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A BOUNDARY ELEMENT PROCEDURE FOR CONTACT PROBLEMS IN PLANE LINEAR ELASTOSTATICS (*)

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Abstract. — *Here we present a new solution procedure for contact problems in plane linear elastostatics* via *boundary integral variational inequalities having as unknowns the trace of the displacement field and its boundary traction. We admit the case of only traction-contact boundary conditions without prescribing the displacements along some part of the boundary of the elasticly deformed body. Without imposing any regularity assumption we establish norm convergence of piecewise polynomial boundary element approximations for mechanically definite problems. In detail we investigate piecewise quadratic and piecewise cubic approximations to the displacement field which lead to nonconform approximation schemes.*

Résumé. — *Nous présentons un nouveau procédé d'approximation pour résoudre des problèmes de contact en élastostatique linéaire plane. En utilisant la méthode intégrale, nous obtenons une inéquation variationnelle ayant comme inconnues la trace du champ des déplacements et le vecteur des contraintes aux limites. Notre travail couvre le cas des problèmes mêlés avec des conditions aux limites du type Neumann et du type Signorini sans imposer aux déplacements d'avoir lieu dans une partie de la frontière du corps élastiquement déformé. Sans avoir besoin d'une hypothèse de régularité, nous établissons la convergence en norme des approximations pour des éléments de frontière polynomiaux par morceaux. Nous étudions en détail les cas quadratique et cubique par morceaux ce qui conduit à des schémas d'approximation non conforme.*

1. INTRODUCTION

This paper presents a new boundary integral procedure and a convergence analysis for the resulting boundary element approximation for the solution of a class of unilateral problems in linear elastostatics, which were initiated by Signorini [30] over fifty years ago. In particular we address those contact

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problems, which are the most interesting ones from the view of applications and the most delicate ones from the view of mathematics where the linear-elastic body, which is supported by a rigid foundation, is only subjected to surface tractions, but is not fixed along some part of its boundary.

After describing the variational formulation of the contact problem considered we transform it into an equivalent boundary integral variational inequality. This solution procedure works for arbitrary Lipschitz domains ; in contrary to [28] we need no knowledge of Green's function. The resulting boundary bilinear form is shown to satisfy a Gårding inequality with a compact perturbation term. Therefore we can apply the recent abstract discretization theorem of [12] that extends the discretization theory of Glowinski [9] from elliptic bilinear forms to the more general semicoercive case. Based on this result we establish norm convergence of piecewise polynomial boundary element approximations for mechanically definite problems without imposing any regularity assumption. In detail we investigate piecewise quadratic and piecewise cubic approximations to the displacement field which lead to nonconform approximation schemes.

Thus we extend the recent convergence analysis given by Han [14] for the harmonic Dirichlet-Signorini boundary value problem in several respects. For further information concerning contact problems and related unilateral problems in mechanics, respectively the numerical solution of variational inequalities we also refer to Duvaut and Lions [4], Hlavaček, Haslinger, Nečas and Lovišek [15], Kikuchi and Oden [18], Panagiotopoulos [27], respectively to Glowinski, Lions and Trémolières [10].

2. THE VARIATIONAL FORMULATION OF THE CONTACT PROBLEM IN PLANE LINEAR ELASTOSTATICS

In this section we describe the variational formulation of the contact problem [30, 18, 27] within the range of plane linear elastostatics, assuming Hooke's law and small deformations of a homogeneous, isotropic body. So let $\Omega \subset \mathbb{R}^2$ be a bounded plane domain with the Lipschitz boundary Γ [23, 11], occupied by an elastic body, and let $\underline{x} = (x_1, x_2)$ be a Cartesian coordinate system. Then $\underline{n} = (n_1, n_2)$, the outward normal to Γ, exists almost everywhere and $\underline{n} \in \mathbb{L}^\infty(\Gamma) := [L^\infty(\Gamma)]^2$ (see [23, Lemma 2.4.2]). Likewise we write $\mathbb{H}^1(\Omega)$ for $[H^1(\Omega)]^2$, $\mathbb{H}^{1/2}(\Gamma)$ for $[H^{1/2}(\Gamma)]^2$, etc.

Now the displacement field $\underline{u} = (u_1, u_2) \in \mathbb{H}^1(\Omega)$ has to satisfy

$$\Delta^* \underline{u} := \mu\, \Delta \underline{u} + (\lambda + \mu)\, \text{grad div}\, \underline{u} = \underline{0} \quad \text{in} \quad \Omega\,, \tag{2.1}$$

where $\mu > 0$, $\lambda > -\mu$ are the given Lamé constants. Moreover we introduce the traction operator $\mathcal{T}$ on Γ by

$$\underline{\varphi} := \mathcal{T}(\underline{u}) := \lambda\, (\text{div}\, \underline{u})\, \underline{n} + 2\, \mu\, \frac{\partial \underline{u}}{\partial \underline{n}} + \mu\, \underline{n} \times \text{curl}\, \underline{u}\,,$$

where curl $\underline{u} := \text{curl}\,(u_1, u_2, 0)$. The traction $\underline{\varphi}$ is decomposed into the normal, respectively the tangential component

$$\varphi_n := \underline{\varphi} \cdot \underline{n}\,; \qquad \varphi_t := \underline{\varphi} \cdot \underline{t}\,,$$

using the unit tangential vector $\underline{t} = (t_1, t_2) = (-n_2, n_1)$. Similarly, we decompose the trace $\underline{u} \mid \Gamma$ of the displacement vector $\underline{u}$:

$$u_n := \underline{n} \,.\, \underline{u} \mid \Gamma\,; \qquad u_t := \underline{t} \,.\, \underline{u} \mid \Gamma\,.$$

To formulate the boundary conditions, let $\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_N \cup \bar{\Gamma}_S$, where the open parts Γ_D, Γ_N, and Γ_S are mutually disjoint. Zero displacements, respectively tractions $\underline{g} \in \mathbb{H}^{-1/2}(\Gamma_N)$ are prescribed on Γ_D, resp. Γ_N, i.e.

$$\underline{u} = \underline{0} \quad \text{on} \quad \Gamma_D\,, \tag{2.2}$$

$$\mathcal{T}(\underline{u}) = \underline{g} \quad \text{on} \quad \Gamma_N\,. \tag{2.3}$$

On the remaining part Γ_S, the Signorini's conditions are imposed, i.e.

$$u_n \leq 0\,, \quad \varphi_n \leq h_n\,, \quad u_n(\varphi_n - h_n) = 0\,, \quad \varphi_t = h_t\,, \tag{2.4}$$

where $\underline{h} \in \mathbb{H}^{-1/2}(\Gamma_S)$ is given and $h_n := \underline{h} \,.\, \underline{n}$, $h_t := \underline{h} \,.\, \underline{t}$. To make the contact problem meaningful we assume meas $(\Gamma_S) > 0$, but we do not require meas $(\Gamma_D) > 0$. Note there is no loss of generality to assume zero displacements and zero body forces. Indeed, more general conditions can be reduced to the form given above by a superposition argument that uses the solution of the linear boundary value problem

$$\Delta^* \underline{u} = \underline{f} \quad \text{in} \quad \Omega$$

$$\underline{u} = \underline{u}_D^0 \quad \text{on} \quad \Gamma_D\,,\; \mathcal{T}(\underline{u}) = \underline{0} \quad \text{on} \quad \Gamma_N\,,\; \underline{u} = \underline{u}_S^0 \quad \text{on} \quad \Gamma_S$$

and an appropriately redefined right hand side $\underline{h}$ in (2.4). To give the variational formulation of the boundary value problem (2.1)-(2.4) we introduce the bilinear form (see [20])

$$\beta(\underline{v}, \underline{w}) := (\lambda + \mu) \int_\Omega \operatorname{div} \underline{v} \operatorname{div} \underline{w}\, dx + $$

$$+ \frac{\mu}{2} \sum_{j \neq k = 1}^{2} \int_\Omega \left(\frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial x_j} \right) \left(\frac{\partial w_j}{\partial x_k} + \frac{\partial w_k}{\partial x_j} \right) dx$$

$$+ \frac{\mu}{2} \sum_{j, k = 1}^{2} \int_\Omega \left(\frac{\partial v_j}{\partial x_k} - \frac{\partial v_k}{\partial x_j} \right) \left(\frac{\partial w_j}{\partial x_k} - \frac{\partial w_k}{\partial x_j} \right) dx$$

representing the strain energy [18, 27] and the linear form

$$\ell(\underline{v}) := \int_{\Gamma_N} \underline{g} \cdot \underline{v}\, ds + \int_{\Gamma_S} \underline{h} \cdot \underline{v}\, ds$$

representing the work done by the exterior forces [18, 27] on the function space

$$\overset{*}{\mathbb{H}}{}^1(\Omega) := \{\underline{v} \in \mathbb{H}^1(\Omega) : \underline{v} = \underline{0} \text{ on } \Gamma_D\}$$

and the convex closed cone

$$\mathcal{K} := \{v \in \overset{*}{\mathbb{H}}{}^1(\Omega) : \underline{v} \cdot \underline{n} \leq 0 \text{ on } \Gamma_S\} .$$

Then the variational formulation of (2.1)-(2.4) reads :

Find $\underline{u} \in \mathcal{K}$ such that for all $\underline{v} \in \mathcal{K}$

$$\beta(\underline{u}, \underline{v} - \underline{u}) \geq \ell(\underline{v} - \underline{u}) . \tag{2.5}$$

This variational inequality (2.5) on the domain Ω can be obtained from the boundary value problem (2.1)-(2.4) in a standard fashion using Green's formula (see e.g. [4], [18]).

