M2AN - MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

HUNG-JU KUO
N. S. TRUDINGER

On the discrete maximum principle for parabolic difference operators

M2AN - Modélisation mathématique et analyse numérique, tome 27, n° 6 (1993), p. 719-737

<http://www.numdam.org/item?id=M2AN_1993__27_6_719_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE
(Vol. 27, n° 6, 1993, p. 719 à 737)

ON THE DISCRETE MAXIMUM PRINCIPLE FOR PARABOLIC DIFFERENCE OPERATORS (*)

by HUNG-JU KUO ([1], [2]) and N. S. TRUDINGER ([3])

Communicated by R. TEMAM

Abstract. — *We derive a discrete analogue for parabolic difference inequalities of the Krylov maximum principle for parabolic differential inequalities. The result embraces both explicit and implicit difference schemes and extends to the parabolic case our previous work on linear elliptic difference inequalities with random coefficients.*

Résumé. — *Pour des inégalités aux différences paraboliques, on établit l'analogue discret du principe du maximum de Krylov connu pour les inégalités paraboliques. Le résultat obtenu concerne à la fois des schémas aux différences explicites et implicites et est une généralisation au cas parabolique de notre travail antérieur concernant les inégalités aux différences elliptiques linéaires à coefficients aléatoires.*

AMS MOS : Primary : 65M06, 35K20, 39A70 ; Secondary : 35A15, 65M12, 39A10.

1. INTRODUCTION

In this paper we establish discrete versions of the Krylov maximum principle [4, 5] for linear parabolic partial differential operators of the form

$$\mathscr{L}u = a^{ij} D_{ij} u + b^i D_i u + cu - D_t u \tag{1.1}$$

in cylindrical regions $Q^+ = \Omega \times \mathbb{R}^+ \subset \mathbb{R}^{n+1}$, where Ω is a domain in Euclidean n-space, $\mathbb{R}^n$. Our results are analogues of the discrete version of the Aleksandrov maximum principle for elliptic operators,

$$Lu = a^{ij} D_{ij} u + b^i D_i u + cu , \tag{1.2}$$

(*) Received July 27, 1992, revised February 4, 1993.

([1]) Reasearch supported by the NSC of the Republic of China under Grant No. NSC80-0208-M005-01.

([2]) National Chung-Hsing University Taiwan.

([3]) Australian National University Australia and Northwestern University, U.S.A.

established in our previous work [6] (Theorem 2.1). As in [6], our estimates are formulated in such a way that their continuous versions follow via Taylor's formula.

Our discretizations of the operator (1.1) will involve linear difference operators, essentially of positive type, acting on space-time mesh functions. We will consider meshes in $\mathbb{R}^{n+1}$ of the form,

$$\begin{aligned} E = E_{h,\tau} &= \mathbb{Z}_h^n \times \mathbb{Z}_\tau \\ &= \left\{(x, t) \in \mathbb{R}^{n+1} \middle| x = (m_1, \ldots, m_n)\, h,\ t = m\tau,\ m_i, m \in \mathbb{Z}\right\} \end{aligned} \tag{1.3}$$

with spatial mesh length $h > 0$ and time step $\tau > 0$. A real-valued function on E is called a mesh function and, for fixed $y \neq 0, \in \mathbb{Z}_h^n$, we define the following basic difference operators, acting on the linear space of mesh functions, $\mathcal{M}(E)$:

$$\begin{aligned} \delta_y^+ u(x, t) &= \frac{1}{|y|} \{u(x + y, t) - u(x, t)\}\ , \\ \delta_y^- u(x, t) &= \frac{1}{|y|} \{u(x, t) - u(x - y, t)\}\ , \\ \delta_y u(x, t) &= \frac{1}{2} (\delta_y^+ + \delta_y^-)\, u(x, t) = \frac{1}{2|y|} \{u(x + y, t) - u(x - y, t)\}\ , \\ \delta_y^2 u(x, t) &= \delta_y^+ \delta_y^- u(x, t) \\ &= \frac{1}{|y|^2} \{u(x + y, t) - 2\, u(x, t) + u(x - y, t)\}\ , \\ \delta_t^- u(x, t) &= \frac{1}{\tau} \{u(x, t) - u(x, t - \tau)\}\ . \end{aligned} \tag{1.4}$$

The spatial part of our difference operators will be determined by second order difference operators of the form

$$\begin{aligned} L_h u(x, t) = \sum_y a(x, t, y)\, \delta_y^2 u(x, t) + \\ + \sum_y b(x, t, y)\, \delta_y u(x, t) + c(x, t)\, u(x, t) \end{aligned} \tag{1.5}$$

with real coefficients a, b, c, having compact support with respect to y, and satisfying as in [6], the condition

$$a(x, t, y) - \frac{1}{2} |y|\ |b(x, t, y)| \geqslant 0\ . \tag{1.6}$$

Such operators can be used to approximate uniformly elliptic differential

operators of the form (1.2) ([6, 8]). From the operators (1.5) we construct parabolic difference operators of the form

$$\mathscr{L}^{(\alpha)} u(x, t) = \mathscr{L}_{h,\tau}^{(\alpha)} u(x, t)$$
$$= (1 - \alpha) L_h u(x, t) + \alpha L_h u(x, t - \tau) - \delta_t^- u(x, t) \quad (1.7)$$

where α is a fixed number satisfying $0 \leqslant \alpha \leqslant 1$. When $\alpha = 0$, the operator $\mathscr{L}$ is called *implicit*, when $\alpha = 1$, the operator $\mathscr{L}$ is called *explicit* and the general case is refered to as *explicit-implicit* in correspondence with the resultant difference schemes. The operators (1.7) are of *positive type* (as defined by Motzkin and Wasow [8], see also [7]), if, as well as (1.6) holding, we have

$$\alpha\tau \left(2 \sum_y \frac{a(x, t, y)}{|y|^2} - c(x, t) \right) \leqslant 1 , \quad (1.8)$$

$$(1 - \alpha) c(x, t) + \alpha c(x, t - \tau) \leqslant 0 . \quad (1.9)$$

In the parabolic case, (1.9) can be achieved by replacement of u by $e^{Ct} u$ for an appropriate constant C, provided τ is sufficiently small. Consequently only (1.6) and (1.8) will be essential for us with (1.9) being replaced by a weaker condition (see (2.6)).

