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ON THE MAXWELL'S SYSTEM IN COMPOSITE MEDIA (*)

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Abstract. — *The Maxwell's system is considered in a composite medium with a periodic structure, and the solution is derived by means of Bloch expansion techniques. The behaviour of the homogenized medium is obtained as a limiting case of this solution. Some numerical results on stratified media are presented.*

Résumé. — *On considère les équations de Maxwell dans un milieu composite à structure périodique. La solution est construite au moyen des techniques de développement de Bloch. De cette expression est déduit le cas statique limite qui fournit le comportement du milieu homogénéisé. Quelques résultats numériques sur des milieux stratifiés sont présentés.*

1. INTRODUCTION

With the increasing use of composite materials in a lot of technological domains, the studies on periodic structures present a real interest. Optical media are one of these new materials. Roughly speaking the periodic optical medium may globally be considered as an homogeneous, anisotropic material. But to understand the precise nature of electromagnetic wave propagation, the periodicity of the microstructure has to be taken into account. In that way, some interesting filtration properties of these media can be explained. A theoretical tool of investigation is the concept of Bloch expansion. It was introduced by F. Bloch in the quantum theory of electrons in crystals [2]. Since then, it has been applied many times, and recently, it has been used in research on elastic [8] or piezoelectric composites [9].

This paper described the effective dynamical properties of optical media by means of Bloch expansions. First, the problem is set in an appropriate functional framework from which existence and uniqueness of the solution are deduced. The Bloch expansion of the solution is then constructed. As an

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application, the macroscopic behaviour of the homogenized medium is reached when the data are slowly variable functions compared to the period of the material. This coincide with the results of [7], obtained from a two scale method. In the last section, some numerical results on stratified media are presented.

2. STATEMENT OF THE PROBLEM

2.1. Local equations

We consider the Maxwell's system in an unbounded, nonhomogeneous medium, with a periodic structure. We assume that there are no source terms. The electric and magnetic fields **E** and **H**, the electric and magnetic inductions **D** and **B** and the electric current **J** are related by equations (1) (2) and constitutive laws (3) [3] :

$$\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} - \mathbf{rot}\,\mathbf{H} = 0 \quad \operatorname{div}\mathbf{D} = q \tag{1}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{rot}\,\mathbf{E} = 0 \quad \operatorname{div}\mathbf{B} = 0 \tag{2}$$

$$\mathbf{D} = \eta\mathbf{E} \quad \mathbf{B} = \mu\mathbf{H} \quad \mathbf{J} = \sigma\mathbf{E}\,. \tag{3}$$

The relation div $\mathbf{D} = q$ has to be considered as a definition of the electric load q.

The characteristic coefficients : η the dielectric constant, μ the magnetic permeability and σ the resistivity, are assumed to be bounded functions on $2\,\pi Y$, $Y = (]0, 1[)^3$ (i.e. periodic functions of the variables y_k with period $2\,\pi$, [1], [6]), and such that :

$$0 < \alpha \leqslant \eta,\ \mu,\ \sigma \leqslant \alpha' \quad \alpha,\ \alpha' \text{ constants}\,. \tag{4}$$

We also assume that η *is a regular function.*

The following initial conditions are given :

$$\mathbf{E}(y, 0) = \mathbf{E}_0(y) \quad \mathbf{H}(y, 0) = \mathbf{H}_0(y)\,. \tag{5}$$

Note : The initial condition (5) on **H** must satisfy div $(\mu\mathbf{H}_0) = 0$. From (2) this property holds for the solution $\mathbf{H}(t)$, for every time t.

2.2. Global equations

We introduce the following functional spaces :

$$H_\eta(\mathbb{R}^3)\ (\text{resp. } \mu) = (L^2(\mathbb{R}^3))^3$$

with the inner product $(.,.)_\eta$, with weight η (resp. μ) and

$$V(\mathbb{R}^3) = \left\{\mathbf{F},\ \mathbf{F} \in (L^2(\mathbb{R}^3))^3,\ \mathbf{rot}\,\mathbf{F} \in (L^2(\mathbb{R}^3))^3\right\}.$$

The local equations (1) (2) are multiplied by suitable test functions and the problem can be reformulated :
find (**E**, **H**), function of t with values in $H_\eta(\mathbb{R}^3) \times H_\mu(\mathbb{R}^3)$, such that :

$$\begin{cases} \dfrac{d\mathbf{E}}{dt} + \Sigma\mathbf{E} + C\mathbf{H} = 0 \\ \dfrac{d\mathbf{H}}{dt} + D\mathbf{E} = 0 \\ \mathbf{E}(0) = \mathbf{E}_0 \quad \mathbf{H}(0) = \mathbf{H}_0 . \end{cases} \tag{6}$$

The operators Σ, C and D in (6) are respectively defined (where « . » represents the usual inner product in $\mathbb{R}^3$), by :

$$(\mathbf{E}, \mathbf{F}) \in H_\eta(\mathbb{R}^3) \times H_\eta(\mathbb{R}^3) \quad (\Sigma\mathbf{E}, \mathbf{F})_\eta = \int_{\mathbb{R}^3} \sigma\mathbf{E} \,.\, \bar{\mathbf{F}}\, dy$$

$$(\mathbf{H}, \mathbf{F}) \in V(\mathbb{R}^3) \times V(\mathbb{R}^3) \quad (C\mathbf{H}, \mathbf{F})_\eta = -\int_{\mathbb{R}^3} \mathbf{rot}\,\mathbf{H} \,.\, \bar{\mathbf{F}}\, dy$$

$$(\mathbf{E}, \mathbf{G}) \in V(\mathbb{R}^3) \times V(\mathbb{R}^3) \quad (D\mathbf{E}, \mathbf{G})_\mu = \int_{\mathbb{R}^3} \mathbf{rot}\,\mathbf{E} \,.\, \bar{\mathbf{G}}\, dy \,.$$

Assuming that :

$$\mathbf{E}_0 \in H_\eta(\mathbb{R}^3), \quad \mathbf{H}_0 \in H_\mu(\mathbb{R}^3)$$

then the hyperbolic system (6) has a solution and only one [3].

