \theta_{j+\frac{1}{2}}^n\ .$$

Hence, using (4.4) we therefore conclude that

$$\sum_i I_2^i \leqslant C h [\Delta t_n + h] \leqslant C h\, \Delta t_n\,, \tag{4.16}$$

where C is independent of h.

We can treat I_3^i by the same argument and obtain that

$$\sum_i \left| I_3^i \right| \leqslant C h\, \Delta t_n\,,$$

where C is independent of h.

With the above estimates we conclude that

$$\sum_i \left| \left\langle U^{n+1} - T(\Delta t_n)\, U^n,\ \phi_i \right\rangle \right| \leqslant C h\, \Delta t_n ,$$

which implies that

$$\begin{aligned} V(U^{n+1}) \leqslant \Bigg| \sum_{l,\,(i=2l)} \Bigg\{ \Bigg| & \max_{x \in [x_{i+\frac{5}{2}},\, x_{i+\frac{3}{2}})} T(\Delta t_n)\, U^n(x) \\ & - \min_{x \in [x_{i+\frac{3}{2}},\, x_{i+\frac{1}{2}})} T(\Delta t_n)\, U^n(x) \Bigg| \\ & + \Bigg| \min_{x \in [x_{i+\frac{3}{2}},\, x_{i+\frac{1}{2}})} T(\Delta t_n)\, U^n(x) \\ & - \max_{x \in [x_{i+\frac{1}{2}},\, x_{i-\frac{1}{2}})} T(\Delta t_n)\, U^n(x) \Bigg| \Bigg\} + C\, \Delta t_n \\ & \leqslant V(T(\Delta t_n)\, U^n) + C\, \Delta t_n \\ & \leqslant V(U^n) + C\, \Delta t_n , \end{aligned}$$

from which we get (4.5).

We now prove (4.6). We notice that, for each i in (3.14), $\tilde{\Phi}_i^n$ is zero outside the interval $[x_{i-k-\frac{1}{2}}, x_{i+k+\frac{1}{2}}]$, where $k = [AD] + 1$. Thus

$$\left| U_i^{n+1} - U_i^n \right| \Delta x_i \leqslant A\, \Delta t_n\, V(\tilde{u}^n)[x_{i-k-\frac{1}{2}}, x_{i+k+\frac{1}{2}}] ,$$

or,

$$\sum_i \left| U_i^{n+1} - U_i^n \right| \Delta x_i \leqslant (2k+1)\, A\, \Delta t_n\, V(\tilde{u}^n) ,$$

from which we get (4.6). We have completed the proof.

Remark 3 : The results in this section can be easily generalised to the case where the initial datum has unbounded support.

5. DISCONTINUOUS LINEAR RECOVERY

In this section, following the idea of van Leer [18] we discuss another recovery technique, namely, discontinuous linear recovery. Suppose that U^n is the approximation at time t_n :

$$U^n(x) = U_j^n \,, \quad x \in [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}) \,.$$

Then we define the recovery function $\tilde{u}^n(x)$ (*cf.* van Leer [18]) to be

$$\tilde{u}^n(x) = U_j^n + (x - x_j)\, \delta_j^n, \quad x \in [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}) \,, \tag{5.1}$$

where $x_j = \dfrac{1}{2}\,(x_{j+\frac{1}{2}} + x_{j+\frac{1}{2}})$ and

$$\delta_j^n = \begin{cases} \operatorname{sgn}\,(U_{j+1}^n - U_j^n) \min \left\{ \dfrac{\left| U_{j+1}^n - U_j^n \right|}{\Delta x_j} \,,\, \dfrac{\left| U_j^n - U_{j-1}^n \right|}{\Delta x_j} \right\} \\ \quad \text{if} \quad \operatorname{sgn}\,(U_{j+1}^n - U_j^n) = \operatorname{sgn}\,(U_j^n - U_{j-1}^n) \\ 0 \quad \text{otherwise}\,. \end{cases} \tag{5.2}$$

This choice, including the switching off of the modification at a peak or a trough, corresponds to the device used by van Leer [18] ; other choices of discontinuous linear recovery are also possible.

