M2AN - MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

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M2AN - Modélisation mathématique et analyse numérique, tome 27, n° 4 (1993), p. 421-456

<http://www.numdam.org/item?id=M2AN_1993__27_4_421_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

(Vol. 27, n° 4, 1993, p. 421 à 456)

HOMOGENIZATION AND EFFECTIVE PROPERTIES OF PLATES WEAKENED BY PARTIALLY PENETRATING FISSURES : CONVERGENCE AND DUALITY (*)

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Communicated by P. G. CIARLET

Abstract. — *In the present paper the asymptotic method used in [1] is justified by means of the method of epi-convergence. Next, the dual homogenization is performed. The explicit form of the homogenized complementary potential is derived.*

Résumé. — *On justifie dans cet article la méthode asymptotique de [1] par la méthode de l'épi-convergence. On applique ensuite l'homogénéisation duale. On obtient une forme explicite du potentiel complémentaire homogénéisé.*

1. INTRODUCTION

For a two-layer plate model the problem of finding effective properties was studied in a previous paper [1], provided that one of the layers is weakened by periodically distributed fissures. To simplify the presentation of basic ideas it was assumed that the plate material is homogeneous. Inhomogeneities are introduced by the fissures. By using the method of two-scale asymptotic expansions the overall behaviour of such a fissured plate was investigated. It turns out that the homogenized plate is nonlinear, hyperelastic and without fissures, which are smeared-out by the process of homogenization. The macroscopic behaviour is still elastic because friction was neglected. The presence of friction at the microscopic level results in inelastic macroscopic response (*cf.* Refs. [2, 4]).

(*) Manuscrit reçu le 25 janvier 1991 et sous forme révisée le 25 mai 1992.
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The purpose of the present paper is twofold. Firstly, by using the method of epi-convergence [3] we shall justify the results obtained in [1] by the formal method of asymptotic expansions. It is also concluded that the total potential energy J^{ε} of the fissured plate ($\varepsilon > 0$) tends to the total potential energy of the homogenized plate as $\varepsilon \to 0$. Secondly, the dual homogenization problem is solved. Consequently, the homogenization problem in terms of generalized forces is resolved.

Before passing to the study of epi-convergence and duality we adduce indispensable notions and results, thus facilitating the reading of the paper. Mechanical aspects of the problem studied as well as the local one and its properties are discussed in the paper [1] (*cf.* also Ref. [4]). Most essential results presented here were primarily obtained in [4]. Sometimes we shall use Roman numerals, which refer to reference [1].

2. ELEMENTS OF THE THEORY OF EPI-CONVERGENCE. EPI-CONVERGENCE AND DUALITY

2.1. Epi-convergence

Detailed presentation of the theory of epi-convergence, which is a particular case of so called Γ-convergence, is available in the book by Attouch [3], see also references [5, 6, 7].

DEFINITION 2.1 : *Let* (X, τ) *be a metrisable topological space and* $\{G_{\varepsilon}\}_{\varepsilon > 0}$ *a sequence of functionals form* X *into* $\bar{\mathbb{R}}$, *the extended reals.*

a) *The* τ*-epi-limit inferior* $\tau - li_e\, G_{\varepsilon}$, *denoted also by* G^i, *is the functional on* X *defined by*

$$G^i(u) = \tau - li_e\, G_{\varepsilon}(u) = \min u_{\varepsilon} \xrightarrow{\tau} u \liminf_{\varepsilon} G_{\varepsilon}(u_{\varepsilon})\,.$$

b) *The* τ*-epi-limit superior* $\tau - ls_e\, G_{\varepsilon}$, *denoted also by* G^s, *is the functional on* X *defined by*

$$G^s(u) = \tau - ls_e\, G_{\varepsilon}(u) = \min u_{\varepsilon} \xrightarrow{\tau} u \liminf_{\varepsilon} G_{\varepsilon}(u_{\varepsilon})\,.$$

c) *The sequence* $\{G_{\varepsilon}\}_{>0}$ *is said to be* τ*-epi-convergent if* $G^i = G^s$. *Then we write*

$$G = \tau - \lim_e G_{\varepsilon}\,.$$

PROPERTIES : *Let* $G_{\varepsilon} : (X, \tau) \to \bar{\mathbb{R}}$ *be a sequence of functionals which is* τ*-epi-convergent ;* $G = \tau - \lim_e G_{\varepsilon}$. *Then the following properties hold :*

(i) *the functionals* G^i *and* G^s *are* τ*-lower semicontinuous* (τ*-l.s.c.*).

(ii) *If the functionals* G_ε *are convex then* $G^s = \tau - ls_e\, G_\varepsilon$ *is also convex. Hence the epi-limit* $G = \tau - \lim_e G_\varepsilon$ *is a* τ*-closed* (τ*-l.s.c.*) *convex functional.*

(iii) *If* $\Phi : X \to \mathbb{R}$ *is a* τ*-continuous functional, called perturbation functional, then*

$$\tau - \lim_e (G_\varepsilon + \Phi) = \tau - \lim_e G_\varepsilon + \Phi = G + \Phi .$$

(iv)

$$G(u) = \tau - \lim_e G_\varepsilon(u) \Leftrightarrow \begin{cases} \forall u_\varepsilon \underset{\varepsilon \to 0}{\overset{\tau}{\to}} u,\ G(u) \leqslant \liminf\limits_{\varepsilon \to 0} G_\varepsilon(u_\varepsilon),\ u \in X\,; \\ \forall u \in X\ \exists u_\varepsilon \underset{\varepsilon \to 0}{\overset{\tau}{\to}} u \quad \text{such that} \\ G(u) \geqslant \limsup\limits_{\varepsilon \to 0}\ G_\varepsilon(u_\varepsilon)\,. \end{cases}$$

In practical situations the last property plays an essential role. Very useful is also the following theorem.