We shall formulate a discrete version of the Krylov maximum principle in the next section for operators $\mathscr{L}^{(\alpha)}$ satisfying stricter conditions than (1.6) and (1.8) corresponding to the non-degeneracy condition assumed in [6]. In Section 3 we provide some basic inequalities from Krylov's paper [4] which are used in our proof which is supplied in Section 4. Finally in Section 5 we relate the discrete and continuous versions of the maximum principle. In an ensuing paper, we shall apply Theorem 2.1 to the derivation of local estimates, corresponding to those in the elliptic case [6].

2. THE DISCRETE MAXIMUM PRINCIPLE

In this section we formulate a discrete maximum principle for the operator $\mathscr{L}^{(\alpha)}$, Theorem 2.1. The spatial operators L_h in (1.5) will be subjected to the same non-degeneracy condition as in [6]. That is, as well as (1.6), we assume for each point $x \in \Omega_h$ and $t = m\tau$, $m = 0, 1, \ldots, N$ for some $N \in \mathbb{N}$, there exists an orthogonal set of vectors $y^1, \ldots, y^n \in \mathbb{Z}_h^n$ such that

$$(1 - \alpha) \left\{ a(x, t, y^i) - \frac{|y^i|}{2} |b(x, t, y^i)| \right\} +$$
$$+ \alpha \left\{ a(x, t - \tau, y^i) - \frac{|y^i|}{2} |b(x, t - \tau, y^i)| \right\} \geqslant \lambda_i(x, t) > 0 , \quad (2.1)$$

$i = 1, \ldots, n$. Furthermore we will assume the coefficients a vanish whenever

$$|y| = |y|_2 > Kh$$

for some fixed $K \in \mathbb{N}$ and write

$$\mathscr{D} = \mathscr{D}(x, t) = \prod_{i=1}^{n} \lambda_i(x, t) \tag{2.2}$$

$$a(x, t) = \sum_y a(x, t, y), \quad b(x, t) = \sum_y |b(x, t, y)|,$$

$$a_0 = \max a(x, t), \quad b_0 = \max b(x, t), \quad c_0^{\pm} = \max c^{\pm}(x, t).$$

We also need to assume non-degeneracy with respect to the time variable and this we do by strengthening the condition (1.8) to

$$\alpha\tau\left(2\sum_y \frac{a(x, t, y)}{|y|^2} - c(x, t)\right) \leqslant 1 - \gamma(x, t) \tag{2.3}$$

for some positive function γ. Note that (2.3) will be satisfied if the time step τ and the ratio τ/h^2 are sufficiently small, in particular if

$$2\alpha a_0 K\tau/h^2 + \alpha\tau c_0^- < 1. \tag{2.4}$$

When the operator $\mathscr{L}$ is implicit, $\alpha = 0$ and (2.3), (2.4) are automatically satisfied. Combining (2.2) with (2.3), we also write

$$\tilde{\mathscr{D}}(x, t) = \gamma(x, t - \tau)\mathscr{D}(x, t), \quad \mathscr{D}^* = (\tilde{\mathscr{D}})^{\frac{1}{n+1}}. \tag{2.5}$$

In accordance with our remarks after condition (1.9), we shall replace (1.9) by the condition

$$(1 - \alpha)\tau c \leqslant 1 - \mu \tag{2.6}$$

where μ is a positive constant. Writing $T = \tau N$, we distinguish interior and boundary points in the discrete cylinder

$$Q_{h,\tau} = \Omega_h \times (\mathbb{Z}_\tau \cap [0, T]),$$

corresponding to the operators $\mathscr{L}_{h,\tau}^{(\alpha)}$. First we define the discrete interiors and boundaries of the set Ω_h corresponding to the operators $\mathscr{L}^{(\alpha)}$ by

$$\Omega_h^0(t) = \{x \in \Omega_h \mid (1 - \alpha)a(x, t, y) + \alpha a(x, t - \tau, y) = 0, \ \forall x + y \notin \Omega_h\}$$
$$\Omega_h^b(t) = \Omega_h - \Omega_h^0(t).$$

The discrete *parabolic interior* and *boundary* of the cylinder $Q_{h,\tau}$ are then defined by

$$Q^0_{h,\tau} = \bigcup_{m=1}^{N} \Omega^0_h(m\tau) \times \{m\tau\}$$
$$= \{(x, t) \in Q_{h,\tau} \mid (1-\alpha)\, a(x, t, y) + \alpha a(x, t-\tau, y) = 0, \quad \forall (x+y, t) \notin Q_{h,\tau}, t > 0\}$$

$$Q^b_{h,\tau} = Q_{h,\tau} - Q^0_{h,\tau}$$
$$= \bigcup_{m=1}^{N} [\Omega^b_h(m\tau) \times \{m\tau\}] \cup [\Omega_h \times \{0\}].$$

We also need to recall the notion of upper contact set, as used in [6]. Namely, if u is a spatial mesh function defined on Ω_h, we define the *upper contact set* of u, $\Gamma^+ = \Gamma^+(u)$ to be the subset where u is concave, that is Γ^+ consists of those points x in Ω_h for which there exists a hyperplane $\mathbb{P} = \mathbb{P}(x)$ in $\mathbb{R}^{n+1}$ passing through $(x, u(x))$ and lying above the graph of u. For a space-time mesh function u defined on the cylinder $Q_{h,\tau}$ we then define its spatial *upper contact set* by

$$\Gamma^+ = \Gamma^+(u) = \{(x, t) \in Q_{h,\tau} \mid x \in \Gamma^+_t\}$$

where Γ^+_t denotes the upper contact set on Ω_h of the spatial mesh function, $u^t(x) = u(x, t)$. We also define the *increasing set* of a space-time mesh function u by

$$I = I(u) = \{(x, t) \in Q_{h,\tau} \mid u(x, t) > u(x, s), \text{ for all } s,\ 0 \leq s < t\}$$

and let

$$\mathcal{S} = \mathcal{S}(u) = \Gamma^+ \cap I$$

denote the *increasing-upper contact set* of u. We can now state the following discrete analogues of the Krylov maximum principle corresponding to the operator $\mathcal{L}^{(\alpha)}$, $0 \leq \alpha \leq 1$.