The ECG scheme is then given by (2.13), with $\tilde{\Phi}_i^n$ given by (1.9) using (5.1) to define $\tilde{u}^n(x)$. Clearly, the projection property (2.14) is satisfied. We also have from Theorem 3.1 that

$$\left\langle U^{n+1}, \phi_i \right\rangle = \left\langle T(\Delta t_n)\, \tilde{u}^n, \phi_i \right\rangle \quad \forall i \,, \tag{5.3}$$

where $T(\,.\,)$ is the transport-collapse operator.

The following theorem is obvious, since the recovery in this case does not increase the total variation.

THEOREM 5.1 : *Suppose the mesh is quasiregular. For the ECG scheme defined with recovery* (5.1) *and* (5.2) *we have that*

$$\|U^n\|_{L^\infty} = \|\tilde{u}^n\|_{L^\infty} \leqslant \|U^0\|_{L^\infty} \quad \forall n \,, \tag{5.4}$$

$$V(U^{n+1}) \leqslant V(U^0) \quad \forall n \,, \tag{5.5}$$

$$\sum_j \left| U_j^n - U_j^m \right| \Delta x_j \leqslant K(t_n - t_m) \,, \tag{5.6}$$

where $0 \leqslant t_m < t_n < \infty$, *and* K *is a constant independent of* n *and* m.

Remark : (5.6) can be derived from the properties of the transport-collapse operator, namely,

$$\|U^{n+1} - U^n\|_{L^1} \leq \|T(\Delta t_n)\, \tilde{u}^n - \tilde{u}^n\|_{L^1} \leq V(\tilde{u}^n)\, \Delta t_n\,.$$

Moreover, in doing so the mesh condition of quasiregularity can be dropped.

Numerical experiments : Here we present some computational results using the discontinuous linear recovery described above. Our purpose is to compare them with the results given for the continuous linear recovery by Morton and Sweby [8], Childs and Morton [2] obtained with the continuous linear recovery scheme of Section 4 (we remind the reader that *figs.* 13 and 14 in [2] are upside-down).

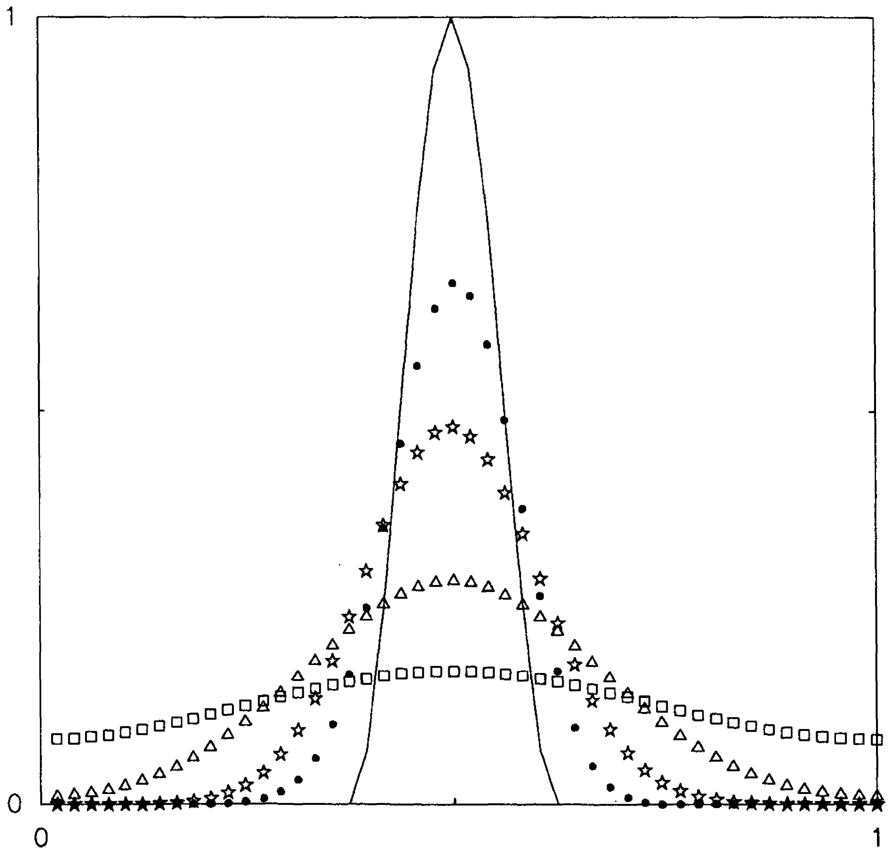

Figure 1. — Linear advection without recovery for mesh ratios 0.4875(square), 1.4444(triangle), 4.3333(star), 7.8(dot).