THEOREM 2.1 : *Let* $G = \tau - \lim_e G_\varepsilon$ *and suppose that there exists a* τ*-relatively compact subset* $X_0 \subset X$ *such that* $\inf\limits_{X_0} G_\varepsilon = \inf\limits_{X} G_\varepsilon (\forall \varepsilon > 0)$. *Then* $\inf\limits_X G = \lim\limits_{\varepsilon \to 0} \left(\inf\limits_X G_\varepsilon \right)$. *Moreover, if* $\{u_\varepsilon\}_{\varepsilon > 0}$ *is such that* $G_\varepsilon(u_\varepsilon) - \inf\limits_X G_\varepsilon \underset{\varepsilon \to 0}{\to} 0$, *then every* τ*-cluster point of the sequence* $\{u_\varepsilon : \varepsilon \to 0\}$ *minimizes* G *on* X. ∎

Remark 2.1 : From a practical viewpoint the following sufficient condition of existence of a compact set X_0 is very convenient :

If X is a Banach space with τ-compact balls, then a sufficient condition of existence of a compact set X_0 is that the sequence $\{G_\varepsilon\}_{\varepsilon > 0}$ satisfies **the condition of equicoercivity :**

$$\limsup_\varepsilon\ G_\varepsilon(u_\varepsilon) < +\infty \Rightarrow \limsup_\varepsilon\ \|u_\varepsilon\| < \infty\,. \qquad (2.1)$$

Remark 2.2 : If the topology τ is not metrisable then the notion of sequential epi-convergence is used [3, 8].

DEFINITION 2.2 : *A sequence* $\{G_\varepsilon\}_{\varepsilon > 0}$ τ*-epi-converges sequentially to* G *if and only if in every point* $u \in X$ *one has*

1) $u_\varepsilon \overset{\tau}{\to} u \Rightarrow \liminf\limits_{\varepsilon \to 0} G_\varepsilon(u_\varepsilon) \geqslant G(u)$.

2) $\forall \varepsilon > 0 \ \exists u_\varepsilon \overset{\tau}{\to} u$ *such that*

$$\limsup_{\varepsilon} G_\varepsilon(u_\varepsilon) \leq G(u) + \varepsilon .$$

Then we write $G = \text{seq } \tau - \lim_{e \atop \varepsilon \to 0} G_\varepsilon$. ■

The notions of epi-convergence (definition 2.1) and sequential epi-convergence (definition 2.2) are equivalent, provided that τ is metrisable. Moreover, the condition (2) above is satisfied for $\varepsilon = 0$, see the property (iv) formulated earlier.

In practical situation of interest is the case when (X, τ) is the dual space, say E^*, of a separable Banach space E, equiped with the weak-* topology. For instance, $E^* = \mathbb{M}^1(\Omega)$ and $\tau = \sigma(\mathbb{M}^1(\Omega), C_0(\Omega))$; $\mathbb{M}^1(\Omega)$ is the space of bounded measures [8].

The following result relates the topological and sequential notions of epi-convergence, where σ denotes a weak-* topology on E^*.

THEOREM 2.2 [9] : *Let E be a separable Banach space and* E^* *its dual. If* $\{G_\varepsilon\}_{\varepsilon > 0}$ *is defined and equi-coercive on* E^* *then the following statements are equivalent :*

(α) $$G = \sigma - \lim_e G_\varepsilon ,$$

(β) $$G = \text{seq } \sigma - \lim_e G_\varepsilon .$$ ■

We observe that for σ non-metrisable, the σ-epi-limit in the statement (α) is to be understood in the topological sense, see [3]. A consequence of theorem 2.2 is that if $\{G_\varepsilon\}_{\varepsilon > 0}$ is equi-coercive on E^* and G is its sequential σ-epi-limit then G is σ-l.s.c.

To end up with notions of epi-convergence let us adduce the topological one [3].

DEFINITION 2.3 : *Let* (X, τ) *be a topological space and* $N_\tau(x)$ *a set of neighborhoods of* $x \in X$. *Further, let* $\{G_\varepsilon\}_{\varepsilon > 0}$ *be a sequence of functionals,* $G_\varepsilon \to (-\infty, +\infty]$. *The* τ*-epi-limit inferior and the* τ*-epi-limit superior are defined by*

$$G^i(x) = \tau - li_e\, G_\varepsilon(x) = \sup_{V \in N_\tau(x)} \ \liminf_{\varepsilon \to 0} \ \inf_{u \in V} G_\varepsilon(u) ,$$

$$G^s(x) = \tau - ls_e\, G_\varepsilon(x) = \sup_{V \in N_\tau(x)} \ \limsup_{\varepsilon \to 0} \ \inf_{u \in V} G_\varepsilon(u) ,$$

respectively. If $G^i(x) = G^s(x)$, *for each* $x \in X$, *then we write* $G = \tau - \lim_{e \atop \varepsilon \to 0} G_\varepsilon$ (*cf.* definition 2.1). In a metrisable topological space the definitions 2.1 and 2.3 coincide. For instance, in a general topological space we have

$$\inf_{x_\varepsilon \xrightarrow{\tau} x} \liminf_{\varepsilon} G_\varepsilon(x_\varepsilon) \geqslant \sup_{V \in N_\tau(x)} \liminf_{\varepsilon} \inf_{u \in V} G_\varepsilon(u) . \tag{2.2}$$

If (X, τ) is a metrisable space then in (2.2) equality holds.

2.2. Epi-convergence and duality

Having a sequence of functionals $\{G_\varepsilon\}_{\varepsilon > 0}$ one can construct the sequence of conjugate funtionals G_ε^* by using the Fenchel transformation

$$G_\varepsilon^*(u^*) = \sup_{u \in X} \{\langle u^*, u\rangle - G_\varepsilon(u)\} , \quad u^* \in X^* . \tag{2.3}$$

As usual, $(X^*, X, \langle . , . \rangle)$ is a dual pair (*cf.* Ref. [10]).