3. BLOCH EXPANSION OF THE SOLUTION

3.1. Problems defined on the basic cell

In agreement with the results obtained with other classical operators (elasticity [8], piezoelectricity [9]), the periodicity of the characteristic coefficients η, μ, σ, involves that the solution (**E**, **H**) of the problem (6) has the form :

$$\begin{cases} \mathbf{E}(y, t) = \displaystyle\int_Y e^{i\mathbf{k}\,.\,y}\, \tilde{\mathbf{E}}(y, \mathbf{k}, t)\, d\mathbf{k} \quad (\mathbf{k}\,.\,y = k_\ell\, y_\ell) \\ \mathbf{H}(y, t) = \displaystyle\int_Y e^{i\mathbf{k}\,.\,y}\, \tilde{\mathbf{H}}(y, \mathbf{k}, t)\, d\mathbf{k} \end{cases} \tag{7}$$

where the vectors $\tilde{\mathbf{E}}$, $\tilde{\mathbf{H}}$, as functions of y belong to $2\,\pi Y$. These expressions lead us to introduce operators, depending on the parameter **k**, $\mathbf{k} \in Y$, defined on the basic cell $2\,\pi Y$.

Interesting properties arise in the following special case :

$$\sigma = 0 \quad \operatorname{div}\,(\eta\mathbf{E}_0) = 0 \,. \tag{8}$$

From equations (1) (2), we obtain a system with one unknown function **E** :

$$\eta \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mathbf{rot}\ (\mu^{-1}\ \mathbf{rot}\ \mathbf{E}) = 0 \quad \operatorname{div}\ (\eta \mathbf{E}) = 0\,. \tag{9}$$

Let us introduce the functional space :

$$V_\eta(\mathbb{R}^3) = \left\{\mathbf{F},\ \mathbf{F} \in (L^2(\mathbb{R}^3))^3,\ \mathbf{rot}\ \mathbf{F} \in (L^2(\mathbb{R}^3))^3,\ \operatorname{div}\ (\eta \mathbf{F}) = 0\right\}.$$

Then the problem in **E** can be written :

$$\begin{cases} \dfrac{d^2\mathbf{E}}{dt^2} + A\mathbf{E} = 0 \\ \mathbf{E}(0) = \mathbf{E}_0 \quad \dfrac{d\mathbf{E}}{dt}(0) = -C\mathbf{H}_0 = \mathbf{E}_1 \end{cases} \tag{10}$$

where A denotes the operator defined by :

$$(\mathbf{E},\ \mathbf{F}) \in V_\eta(\mathbb{R}^3) \times V_\eta(\mathbb{R}^3),\ (A\mathbf{E},\ \mathbf{F})_\eta = \int_{\mathbb{R}^3} \mu^{-1}\ \mathbf{rot}\ \mathbf{E}\ .\ \overline{\mathbf{rot}\ \mathbf{F}}\ dy\,.$$

From (7), we note that the derivation $\partial/\partial y_j$ on **E** corresponds to the operation $\partial/\partial y_j + ik_j$ on $\tilde{\mathbf{E}}$. So we are led to introduce the following functional spaces :

$$H_\eta = (L^2(2\ \pi Y))^3$$

with its defined inner product with weight η

$$V_\eta(\mathbf{k}) = \{\mathbf{E},\ \mathbf{E} \in (L^2(2\ \pi Y))^3,\ \mathbf{rot}\ \mathbf{E} \in (L^2(2\ \pi Y))^3,$$
$$\operatorname{div}\ (\eta \mathbf{E}) + i\eta \mathbf{k}\ .\ \mathbf{E} = 0\}\,. \tag{11}$$

The operators $A(\mathbf{k})$, $\mathbf{k} \in Y$, are defined by :

$$(\mathbf{E},\ \mathbf{F}) \in V_\eta(\mathbf{k}) \times V_\eta(\mathbf{k})$$

$$(A(\mathbf{k})\ \mathbf{E},\ \mathbf{F})_\eta = \int_{2\ \pi Y} \mu^{-1}(\mathbf{rot}\ \mathbf{E} + \mathbf{ik} \wedge \mathbf{E})\ .\ (\overline{\mathbf{rot}\ \mathbf{F} + \mathbf{ik} \wedge \mathbf{F}}\ )\ dy\,. \tag{12}$$

Note : for each $\mathbf{k} \in Y$, the space $V_\eta(\mathbf{k})$ is a subspace of $(H^1(2\ \pi Y))^3$. The domain $D(A(\mathbf{k}))$ of the operator $A(\mathbf{k})$, $\mathbf{k} \in Y$, is made up of elements **E** that belong to $(H^1_p(2\ \pi Y))^3$ and such that $\mathbf{n} \wedge \mathbf{rot}\ \mathbf{E}$ is antiperiodic. $(H^1_p(2\ \pi Y))^3$ denotes the space of the function in $(H^1(2\pi Y))^3$ that take equal values at two opposite points of two opposite sides of the cell $2\ \pi Y$. n is the outer normal to $\partial(2\ \pi Y)$.

3.2. Properties of the operators A(k)

PROPOSITION : The operator $A(\mathbf{k})$ is, for each $\mathbf{k} \in Y$, a positive, selfadjoint operator with compact resolvent ([7]).

The first two properties immediately result from the definition (12). From assumption (4), it follows that :

$$(A(\mathbf{k})\,\mathbf{E},\,\mathbf{E})_\eta \geqslant \beta \int_{2\pi Y} [\,|\mathbf{rot}\,\mathbf{E}\,|^2 + (\mathbf{ik} \wedge \mathbf{E}) \,.\, \overline{\mathbf{rot}\,\mathbf{E}} + \mathbf{rot}\,\mathbf{E} \,.\, (\overline{\mathbf{ik} \wedge \mathbf{E}}) + |\,\mathbf{k} \wedge \mathbf{E}\,|^2]\,dy$$

(with β positive constant). The second and third terms of the integral are underestimated, using the identity :

$$z\bar{Z} + \bar{z}Z \geqslant -2\,|z\,|^2 - \frac{|Z|^2}{2}\,.$$

And since $\mathbf{k}$ belongs to Y, we have :

$$|\,\mathbf{k} \wedge \mathbf{E}\,|^2 \leqslant |\,\mathbf{E}\,|^2 \quad (\text{norms in } \mathbb{R}^3)\,.$$

Finally, we obtain the inequality :

$$(A(\mathbf{k})\,\mathbf{E},\,\mathbf{E})_\eta \geqslant \frac{\beta}{2}\,\|\mathbf{rot}\,\mathbf{E}\,\|^2_{(L^2(2\pi Y))^3} - \beta\,\|\mathbf{E}\,\|^2_{(L^2(2\pi Y))^3} \tag{13}$$

where β is a positive constant that does not depend on $\mathbf{k}$.