THEOREM 2.1 : *Let u be a space-time mesh function on the cylinder $Q_{h,\tau}$ satisfying the difference inequality,*

$$\mathcal{L}^{(\alpha)} u \geq f \quad \text{in} \quad Q^0_{h,\tau}, \tag{2.7}$$

together with the boundary condition,

$$u \leq 0 \quad \text{in} \quad Q^b_{h,\tau}, \tag{2.8}$$

Then we have the estimate

$$\max_{Q_{h,\tau}} u \leq CR^{\frac{n}{n+1}} \|f/\mathscr{D}^*\|_{L^{n+1}(\mathscr{S})}, \tag{2.9}$$

where C is a constant depending only on n, K, μ, $\dfrac{b_0 T}{R}$, $c_0^+ T$, $R = \operatorname{diam} \Omega$ and $\mathscr{S}$ is the increasing-upper contact set of u.

We remark that, as with the continuous case [4, 5], the estimate (2.8) is one of several variants which stem from the special case $b_0 = c_0^+ = 0$. A more explicit form of the constant C, with an exponential dependence on the quantities b_0 and c_0^+ will be given in the course of the proof of Theorem 2.1. With the integral norm in (2.9) taken over $\mathscr{S}$, rather than $Q_{h,\tau}$, the estimate (2.9) actually corresponds to refinements of the original Krylov estimate due to Nazarov and Ural'tseva [9], Reye [10] and Tso [11]. The form of estimate (2.9), involving the integral L^{n+1} norm, rather than a sup norm, will be crucial in deriving local estimates and subsequent stability results in our ensuing paper (*cf.* [6, 7]).

3. PRELIMINARIES

Our proof of the maximum principle, Theorem 2.1 is based on certain inequalities of Krylov [4], together with the discrete adaptation in [6] of the geometric argument of Aleksandrov. As in [6], the notion of *normal mapping* (or supergradient) is crucial, the normal mapping of a spatial mesh function u on the set Ω_h being defined by

$$\chi(x) = \chi_u(x) = \{p \in \mathbb{R}^n \,|\, u(z) \leq u(x) + p \cdot (z - x), \quad \forall z \in \Omega_h\}. \tag{3.1}$$

The upper contact set Γ^+ of u is thus the subset of Ω_h where χ_u is non-empty. Note that in the discrete case, $\chi(\Omega_h) = \mathbb{R}^n$ and that $\chi(x)$ is unbounded whenever x is an extreme point of the convex hull of Ω_h, which we denote as $\hat{\Omega}_h$. The basic inequalities we need are encompassed in the following lemmas, which correspond to special cases of [4], Corollary 1.

LEMMA 3.1 : *Let u and v be mesh functions on Ω_h, vanishing at extreme points of $\hat{\Omega}_h$ and satisfying $u \geq v$ on Ω_h. Then we have the inequalities,*

$$0 \leq \sum_{x \in \Omega_h} (u(x)|\chi_u(x)| - v(x)|\chi_v(x)|) \leq$$

$$\leq (n+1) \sum_{x \in \Omega_h} (u - v)(x)|\chi_u(x)|. \tag{3.2}$$

Note that in (3.2), we use $|S|$ to denote the n dimensional Lebesgue measure of a measurable set S in $\mathbb{R}^n$ and the terms in the above sums are understood to vanish whenever u and v vanish, in particular at the extreme points of the convex set $\hat{\Omega}_h$. In fact there is no loss of generality in replacing the functions u and v in Lemma 3.1 by their concave envelopes which then vanish on $\partial\hat{\Omega}_h$. In this form, Lemma 3.1, is directly covered by Krylov [4].

LEMMA 3.2 : *Let u be a mesh function of Ω_h, vanishing at extreme points of $\hat{\Omega}_h$. Then for any $z \in \Omega_h$, we have the estimate,*

$$u(z) \leq \left\{ \frac{d_z^n}{\omega_n} \sum_{x \in \Omega_h} u(x) |\chi_u(x)| \right\}^{\frac{1}{n+1}}, \tag{3.3}$$

where $d_z = \max_{x \in \Omega_h} |z - x|$.

For completeness, we shall describe the proofs of Lemmas 3.1, 3.2 following Krylov [4]. First we need their analogues for smooth functions. Setting $Q_T = \Omega \times (0, T)$ and assuming $u \in C^\infty(\mathbb{R}^{n+1})$, with $u = 0$ on $\partial\Omega \times (0, T)$, we have, by integration by parts,

$$\iint_{Q_T} D_t u (\det D^2 u)\, dx\, dt = - \iint_{Q_T} u D_t (\det D^2 u)\, dx\, dt + \int_\Omega \{u(x, T) \det D^2 u(x, T) - u(x, 0) \det D^2 u(x, 0)\}\, dx .$$

Letting $[u^{ij}]$ denote the cofactor matrix of the Hessian matrix $D^2 u$, we then have

$$\begin{aligned} \iint_{Q_T} u D_t (\det D^2 u)\, dx\, dt &= \iint_{Q_T} u u^{ij} D_{ijt} u\, dx\, dt \\ &= \iint_{Q_T} (D_{ij} u) u^{ij} D_t u\, dx\, dt \\ &= n \iint_{Q_T} (\det D^2 u) D_t u\, dx\, dt , \end{aligned}$$

by integration by parts, since $\sum_{i=1}^{n} D_i u^{ij} = 0$, for each j. Accordingly, we obtain the identity,

$$(n+1) \iint_{Q_T} D_t u (\det D^2 u)\, dx\, dt = \int_\Omega \{u(x, T) \det D^2 u(x, T) - u(x, 0) \det D^2 u(x, T)\}\, dx . \tag{3.4}$$