Now a natural question arises : what is a relation between epi-convergence of the sequences $\{G_\varepsilon\}_{\varepsilon > 0}$ and $\{G_\varepsilon^*\}_{\varepsilon > 0}$, respectively ? Existing results are confined to convex problems, see references [3-6, 11-13]. Attouch [3] investigated such an interrelation provided that X is a reflexive separable Banach space. More general results were obtained by Azé [13] who assumes that X is a separable Banach space.

By $\Gamma_0(X)$ we denote the space of convex lower semicontinuous and proper functions (*cf.* Refs. [10, 14]).

THEOREM 2.3 [13] : *Let X be a separable Banach space and $\{G_\varepsilon\}_{\varepsilon > 0} \subset \Gamma_0(X)$. Assume that*

(i) $G = s - \lim_e G_\varepsilon$,

(ii) $\limsup_{\varepsilon} G_\varepsilon^*(u_\varepsilon^*) < +\infty \Rightarrow \sup_{\varepsilon} \|u_\varepsilon^*\|_{X^*} < \infty$.

Then

$$G^* = w^* - \lim_e G_\varepsilon^* . \tag{2.4}$$

■

In the assumption (i) s stands for the strong topology of the space X whereas in (2.4) w^* denotes the weak-* topology of the dual space X^*. Thus in (2.4) the epi-limit is to be understood in the sense of sequential epi-convergence.

Theorem 2.3 plays an important role in homogenization of perfectly plastic solids loaded at the boundary [15] and perfectly plastic plates subjected to boundary bending moments [4, 16]. This theorem is also involved in the formulation of the duality theory proposed by Azé [12]. Azé's theory is convenient for performing dual homogenization.

We pass now to a concise presentation of the Azé's theory of duality. Let X and Z be separable Banach spaces such that $(X^*, X, \langle \cdot , \cdot \rangle_{X^* \times X})$ and $(Z^*, Z, \langle \cdot , \cdot \rangle_{Z^* \times Z})$ are dual pairs. Further, let $\{G_\varepsilon\}_{\varepsilon > 0}$ be a sequence of functionals belonging to $\Gamma_0(X^* \times Z)$. For a fixed $\varepsilon > 0$ the primal problem has the following form :

$$(P_\varepsilon) \qquad \inf \{G_\varepsilon(x^*, 0) \mid x^* \in X^*\} .$$

We see that $z \in Z$ is a perturbation (*cf.* [10, 14]). Let us set

$$h_\varepsilon(z) = \inf \{G_\varepsilon(x^*, z) \mid x^* \in X^*\} , \quad z \in Z . \tag{2.5}$$

Hence

$$h_\varepsilon^*(z^*) = G_\varepsilon^*(0, z^*) , \quad z^* \in Z^* . \tag{2.6}$$

Accordingly, the dual problem is formulated as follows

$$(P_\varepsilon^*) \qquad \sup_{z^* \in Z^*} \{- G_\varepsilon^*(0, z^*)\} , \tag{2.7}$$

or equivalently

$$(P_\varepsilon^*) \qquad \sup_{z^* \in Z^*} \{- h_\varepsilon^*(z^*)\} . \tag{2.8}$$

Now we make the following assumption :

(Z) there exists $r > 0$ such that for each sequence $\{z_\varepsilon\}_{\varepsilon > 0}$ from the ball $B_r = \{z \in Z : \|z\| \leq r\}$ there exists a bounded sequence $\{x_\varepsilon^*\}_{\varepsilon > 0}$ such that $\limsup_\varepsilon G_\varepsilon(x_\varepsilon^*, z_\varepsilon) < + \infty$.

After necessary preparations we can formulate a basic theorem interrelating epiconvergence and duality.

THEOREM 2.4 [12] : *Let X and Z be separable Banach spaces and $\{G_\varepsilon\}_{\varepsilon > 0}$, G functionals from $\Gamma_0(X^* \times Z)$ satisfying the following conditions*

$$G = w^* \times s - \lim_e G_\varepsilon , \tag{2.9}$$

$$(Z) \tag{2.10}$$

$$G_\varepsilon(x^*, 0) \geq m(\|x^*\|), \ \varepsilon > 0, \ x^* \in X^* , \tag{2.11}$$

where m is a coercive, convex and even function.

Then we have

(i) $h = s - \lim_e h_\varepsilon$.

(ii) $h^* = w^* - \lim_e h_\varepsilon^*$.

(iii) $G(\,.\,,0) = w^* - \lim_e G_\varepsilon(\,.\,,0)$.

(iv) If x_ε^* is a minimizer of the problem (P_ε) up to ε, and if z_ε^* is a minimizer of (P_ε^*) up to ε, where $\varepsilon \to 0$, then the sequences $\{x_\varepsilon^*\}_{\varepsilon > 0}$ and $\{z_\varepsilon^*\}_{\varepsilon > 0}$ are bounded. If x^* and z^* are limits of subsequences then :

x^* realizes the infimum of (P),
z^* realizes the supremum of (P^*),

$$\inf P = \sup P^* ,$$

$$\inf P_\varepsilon \underset{\varepsilon \to 0}{\to} \inf P ,$$

$$\sup P_\varepsilon^* \underset{\varepsilon \to 0}{\to} \sup P^* .$$

■

The above theorem requires some comments. The function h in (i) has the form

$$h(z) = \inf \{G(x^*, z) | x^* \in X^*\} ,$$

whereas h^* is the conjugate function, i.e. :

$$h^*(z^*) = G^*(0, z^*),\ z^* \in Z^* .$$

The limit problems (P) and (P^*) have the following form

(P) $$\inf \{G(x^*, 0) | x^* \in X^*\} ,$$

(P^*) $$\sup \{-G^*(0, z^*) | z^* \in Z^*\} ,$$

respectively.

Having in mind applications to homogenization, one can say that (P) is the homogenized problem while (P^*) represents its dual. It is worth noting that theorem 2.4 requires no assumption of periodicity.