Since we do not require that meas $(\Gamma_D) > 0$, Korn's inequality (see [19, 26] for recent proofs in nonsmooth domains) admits rigid body motions which are given (see e.g. [18, Lemma 6.1]) by the subspace

$$\mathcal{R} := \{\underline{r} \in \mathbb{H}^1(\Omega) : \underline{r} = (\omega_1 + \omega_3 x_2, \omega_2 - \omega_3 x_1) ; \omega_1, \omega_2, \omega_3 \in \mathbb{R}\} .$$

Due to Fichera ([8], see also [18, Theorem 6.1]), existence and also uniqueness of the variational solution to (2.1)-(2.4) or to (2.5) are guaranteed, if for all $\underline{r} \in \mathcal{R} \cap \mathcal{K} \setminus \{\underline{0}\}$

$$\ell(\underline{r}) = \int_{\Gamma_N} \underline{g} \cdot \underline{r}\, ds + \int_{\Gamma_S} \underline{h} \cdot \underline{r}\, ds < 0 \tag{2.6}$$

holds. This condition means that the applied forces $\underline{g}$ and $\underline{h}$ should form an obtuse angle with the « directions of escape » of the body, i.e. the rigid body motions away from the obstacle.

3. AN EQUIVALENT BOUNDARY VARIATIONAL INEQUALITY

A fundamental solution of (2.1) is given by the matrix

$$\mathcal{F}(\underline{x}, \underline{y}) := \frac{\lambda + 3\mu}{4\pi\mu(\lambda + 2\mu)} \left\{ \log \frac{1}{|\underline{x} - \underline{y}|} I + \frac{\lambda + \mu}{\lambda + 3\mu} \frac{(\underline{x} - \underline{y})(\underline{x} - \underline{y})^T}{|\underline{x} - \underline{y}|^2} \right\} .$$

Here I is the identity matrix and T denotes the transposed tensor. Correspondingly, the boundary-stress matrix is given by

$$\mathcal{F}_1(\underline{x}, \underline{y}) := (\mathcal{T}_y \mathcal{F}(\underline{x}, \underline{y}))^T$$
$$= \frac{\mu}{2\pi(\lambda + 2\mu)} \left\{ \left[I + \frac{2(\lambda + \mu)}{\mu |\underline{x} - \underline{y}|^2} (\underline{x} - \underline{y})(\underline{x} - \underline{y})^T \right] \frac{\partial}{\partial n_y} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial s_y} \right\} \log |\underline{x} - \underline{y}| .$$

Now let $\underline{u}$ be the variational solution to (2.1)-(2.4) or to (2.5). Then $\underline{u}$ satisfies the differential equation (2.1) in the distributional sense in Ω, and with $\underline{\varphi} = \mathcal{T}(\underline{u})$, we obtain by the Betti representation formula

$$\underline{u}(\underline{x}) = \int_\Gamma \{\mathcal{F}(\underline{x}, \underline{y})\, \underline{\varphi}(\underline{y}) - \mathcal{F}_1(\underline{x}, \underline{y})\, \underline{u}(\underline{y})\}\, ds_y , \quad \underline{x} \in \Omega . \quad (3.1)$$

Here ds stands for the arc length element and y denotes the point of integration. Using the jump relations for elastic potentials [20, 21, 22] one derives the Somigliana identity

$$\frac{1}{2} \underline{u}(\underline{x}) = \int_{\Gamma \setminus \{x\}} \{\mathcal{F}(\underline{x}, \underline{y})\, \underline{\varphi}(\underline{y}) - \mathcal{F}_1(\underline{x}, \underline{y})\, \underline{u}(\underline{y})\}\, ds_y , \quad \underline{x} \in \Gamma , \quad (3.2)$$

and by application of the traction operator $\mathcal{T}_x$

$$\frac{1}{2} \underline{\varphi}(\underline{x}) = \int_{\Gamma \setminus \{x\}} (\mathcal{T}_x \mathcal{F}(\underline{x}, \underline{y}))\, \underline{\varphi}(\underline{y})\, ds_y - \mathcal{T}_x \int_{\Gamma \setminus \{x\}} (\mathcal{T}_y \mathcal{F}(\underline{x}, \underline{y}))^T\, \underline{u}(\underline{y})\, ds_y , \quad \underline{x} \in \Gamma . \quad (3.3)$$

As shown in the Appendix of [32] the latter integral defines the singular integral operator

$$\mathcal{D}_\Gamma(\underline{u})(\underline{x}) := -\mathcal{T}_x \int_{\Gamma \setminus \{x\}} (\mathcal{T}_y \mathcal{F}(\underline{x}, \underline{y}))^T\, \underline{u}(\underline{y})\, ds_y , \quad \underline{x} \in \Gamma$$

on the boundary $\Gamma : \underline{y} = \underline{Z}(s)$, $\underline{x} = \underline{Z}(t)(0 < s, t < L)$ having the special form

$$\mathcal{D}_\Gamma(\underline{u})(\underline{Z}(t)) = \frac{-\mu}{(1-\nu)\pi} I \text{ finite part} \int_{s=0}^{L} \frac{\underline{u}(\underline{Z}(s))}{(s-t)^2}\, ds + + \text{p.v.} \int_{s=0}^{L} \frac{D_1(t, s)}{s-t}\, \underline{u}(\underline{Z}(s))\, ds + \int_{s=0}^{L} D_2(t, s)\, \underline{u}(\underline{Z}(s))\, ds$$

with the Poisson constant

$$\nu = \frac{\lambda}{2(\lambda + \mu)} .$$

Here the first integral on the right hand side is understood as a Hadamard finite part integral (for an introduction to the Hadamard limit, see e.g. [29, § 39, 40]), the second as a Cauchy principal-value integral, whereas the kernel functions D_1 and D_2 are smooth matrix valued functions. To get rid of the non integrable kernel we use partial integration as advocated by Nedelec [25]. As shown in the Appendix of the present paper,

$$- \mathcal{D}_\Gamma(\underline{u})(\underline{x}) = \frac{d}{ds_x} \int_\Gamma \mathcal{F}_0(\underline{x}, \underline{y}) \frac{d\underline{u}(\underline{y})}{ds_y} ds_y , \quad \underline{x} \in \Gamma \tag{3.4}$$

holds with

$$\mathcal{F}_0(\underline{x}, \underline{y}) := \frac{\mu(\lambda + \mu)}{\pi(\lambda + 2\mu)} \left\{ \log \frac{1}{|\underline{x} - \underline{y}|} I + \frac{(\underline{x} - \underline{y})(\underline{x} - \underline{y})^T}{|\underline{x} - \underline{y}|^2} \right\} .$$

Now multiplying (3.2) by a function $\underline{\psi} \in \mathbb{H}^{-1/2}(\Gamma)$ leads to

$$a(\underline{\psi}, \underline{\varphi}) - b(\underline{\psi}, \underline{u}) = 0 , \qquad \forall \underline{\psi} \in \mathbb{H}^{-1/2}(\Gamma) , \tag{3.5}$$

where we introduce

$$a(\underline{\psi}, \underline{\varphi}) := \int_\Gamma \int_\Gamma \mathcal{F}(\underline{x}, \underline{y}) \underline{\varphi}(y) \underline{\psi}(\underline{x}) ds_y \, ds_x ,$$

$$b(\underline{\psi}, \underline{u}) := \frac{1}{2} \int_\Gamma \underline{u}(\underline{x}) \underline{\psi}(\underline{x}) ds_x + \int_\Gamma \int_\Gamma \mathcal{F}_1(\underline{x}, \underline{y}) \underline{u}(\underline{y}) \underline{\psi}(\underline{x}) ds_y \, ds_x ,$$

$$a_0(\underline{\psi}, \underline{\varphi}) := \int_\Gamma \int_\Gamma \mathcal{F}_0(\underline{x}, \underline{y}) \underline{\varphi}(\underline{y}) \underline{\psi}(\underline{x}) ds_y \, ds_x .$$

By Green's formula we obtain with $\Delta^* \underline{u} = 0$, $\underline{\varphi} = \mathcal{T}\underline{u}$

$$\beta(\underline{u}, \underline{v}) = \int_\Gamma \underline{\varphi} \cdot \underline{v} \, ds , \quad \forall \underline{v} \in \mathbb{H}^1(\Omega) . \tag{3.6}$$

Inserting (3.3) into (3.6), using (3.4), and integration by parts results in

$$\begin{aligned} \beta(\underline{u}, \underline{v}) = & \int_\Gamma \mathcal{F}_0(x, \underline{y}) \frac{d\underline{u}(y)}{ds_y} \frac{d\underline{v}(\underline{x})}{ds_x} ds_y \, ds_x \\ & + \int_\Gamma \int_\Gamma (\mathcal{T}_x \mathcal{F}(\underline{x}, \underline{y})) \underline{\varphi}(\underline{y}) \underline{v}(\underline{x}) ds_y \, ds_x + \frac{1}{2} \int_\Gamma \underline{\varphi} \cdot \underline{v} \, ds \\ = & \, a_0\left(\frac{d\underline{u}}{ds}, \frac{d\underline{v}}{ds} \right) + b(\underline{\varphi}, \underline{v}) . \end{aligned}$$

Thus with the convex cone

$$K := \{\underline{v} \in \mathbb{H}^{1/2}(\Gamma) : \underline{v} = 0 \text{ on } \Gamma_D, \underline{v} \,.\, \underline{n} \leqslant 0 \text{ on } \Gamma_S\} ,$$