Now let us assume that Ω is convex and that u and v are convex functions in $C^2(\bar{\Omega})$ vanishing on the boundary $\partial\Omega$, with $u \leqslant v$ in Ω. Setting

$$w(x, t) = (1 - t)\, u(x) + tv(x),$$

we shall apply (3.4) to w on Q_1. Since $D^2 w \geqslant 0$ by convexity and $D_t w = v - u \geqslant 0$, we see immediately that

$$\int_\Omega u \det D^2 u \leqslant \int_\Omega v \det D^2 v . \tag{3.5}$$

Furthermore,

$$\begin{aligned} D_t \left\{ \int_\Omega w_t \det D^2 w \right\} &= \int_\Omega (D_t w)\, w^{ij} D_{ijt} w \\ &= - \int_\Omega w^{ij} (D_{it} w) D_{jt} w \\ &\leqslant 0 \end{aligned}$$

since $[w^{ij}] \geqslant 0$, by virtue of the convexity of w. Hence

$$\begin{aligned} \int_\Omega w_t \det D^2 w &\leqslant \int_\Omega (v - u) \det D^2 w(x, 0) \\ &= \int_\Omega (v - u) \det D^2 u , \end{aligned}$$

so that from (3.4) and (3.5) we conclude

$$0 \leqslant \int_\Omega (v \det D^2 v - u \det D^2 u) \leqslant (n + 1) \int_\Omega (v - u) \det D^2 u , \tag{3.6}$$

which is the smooth version of (3.2). The passage from (3.6) to (3.2) is achieved by approximation. If u is a concave function on Ω, we define a Borel measure ω_u by

$$\omega_u(E) = |\chi_u(E)| \tag{3.7}$$

for any Borel set $E \subset \Omega$. Clearly, ω_u is finite if $u \in C^{0,1}(\bar{\Omega})$ and furthermore, when $u \in C^2(\Omega)$, we have the representation

$$\omega_u(E) = \int_E \det (- D^2 u), \tag{3.8}$$

so that ω_u is absolutely continuous with respect to Lebesgue measure.

Switching from convex to concave functions, it follows that we can write (3.6) in the form

$$0 \leqslant \int_{\Omega} (u\, d\omega_u - v\, d\omega_v) \leqslant (n+1) \int_{\Omega} (u - v)\, d\omega_u ,$$

where u, $v \in C^{0,1}(\bar{\Omega}) \cap C^2(\Omega)$ are concave on Ω, vanish on $\partial\Omega$ and satisfy $u \geqslant v$ in Ω. Using the property that the sequence of measures $\{\omega_{u_m}\}$ converges weakly to ω_u, when the sequence $\{u_m\}$ converges uniformly to u, we may then extend (3.9) to concave functions u, $v \in C^{0,1}(\bar{\Omega})$ and (3.2) thus follows as a special case ; (see [2, 4] for further details). The estimate (3.3) follows from the first inequality in (3.9), as in [4], by taking v to be the conical function

$$v(x) = u(z)\left(1 - \frac{|x - z|}{d_z}\right) ,$$

on the ball $\{|x - z| < d_z\}$.

4. PROOF OF THEOREM 2.1

First we consider the case when $b = c = 0$. It is convenient to consider the difference inequality (2.7) as a system of difference inequalities for the spatial mesh functions u_m, $m = 0, \ldots, N$ given by

$$u_m = u(x, m\tau) . \tag{4.1}$$

Writing

$$a_m(x, y) = a(x, y, m\tau), \quad f_m(x) = f(x, m\tau) ,$$
$$\Omega_m^0 = \Omega_h^0(m\tau), \quad \Omega_m^b = \Omega_h^b(m\tau) ,$$

we see that (2.7), (2.8) are thus equivalent to the system,

$$(1 - \alpha) \sum a_m(x, y)\, \delta_y^2 u_m(x) + \alpha \sum a_{m-1}(x, y)\, \delta_y^2 u_{m-1}(x) - \frac{1}{\tau}(u_m(x) - u_{m-1}(x)) \tag{4.2}$$
$$\geqslant f_m(x) , \quad \text{for} \quad x \in \Omega_m^0 ,$$
$$u_m \leqslant 0 , \quad \text{in} \quad \Omega_m^b ,$$
$$u_0 \leqslant 0 , \quad m = 1, \ldots, N .$$

In order to use Lemma 3.1, we need to replace u_m by an increasing sequence v_m defined by

$$v_m = \max_{j \leq m} u_j^+ .$$

We observe that $v_m = 0$ in $\bigcap_{j \leq m} \Omega_j^b$, $v_0 = 0$, $m = 1, \ldots, N$ and moreover, setting

$$\alpha_m = (1 - \alpha)\, a_m + \alpha a_{m-1} ,$$

it follows that whenever $v_m(x) > v_{m-1}(x)$, then $v_m(x) = u_m(x)$, and we have the difference inequality

$$\begin{aligned}
\sum \alpha_m(x, y)\, \delta_y^2 v_m(x) &\geqslant \\
&\geqslant (1 - \alpha) \sum a_m(x, y)\, \delta_y^2 u_m(x) + \alpha \sum a_{m-1}(x, y)\, \delta_y^2 u_{m-1}(x) \\
&\quad - 2\alpha \sum \frac{a_{m-1}(x, y)}{|y|^2} (u_m(x) - u_{m-1}(x)) \\
&\geqslant \left(\frac{1}{\tau} - 2\alpha \sum \frac{a_{m-1}(x, y)}{|y|^2} \right) (u_m(x) - u_{m-1}(x)) + f_m(x) \\
&\geqslant \frac{\gamma_{m-1}(x)}{\tau} (u_m - u_{m-1})(x) + f_m(x) \qquad (4.3)
\end{aligned}$$

by (4.2), (2.3) where $\gamma_m(x) = \gamma(x, m\tau)$, $m = 1, \ldots, N$. Letting $\Gamma_m^+ = \Gamma^+(v_m)$ and

$$\mathcal{S}_m = \{x \in \Gamma_m^+ \mid v_m(x) > v_{m-1}(x)\}$$

we can now follow our proof of Theorem 2.1 in [6], to obtain

$$\delta_y^2 v_m \leqslant 0 \qquad (4.4)$$

for all $x \in \Gamma_m^+$, $x + y \in \Omega_h$. Consequently, from (4.3), we have

$$\begin{aligned}
&- \sum_{i=1}^{n} \alpha_m(x, y^i)\, \delta_{y^i}^2 v_m(x) + \\
&\qquad + \frac{\gamma_{m-1}(x)}{\tau} (u_m(x) - u_{m-1}(x)) \leqslant - f_m(x) \qquad (4.5)
\end{aligned}$$