3. SOME BASIC RELATIONS

In this section we recall some relations and derive the form of functionals G_ε. Notations used in reference [1] are preserved. As previously $\Omega \subset \mathbb{R}^2$ denotes the upper face of the plate. The plate is parametrized by Cartesian coordinates $\mathbf{x} = (x, x_3)$, $x = (x_\alpha)$, $\alpha = 1,2$. The upper face Ω $(x_3 = 0)$ plays always the role of a reference plane. By a and b we denote the thicknesses of the upper and lower layer, respectively. Moreover, $h = a + b$ is the thickness of the plate.

Let us consider the plate clamped at the boundary and weakened by fissures εF distributed εY-periodically and of constant depth b, see figure I.6. The basic cell Y is two-dimensional and εY is homothetic to Y. We assume that F is of class C^1 and $F = \bar{F} \subset Y$, where $\bar{F}$ denotes the closure of F. We note that F may be a sum of disjoint fissures. The domain $YF = Y \backslash F$ is connected, and F does not intersect the boundary ∂Y of Y. The following notation is introduced for the sum of fissures such that the corresponding εY-cells are contained in the domain Ω

$$F^{\varepsilon} = \bigcup_{i \in I(\varepsilon)} F_{\varepsilon, i}, \quad \Omega^{\varepsilon} = \Omega \backslash F^{\varepsilon}. \tag{3.1}$$

By $\mathbf{r} = (r_{\alpha})$ is denoted the in-plane displacement vector at $x_3 = a$. Further, (φ_{α}), (ψ_{α}) and w stand for the rotations of plate transverse cross-sections in the upper and lower layers and the vertical displacement, respectively. $\mathbf{N} = (N_{\alpha\beta})$ is the membrane force tensor, while $\mathbf{M} = (M_{\alpha\beta})$ and $\mathbf{L} = (L_{\alpha\beta})$ denote moment tensors in the upper and lower layers, respectively. $\mathbf{Q} = (Q_{\alpha})$ and $\mathbf{T} = (T_{\alpha})$ are transverse internal forces in the upper and lower layers, respectively.

For a fixed $\varepsilon > 0$ the functional J^{ε} of the total potential energy of the fissured plate is given by

$$\begin{aligned} J^{\varepsilon}(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}, w) = \frac{1}{2} \int_{\Omega^{\varepsilon}} & [N_{\alpha\beta}(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi})\, e_{\alpha\beta}(\mathbf{r}) + \\ & + M_{\alpha\beta}(\mathbf{r}, \boldsymbol{\varphi})\, \rho_{\alpha\beta}(\boldsymbol{\varphi}) + L_{\alpha\beta}(\mathbf{r}, \boldsymbol{\psi})\, \kappa_{\alpha\beta}(\boldsymbol{\psi}) \\ & + Q_{\alpha}(w, \boldsymbol{\varphi})\, g_{\alpha}(w, \boldsymbol{\varphi}) + T_{\alpha}(w, \boldsymbol{\psi})\, d_{\alpha}(w, \boldsymbol{\psi})]\, dx - \int_{\Omega} pw\, dx, \end{aligned} \tag{3.2}$$

where

$$N_{\alpha\beta}(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}) = C_{\alpha\beta\lambda\mu}\left[h e_{\lambda\mu}(\mathbf{r}) - \frac{a^2}{2} \rho_{\lambda\mu}(\boldsymbol{\varphi}) + \frac{b^2}{2} \kappa_{\lambda\mu}(\boldsymbol{\psi})\right], \tag{3.3}$$

$$M_{\alpha\beta}(\mathbf{r}, \boldsymbol{\varphi}) = C_{\alpha\beta\lambda\mu}\left[-\frac{a^2}{2} e_{\lambda\mu}(\mathbf{r}) + \frac{a^3}{2} \rho_{\lambda\mu}(\boldsymbol{\varphi})\right], \tag{3.4}$$

$$L_{\alpha\beta}(\mathbf{r}, \boldsymbol{\psi}) = C_{\alpha\beta\lambda\mu}\left[\frac{b^2}{2} e_{\lambda\mu}(\mathbf{r}) + \frac{b^3}{3} \kappa_{\lambda\mu}(\boldsymbol{\psi})\right], \tag{3.5}$$

$$Q_{\alpha}(w, \boldsymbol{\varphi}) = aH_{\alpha\beta}\, g_{\beta}(w, \boldsymbol{\varphi}), \quad H_{\alpha\beta} = c_{\alpha 3\beta 3}, \tag{3.6}$$

$$T_{\alpha}(w, \boldsymbol{\psi}) = bH_{\alpha\beta}\, d_{\beta}(w, \boldsymbol{\psi}), \tag{3.7}$$

$$e_{\alpha\beta}(\mathbf{r}) = r_{(\alpha,\beta)} = \left(\frac{\partial r_{\alpha}}{\partial x_{\beta}} + \frac{\partial r_{\beta}}{\partial x_{\alpha}}\right)/2, \quad \rho_{\alpha\beta}(\boldsymbol{\varphi}) = \varphi_{(\alpha,\beta)}, \tag{3.8}$$

$$\kappa_{\alpha\beta}(\boldsymbol{\psi}) = \psi_{(\alpha,\beta)}, \; g_{\alpha}(w, \boldsymbol{\varphi}) = w_{,\alpha} + \varphi_{\alpha}, \; d_{\alpha}(w, \boldsymbol{\psi}) = w_{,\alpha} + \psi_{\alpha}, \tag{3.9}$$

The summation convention is used throughout the paper, unless otherwise stated. Obviously, the plate is subjected to transverse loading p. We assume that $p \in L^2(\Omega)$. The loading functional could include a boundary term. Yet, as we shall soon see, the loading functional will play a role of a perturbation functional in the process of homogenization.