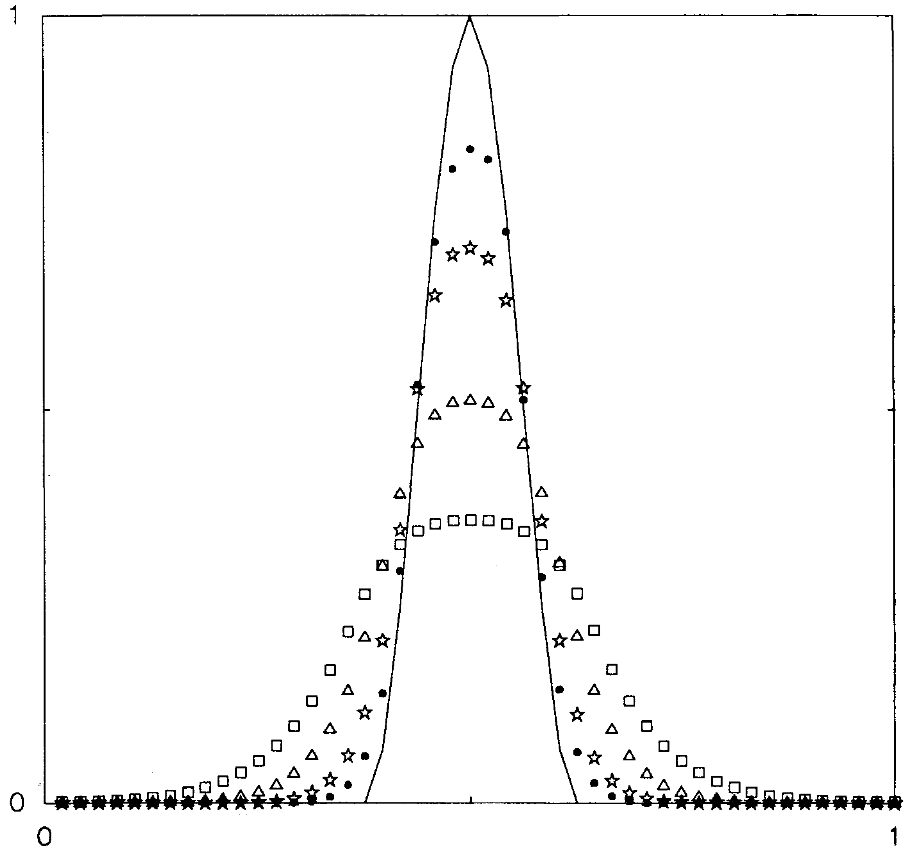

Figure 2. — As figure 1 after discontinuous linear recovery.

We first consider the linear advection equation with periodic initial datum

$$u_t + u_x = 0 ,$$

$$u(x, 0) = \begin{cases} \sin^2 \left[4 \pi \left(x - \frac{1}{4} \right) \right] & x \in \left[\frac{1}{4}, \frac{1}{2} \right] \\ 0 & x \in \left[0, \frac{1}{4} \right) \cup \left(\frac{1}{2}, 1 \right] . \end{cases}$$

We illustrate the solutions after a time 8.125 on a uniform grid on [0, 1] with $\Delta x = 1/48$. We test different mesh ratios $\lambda := \Delta t / \Delta x = 0.4875$ (800 steps), 1.4444(270 steps), 4.3333(90 steps), 7.8(50 steps). Figure 1 shows the results without recovery, while figure 2 shows the results using the discontinuous linear recovery.

We next consider the inviscid Burgers' equation

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0\,,$$

with the following initial datum, consisting of two positive pulses separated by a negative pulse,

$$\begin{aligned}
&u(x) = \sin^2[\pi(x - 0.088)/0.313] \quad \text{if} \quad 0.088 < x < 0.401\,,\\
&u(x) = -1 + 2|x - 0.571|/0.136 \quad \text{if} \quad 0.503 < x < 0.639\,,\\
&u(x) = 1 \quad \text{if} \quad 0.683 < x < 0.7415\,,\\
&u(x) = 0 \quad \text{otherwise}\,.
\end{aligned}$$

We have used a uniform mesh with $\Delta x = 0.01$, and show results for $t = 0.3061$. Figure 3 gives the results of the scheme with and without recovery for the mesh ratio $\lambda = 1.0$ which corresponds to a maximum CFL number of 1.0. Figures 4 and 5 give the corresponding results for $\lambda = 2.5$ and 7.0 respectively.