Now for any $\mathbf{E}$ in $D(A(\mathbf{k}))$, we have the identity (with $E_{j,\,p} = \partial E_j/\partial y_p$) :

$$\int_{2\pi Y} E_{j,\,p}\,\bar{E}_{j,\,p}\,dy = \|\mathrm{div}\,\mathbf{E}\,\|^2_{(L^2(2\pi Y))^3} + \|\mathbf{rot}\,\mathbf{E}\,\|^2_{(L^2(2\pi Y))^3}\,. \tag{14}$$

Indeed, the operator $\mathbf{grad}\,\mathbf{E}$ can be split into a symmetric part $\boldsymbol{\varepsilon}$ and an antisymmetric part $\boldsymbol{\omega}$ and we have :

$$\varepsilon_{jp}\,\bar{\varepsilon}_{jp} + \omega_{jp}\,\bar{\omega}_{jp} = E_{j,\,p}\,\bar{E}_{j,\,p}$$

$$\varepsilon_{jp}\,\bar{\varepsilon}_{jp} - \omega_{jp}\,\bar{\omega}_{jp} = E_{j,\,p}\,\bar{E}_{p,\,j} = (E_j\,\bar{E}_{p,\,j})_{,p} - (E_j\,\bar{E}_{p,\,p})_{,j} + E_{j,\,j}\,\bar{E}_{p,\,p}\,.$$

Then, using the periodicity of $\mathbf{E}$, we get :

$$\int_{2\pi Y} E_{j,\,p}\,\bar{E}_{j,\,p}\,dy = \int_{2\pi Y} E_{j,\,j}\,\bar{E}_{p,\,p}\,dy + 2\int_{2\pi Y} \omega_{j,\,p}\,\bar{\omega}_{j,\,p}\,dy\,.$$

The tensor $\boldsymbol{\omega}$ can be expressed with the help of $\mathbf{rot}\,\mathbf{E}$ and it follows that : $\omega_{jp}\,\bar{\omega}_{jp} = \frac{1}{2}\,|\,\mathbf{rot}\,\mathbf{E}\,|^2$. Equality (14) results.

For any $\mathbf{E} \in D(A(\mathbf{k}))$ and since η is a regular function :

$$\eta\,\mathrm{div}\,\mathbf{E} + \mathbf{E}\,.\,\mathbf{grad}\,\eta + i\,\eta\mathbf{k}\,.\,\mathbf{E} = 0\,.$$

Since $\mathbf{k} \in Y$, we deduce : $(\mathrm{div}\,\mathbf{E})^2 \leqslant \gamma\,|\,\mathbf{E}\,|^2$ (with γ constant, independent of $\mathbf{k}$). And therefore, for $\mathbf{E} \in D(A(\mathbf{k}))$:

$$\|\mathbf{E}\,\|^2_{(H^1(2\pi Y))^3} \leqslant (1+\gamma)\,\|\mathbf{E}\,\|^2_{(L^2(2\pi Y))^3} + \|\mathbf{rot}\,\mathbf{E}\,\|^2_{(L^2(2\pi Y))^3}\,.$$

This relation is injected into (12) and we then get :

$$(A(\mathbf{k})\mathbf{E}, \mathbf{E})_\eta \geq C_1 \|\mathbf{E}\|^2_{(H^1(2\pi Y))^3} - C_2 \|\mathbf{E}\|^2_{(L^2(2\pi Y))^3} \tag{15}$$

with C_1, C_2 positive constants, independent of $\mathbf{k}$.

The embedding of $(H^1(2\pi Y))^3$ into $(L^2(2\pi Y))^3$ beeing compact, the overestimate (15) implies that the operator $A(\mathbf{k}) + C_2 Id$ has a compact inverse ([7]). So :

PROPOSITION : *For each* $\mathbf{k} \in Y$, *there exists a countable sequence of eigenvalues :* $0 \leq \omega_0^2(\mathbf{k}) \leq \omega_1^2(\mathbf{k}) \ldots$, *with corresponding eigenfunctions* $\boldsymbol{\varphi}^0(y, \mathbf{k})$, $\boldsymbol{\varphi}^1(y, \mathbf{k}) \ldots$ *of the operator* $A(\mathbf{k})$. *Moreover, the vectors* $\boldsymbol{\varphi}^m(y, \mathbf{k})$ *form an orthonormal basis in* $H_\eta(2\pi Y)$.

A similar study can be carried on for the unknown function $\mathbf{H}$. The same properties hold for the operators which are then introduced : they are just obtained by changing η into μ and reciprocally. We denote by $\Omega_m^2(\mathbf{k})$ and $\boldsymbol{\psi}^m(y, \mathbf{k})$, $m \in \mathbb{N}$, the eigenelements of these operators.

3.3. Representation of the solution in the conservative case

Assuming (8), we use the basis $\{\boldsymbol{\varphi}^m(y, \mathbf{k})\}$ and $\{\boldsymbol{\psi}^m(y, \mathbf{k})\}$ to define functions $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ in (7).