For the sake of simplicity we assume that the elastic moduli $C_{\alpha\beta\lambda\mu}$ and $H_{\alpha\beta}$ are constants. However, our considerations can readily be generalized to the case when $C_{\alpha\beta\lambda\mu} \in L^\infty(\Omega)$ and $H_{\alpha\beta} \in L^\infty(\Omega)$, thus allowing for two layer plate model made of different materials with perfect bonding at the interface. Anyway, we have

$$\exists \lambda_1 > 0\,, \quad C_{\alpha\beta\lambda\mu}\, t_{\alpha\beta}\, t_{\lambda\mu} \leq \lambda_1 |\mathbf{t}|^2\,, \quad \mathbf{t} \in \mathbb{E}_s^2\,, \tag{3.10}$$

$$\exists c_1 > 0\,, \quad H_{\alpha\beta}\, d_\alpha\, d_\beta \leq c_1 |\mathbf{d}|^2\,, \quad \mathbf{d} \in \mathbb{R}^2\,. \tag{3.11}$$

Further, we make the following assumption

$$\exists \lambda > 0\,, \quad C_{\alpha\beta\lambda\mu}\, t_{\alpha\beta}\, t_{\lambda\mu} \geq \lambda\, |\mathbf{t}|^2\,, \quad \mathbf{t} \in \mathbb{E}_s^2\,, \tag{3.12}$$

$$\exists c_0 > 0\,, \quad H_{\alpha\beta}\, d_\alpha\, d_\beta \geq c_0 |\mathbf{d}|^2\,, \quad \mathbf{d} \in \mathbb{R}^2\,. \tag{3.13}$$

According to our considerations performed in [1], kinematically admissible fields $\mathbf{r}$, $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$ and w are such that

$$\mathbf{r} \in H_0^1(\Omega, \mathbb{R}^2),\ \boldsymbol{\varphi} \in H_0^1(\Omega, \mathbb{R}^2),\ w \in H_0^1(\Omega)\,, \tag{3.14}$$

$$\boldsymbol{\psi} \in K_\varepsilon = \left\{ \mathbf{v} \in H^1(\Omega^\varepsilon, \mathbb{R}^2) \,|\, [\![v_n]\!] \geq 0 \text{ on } F^\varepsilon,\ \mathbf{v} = 0 \text{ on } \partial\Omega \right\}\,. \tag{3.15}$$

Taking account of the relations (3.14) and (3.15) we write

$$J^\varepsilon(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}, w) = G_1^\varepsilon(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}) + \Phi_1(\boldsymbol{\varphi}, \boldsymbol{\psi}, w) + \Phi_2(w)\,, \tag{3.16}$$

where

$$\Phi_1(\boldsymbol{\varphi}, \boldsymbol{\psi}, w) = \\ = \frac{1}{2} \int_\Omega [Q_\alpha(w, \boldsymbol{\varphi})\, g_\alpha(w, \boldsymbol{\varphi}) + T_\alpha(w, \boldsymbol{\psi})\, d_\alpha(w, \boldsymbol{\psi})]\, dx\,, \tag{3.17}$$

$$\Phi_2(w) = -\int_\Omega p w\, dx = -f(w)\,, \tag{3.18}$$

$$G_1^\varepsilon(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}) = \frac{1}{2} \int_\Omega [h C_{\alpha\beta\lambda\mu}\, e_{\alpha\beta}(\mathbf{r})\, e_{\lambda\mu}(\mathbf{r}) - \frac{a^2}{2} C_{\alpha\beta\lambda\mu}\, e_{\alpha\beta}(\mathbf{r})\, \rho_{\lambda\mu}(\boldsymbol{\varphi}) \\ + M_{\alpha\beta}(\mathbf{r}, \boldsymbol{\varphi})\, \rho_{\alpha\beta}(\boldsymbol{\varphi})]\, dx + \frac{1}{2} \int_{\Omega^\varepsilon} \left[\frac{b^3}{3} C_{\alpha\beta\lambda\mu}\, \kappa_{\alpha\beta}(\boldsymbol{\psi})\, \kappa_{\lambda\mu}(\boldsymbol{\psi}) \right. \\ \left. + b^2 C_{\alpha\beta\lambda\mu}\, e_{\alpha\beta}(\mathbf{r})\, \kappa_{\lambda\mu}(\boldsymbol{\psi}) \right] dx\,. \tag{3.19}$$

For $w \in H^1(\Omega)$ we have

$$\Phi_2(w) \leq \|p\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)} = c(p) \|w\|_{L^2(\Omega)} \leq c \|w\|_{1,\Omega}, \quad (3.20)$$

where $c(p) = \|p\|_{L^2(\Omega)}$ and c is a constant which depends on p and Ω.

Thus we conclude that epi-convergence concerns only the sequence of functionals $\{J^\varepsilon(\mathbf{r}, \varphi, ., w)\}_{\varepsilon > 0}$. Further, the relation (3.17) suggests that the functional $\Phi_1(\underset{\sim}{\varphi}, ., w)$ is convex and continuous in the strong topology of $L^2(\Omega, \mathbb{R}^2)$. Thus the functional $\Phi_1 + \Phi_2$ is a perturbation functional.