Conclusions : For the linear equation, the scheme with continuous linear recovery clearly performs better than that with discontinuous linear recovery. This is due to the fact that in the former the linear recovery is still applied at extreme points, while in the latter it is switched off by (5.2) ; it is possible

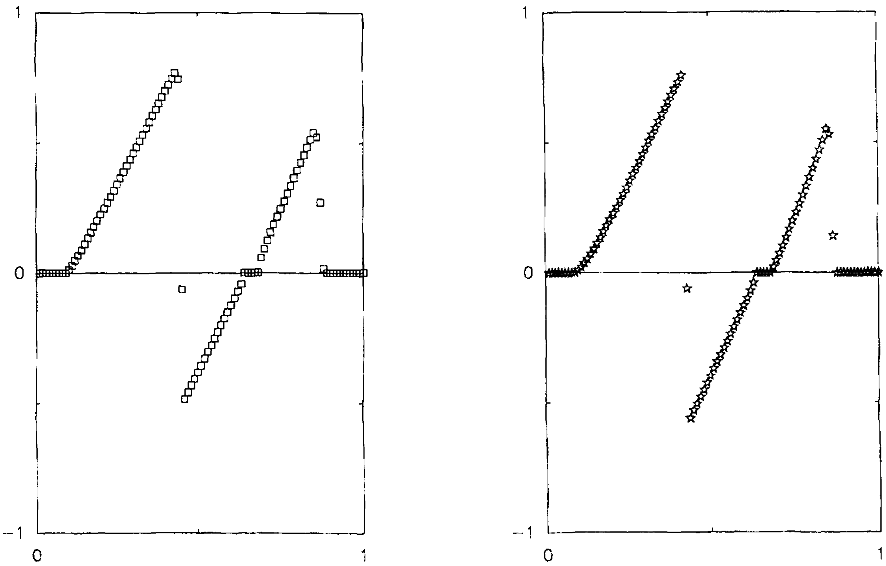

Figure 3. — The ECG scheme for Burgers' equation with mesh ratio 1.0 ; results with no recovery are shown on the left and with discontinuous linear recovery on the right.

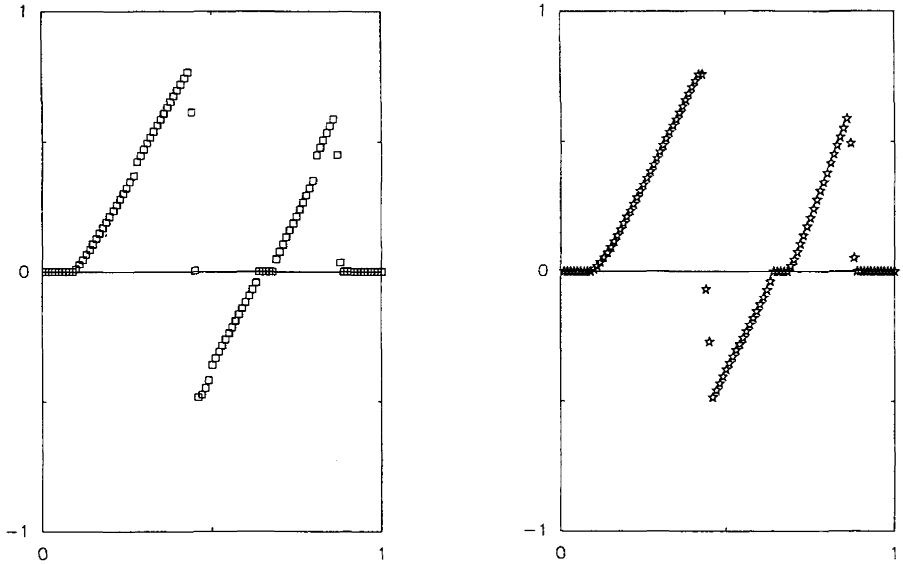

Figure 4. — As figure 3, with ratio 2.5.