for $x \in \mathcal{S}_m$, where $y^i = y^i(x, m\tau)$, is as in (2.1). Letting $\chi_m = \chi_{v_m}$, we can then estimate, as in [6], Theorem 2.1,

$$\begin{aligned}
|\chi_m(x)| &\leqslant \prod_{i=1}^{n} \frac{1}{|y^i|} \{2 v_m(x) - v(x + y^i) - v(x - y^i)\} \\
&= \prod_{i=1}^{n} (- |y^i|\, \delta_{y^i}^2 v_m(x)) , \qquad (4.6)
\end{aligned}$$

so that using the arithmetic geometric mean inequality, we obtain from (4.5),

$$(v_m - v_{m-1})(x)|\chi_m(x)| \leqslant \frac{\tau}{\gamma_{m-1}} \left(\frac{|f_m(x)|}{n+1} \right)^{n+1} \prod_{i=1}^{n} \frac{|y^i|}{\alpha_m}$$

$$\leqslant (Kh)^n \tau \left(\frac{|f_m(x)|}{(n+1)\mathscr{D}_m^*} \right)^{n+1} \tag{4.7}$$

where

$$\mathscr{D}_m^* = \left(\gamma_{m-1} \prod_{i=1}^{n} \lambda_{i,m} \right)^{\frac{1}{n+1}}$$

$$\lambda_{i,m}(x) = \lambda_i(x, m\tau), \quad m = 0, \ldots, N .$$

Since the left hand side of (4.7) vanishes for $x \notin \mathscr{S}_m$, the inequality clearly extends to all $x \in \Omega_h$, with $f_m(x)$ replaced by zero for $x \notin \mathscr{S}_m$. We are now in a position to apply Lemma 3.1. Indeed this yields the estimate

$$\sum_{x \in \Omega_h} v_m(x)|\chi_m(x)| - v_{m-1}(x)|\chi_{m-1}(x)| \leqslant$$

$$\leqslant (n+1) \sum_{x \in \Omega_h} (v_m - v_{m-1})(x)|\chi_m(x)|$$

$$\leqslant \sum_{x \in \mathscr{S}_m} (Kh)^n \tau \left(\frac{|f_m(x)|}{(n+1)\mathscr{D}_m^*} \right)^{n+1} . \tag{4.8}$$

(Note that, by virtue of condition (2.1), the extreme points of $\hat{\Omega}_h$ lie in Ω_m^b, $\forall m$). Summing (4.8) from $m = 1$ to N and using $v_0 = 0$ we thus obtain

$$\sum_{x \in \Omega_h} v_N(x)|\chi_N(x)| \leqslant \sum_{m=1}^{N} \sum_{x \in \mathscr{S}_m} (Kh)^n \tau \left(\frac{|f_m(x)|}{(n+1)\mathscr{D}_m^*} \right)^{n+1} . \tag{4.9}$$

Hence we conclude from Lemma 3.2,

$$\max_{\Omega_h} v_N \leqslant \frac{1}{(n+1)} (\omega_n)^{-\frac{1}{n+1}} (KR)^{\frac{n}{n+1}} \times$$

$$\times \left\{ \sum_{m=1}^{N} \sum_{x \in \mathscr{S}_m} h^n \tau \left(\frac{|f_m(x)|}{\mathscr{D}_m^*} \right)^{n+1} \right\}^{\frac{1}{n+1}} \tag{4.10}$$

and the estimate (2.9) for the case $b = c = 0$.

To treat the general case, we introduce a modified mesh function $\bar{u}$ and corresponding operators $\bar{\mathscr{L}}_\alpha$ defined by

$$\bar{u}(x, t) = e^{-At} u(x, t)$$

$$\bar{\mathscr{L}}_\alpha \bar{u}(x, t) = (1 - \alpha) e^{A\tau} L_h \bar{u}(x, t) + \\ + \alpha L_h \bar{u}(x, t - \tau) - \delta_t^- \bar{u}(x, t), \quad (4.11)$$

where A is a non-negative constant to be determined later. Since

$$L_h \bar{u}(x, t) = e^{-At} L_h u(x, t),$$
$$L_h \bar{u}(x, t - \tau) = e^{-A(t-\tau)} L_h u(x, t - \tau)$$
$$\delta_t^- \bar{u}(x, t) = e^{-A(t-\tau)} \delta_t^- u(x, t) - \frac{1}{\tau} (e^{\tau A} - 1) \bar{u}(x, t)$$

we obtain, from the differential inequality (2.6),

$$\bar{\mathscr{L}}_\alpha \bar{u}(x, t) = e^{-A(t-\tau)} \mathscr{L}_\alpha u(x, t) + \frac{1}{\tau} (e^{\tau A} - 1) \bar{u}(x, t)$$
$$\geqslant - f^-(x, t) + \frac{1}{\tau} (e^{\tau A} - 1) \bar{u}(x, t) \quad (4.12)$$

in $Q^0_{h,\tau}$. By virtue of conditions (1.6), (1.8), it follows that the operator $\tilde{\mathscr{L}}_\alpha$ given by

$$\tilde{\mathscr{L}}_\alpha \bar{u} = \bar{\mathscr{L}}_\alpha \bar{u} - \frac{1}{\tau} (e^{\tau A} - 1) \bar{u}$$

is of positive type provided the analogue of (1.9) holds, that is

$$\tau \{\alpha c(x, t - \tau) + e^{\tau A}(1 - \alpha) c(x, t)\} \leqslant e^{\tau A} - 1. \quad (4.13)$$

Clearly, (4.13) will hold for some constant A if $\tau(1 - \alpha) c < 1$, that is if (2.6) holds. But if (1.6) (1.8) hold and $\mathscr{L}_\alpha$ is nondegenerate in the sense that $a(x, t, y) > 0$ for some y, then $\tau\alpha c > -1$ by (1.8) and this becomes a necessary condition for (4.13). Accordingly, (1.6), (1.8) and (2.6) are both necessary and sufficient conditions for the operator $\tilde{\mathscr{L}}_\alpha$ to be of positive type for some constant A.