Since equation (10) is satisfied by $\mathbf{E}$ of the form (7), it follows that $\tilde{\mathbf{E}}$ is solution of :

$$\frac{d^2\tilde{\mathbf{E}}}{dt^2} + A(\mathbf{k})\tilde{\mathbf{E}} = 0 \tag{16}$$

$$\tilde{\mathbf{E}}\Big|_{t=0} = \tilde{\mathbf{E}}_0 \quad \frac{d\tilde{\mathbf{E}}}{dt}\Big|_{t=0} = \tilde{\mathbf{E}}_1 . \tag{17}$$

Equation (16) is projected on $\boldsymbol{\varphi}^m$, $m \in \mathbb{N}$, and the expression for $\mathbf{E}$ follows :

$$\mathbf{E}(y, t) = \int_Y e^{\mathrm{i}\mathbf{k}\cdot y} \sum_{m=0}^{+\infty} \hat{E}_m(\mathbf{k}, t)\, \boldsymbol{\varphi}^m(y, \mathbf{k}) \tag{18}$$

$$\hat{E}_m(\mathbf{k}, t) = A_m^+(\mathbf{k})\, e^{\mathrm{i}\omega_m(\mathbf{k})t} + A_m^-(\mathbf{k})\, e^{-\mathrm{i}\omega_m(\mathbf{k})t}$$

where the $A_m^\pm(\mathbf{k})$ are determined from the initial conditions (17). A similar expression holds true for $\mathbf{H}(y, t)$ with the help of $\Omega_m(\mathbf{k})$ and $\boldsymbol{\psi}^m(y, \mathbf{k})$.

3.4. Properties of the solution in the dissipative case

From here on we won't use assumption (8). In order to use the previous basis $\{\boldsymbol{\varphi}^m\}$ and $\{\boldsymbol{\psi}^m\}$, we introduce the new unknown function $\mathbf{F}$, $\mathbf{F} \in V_\eta(\mathbb{R}^3)$, defined by means of the convolution :

$$\mathbf{E}(y, t) = f(y, t) * \mathbf{F}(y, t) \quad f(y, t) = 1 - \frac{\sigma}{\eta} e^{-\frac{\sigma}{\eta}t} . \tag{19}$$

F has an expression like (18) in the basis $\{\boldsymbol{\varphi}^m\}$ and, in the same way, **H** in the basis $\{\boldsymbol{\psi}^m\}$. The components $\hat{F}_m(\mathbf{k}, t)$ and $\hat{H}_m(\mathbf{k}, t)$ satisfy :

$$\left\{\begin{array}{l} \dfrac{d\hat{F}_m}{dt} + c_{mn}\,\hat{H}_n = 0 \\ \dfrac{d\hat{H}_n}{dt} + d_{nm}(t)\,\hat{F}_m = 0 \\ \hat{F}_m\Big|_{t=0} = F_m^0 \quad \hat{H}_n\Big|_{t=0} = H_n^0 \end{array}\right. \tag{20}$$

where the coefficients c_{mn} and d_{nm} are defined by :

$$c_{mn} = -\int_{2\pi Y} [\mathbf{rot}\,\boldsymbol{\psi}_n(y, \mathbf{k}) + i\mathbf{k} \wedge \boldsymbol{\psi}^n(y, \mathbf{k})] \,.\, \overline{\boldsymbol{\varphi}^m(y, \mathbf{k})}\, dy$$

$$d_{nm}(t) = \int_{2\pi Y} \{\mathbf{rot}\ [f(y, t) * \boldsymbol{\varphi}^m(y, \mathbf{k})] + \\ + i\mathbf{k} \wedge [f(y, t) * \boldsymbol{\varphi}^m(y, \mathbf{k})] \,.\, \overline{\boldsymbol{\psi}^n(y, \mathbf{k})}\}\, dy\,.$$

The initial conditions F_m^0 and H_m^0 are defined from the initial conditions on **E** and **H**. An expansion of **E** is then deduced from (19). System (20) is obviously a dissipative system since the coefficients d_{nm} depend on t.

4. HOMOGENIZATION FROM THE BLOCH EXPANSION

4.1. Posing the problem

We assume here that the initial conditions are slowly varying functions. Let ε be a small positive given parameter :

$$\mathbf{E}_0 = \mathbf{E}_0(\varepsilon y) \quad \mathbf{H}_0 = \mathbf{H}_0(\varepsilon y)\,. \tag{21}$$

Compared to the data scale, the period of the medium is very small and, when $\varepsilon \to 0$, an approximation is brought by the homogenized medium [7]. In this problem, two space variables appear : a slowly varying one x associated with the initial data and a fastly varying one y associated with the period of the medium. These two quantities are related by : $x = \varepsilon y$. Initially the problem is set up with the variable x. In order to use the previous results, we do the change of variable : $y = x\,\varepsilon^{-1}$ and we take the Laplace transform of system (6) :

$$\left\{\begin{array}{l} (\eta p + \sigma)\,\hat{\mathbf{E}} - \varepsilon^{-1}\,\mathbf{rot}\,\hat{\mathbf{H}} = 0 \\ \mu p\hat{\mathbf{H}} + \varepsilon^{-1}\,\mathbf{rot}\,\hat{\mathbf{E}} = 0\,. \end{array}\right. \tag{22}$$

Previous studies done in elasticity [8] or piezoelectricity [9] suggest, using the periodicity of the medium, that the solution $(\hat{\mathbf{E}}, \hat{\mathbf{H}})$ of (22) should be expanded in the following way, deduced from (7) :

$$\mathbf{k} = \varepsilon\mathbf{K},\ x = \varepsilon y,\ \hat{\mathbf{E}}(x, p) = \int_{\mathbb{R}^3} e^{i\mathbf{K}.x}\, \varepsilon^3 \tilde{\mathbf{E}}(y, \varepsilon\mathbf{K}, p)\, d\mathbf{K} \tag{23}$$

with

$$\varepsilon^3 \tilde{\mathbf{E}} = \tilde{\mathbf{E}}^0(y, \mathbf{K}, p) + \varepsilon\tilde{\mathbf{E}}^1(y, \mathbf{K}, p) + \cdots \tag{24}$$

A similar expression is applied for $\hat{\mathbf{H}}$.

For the solution $(\hat{\mathbf{E}}, \hat{\mathbf{H}})$ we have (22), then the function $(\varepsilon^3 \tilde{\mathbf{E}}, \varepsilon^3 \tilde{\mathbf{H}})$ satisfy :

$$\begin{cases} (\eta p + \sigma)\, \varepsilon^3 \tilde{\mathbf{E}} - \varepsilon^{-1}[\mathbf{rot}\ (\varepsilon^3 \tilde{\mathbf{H}}) + i\varepsilon\tilde{\mathbf{K}} \wedge \varepsilon^3 \tilde{\mathbf{H}}] = 0 \\ \mu p \varepsilon^3 \tilde{\mathbf{H}} + \varepsilon^{-1}[\mathbf{rot}\ (\varepsilon^3 \tilde{\mathbf{E}}) + i\varepsilon\tilde{\mathbf{K}} \wedge \varepsilon^3 \tilde{\mathbf{E}}] = 0\,. \end{cases} \tag{25}$$

The expansions (24) of $\varepsilon^3 \tilde{\mathbf{E}}$ and $\varepsilon^3 \tilde{\mathbf{H}}$ are injected into (25) and we identify the terms of the same power of ε in these equations.