(2.5) leads to the following variational problem : find $[\underline{u}, \underline{\varphi}] \in K \times \mathbb{H}^{-1/2}(\Gamma)$ such that

$$(\pi)\begin{cases} a_0\left(\dfrac{d\underline{u}}{ds}, \dfrac{d\underline{v}}{ds} - \dfrac{d\underline{u}}{ds}\right) + b(\underline{\varphi}, \underline{v} - \underline{u}) \geqslant \ell(\underline{v} - \underline{u}), & \forall \underline{v} \in K ; \\ a(\underline{\psi}, \underline{\varphi}) - b(\underline{\psi}, \underline{u}) = 0 , & \forall \underline{\psi} \in \mathbb{H}^{-1/2}(\Gamma) ; \end{cases}$$

or equivalently

$$B([\underline{u}, \underline{\varphi}], [\underline{v}, \underline{\psi}] - [\underline{u}, \underline{\varphi}]) \geqslant \ell(\underline{v} - \underline{u}), \quad \forall [\underline{v}, \underline{\psi}] \in K \times \mathbb{H}^{-1/2}(\Gamma) \quad (3.7)$$

with the bilinear form

$$B([\underline{u}, \underline{\varphi}], [\underline{v}, \underline{\psi}]) := a_0\left(\frac{d\underline{u}}{ds}, \frac{d\underline{v}}{ds}\right) + a(\underline{\psi}, \underline{\varphi}) + b(\underline{\varphi}, \underline{v}) - b(\underline{\psi}, \underline{u}) .$$

Indeed, since the variational equality in (π) is equivalent to the variational inequality

$$a(\underline{\psi} - \underline{\varphi}, \underline{\varphi}) - b(\underline{\psi} - \underline{\varphi}, \underline{u}) \geqslant 0$$

on the space $\mathbb{H}^{-1/2}(\Gamma)$, the implication $(\pi) \Rightarrow (3.7)$ is immediate. On the other hand, (π) follows from (3.7) by the choices $\underline{\psi} = \underline{0}$, $\underline{v} = \underline{u}$.

Moreover the boundary variational inequality (3.7) is equivalent to the domain variational inequality (2.5). Indeed, let $[\underline{u}, \underline{\varphi}] \in K \times \mathbb{H}^{-1/2}(\Gamma)$ be a solution of (3.7). The choice $\underline{v} = \underline{u}$ in (3.7) leads to (3.5). Therefore the representation formula (3.1) along with (3.2) gives a function $\underline{u} \in \mathbb{H}^1(\Omega)$ such that $\underline{\varphi} = \mathcal{T}\underline{u}$ and $\underline{u} \in \mathcal{K}$. Furthermore the choice $\underline{\psi} = \underline{0}$ in (3.7) shows that

$$\beta(\underline{u}, \underline{v} - \underline{u}) = a_0\left(\frac{d\underline{u}}{ds}, \frac{d\underline{v}}{ds} - \frac{d\underline{u}}{ds}\right) + b(\underline{\varphi}, \underline{v} - \underline{u}) \geqslant \ell(\underline{v} - \underline{u}), \quad \forall \underline{v} \in \mathcal{K} ,$$

thus proving our claim.

Now our aim is to establish a Gårding inequality for the bilinear form $B(.,.)$ in the space

$$V := \mathbb{H}^{1/2}(\Gamma) \times \mathbb{H}^{-1/2}(\Gamma) ,$$

i.e. positive definiteness up to a compact perturbation term. The boundary integral operators that give rise to the bilinear form $B(.,.)$ can be understood as pseudodifferential operators [7]. Since coordinate transformations do not affect their principal symbol [7], thus contribute only to compact perturbation terms (see e.g. [17] for more detailed arguments of this kind) we need only consider the case of a smooth domain in the subsequent reasoning.

LEMMA 3.1 : *There exist a constant* $c > 0$ *and a compact operator* $C : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ *such that*

$$\left\| \frac{dv}{ds} \right\|^2_{-1/2, \Gamma} \geqslant c \, \| v \|^2_{1/2, \Gamma} - \langle Cv, v \rangle_{H^{-1/2} \times H^{1/2}}, \quad \forall v \in H^{1/2}(\Gamma). \tag{3.8}$$

Proof : Let $\theta = 2\pi s/L$, where L is the boundary length, and we can assume without loss of generality that Γ is the unit circle. Then we can argue similar to [14] with the only difference that due to the existent rigid body motions an extra term enters. More detailed using the Fourier expansion for a smooth function v — what by density suffices to consider

$$v = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta),$$

$$\frac{dv}{d\theta} = \sum_{n=1}^{\infty} (nb_n \cos n\theta - na_n \sin n\theta),$$

one finds

$$\| v \|^2_{1/2, \Gamma} = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (1 + n^2)^{1/2} (a_n^2 + b_n^2)$$

$$\left\| \frac{dv}{d\theta} \right\|^2_{-1/2, \Gamma} = \sum_{n=1}^{\infty} (1 + n^2)^{-1/2} n^2 (a_n^2 + b_n^2),$$

$$\geqslant \frac{1}{2} \sum_{n=1}^{\infty} (1 + n^2)^{1/2} (a_n^2 + b_n^2),$$

$$a_0^2 = \left[\frac{1}{2\pi} \int_0^{2\pi} v(\theta) \, d\theta \right]^2 \leqslant c \, \| v \|^2_{0, \Gamma} \quad (c > 0).$$

Hence

$$\left\| \frac{dv}{d\theta} \right\|^2_{-1/2, \Gamma} \geqslant \frac{1}{2} \| v \|^2_{1/2, \Gamma} - \frac{c}{4} \| v \|^2_{0, \Gamma}. \tag{3.9}$$

Since

$$H^{1/2}(\Gamma) \subset H^0(\Gamma) \equiv L^2(\Gamma) \subset H^{-1/2}(\Gamma)$$

forms a Gelfand triple with *compact* and dense embeddings, the last term in (3.9) can be replaced by $\langle Cv, v \rangle$ with $C : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ compact concluding the proof. q.e.d.

LEMMA 3.2 : *The bilinear form $B(\,.\,,\,.\,)$ is bounded in $V \times V$; moreover satisfies a Gårding inequality, i.e. there exist a positive constant c_0 and a compact operator $C : V \to V^*$ such that*

$$B([\underline{v}, \underline{\psi}], [\underline{v}, \underline{\psi}]) + \langle C[\underline{v}, \underline{\psi}], [\underline{v}, \underline{\psi}]\rangle_{V^* \times V} \geq$$
$$\geq c_0 \| [\underline{v}, \underline{\psi}] \|_V^2 := c_0 \left\{ \|\underline{v}\|^2_{\mathbb{H}^{1/2}(\Gamma)} + \|\underline{\psi}\|^2_{\mathbb{H}^{-1/2}(\Gamma)} \right\},$$
$$\forall [\underline{v}, \underline{\psi}] \in V := \mathbb{H}^{1/2}(\Gamma) \times \mathbb{H}^{-1/2}(\Gamma). \tag{3.10}$$

Proof : We have

$$B([\underline{v}, \underline{\psi}], [\underline{v}, \underline{\psi}]) = a_0\left(\frac{d\underline{v}}{ds}, \frac{d\underline{v}}{ds} \right) + a(\underline{\psi}, \underline{\psi}).$$

Note that apart from a different positive constant factor, the matrices $\mathcal{F}$ and $\mathcal{F}_0$ differ in terms on the side diagonal that give rise to a compact perturbation only. By [2, Theorem 1]

$$|a(\underline{\psi}, \underline{\psi})| \leq \text{Const.}\ \|\underline{\psi}\|^2_{\mathbb{H}^{-1/2}(\Gamma)}.$$

Since for any $\underline{v} \in \mathbb{H}^{1/2}(\Gamma)$, $\frac{d\underline{v}}{ds} = \sum_i \frac{\partial \underline{v}}{\partial x_i} \dot{x}_i \in \mathbb{H}^{-1/2}(\Gamma)$, it follows likewise for a_0 that

$$\left| a_0\left(\frac{d\underline{v}}{ds}, \frac{d\underline{v}}{ds} \right) \right| \leq \text{Const.}\ \|\underline{v}\|^2_{\mathbb{H}^{1/2}(\Gamma)}.$$

Therefore it remains to prove (3.10). By [2, Theorem 2] the bilinear form $a(\,.\,,\,.\,)$ satisfies a Gårding inequality on $[\mathbb{H}^{-1/2}(\Gamma)]^2$ in the general case of a Lipschitz domain, i.e.

$$a(\underline{\psi}, \underline{\psi}) \geq c_a \|\underline{\psi}\|^2_{\mathbb{H}^{-1/2}(\Gamma)} - \langle C_A \underline{\psi}, \underline{\psi} \rangle_{\mathbb{H}^{1/2}(\Gamma) \times \mathbb{H}^{-1/2}(\Gamma)},$$
$$\forall \underline{\psi} \in \mathbb{H}^{-1/2}(\Gamma), \tag{3.11}$$

where $c_a > 0$, $C_A : \mathbb{H}^{-1/2}(\Gamma) \to \mathbb{H}^{1/2}(\Gamma)$ is compact. Hence with appropriate $c_0 > 0$, $C_0 : \mathbb{H}^{-1/2}(\Gamma) \to \mathbb{H}^{1/2}(\Gamma)$ being compact we have

$$a_0\left(\frac{d\underline{v}}{ds}, \frac{d\underline{v}}{ds} \right) \geq c_0 \left\| \frac{d\underline{v}}{ds} \right\|^2_{\mathbb{H}^{-1/2}(\Gamma)} - \left\langle C_0 \frac{d\underline{v}}{ds}, \frac{d\underline{v}}{ds} \right\rangle, \quad \forall \underline{v} \in \mathbb{H}^{1/2}(\Gamma). \tag{3.12}$$

Combining (3.11) and (3.12) with Lemma 3.1 yields (3.10). q.e.d.