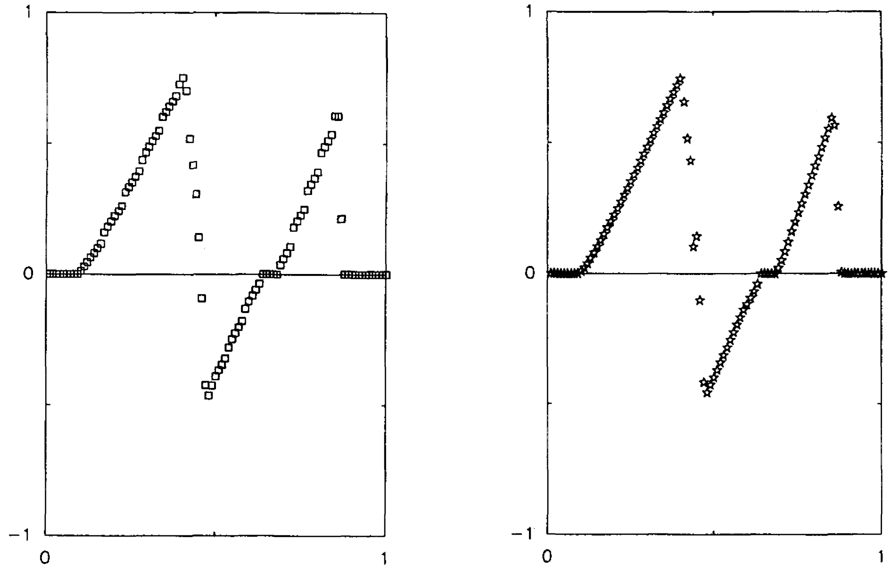

Figure 5. — As figure 3, with ratio 7.0.

that a more careful choice of δ_j^n would make the accuracy of the two schemes more comparable. However, both perform about equally well for Burgers' equation ; there is a slight spreading of the shocks for $\lambda = 1.0$ with the present scheme but improved performance for large λ. Notice that there is a great improvement compared with the scheme having no recovery.

6. ADMISSIBLE SOLUTIONS. ENTROPY INEQUALITY

In this section we shall prove the convergence of the approximation given by Section 4 and Section 5, and that the limit function is an admissible solution of the conservation law.

By the quasiregularity of the mesh, for a given sequence $\{h\}$, $h \to 0$, we have the corresponding discretizations $\{t_n\}$ and $\{x_j\}$ in time and space respectively. With the given mesh, we suppose that $\{U^n\}$ results from the ECG scheme with recovery discussed in Section 4 and Section 5. We then define a sequence of approximations $\{U^h\}$ on $R \times [0, \infty)$ by

$$U^h(x, t) = U_i^n \quad \text{for} \quad (x, t) \in [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times [t_n, t_{n+1}) . \tag{6.1}$$

THEOREM 6.1 : *Suppose that $\{U^n\}$ is the approximation produced by the recovered ECG scheme, as in Theorem 4.1 or Theorem 5.1. Then the sequence $\{U^h\}$ is compact in $L^\infty([0, T] ; L^1(R))$, $0 < T < \infty$, converging to some function u ; and u is an admissible solution of the conservation law. In other words, u satisfies the following entropy inequality ;*

$$\iint [\eta(u)\, \varphi_t + q(u)\, \varphi_x]\, dx\, dt \geq 0 , \tag{6.2}$$

where $\varphi \in C_0^\infty(R \times (0, \infty))$, $\varphi \geq 0$, and where $\eta(u)$ is any Lipschitz continuous convex function, $q(u) = \int_0^u a(\lambda)\, \eta'(\lambda)\, d\lambda$.

Proof : According to a classical result (*cf.* Smoller [16], Chapter 16 § A, for example), from (4.5)-(4.7) or (5.4)-(5.6), it follows that $\{U^h\}$ is compact in $L^\infty([0, T] ; L^1(R))$. We may assume that

$$U^h(x, t) \to u(x, t) \quad \text{in} \quad L^\infty([0, T] ; L^1(R)) .$$

We now prove that u satisfies (6.2). At each time t_n, for any fixed x, by the geometrical structure of the TC operator (see Brenier [1] for details) there are $2P + 1$ values (depending on x, t_n) :

$$u_0, u_1, \ldots, u_{2P}, \quad u_0 \leq u_1 \leq \ldots \leq u_{2P} ,$$

such that

$$(T(\Delta t_n)\,\tilde{u}^n)(x) = \int J\tilde{u}^n(x - a(w)\,\Delta t_n, w)\,dw = \sum_{k=0}^{2P} (-1)^k u_k .$$