4.2. Approximation of the solution

Order ε^{-1}.

The function $\tilde{\mathbf{E}}^0$ belongs to $V_{\eta p + \sigma}(0)$ and the function $\tilde{\mathbf{H}}^0$ to $V_{\mu}(0)$. So, $\tilde{\mathbf{E}}^0$ and $\tilde{\mathbf{H}}^0$ are solutions of :

$$\begin{aligned} &\mathbf{rot}\ \tilde{\mathbf{E}}^0 = 0 \quad \operatorname{div}\ [(\eta p + \sigma)\tilde{\mathbf{E}}^0] = 0 \\ &\mathbf{rot}\ \tilde{\mathbf{H}}^0 = 0 \quad \operatorname{div}\ (\mu\tilde{\mathbf{H}}^0) = 0\,. \end{aligned} \tag{26}$$

In the same way as in [1], we introduce the scalar functions $W^j(\eta p + \sigma)$ and $\chi^j(\mu)$ $(j = 1, 2, 3)$ defined uniquely by :

$$W^j(\eta p + \sigma),\ \chi^j(\mu) \in H^1_p(2\,\pi Y)$$

$$\operatorname{div}\ [(\eta p + \sigma)(e_j + \operatorname{grad} W^j)] = 0 \quad M(W^j) = 0 \tag{27}$$

$$\operatorname{div}\ [\mu(e_j + \operatorname{grad} \chi^j)] = 0 \quad M(\chi^j) = 0 \tag{28}$$

where e_j denotes the j-th vector of the natural basis of $\mathbb{R}^3$ (the ℓ-th component of e_j is $\delta_{j\ell}$) and $M(f)$ is the mean value of the function f on the basic cell $2\,\pi Y$. With the notations of (27) (28), (26) implies :

$$\tilde{\mathbf{E}}^0 = \hat{\mathbf{E}}^0_j(e_j + \operatorname{grad} W^j) \quad \tilde{\mathbf{H}}^0 = \hat{\mathbf{H}}^0_j(e_j + \operatorname{grad} \chi^j)\,.$$

From the properties of the functions W^j and χ^j, we note that :

$$M(\tilde{\mathbf{E}}^0) = \hat{E}_j^0 \, e_j \quad M(\tilde{\mathbf{H}}^0) = \hat{H}_j^0 \, e_j \, . \tag{29}$$

In order to determine the components $\hat{E}_j^0$ and $\hat{H}_j^0$, we use the equations deduced from the identification of the constants terms of (25).

Order ε^0.

$$(\eta p + \sigma)\, \tilde{\mathbf{E}}^0 - i\mathbf{K} \wedge \tilde{\mathbf{H}}^0 - \mathbf{rot}\, \tilde{\mathbf{H}}^1 = 0 \tag{30}$$

$$\mu p \tilde{\mathbf{H}}^0 + i\mathbf{K} \wedge \tilde{\mathbf{E}}^0 + \mathbf{rot}\, \tilde{\mathbf{E}}^1 = 0 \, . \tag{31}$$

These equations are projected on e_j. Let $\mathbf{p}(\sigma, \eta, p)$ and $\mathbf{q}(\mu)$ be the following matrices :

$$p_{ij} = \frac{1}{(2\pi)^3} \int_{2\pi Y} (\eta p + \sigma)(\delta_{ij} + W^j_{,i})\, dy \tag{32}$$

$$q_{ij} = \frac{1}{(2\pi)^3} \int_{2\pi Y} \mu\, (\delta_{ij} + \chi^j_{,i})\, dy \, . \tag{33}$$

With these notations and using expressions (23) and properties (29), we prove that the mean value of the approximation $(\hat{\mathbf{E}}^0, \hat{\mathbf{H}}^0)$ is determined from the system :

$$p(\sigma, \eta, p)\, M(\hat{\mathbf{E}}^0) - \mathrm{rot}\, M(\hat{\mathbf{H}}^0) = 0 \tag{34}$$

$$pq(\mu)\, M(\hat{\mathbf{H}}^0) + \mathrm{rot}\, M(\hat{\mathbf{E}}^0) = 0 \, . \tag{35}$$

4.3. Macroscopic constitutive equations

From (34) and (35), we deduce the homogenized behaviour of the magnetic inductions **D** and **B** and the electric current **J**. Equivalent weak formulations of the definitions (27), (28) of W^j and χ^j allow us to recognize that W^j and χ^j are the functions which are respectively used in the theory of homogenization of the two operators $\partial/\partial y_i[(\eta p + \sigma)\, \partial/\partial y_i]$ and $\partial/\partial y_i(\mu\, \partial/\partial y_i)$ ([7]). In these conditions, from the coefficients in (34) and (35), it follows that :

$$pM(\hat{\mathbf{D}}^0)_i + M(\hat{\mathbf{J}}^0)_i = [(\eta p + \sigma)\, \delta_{ij}]^h\, M(\hat{\mathbf{E}}^0)_j \tag{36}$$

$$M(\hat{\mathbf{B}})_i = (\mu\, \delta_{ij})^h\, M(\hat{\mathbf{H}})^0_j \tag{37}$$

where « h » denotes, as usual, « homogenized » ([7]).

The same result is obtained in [7]. The homogenized behaviours of **D** and **J** in terms of **E** are given by a convolution product.