Returning to the proof of Theorem 2.1, we now define spatial mesh functions $\bar{u}_m$, v_m by

$$\bar{u}_m = u(x, m\tau), \quad v_m(x, m\tau) = \max_{j \leqslant m} (\bar{u}_m)^+(x) \quad (4.14)$$

and let Γ_m^+, $\mathscr{S}_m$ be as before. Setting

$$b_m(x, y) = b(x, y, m\tau), \quad c_m(x) = c(x, m\tau),$$

together with

$$(\alpha_m, \beta_m, \kappa_m) = (1 - \alpha)\, e^{\tau A}(a_m, b_m, c_m) + \alpha\,(a_{m-1}, b_{m-1}, c_{m-1}),$$

we then obtain, in place of (4.3), using the positive type condition (1.6),

$$\sum \alpha_m(x, y)\, \delta_y^2 v_m(x) + \sum \beta_m(x, y)\, \delta_y v_m(x) + \kappa_m(x)\, v_m(x) \geqslant$$

$$\geqslant \bar{\mathscr{L}}_\alpha\, \bar{u}(x, m\tau) + \left\{ \frac{1}{\tau} + \alpha \left(-2 \sum \frac{a_{m-1}(x, y)}{|y|^2} + c_{m-1}(x) \right) \right\} \times$$

$$\times\, (v_m - v_{m-1})(x)$$

$$\geqslant \frac{\gamma_{m-1}(x)}{\tau}\, (v_m - v_{m-1})(x) - f_m^-(x) + \frac{1}{\tau}\, (e^{\tau A} - 1)\, v_m(x), \tag{4.15}$$

whenever $v_m(x) > v_{m-1}(x)$. Now for each $x \in \Gamma_m^+$, there exists a vector $p \in \chi_m$, and consequently for each $y \neq 0$, number $r_y \in [0, 1]$ such that

$$p_y = \frac{p \cdot y}{|y|} = r_y\, \delta_y^+ v_m(x) + (1 - r_y)\, \delta_y^- v_m(x).$$

Thus,

$$\delta_y v_m(x) = \frac{1}{2}\, (\delta_y^+ + \delta_y^-)\, v_m(x)$$

$$= p_y + \left(r_y - \frac{1}{2} \right) (\delta_y^+ + \delta_y^-)\, v_m(x),$$

$$= p_y - \left(r_y - \frac{1}{2} \right) |y|\ \delta_y^2 v_m(x).$$

Therefore, since $\delta_y^2 v_m(x) \leqslant 0$ in Γ_m^+, we obtain from (4.15),

$$\sum \bar{\alpha}_m(x, y)\, \delta_y^2 v_m(x) + \sum \beta_m(x, y)\, p_y + \kappa_m\, v_m(x) \geqslant$$

$$\geqslant \frac{\gamma_{m-1}(x)}{\tau}\, (v_m - v_{m-1})(x) - f_m^-(x) + \frac{1}{\tau}\, (e^{\tau A} - 1)\, v_m(x) \tag{4.16}$$

for all $x \in \mathscr{S}_m$, where we have set

$$\bar{\alpha}_m(x, y) = \alpha_m(x, y) - \frac{|y|}{2}\, \beta_m(x, y) (\geqslant 0 \text{ by } (2.5)).$$

To estimate p_y, we replace the domain Ω above by

$$\tilde{\Omega} = \{x \in \mathbb{R}^n \,|\, \text{dist}\,(x, \Omega) < kR\}$$

where k is a positive constant and R is the diameter of Ω. Extending the mesh function u to vanish outside $\mathcal{Q}_{h,\tau}$, we clearly arrive at (4.16) again, for the corresponding extended functions v_m, with Ω replaced by $\tilde{\Omega}$. But then we can estimate for $x \in \Gamma_m^+$, $p \in \chi_m(x)$,

$$|p| \leqslant \frac{v_m(x)}{\operatorname{dist}(x, \partial\tilde{\Omega})},$$
$$\leqslant \frac{v_m(x)}{kR}.$$

Consequently, from (4.16), we obtain

$$\sum \bar{\alpha}_m(x, y)\, \delta_y^2 v_m(x) - \frac{\gamma_{m-1}(x)}{\tau}(v_m - v_{m-1})(x) + f_m^-(x) \geqslant$$
$$\geqslant \left\{ \frac{1}{\tau}(e^{\tau A} - 1) - \kappa_m(x) - \frac{1}{kR} \sum |\beta_m(x, y)| \right\} v_m(x),$$
$$\geqslant \left\{ \frac{1}{\tau}(e^{\tau A} - 1) - (1 - \alpha) B_0 e^{\tau A} - \alpha B_0 \right\} v_m(x), \quad \left(\text{for } B_0 = \frac{b_0}{kR} + c_0^+ \right),$$
$$\geqslant 0, \tag{4.17}$$

provided

$$(1 - \alpha)\tau B_0 < 1, \quad A \geqslant \frac{B_0}{1 - (1 - \alpha)\tau B_0}. \tag{4.18}$$

Invoking (2.6), we can satisfy (4.18) by fixing k and A such that

$$k \geqslant \frac{(1 + \varepsilon)(1 - \alpha)\tau b_0}{\mu R}, \quad A \geqslant \frac{1 + \varepsilon}{\varepsilon\mu}\left(\frac{b_0}{kR} + c_0^+ \right) \tag{4.19}$$

for some $\varepsilon > 0$. With these choices of k and A, we thus obtain the difference inequality (4.5) with α_m replaced by $\bar{\alpha}_m$ and the estimate (4.10) follows as before. Returning to our original function u, we then conclude the estimate,

$$\max_{\mathcal{Q}_{h,\tau}} u \leqslant \frac{1}{n+1} \omega_n^{-\frac{1}{n+1}} [K(k+1)R]^{\frac{n}{n+1}} e^{AT} \| f/\mathcal{D}^* \|_{L^{n+1}(\mathcal{S})}, \tag{4.20}$$

where k and A are given by (4.19). Furthermore, by appropriate choice $k \geqslant b_0 T/\mu R$, we may write (4.20) in the form

$$\max_{\mathcal{Q}_{h,\tau}} u \leqslant C_0(n) K^{\frac{n}{n+1}} \left(R + \frac{b_0 T}{\mu} \right)^{\frac{n}{n+1}} \times$$
$$\times \exp[C_1 c_0^+ T/\mu] \| f/\mathcal{D}^* \|_{L^{n+1}(\mathcal{S})}, \tag{4.21}$$

where C_0 and C_1 are positive constants. This completes the proof of Theorem 2.1.