Finally we provide sufficient conditions for the density relation

$$\overline{K \cap [C^\infty(\Gamma)]^2} = K. \tag{3.13}$$

This relation is essential for our discretization analysis to come.

LEMMA 3.3 : *Suppose, for the polygonal domain Ω there are only a finite number of « end points » $\bar{\Gamma}_S \cap \bar{\Gamma}_N$, $\bar{\Gamma}_N \cap \bar{\Gamma}_D$, $\bar{\Gamma}_D \cap \bar{\Gamma}_S$. Then there holds* (3.13).

Proof : Since the embedding $H^{1/2}(\Gamma) \subset L^1(\Gamma)$ is continuous and L^1-convergence implies pointwise convergence almost everywhere for a subsequence, K is closed. Therefore it remains to show

$$K \subset \overline{K \cap [C^\infty(\Gamma)]^2}\,.$$

To this end use the continuity and surjectivity of the trace operator γ: $H^1(\Omega) \to H^{1/2}(\Gamma)$ and apply the analogous inclusion

$$\mathcal{K} \subset \overline{\mathcal{K} \cap [C^\infty(\Omega)]^2}\,,$$

which by [16, Theorem 3.2] holds true under the finite number assumption.

q.e.d.

4. DISCRETIZATION OF THE BOUNDARY VARIATIONAL INEQUALITY

In the following we suppose that Ω is polygonal, but not necessarily convex. Let Γ be represented by

$$x_i = Z_i(s)\,, \qquad 0 \leq s \leq L \quad (i = 1, 2)$$

with $Z_i(0) = Z_i(L)(i = 1, 2)$. Using a finite index set J we partition Γ into finitely many segments by the points $P_j = (Z_1(s_j), Z_2(s_j))$, $j = 1, \ldots, |J|$, where the endpoints of $\bar{\Gamma}_D$ and $\bar{\Gamma}_S$ are included and where $s_1 = 0$, $s_{|J|+1} = L$. The partitioning of Γ is characterized by the mesh size

$$h := \max_{j \in J} |s_{j+1} - s_j|\,.$$

Note that the boundary variational inequality (3.7) splits into a variational equality in $\mathbb{H}^{-1/2}(\Gamma)$, which can be discretized in a standard way [24, 31], and a novel variational inequality in the convex closed cone $K \subset \mathbb{H}^{1/2}(\Gamma)$. As an important issue of this paper we want to treat not only piecewise linear, but also piecewise quadratic and piecewise cubic approximations of K. To this end we introduce the space $\mathcal{P}^\kappa$ of polynomials of degree less than or equal to κ ($\kappa = 1, 2, 3$) and the subsequent finite point sets :

$\Sigma_1^h := \{s_j : j \in J\}$,

$\Sigma_2^h := \{s \in (0, L) : s$ is a midpoint of an interval (s_j, s_{j+1}) for some $j \in J\}$,

$\Sigma_3^h := \{s \in (0, L) : s$ divides an interval (s_j, s_{j+1}) by the ratio $1 : 2$ for some $j \in J\}$.

Moreover we let

$$\Pi_1^h := \{P_j : j \in J\} \cap \bar{\Gamma}_S ,$$

where with some appropriate $J_S \subset J$

$$\Pi_1^h = \{P_j : j \in J_S\} ,$$

and for $\kappa = 2, 3$

$$\Pi_\kappa^h := \{P = (Z_1(s), Z_2(s)) : s \in \Sigma_1^h \cup \Sigma_\kappa^h\} \cap \bar{\Gamma}_S .$$

Then the trace space of $\overset{*}{\mathbb{H}}^1(\Omega)$, denoted by $\overset{*}{\mathbb{H}}^{1/2}(\Gamma)$ may be approximated by the finite dimensional subspace

$$U_{\kappa,\mu}^h := \{\underline{v}^h \in [C^\mu(\Gamma)]^2 : v_i^h \circ Z_i | (s_j, s_{j+1}) \in \mathcal{P}^\kappa$$
$$(\forall j \in J ; i = 1, 2) ; \underline{v}^h | \Gamma_D = 0\} ,$$

K by the convex closed cone

$$K_{\kappa,\mu}^h := \{\underline{v}^h \in U_{\kappa,\mu}^h : \underline{v}^h . \underline{n}(P) \leq 0 \ (\forall P \in \Pi_\kappa^h)\} ,$$

and $\mathbb{H}^{-1/2}(\Gamma)$ by the finite dimensional subspace

$$\Phi_{\kappa-1,\mu-1}^h := \{\underline{\psi}^h \in [C^{\mu-1}(\Gamma)]^2 : \psi_i^h \circ Z_i | (s_j, s_{j+1}) \in \mathcal{P}^{\kappa-1}$$
$$(\forall j \in J ; i = 1, 2)\} .$$

Here $\mu \in \mathbb{N}_0$ is fixed with $\mu \leq \kappa - 1$ and $C^{-1}(\Gamma)$ denotes the space of piecewise constant functions. Note that $K_{1,0}^h \subset K$ holds for all $h > 0$.

Thus we are led to the following discretized variational problem : find $[\underline{u}^h, \underline{\varphi}^h] \in K_{\kappa,\mu}^h \times \Phi_{\kappa-1,\mu-1}^h$

$$(\pi_\kappa^h)\begin{cases} a_0\left(\dfrac{d\underline{u}^h}{ds}, \dfrac{d\underline{v}^h}{ds} - \dfrac{d\underline{u}^h}{ds}\right) + b(\underline{\varphi}^h, \underline{v}^h - \underline{u}^h) \geq \ell(\underline{v}^h - \underline{u}^h), \ \forall \underline{v}^h \in K_{\kappa,\mu}^h \\ a(\underline{\psi}^h, \underline{\varphi}^h) = b(\underline{\psi}^h, \underline{u}^h), \ \forall \underline{\psi}^h \in \Phi_{\kappa-1,\mu-1}^h ; \end{cases}$$

or equivalently

$$B([\underline{u}^h, \underline{\varphi}^h], [\underline{v}^h, \underline{\psi}^h] - [\underline{u}^h, \underline{\varphi}^h]) \geq \ell(\underline{v}^h - \underline{u}^h),$$
$$\forall [\underline{v}^h, \underline{\psi}^h] \in K_{\kappa,\mu}^h \times \Phi_{\kappa-1,\mu-1}^h . \quad (4.1)$$

Let us remark that the condition (2.6) guarantees the existence and uniqueness of not only the solution $[\underline{u}, \underline{\varphi}]$ of the problem (π), but also of the solution $[\underline{u}^h, \underline{\varphi}^h]$ of the approximate problems (π_κ^h) because our discretization affects neither the linear form ℓ nor the subspace $\mathcal{R}$ of rigid body motions. Now we can present our main result.

THEOREM 4.1 : *Let solutions* $[\underline{u}, \underline{\varphi}]$ *to* (π) *and* $[\underline{u}_\kappa^h, \underline{\varphi}_\kappa^h]$ *to* (π_κ^h) $(h > 0)$ *exist uniquely. Suppose, for the polygonal domain* Ω, *there are only finitely many end points* $\bar{\Gamma}_S \cap \bar{\Gamma}_D$, $\bar{\Gamma}_D \cap \bar{\Gamma}_N$, $\bar{\Gamma}_N \cap \bar{\Gamma}_S$. *Then for* $\kappa = 1, 2, 3$

$$\lim_{h \to 0} \left\| [\underline{u}_\kappa^h, \underline{\varphi}_\kappa^h] - [\underline{u}, \underline{\varphi}] \right\|_{\mathbb{H}^{1/2}(\Gamma) \times \mathbb{H}^{-1/2}(\Gamma)} = 0 .$$

Proof : In virtue of Lemma 3.2, the bilinear form $B(\,.\,,\,.\,)$ satisfies the Gårding inequality (3.10). Therefore Theorem 4.1 of [12] applies and requires the following hypotheses, due to Glowinski [9, Chapter 1] :

$(H1)$ If $\{\underline{v}^h\}_{h>0}$ weakly converges to $\underline{v}$, where $\underline{v}^h \in K^h := K_{\kappa,\mu}^h$, then $\underline{v} \in K$.

$(H2)$ There exist a subset $M \subset \mathbb{H}^{1/2}(\Gamma)$ such that $\bar{M} = K$ and mappings $\rho^h : M \to U^h := U_{\kappa,\mu}^h$ with the property that, for each $\underline{w} \in M$, $\rho^h \underline{w}$ strongly converges to $\underline{w}$ (as $h \to 0+$) and $\rho^h \underline{w} \in K^h$ for all $0 < h < h_0(\underline{w})$.

We note that the analogous hypotheses for the approximation of $\underline{\psi} \in \mathbb{H}^{-1/2}(\Gamma)$ by $\underline{\psi}^h \in \Phi^h$ are trivially satisfied in view of $\Phi^h \subset \mathbb{H}^{-1/2}(\Gamma)$ and well-known density and approximation properties.

Verification of $(H1)$. Since $K_{1,\mu}^h$ is contained in the weakly closed set K for all $h > 0$, we have only to consider the cases $\kappa = 2$ and $\kappa = 3$ with $\mu \in \mathbb{N}_0$ fixed such that $\mu \leqslant \kappa - 1$.