5. BLOCH WAVES IN STRATIFIED MEDIA

5.1. Floquet problem

The eigenvalues $\omega_n^2(\mathbf{k})$ and eigenfunctions $\boldsymbol{\varphi}^n(y\,;\mathbf{k})$ of the operator $A(\mathbf{k})$, $\mathbf{k} \in Y$, satisfy :

$$\begin{cases} \varphi^n \in D(A(\mathbf{k})) \\ \eta^{-1}(\mathbf{rot}\,.+i\mathbf{k}\wedge .)[\mu^{-1}(\mathbf{rot}\,\boldsymbol{\varphi}^n + i\mathbf{k}\wedge\boldsymbol{\varphi}^n)] = \omega_n^2\boldsymbol{\varphi}^n \\ \operatorname{div}(\eta\boldsymbol{\varphi}^n) + i\eta\mathbf{k}\,.\,\boldsymbol{\varphi}^n = 0\,. \end{cases} \tag{38}$$

This is the Bloch problem set on the basic cell $2\,\pi Y$. The previous theoretical results were obtained with this formulation of the problem.

Let us consider : $\boldsymbol{\psi}^n(y\,;\mathbf{k}) = \boldsymbol{\varphi}^n(y,\mathbf{k})\,e^{i\mathbf{k}\cdot y}$, $\mathbf{k} \in Y$. Then $\omega_n^2(\mathbf{k})$ and $\boldsymbol{\psi}^n(y\,;\mathbf{k})$ are respectively the eigenelements of the problem :

$$\begin{cases} \boldsymbol{\psi}^n e^{-ik\cdot y} \in D(A(\mathbf{k})) \\ \eta^{-1}\,\mathbf{rot}\,(\mu^{-1}\,\mathbf{rot}\,\boldsymbol{\psi}^n) = \omega_n^2\boldsymbol{\psi}^n \\ \operatorname{div}(\eta\boldsymbol{\psi}^n) = 0\,. \end{cases} \tag{39}$$

This is the Floquet problem set on the basic cell $2\,\pi Y$. Numerical results are obtained with this formulation of the problem.

5.2. Numerical results in a two layered medium

When considering a layered medium, the Floquet problem leads to a numerical solution of differential equations.

Let us assume that : $\eta = \eta(y_1)$, $\mu = \mu(y_1)$ and $\mathbf{E}_0 = E_0(y_1)\,e_2$, $\mathbf{H}_0 = H_0(y_1)\,e_3$.

Then the eigenelements required for the study of the solution may be written as :

$$\omega_n^2(\mathbf{k}) = \omega_n^2(k_1), \quad \boldsymbol{\psi}^n(y\,;\mathbf{k}) = \psi^n(y_1\,;k_1)\,e_2\,.$$

In the following, we shall denote them $\omega_n^2(k)$ and $\psi^n(y_1\,;k)$.

We now consider a stratified medium with a basic period composed of two homogeneous and isotropic layers with thicknesses a and b :

$$\forall y_1 \in\,]0, a[\quad \eta(y_1) = \eta_a,\ \mu(y_1) = \mu_a$$

$$\forall y_1 \in\,]a, \ell[\quad \eta(y_1) = \eta_b,\ \mu(y_1) = \mu_b\,.$$

The eigenelements $\omega^2(k)$ and $\psi(k)$, $k \in 2\,\pi(a+b)^{-1}$ are solutions of the Floquet's problem set on $]0, \ell[$, $\ell = a + b$:

$$\frac{d}{dy_1}\left(\mu_a^{-1}\frac{d\psi}{dy_1}\right) = -\,\omega^2\eta_a\,\psi \quad \forall y_1 \in [0, a] \tag{40}$$

$$\frac{d}{dy_1}\left(\mu_b^{-1}\frac{d\psi}{dy_1}\right) = -\,\omega^2\eta_b\,\psi \quad \forall y_1 \in [a, \ell] \tag{41}$$

$$[\![\psi]\!]_{y_1 = a} = 0, \quad \left[\!\!\left[\mu^{-1} \frac{d\psi}{dy_1} \right]\!\!\right]_{y_1 = a} = 0 \tag{42}$$

$$\psi(a + b) = \psi(0)\, e^{ik\ell}, \quad \mu^{-1} \frac{d\psi}{dy_1}(\ell) = \mu^{-1} \frac{d\psi}{dy_1}(0)\, e^{ik\ell}. \tag{43}$$

From equations (40), (41), we obtain with $k_a = \omega(\eta_a \mu_a)^{1/2}$ and $k_b = \omega(\eta_b \mu_b)^{1/2}$:

$$\forall y_1 \in]0, a[, \quad \psi(y_1) = A_1 e^{ik_a y_1} + A_2 e^{ik_a y_1}$$
$$\forall y_1 \in]a, \ell[, \quad \psi(y_1) = B_1 e^{ik_b y_1} + B_2 e^{ik_b y_1}.$$

Let us introduce the following notations :

$$S = e^{ik_a a},\ T = e^{ik_b b},\ T_a = e^{ik_b a},\ T_\ell = e^{ik_b \ell},\ Z = e^{ik_a \ell},$$
$$I_a = (\eta_a/\mu_a)^{1/2},\ I_b = (\eta_b/\mu_b)^{1/2}.$$

The interface conditions (42) and the properties of periodicity (43) imply :

$$A_1 S^{-1} + A_2 S = B_1 T_a^{-1} + B_2 T_a$$
$$A_1 I_a S^{-1} - A_2 I_a S = B_1 I_a T_a^{-1} - B_2 I_b T_a$$
$$B_1 T_\ell^{-1} + B_2 T_\ell = A_1 Z^{-1} + A_2 Z$$
$$B_1 I_b T_\ell^{-1} - B_2 I_b T_\ell = A_1 I_a Z^{-1} - A_2 I_b Z.$$

This linear, homogeneous system has a non trivial solution if its determinant is zero valued. This happens (see Naciri's calculations [4]) when :

$$Z^2 + 1 = \frac{Z}{4\alpha}\left[(1 + \alpha)^2 (ST + S^{-1}T^{-1}) - (1 - \alpha)^2 (ST^{-1} + S^{-1}T)\right] \tag{44}$$

with $\alpha = I_a/I_b$. For real values of k, k_a, k_b, this relation is equivalent to :

$$\cos(k\ell) = \cos\left[\omega a(\eta_a \mu_a)^{1/2} + \omega b(\eta_b \mu_b)^{1/2}\right] - \\ - \frac{(1 - \alpha)^2}{2\alpha} \sin\left[\omega a(\eta_a \mu_a)^{1/2}\right] \sin\left[\omega b(\eta_b \mu_b)^{1/2}\right]. \tag{45}$$