Remarks :

(i) It is clear from the above proof that the constant K in the estimates (2.9), (4.10), (4.20), (4.21) can be replaced by

$$K = \max_{Q_{h,\tau}} |y^i|$$

$$i = 1, \ldots, n$$

(ii) When $\mathscr{Z}$ is explicit, that is $\alpha = 1$, we can take $C_1 = \mu = 1$ in (4.21).

(iii) When $b_0 = 0$, we can take $k = 0$, $\tilde{\Omega} = \Omega$, $C_1 = 1$ so that the estimate (4.21) reduces to

$$\max_{Q_{h,\tau}} u \leq C_0(n)(KR)^{\frac{n}{n+1}} \exp[c_0^+ T/\mu] \|f/\mathscr{D}^*\|_{L^{n+1}(\mathscr{S})}. \tag{4.22}$$

(iv) Utilizing the above case, $b_0 = 0$, and following the proof of Theorem 2.1, we obtain in the general case,

$$\max_{Q_{h,\tau}} u \leq C_0(n)(K(k+1)R)^{\frac{n}{n+1}} \exp[c_0 T/\mu] \cdot$$

$$\cdot \left(\|f/\mathscr{D}^*\|_{L^{n+1}(\mathscr{S})} + \frac{\max u^+}{kR} \|b/\mathscr{D}^*\|_{L^{n+1}(\mathscr{S})} \right). \tag{4.23}$$

By appropriate choice of k, we then conclude the estimate

$$\max_{Q_{h,\tau}} u \leq C \left(R^{\frac{n}{n+1}} + \|b/\mathscr{D}^*\|^n_{L^{n+1}(\mathscr{S})} \right) \|f/\mathscr{D}^*\|_{L^{n+1}(\mathscr{S})} \tag{4.24}$$

where now C depends on n, K and $c_0 T/\mu$.

5. SEMIDISCRETE SCHEMES

By letting either of the parameters h, τ tend to zero, we can recover estimates for semidiscrete schemes. For a continuous time scheme, we consider the spatially discrete mesh $E = E_h = \mathbb{Z}_h^n \times \mathbb{R}$, with mesh function $u \in \mathscr{M}(E)$ assumed to be absolutely continuous with respect to time. The operator $\mathscr{Z} = \mathscr{Z}_h$ is defined by

$$\mathscr{Z} u(x, t) = L_h u(x, t) - D_t u, \tag{5.1}$$

for almost all t, and our hypotheses on $\mathscr{L}$ reduce to (1.6), (2.1) with $\mathscr{D}^* = \mathscr{D}_0^*$ defined by

$$\mathscr{D}^*(x, t) = \left(\prod_{i=1}^{n} \lambda_i(x, t) \right)^{1/(n+1)}$$

The discrete parabolic interior and boundary of the cylinder $Q_h = Q_h \times (0, T]$, corresponding to $\mathscr{L}$, are then defined by

$$Q_h^0 = \{(x, t) \in Q_h | a(x, t, y) = 0 \ \forall x + y \notin \Omega_h\}$$
$$Q_h^b = \bar{Q}_h - Q_h^0 .$$

The increasing upper contact set $\mathscr{S}$ of the mesh function u is defined as before. Corresponding to Theorem 2.1, we then have the following estimate.

THEOREM 5.1 : *Let u be a space-time mesh function in the cylinder Q_h, satisfying the differential-difference inequality,*

$$\mathscr{L} u \geqslant f \quad \text{in} \quad Q_h^0 , \tag{5.2}$$

together with the boundary conditions,

$$u \leqslant 0 \quad \text{in} \quad Q_h^b . \tag{5.3}$$

where f is a mesh function satisfying $f/\mathscr{D}^* \in L^{n+1}(\mathscr{S})$. Then we have the estimate,

$$\max_{Q_h} u \leqslant C R^{\frac{n}{n+1}} \| f/\mathscr{D}^* \|_{L^{n+1}(\mathscr{S})} , \tag{5.4}$$

where C depends on n, K, $b_0 T/R$, $c_0^+ T$ and $R = \operatorname{diam} \Omega$.

The norm in (5.4) is defined by

$$\| g \|_{L^p(\mathscr{S})} = \left\{ \int_0^T \sum_{x \in \Omega_h} h^n | \chi_{\mathscr{S}} g(x, \tau) |^p \, dx \right\}^{1/p}$$

for $g = f/\mathscr{D}^*$, $p = n + 1$, where $\chi_{\mathscr{S}}$ denotes the characteristic function of $\mathscr{S}$ in Q_h. Theorem 5.1 follows from Theorem 2.1 with $\alpha = 0$, by sending τ to zero and observing that we may express (5.2) in the integral form,

$$\frac{1}{\tau} \int_{t-\tau}^{t} L_h u(x, s) \, ds - \delta_t^- u(x, t) \geqslant \frac{1}{\tau} \int_{t-\tau}^{t} f(x, s) \, ds \tag{5.5}$$

for all $(x, t) \in Q_h^0$, $t \geqslant \tau > 0$. Alternatively, it can be proved directly from the spatially discrete version of the identity (3.4) and Lemma 3.2. The remarks at the end of Section 4, with $\alpha = 0$, $\tau \to 0$ also apply here.