Let the polygonal boundary part $\bar{\Gamma}_S$ be partitioned by

$$\bar{\Gamma}_S = \bigcup_{j \in J_S} [P_j, P_{j+1}] ,$$

where the closed line segment $[P_j, P_{j+1}]$ has the intermediate point $P_{j+\frac{1}{2}} \in \Pi_2^h$, respectively the two intermediate points $P_{j+\frac{1}{3}}, P_{j+\frac{2}{3}} \in \Pi_3^h$. For any $\psi \in C^0(\bar{\Gamma}_S)$ with $\psi \geqslant 0$ we define

$$\psi^h = \sum_{j \in J_S} \psi(P_{j+\frac{1}{2}})\, \chi_{j+\frac{1}{2}} .$$

where $\chi_{j+\frac{1}{2}}$ denotes the characteristic function of the open segment $]P_j, P_{j+1}[$. Then $\psi^h \geqslant 0$ on Γ_S $(\kappa = 2, 3)$ and by the uniform continuity of ψ on $\bar{\Gamma}_S$

$$\lim_{h \to 0} \| \psi^h - \psi \|_{L^\infty(\Gamma_S)} = 0 . \tag{4.2}$$

Now let $\{\underline{v}^h\}_{j>0}$ be a family weakly convergent to $\underline{v} \in \mathbb{H}^1(\Gamma)$, where $\underline{v}^h \in K^h_{\kappa,\mu}$ ($\forall h > 0$; $\kappa = 2$ or $\kappa = 3$). Since the embedding $H^{1/2}(\Gamma) \subset L^1(\Gamma_S)$ is weakly continuous, the normal components v_n^h converge weakly to v_n in $L^1(\Gamma_S)$ and are norm bounded. Therefore by the estimate

$$\left| \int_{\Gamma_S} (v_n^h \psi^h - v_n \psi)\, ds \right| \leqslant$$

$$\leqslant \|v_n^h\|_{L^1(\Gamma_S)} \|\psi^h - \psi\|_{L^\infty(\Gamma_S)} + \left| \int_{\Gamma_S} (v_n^h - v_n)\, \psi\, ds \right| ,$$

using (4.2) and $\psi \in L^\infty(\Gamma_S) = (L^1(\Gamma_S))^*$, we obtain that

$$\lim_{h \to 0} \int_{\Gamma_S} v_n^h \psi^h\, ds = \int_{\Gamma_S} v_n\, \psi\, ds\,. \tag{4.3}$$

From Simpson's rule it follows for $\underline{v}^h \in K^h_{2,\mu}$ and all $\psi \in C^0(\Gamma)$ with $\psi \geqslant 0$ that

$$\begin{aligned} \int_{\Gamma_S} v_n^h \psi^h\, ds &= \\ &= \sum_{j \in J_S} \int_{s_j}^{s_{j+1}} \psi(P_{j+\frac{1}{2}}) \sum_{i=1}^{2} (v_i^h \circ Z_i)(s)\, n_i\, ds \\ &= \frac{1}{6} \sum_{j \in J_S} \psi(P_{j+\frac{1}{2}})(s_{j+1} - s_j)\left[v_n^h(P_j) + 4\, v_n^h(P_{j+\frac{1}{2}}) + v_n^h(P_{j+1}) \right] \\ &\leqslant 0\,, \end{aligned} \tag{4.4}$$

whereas from Newton's pulcherrima quadrature rule (see e.g. [13, § 7.1.5]) for $\underline{v}^h \in K^h_{3,\mu}$

$$\begin{aligned} \int_{\Gamma_S} v_n^h \psi^h\, ds &= \frac{1}{8} \sum_{j \in J_S} \psi(P_{j+\frac{1}{2}})(s_{j+1} - s_j)\left[v_n^h(P_j) + 3\, v_n^h(P_{j+\frac{1}{3}}) \right. \\ &\quad \left. + 3\, v_n^h(P_{j+\frac{2}{3}}) + v_n^h(P_{j+1}) \right] \\ &\leqslant 0\,. \end{aligned} \tag{4.5}$$

Combining (4.3) and (4.4), respectively (4.5) we obtain that for all $\psi \in C^0(\bar{\Gamma}_S)$ with $\psi \geqslant 0$

$$\int_{\Gamma_S} v_n\, \psi\, ds \leqslant 0\,,$$

hence $v_n \leqslant 0$ almost everywhere on Γ_S or $\underline{v} \in K$. This proves ($H1$).

Verification of (*H*2). In virtue of Lemma 3.3, we can take $M = K \cap [C^{\infty}(\Gamma)]^2$. Clearly for any $\underline{w} \in M$, $w_n = \underline{w} \cdot \underline{n} \in C^{\infty}]P_j, P_{j+1}[$ $(j \in J)$, in particular $w_n \geq 0$ pointwise on $]P_j, P_{j+1}[$ $(j \in J_S)$. By defining the normal vector at a possible corner point P_j as a certain convex combination (e.g. the average) of the one-sided limits

$$\lim_{\substack{P \to P_j \\ P \in]P_j, P_{j+1}[}} \underline{n}(P) \quad \text{resp.} \quad \lim_{\substack{P \to P_j \\ P \in]P_{j-1}, P_j[}} \underline{n}(P)$$

we can even ensure that $w_n \geq 0$ holds pointwise throughout Γ_S. Now we define $\rho_\kappa^h : \mathbb{H}^{1/2}(\Gamma) \cap [C^{\infty}(\Gamma)]^2 \to U^h_{\kappa, \kappa - 1} \subseteq U^h_{\kappa,\mu}$ by L-periodic spline interpolation subordinated to the partitioning of Γ. Thus in particular

$$\rho_\kappa^h \underline{w}(P) = \underline{w}(P), \quad \forall P \in \Pi_\kappa^h \quad (\kappa = 1, 2, 3).$$

Hence by the definition above of the normal at corner points, $\rho_\kappa^h \underline{w}$ belongs to $K^h_{\kappa, \kappa - 1} \subseteq K^h_{\kappa, \mu}$ for any $\underline{w} \in M$, since $\mu \leq \kappa - 1$. Moreover by spline interpolation theory (see e.g. [5, Lemma 4.1]), $U_{\kappa, \kappa - 1}$ is a regular family of finite elements in the sense of Babuška and Aziz [1] and therefore we have

$$\|\underline{w} - \rho_\kappa^h \underline{w}\|_{\mathbb{H}^{1/2}(\Gamma)} \leq \mathrm{c}h^{\kappa - 1/2} \|\underline{w}\|_{\mathbb{H}^{\kappa}(\Gamma)} \quad (\kappa = 1, 2, 3)$$

with $c > 0$ independent of h and $\underline{w}$. Hence we conclude that

$$\lim_{h \to 0} \|\underline{w} - \rho_\kappa^h \underline{w}\|_{\mathbb{H}^{1/2}(\Gamma)} = 0, \quad \forall \underline{w} \in M; \quad \kappa = 1, 2, 3.$$

q.e.d.

Remark : By the proof above (see in particular the estimates (4.4) and (4.5)) we have shown that boundary element convergence holds true for arbitrary piecewise polynomial approximations as long as the corresponding Newton-Cotes quadrature formula has positive weights. This is a reasonable restriction for practical computations and is satisfied for the Newton-Cotes formulae up to the order $\kappa = 8$ [6, § 6.2.1].

APPENDIX

The following computations are based on the well-known equivalence between plane elasticity and the biharmonic equation for the Airy stress function which in particular provides an explicit formula for the boundary tractions, see (4.79) in Chapter 11 of [3].

For the fundamental solution we have

$$(\mathcal{F}(\underline{x}, \underline{y}))_{jl} = \frac{\lambda + 3\mu}{4\pi\mu(\lambda + 2\mu)} f_{jl},$$

where the matrix (f_{jl}) is given by

$$\begin{pmatrix} \rho \dfrac{(y_1 - x_1)^2}{|\underline{y} - \underline{x}|^2} - \log|\underline{y} - \underline{x}| & \rho \dfrac{(y_1 - x_1)(y_2 - x_2)}{|\underline{y} - \underline{x}|^2} \\ \rho \dfrac{(y_1 - x_1)(y_2 - x_2)}{|\underline{y} - \underline{x}|^2} & \rho \dfrac{(y_2 - x_2)^2}{|\underline{y} - \underline{x}|^2} - \log|\underline{y} - \underline{x}| \end{pmatrix}$$

and the constant

$$\rho := \frac{\lambda + \mu}{\lambda + 3\mu}.$$

In the first step we represent the two column vectors of $\mathcal{F}$ as two displacement vectors with a possible rigid body motion part. Thus following [3], Chapter 11, (4.76) we use the ansatz

$$u_1 = \frac{1}{2\mu}[\rho^{-1}\gamma_1 - y_1\gamma_{1,1} - y_2\gamma_{2,1} - \chi_{,1}] + a + by_2,$$
$$u_2 = \frac{1}{2\mu}[\rho^{-1}\gamma_2 - y_1\gamma_{1,2} - y_2\gamma_{2,2} - \chi_{,2}] + d - by_1$$

for a displacement vector $(u_1, u_2)^T$ that involves the real part γ_1 and the imaginary part γ_2 of a holomorphic function and a harmonic function χ in the complex variable $y_1 + iy_2$, where the l-th partial derivative is denoted by the subscript $_{,l}$. This gives

$$u_{1,1} = \frac{1}{2\mu}[\rho^{-1}\gamma_{1,1} - \gamma_{1,1} - y_1\gamma_{1,11} - y_2\gamma_{2,11} - \chi_{,11}],$$
$$u_{1,2} = \frac{1}{2\mu}[\rho^{-1}\gamma_{1,2} - y_1\gamma_{1,12} - \gamma_{2,1} - y_2\gamma_{2,12} - \chi_{,12}] + b,$$
$$u_{2,1} = \frac{1}{2\mu}[\rho^{-1}\gamma_{2,1} - \gamma_{1,2} - y_1\gamma_{1,21} - y_2\gamma_{2,21} - \chi_{,21}] - b,$$
$$u_{2,2} = \frac{1}{2\mu}[\rho^{-1}\gamma_{2,2} - y_1\gamma_{1,22} - \gamma_{2,2} - y_2\gamma_{2,22} - \chi_{,22}].$$