On the other hand, it follows from Theorem 3.1 that

$$\eta\,(U_i^{n+1}) \leqslant \frac{1}{\Delta x_i} \int \phi_i\ \eta\,(T(\Delta t_n)\,\tilde{u}^n)\,dx \leqslant \frac{1}{\Delta x_i} \int \phi_i \sum_{k=0}^{2P} (-1)^k\ \eta\,(u_k)\,dx\,,$$

where we have used Jensen's inequality and the inequality

$$\eta\left(\sum_{k=0}^{2P} (-1)^k u_k\right) \leqslant \sum_{k=0}^{2P} (-1)^k\ \eta\,(u_k)\,.$$

The definition of Ju also implies that

$$\sum_{k=0}^{2P} (-1)^k\ \eta\,(u_k) = \int \eta'(w)\,J\tilde{u}^n(x - a(w)\,\Delta t_n, w)\,dw + \eta\,(0)\,.$$

Thus,

$$\begin{aligned}
\eta\,(U_i^{n+1})\,\Delta x_i &\leqslant \int dx \int \phi_i(x)\ \eta'(w)\,J\tilde{u}^n(x - a(w)\,\Delta t_n, w)\,dw + \eta\,(0)\,\Delta x_i \\
&= \int dx \int \phi_i(x + a(w)\,\Delta t_n)\ \eta'(w)\,J\tilde{u}^n(x, w)\,dw + \eta\,(0)\,\Delta x_i \\
&= \int dx \int_0^{\tilde{u}^n(x)} \phi_i(x + a(w)\,\Delta t_n)\ \eta'(w)\,dw + \eta\,(0)\,\Delta x_i\,.
\end{aligned}$$

Therefore, for any $\varphi \geqslant 0$, $\varphi \in C_0^\infty(R \times (0, \infty))$, we have that

$$\begin{aligned}
&\int \eta\,(U^{n+1})\,\varphi\,(x, t_n)\,dx \leqslant \\
&\qquad \leqslant \int dx \int_0^{\tilde{u}^n(x)} \varphi\,(x + a(w)\,\Delta t_n)\ \eta'(w)\,dw + \eta\,(0) \int \varphi\,(x, t_n)\,dx \\
&\qquad \leqslant \Delta t_n \int D_x\,\varphi\,(x, t_n)\,dx \int_0^{\tilde{u}^n(x)} a(w)\ \eta'(w)\,dw \\
&\qquad\quad + \int \varphi\,(x, t_n)\ \eta\,(\tilde{u}^n(x))\,dx + O\,((\Delta t_n)^2),
\end{aligned}$$

and hence that

$$\sum_n \left[\int \eta(U^{n+1})\, \varphi(x, t_n)\, dx - \int \eta(U^n)\, \varphi(x, t_n)\, dx \right] \leq$$

$$\leq \sum_n \int_{t_n}^{t_{n+1}} dt \int D_x\, \varphi(x, t_n)\, q(\tilde{u}^n(x))\, dx$$

$$+ \sum_n \int \varphi(x, t_n)[\eta(\tilde{u}^n) - \eta(U^n)]\, dx + O(h)\,.$$

It is obvious that (6.2) follows, if we can prove that

$$\Delta := \sum_n \int \varphi(x, t_n)[\eta(\tilde{u}^n) - \eta(U^n)]\, dx \to 0\,, \quad \text{as} \quad h \to 0\,. \tag{6.3}$$

In fact, by Taylor series expansion ;

$$|\Delta| \leq \left| \sum_{n,i} a_i^n \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} [\eta(\tilde{u}^n) - \eta(U_i^n)]\, dx \right|$$

$$+ C \sum_n \int_{t_n}^{t_{n+1}} dt \int_{\operatorname{supp}\{\varphi\}} |\tilde{u}^n - U^n|\, dx$$

$$= \left| \sum_{n,i} \frac{1}{2} a_i^n \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \eta''(\xi_i^n(x))(\tilde{u}^n - U_i^n)^2\, dx \right|$$

$$+ C \sum_n \int_{t_n}^{t_{n+1}} dt \int_{\operatorname{supp}\{\varphi\}} |\tilde{u}^n - U^n|\, dx\,,$$

where $\xi_i^n(x)$ is between $\tilde{u}^n$ and U_i^n.