We set

$$j[\mathbf{e}(\mathbf{r}), \boldsymbol{\rho}(\boldsymbol{\varphi}), \boldsymbol{\kappa}(\boldsymbol{\psi})] = \frac{1}{2} [N_{\alpha\beta}(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi})\, e_{\alpha\beta}(\mathbf{r}) + M_{\alpha\beta}(\mathbf{r}, \boldsymbol{\varphi})\, \rho_{\alpha\beta}(\boldsymbol{\varphi}) +$$

$$+ L_{\alpha\beta}(\mathbf{r}, \boldsymbol{\psi})\, \kappa_{\alpha\beta}(\boldsymbol{\psi})] = j_0[\mathbf{e}(\mathbf{r}), \boldsymbol{\rho}(\boldsymbol{\varphi})] + j_1[\mathbf{e}(\mathbf{r}), \boldsymbol{\kappa}(\boldsymbol{\psi})], \quad (3.21)$$

where

$$j_0[\mathbf{e}(\mathbf{r}), \boldsymbol{\rho}(\boldsymbol{\varphi})] = \frac{1}{2} [h C_{\alpha\beta\lambda\mu}\, e_{\alpha\beta}(\mathbf{r})\, e_{\lambda\mu}(\mathbf{r}) -$$

$$- \frac{a^2}{2} C_{\alpha\beta\lambda\mu}\, \mathbf{e}_{\alpha\beta}(\mathbf{r})\, \rho_{\lambda\mu}(\boldsymbol{\varphi}) + M_{\alpha\beta}(\mathbf{r}, \boldsymbol{\varphi})\, \rho_{\alpha\beta}(\boldsymbol{\varphi}) \Big], \quad (3.22)$$

$$j_1[\mathbf{e}(\mathbf{r}), \boldsymbol{\kappa}(\boldsymbol{\psi})] = \frac{1}{2} \Big[\frac{b^3}{3} C_{\alpha\beta\lambda\mu}\, \kappa_{\alpha\beta}(\boldsymbol{\psi})\, \kappa_{\lambda\mu}(\boldsymbol{\psi}) +$$

$$+ b^2 C_{\alpha\beta\lambda\mu}\, e_{\alpha\beta}(\mathbf{r})\, \kappa_{\lambda\mu}(\boldsymbol{\psi})]. \quad (3.23)$$

The properties of the matrix $\mathbf{D}$, given by (I.2.26), and the inequalities (3.10) and (3.12) imply, (*cf.* (I.2.28))

$$\exists c_1 \geq c_0 > 0,$$

$$(|\mathbf{e}|^2 + |\boldsymbol{\rho}|^2 + |\boldsymbol{\kappa}|^2) \leq j(\mathbf{e}, \boldsymbol{\rho}, \boldsymbol{\kappa}) \leq c_1(|\mathbf{e}|^2 + |\boldsymbol{\rho}|^2 + |\boldsymbol{\kappa}|^2), \quad (3.24)$$

for each $\mathbf{e}, \boldsymbol{\rho}, \boldsymbol{\kappa} \in \mathbb{E}^2_s$. Obviously, the constant c_0 and c_1 are not the same ones as in (3.11) and (3.13).

The relation (3.19) suggests that the epi-convergence will involve only the integral over Ω^ε. However, in order to pass to the limit in the sense of epi-convergence non-negativeness of an integrand will be used. Thus we have to work with the function j, and not only with j_1.

Let us set

$$\mathbf{G}^\varepsilon(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}) = \begin{cases} G_1^\varepsilon(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}), & \text{if } \mathbf{r}, \boldsymbol{\varphi} \in H^1(\Omega, \mathbb{R}^2), \quad \boldsymbol{\psi} \in K^\varepsilon; \\ +\infty, & \text{otherwise}, \end{cases} \quad (3.25)$$

where

$$K^{\varepsilon} = \{v \in H^1(\Omega^{\varepsilon}, \mathbb{R}^2) \mid [\![v_n]\!] \geq 0, \text{ on } F^{\varepsilon}\} . \tag{3.26}$$

In the sequel epi-convergence of the functionals $G^{\varepsilon}(\mathbf{r}, \boldsymbol{\varphi}, .\,)$ will be investigated.

4. EPI-CONVERGENCE OF SEQUENCE OF FUNCTIONALS $\{G^{\varepsilon}(\mathbf{r}, \boldsymbol{\varphi}, .\,)\}_{\varepsilon > 0}$

Let us recall the definition of Y-periodic functions defined on $Y \backslash F$:

$$H^1_{\text{per}}(Y \backslash F) = \\ = \{v \in H^1(Y \backslash F) \mid \text{traces of } v \text{ are equal at opposite sides of } Y\} . \tag{4.1}$$

We shall also use the notation $\mathbf{E} = \mathbf{e}^h$, $\boldsymbol{\chi} = \boldsymbol{\kappa}^h$, etc.

The main result of this section is given in the form of the following theorem.

THEOREM 4.1 : *The sequence of functionals* $\{G^{\varepsilon}(\mathbf{r}, \boldsymbol{\varphi}, .\,)\}_{\varepsilon > 0}$ *defined by* (3.25) *epi-converges in the strong topology of* $L^2(\Omega, \mathbb{R}^2)$ to

$$G(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}) = \\ = \begin{cases} \int_{\Omega} j_0[\mathbf{e}(\mathbf{r}), \boldsymbol{\rho}(\boldsymbol{\varphi})]\, dx + \int_{\Omega} W_1[\mathbf{e}(\mathbf{r}), \boldsymbol{\kappa}(\boldsymbol{\psi})]\, dx\,, & \text{if } \mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi} \in H^1(\Omega, \mathbb{R}^2)\,; \\ +\infty\,, & \textit{otherwise}\,, \end{cases} \tag{4.2}$$

where

$$W_1(\mathbf{E}, \boldsymbol{\chi}) = \inf_{\mathbf{v} \in K_{YF}} \frac{1}{|Y|} \int_{Y \backslash F} j_1(\mathbf{E}, \boldsymbol{\kappa}_y(\mathbf{v}) + \boldsymbol{\chi})\, dy\,, \tag{4.3}$$

and

$$j_1(\mathbf{E}, \boldsymbol{\kappa}_y(\mathbf{v}) + \boldsymbol{\chi}) = \frac{1}{2}\left[\frac{b^3}{3} C_{\alpha\beta\lambda\mu}(\kappa_{y\alpha\beta}(\mathbf{v}) + \chi_{\alpha\beta})(\kappa_{y\lambda\mu}(\mathbf{v}) + \chi_{\lambda\mu}) + \right. \\ \left. + b^2 C_{\alpha\beta\lambda\mu}(\kappa_{y\alpha\beta}(\mathbf{v}) + \chi_{\alpha\beta}) E_{\lambda\mu}\right]\,, \tag{4.4}$$

$$K_{YF} = \{\mathbf{v} \in H^1_{\text{per}}(Y \backslash F, \mathbb{R}^2) \mid [\![v_N]\!] \geq 0 \text{ on } F\}\,, \tag{4.5}$$

$$\kappa_{y\alpha\beta}(\mathbf{v}) = \left(\frac{\partial v_\alpha}{\partial y_\beta} + \frac{\partial v_\beta}{\partial y_\alpha}\right)/2\,. \tag{4.6}$$