In virtue of

$$\chi_{,12} = \chi_{,21}, \quad \Delta\chi = \Delta\gamma_1 = \Delta\gamma_2 = 0,$$
$$\gamma_{1,1} = \gamma_{2,2}, \quad \gamma_{1,2} = -\gamma_{2,1}$$

one derives

$$u_{1,1} + u_{2,2} = \frac{\rho^{-1} - 1}{\mu} \gamma_{1,1} = \frac{1-\rho}{\mu \rho} \gamma_{2,2}, \tag{A.1}$$

$$u_{1,2} - u_{2,1} = \frac{\rho^{-1} + 1}{\mu} \gamma_{1,2} + 2b = 2b - \frac{1+\rho}{\mu \rho} \gamma_{2,1}. \tag{A.2}$$

On the other hand, from the first column of (f_{jl})

$$f_{11,1} = -\frac{y_1 - x_1}{|\underline{y} - \underline{x}|^2} + 2\rho \frac{(y_1 - x_1)(y_2 - x_2)^2}{|\underline{y} - \underline{x}|^4},$$

$$f_{11,2} = -\frac{y_2 - x_2}{|\underline{y} - \underline{x}|^2} - 2\rho \frac{(y_2 - x_2)(y_1 - x_1)^2}{|\underline{y} - \underline{x}|^4},$$

$$f_{21,1} = \rho (y_2 - x_2) \frac{(y_2 - x_2)^2 - (y_1 - x_1)^2}{|\underline{y} - \underline{x}|^4},$$

$$f_{21,2} = \rho (y_1 - x_1) \frac{(y_1 - x_1)^2 - (y_2 - x_2)^2}{|\underline{y} - \underline{x}|^4},$$

what results in

$$f_{11,1} + f_{21,2} = -(1-\rho) \frac{y_1 - x_1}{|\underline{y} - \underline{x}|^2}, \tag{A.3}$$

$$f_{11,2} - f_{21,1} = -(1+\rho) \frac{y_2 - x_2}{|\underline{y} - \underline{x}|^2}. \tag{A.4}$$

Likewise from the second column of (f_{jl})

$$f_{12,1} = \rho (y_2 - x_2) \frac{(y_2 - x_2)^2 - (y_1 - x_1)^2}{|\underline{y} - \underline{x}|^4},$$

$$f_{12,2} = \rho (y_1 - x_1) \frac{(y_1 - x_1)^2 - (y_2 - x_2)^2}{|\underline{y} - \underline{x}|^4},$$

$$f_{22,1} = -\frac{y_1 - x_1}{|\underline{y} - \underline{x}|^2} - 2\rho \frac{(y_1 - x_1)(y_2 - x_2)^2}{|\underline{y} - \underline{x}|^4},$$

$$f_{22,2} = -\frac{y_2 - x_2}{|\underline{y} - \underline{x}|^2} + 2\rho \frac{(y_1 - x_1)^2 (y_2 - x_2)}{|\underline{y} - \underline{x}|^4},$$

what results in

$$f_{12,1} + f_{22,2} = (\rho - 1)\frac{y_2 - x_2}{|\underline{y} - \underline{x}|^2}, \tag{A.5}$$

$$f_{12,2} - f_{22,1} = (1 + \rho)\frac{y_1 - x_1}{|\underline{y} - \underline{x}|^2}. \tag{A.6}$$

Comparing (A.1), (A.2) to (A.3), (A.4) we obtain for the first column referred to by the superscript (1)

$$\gamma_{1,1}^{(1)} = -\mu\rho\frac{y_1 - x_1}{|\underline{y} - \underline{x}|^2},$$

$$\gamma_{1,2}^{(1)} = -\mu\rho\frac{y_2 - x_2}{|\underline{y} - \underline{x}|^2} + \frac{2\mu\rho}{1+\rho}b^{(1)},$$

$$\gamma_{2,1}^{(1)} = \mu\rho\frac{y_2 - x_2}{|\underline{y} - \underline{x}|^2} - \frac{2\mu\rho}{1+\rho}b^{(1)},$$

$$\gamma_{2,2}^{(1)} = -\mu\rho\frac{y_1 - x_1}{|\underline{y} - \underline{x}|^2},$$

what leads to

$$\gamma_1^{(1)} = -\mu\rho\log|\underline{y} - \underline{x}| + \frac{2\mu\rho}{1+\rho}b^{(1)}y_2, \tag{A.7}$$

$$\gamma_2^{(1)} = \mu\rho\arctan\frac{y_1 - x_1}{y_2 - x_2} - \frac{2\mu\rho}{1+\rho}b^{(1)}y_1, \tag{A.8}$$

moreover by the ansatz,

$$\begin{aligned}\chi_{,1}^{(1)} = {} & \mu\log|\underline{y} - \underline{x}| - \mu(\rho + 2a^{(1)}) \\ & - \frac{\mu\rho}{|\underline{y} - \underline{x}|^2}[x_2(y_2 - x_2) - x_1(y_1 - x_1)],\end{aligned} \tag{A.9}$$

$$\begin{aligned}\chi_{,2}^{(1)} = {} & \mu\arctan\frac{y_1 - x_1}{y_2 - x_2} - 2\mu d^{(1)} \\ & + \frac{\mu\rho}{|\underline{y} - \underline{x}|^2}[x_1(y_2 - x_2) + x_2(y_1 - x_1)].\end{aligned} \tag{A.10}$$

Similarly comparing (A.1), (A.2) to (A.5), (A.6) we obtain for the second column referred to by the superscript (2)

$$\gamma_{1,1}^{(2)} = -\mu\rho\,\frac{y_2 - x_2}{|\underline{y} - \underline{x}|^2},$$

$$\gamma_{1,2}^{(2)} = -\mu\rho\,\frac{y_1 - x_1}{|\underline{y} - \underline{x}|^2} + \frac{2\mu\rho}{1+\rho}\,b^{(2)},$$

$$\gamma_{2,1}^{(2)} = -\mu\rho\,\frac{y_1 - x_1}{|\underline{y} - \underline{x}|^2} - \frac{2\mu\rho}{1+\rho}\,b^{(2)},$$

$$\gamma_{2,2}^{(2)} = -\mu\rho\,\frac{y_2 - x_2}{|\underline{y} - \underline{x}|^2},$$

what leads to

$$\gamma_1^{(2)} = \mu\rho\,\arctan\frac{y_2 - x_2}{y_1 - x_1} + \frac{2\mu\rho}{1+\rho}\,b^{(2)}\,y_2\,, \tag{A.11}$$

$$\gamma_2^{(2)} = -\mu\rho\,\log|\underline{y} - \underline{x}| - \frac{2\mu\rho}{1+\rho}\,b^{(2)}\,y_1\,, \tag{A.12}$$

moreover by the ansatz,

$$\chi_{,1}^{(2)} = \mu\,\arctan\frac{y_2 - x_2}{y_1 - x_1} - 2\,\mu a^{(2)} + \frac{\mu\rho}{|\underline{y} - \underline{x}|^2}\,[x_1(y_2 - x_2) + x_2(y_1 - x_1)]\,, \tag{A.13}$$

$$\chi_{,2}^{(2)} = \mu\,\log|\underline{y} - \underline{x}| - \mu(\rho + 2\,d^{(2)}) - \frac{\mu\rho}{|\underline{y} - \underline{x}|^2}\,[x_1(y_1 - x_1) - x_2(y_2 - x_2)]\,. \tag{A.14}$$