Relation (45) implies that $\omega(2\pi/\ell - k) = \omega(k)$. From numerical solution of (44) and (45), we obtain the following figures :

Figure 1 shows the forbidden frequencies phenomena that characterize periodic media : there are angular frequencies which are not of the form $\omega_n(k)$ [6]. In these bands, the right hand term of equation (45) is greater than 1 in absolute value, and we shall build a solution of (44) with complex values

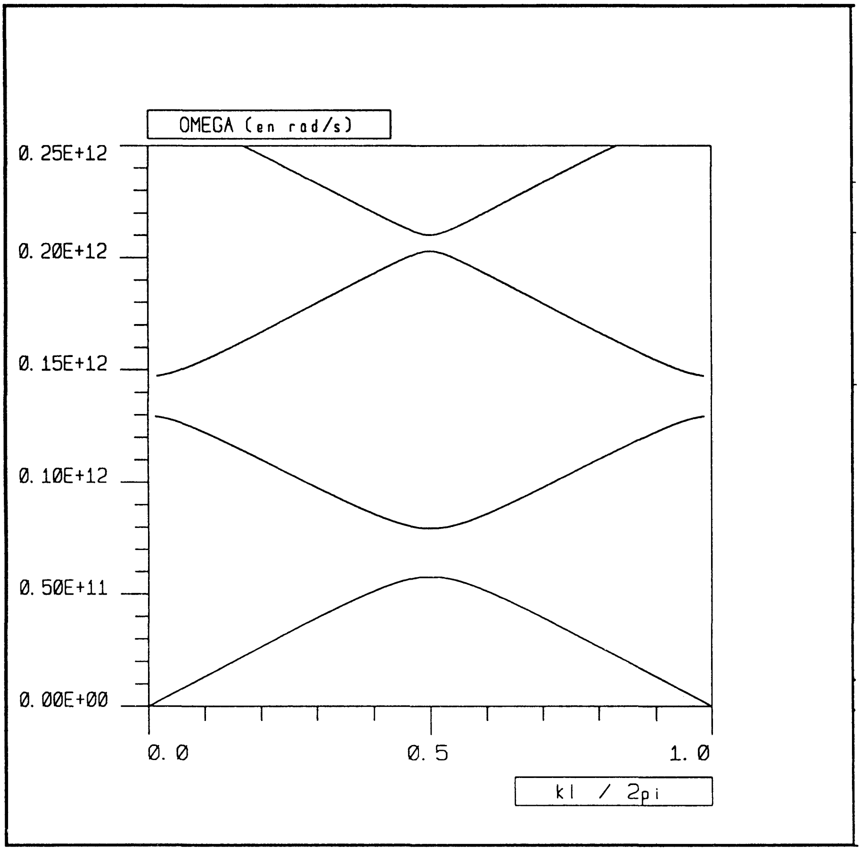

Figure 1. — The first three eigenfrequencies for the following data :
$a = b = 5$ mm, $\mu_a = \mu_b = \mu_0$, $\eta_a = \eta_0 = 3\,\eta_0$, $\sigma_a = \sigma_b = 0$.

(μ_0, η_0 are vacum's characteristics).

of k. We can note that $\omega_n(k)$ are continuous functions of k, in good agreement with [6] and [10].

Floquet theory may also be applied to the study of plane waves propagating normally to the layers [11]. The waves have the form : $E(y\,;t) = \mathrm{Re}\,[\phi\,(y_1)\,e^{i(\omega t - ky_2)}]$ where ϕ is periodic and $k = k' - ik''$ and (44) is the dispersion relation. For each $\omega > 0$, this equation does not define an unique k. If k_0 is solution of (44), then $k_p = k_0 + 2\,p\pi$, $p \in Z$ are also solutions of (44). We keep the determination of $k(\omega)$ which is a continuous and increasing function of ω, which is consistent with the homogeneous case when $\eta_a \to \eta_b$ and $\mu_a \to \mu_b$.

On figures (2) and (3), the dispersion curve, already obtained in [11], is compared with the homogenization theory. Again we obtain the forbidden frequencies. They correspond, in figure 1, to a passage from one mode of transmission to another. For these frequencies k' remains constant and k'' is not zero valued. The waves are sharply attenuated in space, so the medium becomes opaque for those frequencies, although it does not absorb energy. Periodic homogenization does not predict this phenomenon, but it is in very good agreement when $k\ell/2\pi < 0.3$, i.e. for low frequencies waves.

Equation (44) can also handle the case of dissipative layers ($\sigma(y_1) \neq 0$). In that case, $k(y_1)$ is complex valued and the waves are naturally attenuated, the energy beeing dissipated by Joule's effect.

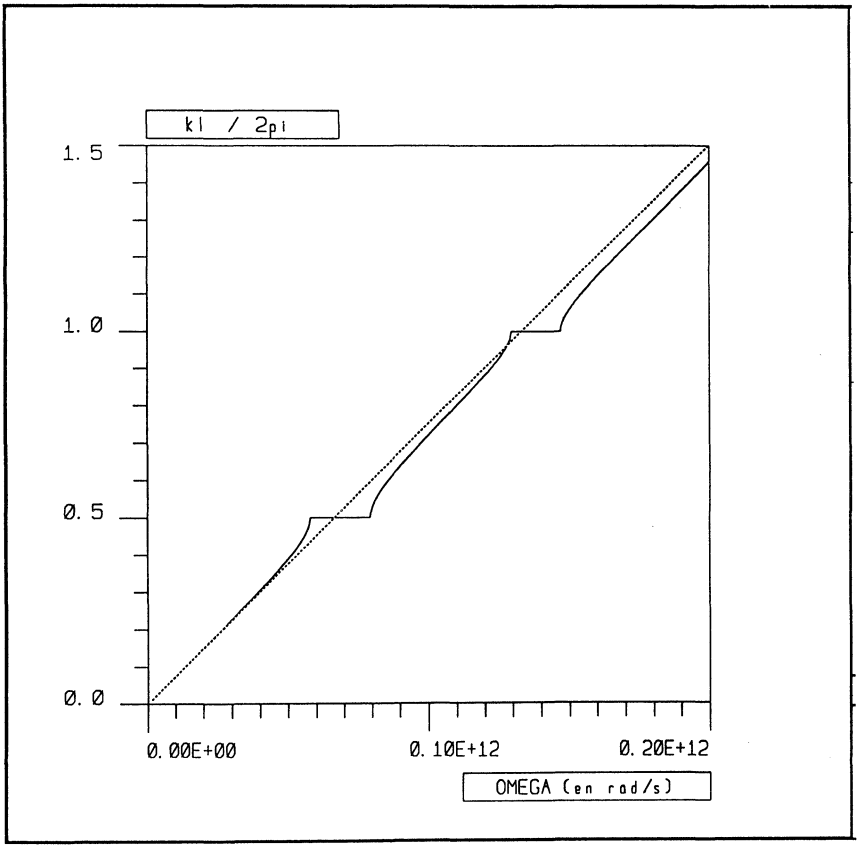

Figure 2.