For the continuous linear recovery, (6.3) holds, because of (4.10). For the discontinuous linear recovery, we notice that

$$\tilde{u}^n - U_i^n = \delta_i^n(x - x_i)\,, \quad x \in [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]\,,$$

and hence that

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (\tilde{u}^n - U_i^n)^2\, dx \leq C\, (\delta_i^n\, \Delta x_i)\, h^2\,.$$

Thus,

$$\left| \sum_{n,i} \frac{1}{2} a_i^n \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \eta''(\xi_i^n(x))(\tilde{u}^n - U_i^n)^2\, dx \right| \leq C \sum_n h^2 \sum_i \delta_i^n\, \Delta x_i \to 0\,,$$

where we have used the inequality $\sum_i \delta_i^n \Delta x_i \leq 2\, V(U^n)$, due to the definition of δ_i^n in (5.2).

7. APPENDIX : RIEMANN-STIELTJES INTEGRAL

We review briefly the theory of the Stieltjes integral. For more details, see Natanson [13] and Volpert [19]. First we recall the definition of functions of bounded variation in one dimension. Suppose f is a function defined on the interval $[a, b] \subset R$, and let $\{x_i\}$, $i = 0, 1, 2, \dots, n$, be a partition of $[a, b]$, that is, $a = x_0 < x_1 < \cdots < x_n = b$. We define

$$V(f)[a, b] = \max \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| ,$$

where the maximum is taken through all possible partitions $\{x_i\}$. If $V(f)[a, b] < \infty$, we say that f has bounded variation on $[a, b]$ and $V(f)[a, b]$ is the total variation of f on $[a, b]$.

The R-S integral is defined as follows. Suppose f and g are two functions defined on $[a, b]$, and let $\{x_i\}$ be a partition of $[a, b]$. We define

$$\sigma = \sum_{k=0}^{n-1} f(\xi_k)[g(x_{k+1}) - g(x_k)] ,$$

where ξ_k, $x_k \leq \xi_k \leq x_{k+1}$, is arbitrary. If as $\lambda := \max\limits_{0 \leq k \leq n-1} (x_{k+1} - x_k) \to 0$, the limit of σ exists and is independent of the partition $\{x_i\}$, we say that $I = \lim\limits_{\lambda \to 0} \sigma$ is the Riemann-Stieltjes integral (or R-S integral) of f related to g, and we write

$$I := \int_a^b f(x)\, dg(x) .$$

We denote by $BV[a, b]$ the set of all functions having bounded variation on $[a, b]$. It is proved in [13] that if $f \in C[a, b]$, $g \in BV[a, b]$, then $\int_a^b f(x)\, dg(x)$ and $\int_a^b g(x)\, df(x)$ exist, and

$$\int_a^b f(x)\, dg(x) + \int_a^b g(x)\, df(x) = f(b)\, g(b) - f(a)\, g(a) ,$$

$$\left| \int_a^b f(x)\, dg(x) \right| \leq \|f\|_C\, V(g)[a, b] .$$

It is also clear that if g' is Riemann integrable on $[a, b]$, then

$$\int_a^b f(x)\, dg(x) = \int_a^b f(x)\, g'(x)\, dx ,$$

where the right-hand side is a Riemann integral.

In the following we list some results used repeatedly in this paper.

THEOREM 7.1 : *Let* $f \in C[a, b]$, $g \in C[a, b] \cap BV[a, b]$.

a) *If* $F(\xi)$ *is Lipschitz continous in* ξ *and* $F'(g(\,.\,)) \in BV[a, b]$, *we then have that*

$$\int_a^b f(x)\, dF(g(x)) = \int_a^b f(x)\, F'(g(x))\, dg(x) .$$

b) *If* g_n *converges pointwise to* g *on* $[a, b]$, *and* $V(g_n)[a, b] \leq K < \infty$, *then*

$$\lim_{n \to \infty} \int_a^b f(x)\, dg_n(x) = \int_a^b f(x)\, dg(x) .$$

c) *If* $f_n \in C[a, b]$ *converges uniformly to* f *on* $[a, b]$, *then*

$$\lim_{n \to \infty} \int_a^b f_n(x)\, dg(x) = \int_a^b f(x)\, dg(x) .$$

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