By (4.79) in [3] we have

$$(4\,\pi\mu)\,\frac{\lambda + 2\,\mu}{\lambda + 3\,\mu}\,(\mathcal{T}_y\,\mathcal{F}(\underline{x}, \underline{y}))_{jl} =: \mu\rho\,\frac{d}{ds_y}\,(g_{jl})\,, \tag{A.15}$$

where from (A.7)-(A.10), (A.11)-(A.14)

$$g_{11} = (\mu\rho)^{-1}\left[\gamma_2^{(1)} + y_1\,\frac{\partial\gamma_1^{(1)}}{\partial y_2} + y_2\,\frac{\partial\gamma_2^{(1)}}{\partial y_2} + \frac{\partial\chi^{(1)}}{\partial y_2}\right]$$
$$= (1 + \rho^{-1})\arctan\frac{y_1 - x_1}{y_2 - x_2} - 2\,\frac{(y_1 - x_1)(y_2 - x_2)}{|\underline{y} - \underline{x}|^2} - 2\,d^{(1)}\rho^{-1}\,,$$

$$g_{21} = -(\mu\rho)^{-1}\left[\gamma_1^{(1)} + y_1\,\frac{\partial\gamma_1^{(1)}}{\partial y_1} + y_2\,\frac{\partial\gamma_2^{(1)}}{\partial y_1} + \frac{\partial\chi^{(1)}}{\partial y_1}\right]$$
$$= (1 - \rho^{-1})\log|\underline{y} - \underline{x}| + 2\,\frac{(y_1 - x_1)^2}{|\underline{y} - \underline{x}|^2} + 2\,a^{(1)}\rho^{-1}\,,$$

$$g_{12} = (\mu\rho)^{-1}\left[\gamma_2^{(2)} + y_1 \frac{\partial \gamma_1^{(2)}}{\partial y_2} + y_2 \frac{\partial \gamma_2^{(2)}}{\partial y_2} + \frac{\partial \chi^{(2)}}{\partial y_2}\right]$$

$$= -(1-\rho^{-1}) \log |\underline{y} - \underline{x}| - 2 \frac{(y_2 - x_2)^2}{|\underline{y} - \underline{x}|^2} - 2\, d^{(2)} \rho^{-1},$$

$$g_{22} = -(\mu\rho)^{-1}\left[\gamma_1^{(2)} + y_1 \frac{\partial \gamma_1^{(2)}}{\partial y_1} + y_2 \frac{\partial \gamma_2^{(2)}}{\partial y_1} + \frac{\partial \chi^{(2)}}{\partial y_1}\right]$$

$$= -(1+\rho^{-1}) \arctan \frac{y_2 - x_2}{y_1 - x_1} + 2 \frac{(y_1 - x_1)(y_2 - x_2)}{|\underline{y} - \underline{x}|^2} + 2\, a^{(2)} \rho^{-1}.$$

For $\underline{v}(\underline{y}) = (v_1(\underline{y}), v_2(\underline{y}))^T$, $v_j' := \dfrac{dv_j(\underline{y})}{ds_y}$ it follows from (A.15) that

$$\frac{4\pi}{\rho} \frac{\lambda + 2\mu}{\lambda + 3\mu} \left(\int_\Gamma (\mathcal{T}_y \mathcal{F}(\underline{x}, \underline{y}))^T \underline{v}(\underline{y})\, ds_y \right)_j =$$

$$= \int_\Gamma \sum_{l=1}^{2} \left(\frac{d}{ds_y} g_{jl}(\underline{x}, \underline{y}) \right)^T v_l(\underline{y})\, ds_y$$

$$= \int_\Gamma \sum_{l=1}^{2} (-g_{lj}(\underline{x}, \underline{y}))\, v_l'\, ds_y =: h_j(\underline{x}) \tag{A.16}$$

using partial integration. Hence for $\underline{h}(\underline{x}) := (h_1(\underline{x}), h_2(\underline{x}))^T$ we have

$$h_1(\underline{x}) = \int_\Gamma \{-g_{11} v_1' - g_{21} v_2'\}\, ds_y =$$

$$= -\int_\Gamma \left\{ \left[(1+\rho^{-1}) \arctan \frac{y_1 - x_1}{y_2 - x_2} - 2 \frac{(y_1 - x_1)(y_2 - x_2)}{|\underline{y} - \underline{x}|^2} - 2\, d^{(1)} \rho^{-1} \right] v_1' + \right.$$

$$\left. + \left[(1-\rho^{-1}) \log |\underline{y} - \underline{x}| + 2 \frac{(y_1 - x_1)^2}{|\underline{y} - \underline{x}|^2} + 2\, a^{(1)} \rho^{-1} \right] v_2' \right\} ds_y,$$

$$h_2(\underline{x}) = \int_\Gamma \{-g_{12} v_1' - g_{22} v_2'\}\, ds_y$$

$$= \int_\Gamma \left\{ \left[(1-\rho^{-1}) \log |\underline{y} - \underline{x}| + 2 \frac{(y_2 - x_2)^2}{|\underline{y} - \underline{x}|^2} + 2\, d^{(2)} \rho^{-1} \right] v_1' + \right.$$

$$\left. + \left[(1+\rho^{-1}) \arctan \frac{y_2 - x_2}{y_1 - x_1} - 2 \frac{(y_1 - x_1)(y_2 - x_2)}{|\underline{y} - \underline{x}|^2} - 2\, a^{(2)} \rho^{-1} \right] v_2' \right\} ds_y.$$

Now in the second step we interpret $\underline{h}(\underline{x})$ as a displacement vector and use the same ansatz as in the first step. To compute the functions γ_1, γ_2, $\tilde{\chi}$ now in the complex variable $x_1 + ix_2$ we need the partial derivatives $h_{j,l} := \dfrac{\partial h_j}{\partial x_l}$ which are given by

$$h_{1,1}(\underline{x}) = \\ = \int_\Gamma \left\{ \left[-(1+\rho^{-1}) \frac{x_2 - y_2}{|\underline{x} - \underline{y}|^2} + 2(x_2 - y_2) \frac{(x_2 - y_2)^2 - (x_1 - y_1)^2}{|\underline{x} - \underline{y}|^4} \right] v_1' - \left[(1-\rho^{-1}) \frac{x_1 - y_1}{|\underline{x} - \underline{y}|^2} + 4 \frac{(x_1 - y_1)(x_2 - y_2)^2}{|\underline{x} - \underline{y}|^4} \right] v_2' \right\} ds_y ,$$

$$h_{1,2}(\underline{x}) = \\ = \int_\Gamma \left\{ \left[(1+\rho^{-1}) \frac{x_1 - y_1}{|\underline{x} - \underline{y}|^2} + 2(x_1 - y_1) \frac{(x_1 - y_1)^2 - (x_2 - y_2)^2}{|\underline{x} - \underline{y}|^4} \right] v_1' - \left[(1-\rho^{-1}) \frac{x_2 - y_2}{|\underline{x} - \underline{y}|^2} - 4 \frac{(x_2 - y_2)(x_1 - y_1)^2}{|\underline{x} - \underline{y}|^4} \right] v_2' \right\} ds_y ,$$

$$h_{2,1}(\underline{x}) = \\ = \int_\Gamma \left\{ \left[(1-\rho^{-1}) \frac{x_1 - y_1}{|\underline{x} - \underline{y}|^2} - 4 \frac{(x_1 - y_1)(x_2 - y_2)^2}{|\underline{x} - \underline{y}|^4} \right] v_1' - \left[(1+\rho^{-1}) \frac{x_2 - y_2}{|\underline{x} - \underline{y}|^2} + 2(x_2 - y_2) \frac{(x_2 - y_2)^2 - (x_1 - y_1)^2}{|\underline{x} - \underline{y}|^4} \right] v_2' \right\} ds_y ,$$

$$h_{2,2}(\underline{x}) = \\ = \int_\Gamma \left\{ \left[(1-\rho^{-1}) \frac{x_2 - y_2}{|\underline{x} - \underline{y}|^2} + 4 \frac{(x_2 - y_2)(x_1 - y_1)^2}{|\underline{x} - \underline{y}|^4} \right] v_1' + \left[(1+\rho^{-1}) \frac{x_1 - y_1}{|\underline{x} - \underline{y}|^2} - 2(x_1 - y_1) \frac{(x_1 - y_1)^2 - (x_2 - y_2)^2}{|\underline{x} - \underline{y}|^4} \right] v_2' \right\} ds_y .$$

Analogously to (A.1), (A.2), respectively (A.3), (A.4) we derive

$$h_{1,1} + h_{2,2} = 2(1-\rho^{-1}) \int_\Gamma \left\{ \frac{x_2 - y_2}{|\underline{x} - \underline{y}|^2} v_1' - \frac{x_1 - y_1}{|\underline{x} - \underline{y}|^2} v_2' \right\} ds_y \\ = \frac{\rho^{-1} - 1}{\mu} \gamma_{1,1} = \frac{1 - \rho}{\mu\rho} \gamma_{2,2} ,$$

$$h_{1,2} - h_{2,1} = 2(1+\rho^{-1}) \int_\Gamma \left\{ \frac{x_1 - y_1}{|\underline{x} - \underline{y}|^2} v_1' + \frac{x_2 - y_2}{|\underline{x} - \underline{y}|^2} v_2' \right\} ds_y$$

$$= \frac{\rho^{-1}+1}{\mu} \gamma_{1,2} + 2b = 2b - \frac{1+\rho}{\mu\rho} \gamma_{2,1} .$$

Thus we obtain

$$\gamma_1 = -2\mu \int_\Gamma \left\{ \arctan \frac{x_1 - y_1}{x_2 - y_2} v_1' - \log |\underline{x} - \underline{y}| \, v_2' \right\} ds_y + 2b \frac{\rho\mu}{1+\rho} x_2 ,$$

$$\gamma_2 = -2\mu \int_\Gamma \left\{ \log |\underline{x} - \underline{y}| \, v_1' - \arctan \frac{x_2 - y_2}{x_1 - y_1} v_2' \right\} ds_y - 2b \frac{\rho\mu}{1+\rho} x_1 .$$

By (A.16) we have

$$-\frac{4\pi}{\rho} \frac{\lambda + 2\mu}{\lambda + 3\mu} \mathcal{T}_x \int_\Gamma ((\mathcal{T}_y \mathcal{F}(\underline{x}, \underline{y}))^T) \underline{v}(\underline{y}) \, ds_y = -\mathcal{T}_x \underline{h}(\underline{x}) .$$