Figures 2 and 3. — Dispersion relation with Floquet theory : — Floquet theory ; ... Homogenization theory.

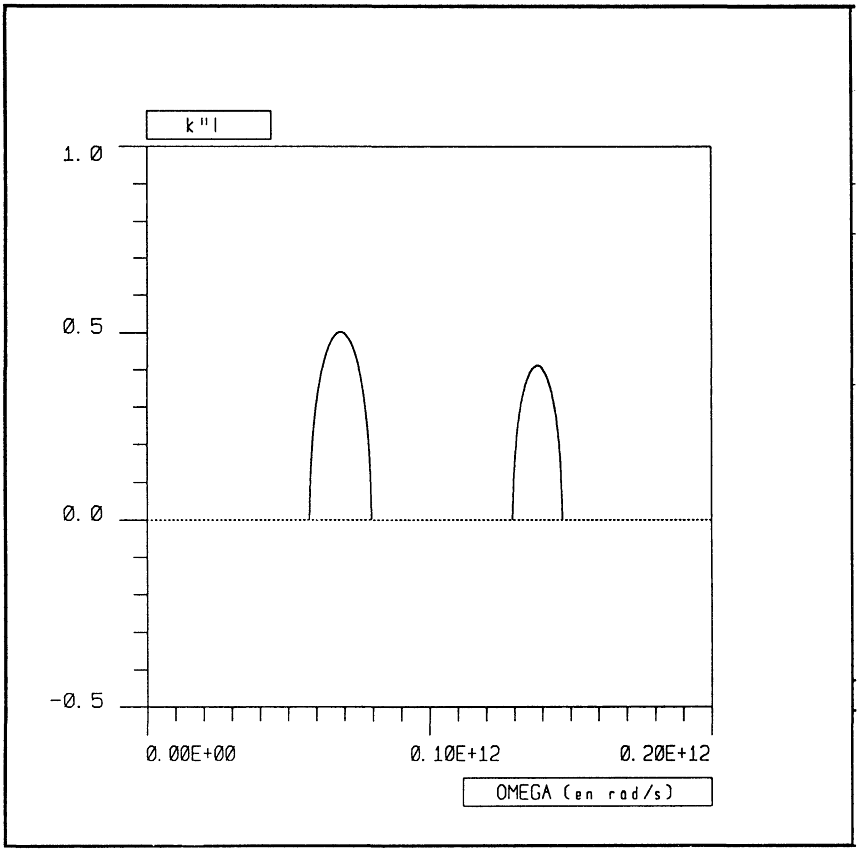

Figure 3.

Figure 4 shows the dispersion curves obtained with Floquet theory (— curves) compared with the ones obtained with periodic homogenization theory (... curves). We note a very good agreement when : $|k\ell| < 0.5$.

The method developed in this paper will be applied to the three dimensional case. It requires the use of $3\,D$ finite element [5] to solve Floquet problem.

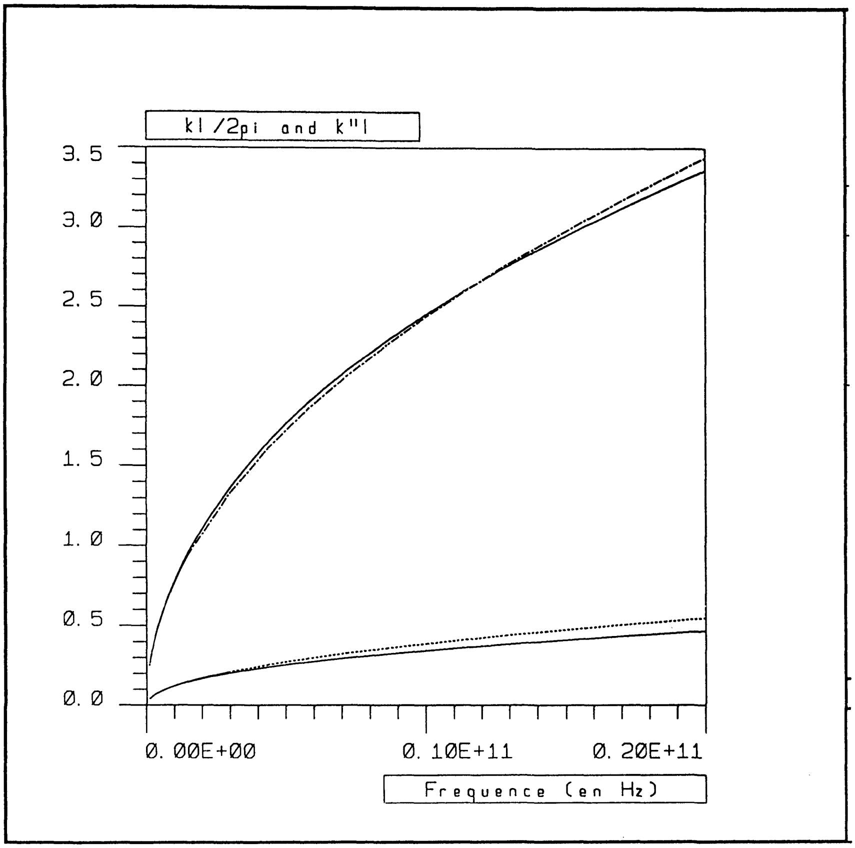

Figure 4.

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