Proof : Towards this end we follow the approach proposed primarily by Attouch [3] for scalar elliptic problems and next used by Attouch and Murat [17], also in the scalar case, but in the presence of fissures. However, we do not follow precisely this approach, because some ideas due to Bouchitté [8] are also exploited.

According to the property (iv) of epi-limit (*cf.* Section 2), we have to demonstrate that :

a) For any $\boldsymbol{\psi} \in H^1(\Omega, \mathbb{R}^2)$ there exists a sequence $\{\boldsymbol{\psi}^\varepsilon\}_{\varepsilon > 0} \subset K^\varepsilon$ strongly convergent to $\boldsymbol{\psi}$ in the strong topology of $L^2(\Omega, \mathbb{R}^2)$ such that

$$G(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}) \geqslant \limsup_{\varepsilon \to 0} G^\varepsilon(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}^\varepsilon) . \tag{4.7}$$

b) For any sequence $\{\boldsymbol{\psi}^\varepsilon\}_{\varepsilon > 0} \subset K^\varepsilon$ such that $\boldsymbol{\psi}^\varepsilon \underset{\varepsilon \to 0}{\to} \boldsymbol{\psi}$ in the strong topology of $L^2(\Omega, \mathbb{R}^2)$, the following inequality holds

$$G(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}) \leqslant \liminf_{\varepsilon \to 0} G^\varepsilon(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}^\varepsilon) . \tag{4.8}$$

The proof is performed in several steps. In steps 1 and 2 we shall prove (4.7).

Step 1.

Let $\{\Omega_i\}_{i \in J}$ be a finite partition of Ω by polyhedral sets. Such a partition enables one to use the local character of the functionals $G^\varepsilon(\mathbf{r}, \boldsymbol{\varphi}, \,.\,)$. We set

$$\Omega_i^\delta = \{x \in \Omega_i \mid \text{dist}\,(x, \partial\Omega_i) > \delta\} , \quad \delta > 0 ,$$

and let $\varphi_i^\delta \in \mathcal{D}(\Omega_i)$ be such that $0 \leqslant \varphi_i^\delta \leqslant 1$, $\varphi_i^\delta|_{\Omega_i^\delta} = 1$. $\mathcal{D}(\Omega_i)$ is the space of functions of class C^∞ with compact support in Ω_i.

Let $\boldsymbol{\psi} \in H^1(\Omega, \mathbb{R}^2)$ be a piecewise affine continuous function, that is

$$\psi_\alpha(x) = \chi^i_{\alpha\beta}\, x_\beta + z^i_\alpha , \quad \boldsymbol{\chi}^i \in \mathbb{E}^2_s , \quad \mathbf{z}^i \in \mathbb{R}^2 , \quad \forall x \in \Omega_i , \tag{4.9}$$

where $i \in J$. Hence we have

$$\boldsymbol{\kappa}(\boldsymbol{\psi}(x)) = \boldsymbol{\chi}^i , \quad x \in \Omega_i , \quad i \in J . \tag{4.10}$$

With every family of functions $\{\mathbf{v}^i\}_{i \in J} \subset K_{YF}$ we associate the sequence

$$\boldsymbol{\psi}^{\varepsilon,\,\delta}(x) = \boldsymbol{\psi}(x) + \varepsilon \sum_{i \in J} \varphi_i^\delta(x)\, \mathbf{v}^i\left(\frac{x}{\varepsilon}\right) , \quad \varepsilon > 0 . \tag{4.11}$$

Since

$$\int_{\Omega_i} \left|\mathbf{v}^i\left(\frac{x}{\varepsilon}\right)\right|^2 dx \leqslant n_0^2 \int_Y |\mathbf{v}_i(y)|^2\, dy ,$$

where $n_0(\varepsilon)$ is sufficiently large [18], therefore we have

$$\boldsymbol{\psi}^{\varepsilon,\,\delta}(x) \to \boldsymbol{\psi}(x)\,,$$

in the strong topology of $L^2(\Omega, \mathbb{R}^2)$.

Further, equation (4.11) yields

$$[\![\psi_n^{\varepsilon,\,\delta}]\!]_{F^\varepsilon} = \varepsilon \sum_{i \in J} \varphi_i^\delta [\![v_n^i]\!]_{F^\varepsilon} \geq 0\,. \tag{4.12}$$

Let $t < 1$ (intended to go to 1) and set $\omega_i^\varepsilon = \Omega^\varepsilon \cap \Omega_i$. We note that $t\varphi_i^\delta + t(1-\varphi_i^\delta) + (1-t) = 1$. The function $j_1(\mathbf{e}, .\,)$ is convex. Exploiting these facts we obtain

$$G^\varepsilon(\mathbf{r}, \boldsymbol{\varphi}, t\boldsymbol{\psi}^{\varepsilon,\,\delta}) = \sum_{i \in J} \int_{\omega_i^\varepsilon} j\Big[\mathbf{e}(\mathbf{r}(x)), \boldsymbol{\rho}(\boldsymbol{\varphi}(x)), t\varphi_i^\delta(x)\Big(\boldsymbol{\chi}^i + \boldsymbol{\kappa}\Big(\mathbf{v}^i\Big(\frac{x}{\varepsilon}\Big)\Big)\Big)$$