When we let the spatial mesh length h tend to zero, we obtain a system of elliptic operators of the form

$$\mathscr{L}_\alpha u(x, t) = (1 - \alpha) Lu(x, t) + \alpha Lu(x, t - \tau) - \delta_t^- u(x, t) \quad (5.6)$$

where now $E = \mathbb{R}^n \times \mathbb{Z}_\tau$ and

$$Lu(x, t) = \sum_y a(x, y, t) D_{\hat{y}}^2 u(x, y) + \\ + \sum_y b(x, y, t) D_{\hat{y}} u(x, t) + c(x, t) u(x, t), \quad (5.7)$$

where $\hat{y} = y/|y|$ and the first and second spatial derivatives of the mesh function u are assumed to exist in a reasonable sense, for example $u_m \in C^2(\mathbb{R}^n)$ for each m, where $u_m(x) = u(x, m\tau)$. In order to fulfill conditions (1.6), (2.1), (2.3), (2.6) as $h \to 0$, we clearly must restrict to the implicit case $\alpha = 0$, whence we take

$$\mathscr{L} u(x, t) = Lu(x, t) - \delta_t^- u(x, t) \quad (5.8)$$

and conditions (1.6), (2.1), (2.6) reduce to

$$a(x, y, t) \geqslant 0, \quad a(x, y, t) \geqslant \lambda_1(x, t) > 0, \quad (5.9)$$

$$\tau c(x, t) \leqslant 1 - \mu. \quad (5.10)$$

Utilizing the representation of Motzkin-Wasow [8] (see also [6]), we may replace (5.7) by any *uniformly* elliptic operator L of the form

$$Lu(x, t) = a^{ij}(x, t) D_{ij} u(x, t) + b^i(x, t) D_i u(x, t) + c(c, t) u(x, t) \quad (5.11)$$

satisfying (5.10) and deduce an appropriate analogue of Theorem 2.1, by letting $h \to 0$. However the proof of Theorem 2.1 can be applied directly to the operators $\mathscr{L}$, similar to the approach in [4], to get results for non-uniformly parabolic operators. Letting $Q_\tau = \Omega \times \{m\tau\}_{m=1}^N$, we assume that the operator L is elliptic in Q_τ for each t, that is the coefficient matrix $[a^{ij}]$ is positive for all x, $t \in Q_\tau$, and write

$$\mathscr{D}^* = (\det [a^{ij}])^{\frac{1}{n+1}}, \quad b_0 = \max_{Q_\tau} |b|, \quad c_0^+ = \max_{Q_\tau} c^+.$$

Corresponding to Theorem 2.1, 5.1, we now have the following discrete time estimate.

THEOREM 5.2 : *Let u, f be mesh functions on Q_τ, with $u_m \in C^2(\Omega) \cap C^0(\bar{\Omega})$, $f_m \in L^{n+1}(\Omega)$ for each $n = 1, \ldots, N$, satisfying the differential-difference inequality*

$$\mathscr{L} u \geqslant f \quad \text{in} \quad Q_\tau , \tag{5.12}$$

together with the boundary condition

$$u_m \leqslant 0 \quad \text{on} \quad \partial\Omega , \quad u_0 \leqslant 0 , \quad m = 1, \ldots, N . \tag{5.13}$$

Then we have the estimate

$$\sup_{Q_\tau} u \leqslant C R^{\frac{n}{n+1}} \| f/\mathscr{D}^* \|_{L^{n+1}(\mathscr{S})} \tag{5.14}$$

where C is a constant depending only on n, μ, $b_0 T/R$, $c_0^+ R$, $\mathscr{S}$ denotes the increasing-upper contact set of u and $R = \operatorname{diam} \Omega$.

The norm in (5.14) is defined by

$$\| g \|_{L^p(\mathscr{S})} = \left\{ \int_\Omega \sum_{m=1}^{N} \tau \, | \chi_{\mathscr{S}} \, g(x, m\tau) |^p \, dx \right\}^{1/p}$$

for $g = f/\mathscr{D}^*$, $p = n + 1$, where $\chi_{\mathscr{S}}$ denotes the characteristic function of $\mathscr{S}$ in Q_τ. Corresponding remarks to Remarks (iii) and (iv) at the end of Section 4 also apply here. In order to directly adapt our previous proof, we also use that the concave hull of a function $u \in C^2(\Omega)$ belongs to $C^{1,1}(\Omega)$, ([9], [10]). By sending $\tau \to 0$ in Theorem 5.2 we obtain the estimates of [4, 9, 10, 11], but these can also be deduced directly from (3.4) and Lemma 3.2.

REFERENCES

[1] A. D. ALEKSANDROV, *Uniqueness conditions and estimates for the solution of the Dirichlet problem,* Vestnik Leningrad. Univ 18, 1963, no. 3, pp. 5-29, English transl., *Amer Math Soc Transl* 1968, 2, 68, pp. 89-119.

[2] I. YA BAKEL'MAN, *Geometric methods for solving elliptic equations,* Nauka, Moscow, 1965 (In Russian)

[3] D. GILBARG and N. S. TRUDINGER, *Elliptic partial differential equations of second order,* 2nd ed., Springer-Verlag, Berlin, Heidelberg, New York and Tokyo, 1983.

[4] N. V. KRYLOV, *Sequence of convex functions and estimates of maximum of the solution of a parabolic equation,* Sibirsk *Mat Ž* 1976, 17, pp. 290-303 : English translation in *Siberian Math J* 1976, 17, pp 226-237.

[5] N. V. KRYLOV, *Nonlinear elliptic and parabolic equations of the second order,* Nauka, Moscow, 1985 (In Russian). English translation by D. Reidel Publishing Company, Dordrecht, Holland, 1987.

[6] H. J. KUO and N. S. TRUDINGER, Linear elliptic difference inequalities with random coefficients, *Math. Comp.* 1990, 55, pp. 37-53.

[7] H. J. KUO and N. S. TRUDINGER, Discrete methods for fully nonlinear elliptic equations, *SIAM J. on Numer. Anal.* 1992, 29, pp. 123-135.

[8] T. MOTZKIN and W. WASOW, On the approximation of linear elliptic differential equations by difference equations with positive coefficients, *J. Math. Phys.* 1952, 31, pp. 253-259.

[9] A. I. NAZAROV and N N. URAL'TSEVA, Convex monotone hulls and estimaties of the maximum of the solution of parabolic equations, *Zap. Nauchn Sem. LOMI,* 1985, 147, pp. 71-86 (In Russian).

[10] S. J. REYE, *Harnack inequalities for parabolic equations in general form with bounded measurable coefficients,* Research Report R44-84, Centre for Math. Anal. Aust. Nat. Univ. (1984) (see also Doctoral dissertation : Fully non-linear parabolic differential equations of second order, Aust. Nat. Univ. 1985).

[11] K. TSO, On an Aleksandrov-Bakel'man type maximum principle for second order parabolic equations, *Comm. Partial Differential Equations* 1985, 10, pp. 543-553.