By (4.79) in [3] it follows for $\underline{w}(\underline{x}) = (w_1(\underline{x}), w_2(\underline{x}))^T$, $w_j' := \dfrac{dw_j(\underline{x})}{ds_x}$ that

$$-\frac{4\pi}{\rho} \frac{\lambda + 2\mu}{\lambda + 3\mu} \int_\Gamma \mathcal{T}_x \left(\int_\Gamma (\mathcal{T}_y \mathcal{F}(\underline{x}, \underline{y}))^T \underline{v}(\underline{y}) \, ds_y \right) \underline{w}(\underline{x}) \, ds_x =$$

$$-\int_\Gamma \mathcal{T}_x \underline{h} \cdot \underline{w}(\underline{x}) \, ds_x =: -\int_\Gamma \left(\frac{d}{ds_x} \underline{k} \right) \cdot \underline{w}(\underline{x}) \, ds_x = \int_\Gamma \sum_{j=1}^{2} k_j(\underline{x}) \, w_j'(\underline{x}) \, ds_x \tag{A.17}$$

using partial integration. Here using the ansatz we obtain

$$k_1(\underline{x}) = \gamma_2 + x_1 \frac{\partial \gamma_1}{\partial x_2} + x_2 \frac{\partial \gamma_2}{\partial x_2} + \frac{\partial \tilde{\chi}}{\partial x_2} = (1+\rho^{-1}) \gamma_2 - 2\mu h_2 + 2\mu (d - bx_1)$$

$$= -4\mu \int_\Gamma \left\{ \left[\log |\underline{x} - \underline{y}| + \frac{(x_2 - y_2)^2}{|\underline{x} - \underline{y}|^2} + d^{(2)} \rho^{-1} \right] v_1' - \left[\frac{(x_1 - y_1)(x_2 - y_2)}{|\underline{x} - \underline{y}|^2} + a^{(2)} \rho^{-1} \right] v_2' \right\} ds_y - 2\mu d ,$$

$$k_2(\underline{x}) = -\gamma_1 - x_1 \frac{\partial \gamma_1}{\partial x_1} - x_2 \frac{\partial \gamma_2}{\partial x_1} - \frac{\partial \tilde{\chi}}{\partial x_1}$$

$$= -(1+\rho^{-1}) \gamma_1 + 2\mu h_1 - 2\mu (a + bx_2)$$

$$= 4\mu \int_\Gamma \left\{ \left[\frac{(x_1 - y_1)(x_2 - y_2)}{|\underline{x} - \underline{y}|^2} - d^{(1)} \rho^{-1} \right] v_1' - \left[\log |\underline{x} - \underline{y}| + \frac{(x_1 - y_1)^2}{|\underline{x} - \underline{y}|^2} + a^{(1)} \rho^{-1} \right] v_2' \right\} ds_y + 2\mu a .$$

Choosing the constants

$$d^{(2)} = -\rho,\ a^{(2)} = 0,\ d = 0,\ d^{(1)} = 0,\ a^{(1)} = -\rho,\ a = 0$$

and using the definition of ρ we obtain finally from (A.17)

$$-\int_\Gamma \mathcal{T}_x \left(\int_\Gamma (\mathcal{T}_y \mathcal{F}(\underline{x}, \underline{y}))^T \underline{v}(\underline{y})\, ds_y \right) \underline{w}(\underline{x})\, ds_x =$$

$$= \int_\Gamma \int_\Gamma \mathcal{F}_0(\underline{x}, \underline{y}) \frac{d\underline{v}(\underline{y})}{ds_y} \frac{d\underline{w}(\underline{x})}{ds_x}\, ds_y\, ds_x ,$$

where

$$(\mathcal{F}_0(\underline{x}, \underline{y}))_{jl} = \frac{\lambda + \mu}{\lambda + 2\mu} \frac{\mu}{\pi} e_{jl} ,$$

and the matrix (e_{jl}) is given by

$$\begin{pmatrix} \dfrac{(x_1 - y_1)^2}{|\underline{x} - \underline{y}|^2} - \log |\underline{x} - \underline{y}| & \dfrac{(x_1 - y_1)(x_2 - y_2)}{|\underline{x} - \underline{y}|^2} \\ \dfrac{(x_1 - y_1)(x_2 - y_2)}{|\underline{x} - \underline{y}|^2} & \dfrac{(x_2 - y_2)^2}{|\underline{x} - \underline{y}|^2} - \log |\underline{x} - \underline{y}| \end{pmatrix}$$

as is claimed in (3.4).

REFERENCES

[1] I. BABUŠKA and A. K. AZIZ, *Survey lectures on the mathematical formulation of the finite element method,* in *The Mathematical Foundation of the Finite Element Method* (A. K. Aziz, ed.) Academic Press, New York, 1972, pp. 3-359.

[2] M. COSTABEL, Boundary integral operators on Lipschitz domains : Elementary results, *SIAM J. Math. Anal. 19,* 1988, pp. 613-626.

[3] R. DAUTRAY and J. L. LIONS, *Mathematical Analysis and Numerical Methods for Science and Technology,* Vol. 4, *Integral Equations and Numerical Methods,* Springer, Berlin, 1990.

[4] G. DUVAUT and J. L. LIONS, *Inequalities in Mechanics and Physics,* Springer, Berlin, 1976.

[5] J. ELSCHNER, On spline approximation for a class of integral equations, I : Galerkin and collocation methods with piecewise polynomials, *Math. Methods in the Applied Sciences 10,* 1988, pp. 543-559.

[6] H. ENGELS, *Numerical Quadrature and Cubature,* Academic Press, New York, 1980.

[7] G. I. ÈSKIN, *Boundary Value Problems for Elliptic Pseudodifferential Equations, Translations of Mathematical Monographs,* Vol. 52, American Mathematical Society, Providence, 1981.

[8] G. FICHERA, *Boundary value problems of elasticity with unilateral constraints,* in *Handbuch der Physik — Encyclopedia of Physics,* Band VI a/2 Festkörpermechanik II, Springer, Berlin, 1972, pp. 391-424.

[9] R. GLOWINSKI, *Numerical Methods for Nonlinear Variational Problems,* Springer, New York, 1984.

[10] R. GLOWINSKI, J. L. LIONS and R. TRÉMOLIÈRES, *Numerical Analysis of Variational Inequalities,* North-Holland, Amsterdam, 1981.

[11] P. GRISVARD, *Elliptic Problems in Nonsmooth Domains,* Pitman, Boston, 1985.

[12] J. GWINNER, Discretization of semicoercive variational inequalities, *Aequationes Mathematicae 42,* 1991, pp. 72-79.

[13] G. HÄMMERLIN and K. H. HOFFMANN, *Numerische Mathematik,* Springer, 1989.

[14] H. HAN, A direct boundary element method for Signorini problems, *Math. Computation 55,* 1990, pp. 115-128.

[15] I. HLAVAČEK, J. HASLINGER, J. NEČAS and J. LOVIŠEK, *Solution of Variational Inequalities in Mechanics,* Springer, Berlin, 1988.

[16] I. HLAVAČEK and J. LOVIŠEK, A finite element analysis for the Signorini problem in plane elastostatics, *Aplikace Mat. 22,* 1977, pp. 215-228.

[17] G. C. HSIAO, E. P. STEPHAN, W. L. WENDLAND, On the Dirichlet problem in elasticity for a domain exterior to an arc, *J. Computational Appl. Mathematics 34,* 1991, pp. 1-19.

[18] N. KIKUCHI and J. T. ODEN, *Contact Problems in Elasticity : a Study of Variational Inequalities and Finite Element Methods,* SIAM, Philadelphia, 1988.

[19] V. A. KONDRATIEV and O. A. OLEINIK, On Korn's inequalities, *C.R. Acad. Sci. Paris I 308,* 1989, pp. 483-487.

[20] V. D. KUPRADZE, *Potential Methods in the Theory of Elasticity,* Israel Program for Scientific Translations, Jerusalem, 1965.

[21] V. D. KUPRADZE, T. G. GEGELIA, M. O. BASHELEISHVILI and T. V. BURCHULADZE, *Three-Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity,* North-Holland, Amsterdam, 1979.

[22] N. I. MUSKHELISHVILI, *Some Basic Problems of the Mathematical Theory of Elasticity,* Noordhoff, Groningen, 1963.

[23] J. NEČAS, *Les méthodes directes en théorie des équations elliptiques,* Academia, Masson, Prague, Paris, 1967.

[24] J. C. NEDELEC, *Approximation des Équations Intégrales en Mécanique et en Physique,* Lecture Notes, Centre Math. Appl., École polytechnique, Palaiseau, France 1977.

[25] J. C. NEDELEC, Integral equations with non integrable kernels, *Integral Equations and Operator Theory 5,* 1982, pp. 562-582.

[26] J. A. NITSCHE, On Korn's second inequality, *R.A.I.R.O. Anal. Numér. 15,* 1981, pp. 237-248.

[27] P. D. PANAGIOTOPOULOS, *Inequality Problems in Mechanics and Applications,* Birkhäuser, Basel, 1985.

[28] P. D. PANAGIOTOPOULOS, Boundary integral « equation » methods for the Signorini-Fichera problem, in *Boundary Elements 7,* vol. 2, exp. No. *12,* 1985, pp. 73-83.

[29] R. SAUER, *Anfangswertprobleme bei partiellen Differentialgleichungen,* Springer, Berlin, 1952.

[30] A. SIGNORINI, *Sopra alcune questioni di elastostatica,* Atti della Società Italiana per il Progresso della Scienze, 1933.

[31] W. L. WENDLAND, *On some mathematical aspects of boundary element methods for elliptic problems,* in MAFELAP V (J. R. Whiteman, ed.), Academic Press, New York, 1985, pp. 193-227.

[32] W. L. WENDLAND and E. P. STEPHAN, A hypersingular boundary integral method for two-dimensional screen and crack problems, *Arch. Rational Mech. Anal. 112,* 1990, pp. 363-390.