$$+ t(1-\varphi_i^\delta(x))\,\boldsymbol{\chi}^i + (1-t)\frac{\varepsilon t}{1-t}\,\mathrm{sym}\,\Big(\mathbf{v}^i\Big(\frac{x}{\varepsilon}\Big) \otimes \nabla\varphi_i^\delta(x)\Big)\Big]\,dx \leq$$

$$\leq \sum_{i \in J} \Big\{\int_{\omega_i^\varepsilon} j\Big[\mathbf{e}(\mathbf{r}(x)), \boldsymbol{\rho}(\boldsymbol{\varphi}(x)), \boldsymbol{\chi}^i + \boldsymbol{\kappa}\Big(\mathbf{v}^i\Big(\frac{x}{\varepsilon}\Big)\Big)\Big]\,dx +$$

$$+ \int_{\omega_i^\varepsilon} (1-\varphi_i^\delta(x))\,j\,[\mathbf{e}(\mathbf{r}(x)), \boldsymbol{\rho}(\boldsymbol{\varphi}(x)), \boldsymbol{\chi}^i]\,dx$$

$$+ (1-t)\int_{\omega_i^\varepsilon} j\Big[\mathbf{e}(\mathbf{r}(x)), \boldsymbol{\rho}(\boldsymbol{\varphi}(x)), \frac{\varepsilon t}{1-t}\,\mathrm{sym}\,\Big(\mathbf{v}^i\Big(\frac{x}{\varepsilon}\Big) \otimes \nabla\varphi_i^\delta(x)\Big)\Big]\,dx\Big\}$$

$$\leq \sum_{i \in J} \Big\{\int_{\omega_i^\varepsilon} j\Big[\mathbf{e}(\mathbf{r}), \boldsymbol{\rho}(\boldsymbol{\varphi}), \boldsymbol{\chi}^i + \boldsymbol{\kappa}\Big(\mathbf{v}^i\Big(\frac{x}{\varepsilon}\Big)\Big)\Big]\,dx +$$

$$+ \lambda_1 \int_{\omega_i^\varepsilon} (1-\varphi_i^\delta(x))[|\mathbf{e}(\mathbf{r}(x))|^2 + |\boldsymbol{\rho}(\boldsymbol{\varphi}(x))|^2 + |\boldsymbol{\chi}^i|^2]\,dx$$

$$+ \lambda_1(1-t)\int_{\omega_i^\varepsilon} [|\mathbf{e}(\mathbf{r}(x))|^2 + |\boldsymbol{\rho}(\boldsymbol{\varphi}(x))|^2$$

$$+ \Big|\frac{\varepsilon t}{1-t}\,\mathrm{sym}\,\Big(\mathbf{v}^i\Big(\frac{x}{\varepsilon}\Big) \otimes \nabla\varphi_i^\delta(x)\Big)\Big|^2\Big]\,dx\Big\}\,,$$

since j is non-negative. Here we have used the following notation

$$[\mathrm{sym}\,(\mathbf{v}^i \otimes \nabla\varphi_i^\delta)]_{\alpha\beta} = \frac{1}{2}\left(v_\alpha \frac{\partial\varphi_i^\delta}{\partial x_\beta} + v_\beta \frac{\partial\varphi_i^\delta}{\partial x_\alpha}\right)\,. \tag{4.13}$$

The function $j\left[\mathbf{e}(\mathbf{r}), \boldsymbol{\rho}(\boldsymbol{\varphi}), \boldsymbol{\chi}^i + \boldsymbol{\kappa}\left(\mathbf{v}^i\left(\frac{\cdot}{\varepsilon}\right)\right)\right]$ is εY-periodic, hence we arrive at the following inequality

$$\limsup_{\substack{\delta \to 0 \\ t \to 1^-}} \limsup_{\varepsilon \to 0} G^\varepsilon(\mathbf{r}, \boldsymbol{\varphi}, t\boldsymbol{\psi}^{\varepsilon, \delta}) \leq$$

$$\leq \sum_{i \in J} \int_{\Omega_i} \left\{ \frac{1}{|Y|} \int_{Y \setminus F} j[\mathbf{e}(\mathbf{r}(x), \boldsymbol{\rho}(\boldsymbol{\varphi}(x)), \mathbf{x}^i + \boldsymbol{\kappa}_y(\mathbf{v}^i(y))] \, dy \right\} dx . \quad (4.14)$$

After Attouch [3, p. 33, Corollary 1.16] we formulate

LEMMA 4.1 : *Let* $\{\mathbf{a}_{A, B} \mid A = 1, 2, \dots ; B = 1, 2, \dots\}$ *be a doubly indexed family in* $\bar{\mathbb{R}}$. *Then there exists a mapping* $A \to B(A)$, *increasing to* $+\infty$, *such that*

$$\limsup_{A \to \infty} a_{A, B(A)} \leq \limsup_{B \to \infty} \left(\limsup_{A \to \infty} a_{A, B} \right) . \qquad \blacksquare$$

By using this lemma we see that one can construct a mapping $\varepsilon \to (t(\varepsilon), \delta(\varepsilon))$ with $(t(\varepsilon), \delta(\varepsilon)) \to (1^-, 0)$, $\varepsilon \to 0$, such that setting

$$\psi^\varepsilon = t(\varepsilon) \, \psi^{\varepsilon, \delta(\varepsilon)}, \quad (4.15)$$

(thus $\boldsymbol{\psi}^\varepsilon \underset{\varepsilon \to 0}{\to} \boldsymbol{\psi}$ strongly in $L^2(\Omega, \mathbb{R}^2)$) from (4.14) we deduce that

$$\limsup_{\varepsilon \to 0} G^\varepsilon(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}^\varepsilon) \leq$$

$$\leq \sum_{i \in J} \int_{\Omega_i} \left\{ \frac{1}{|Y|} \int_{Y \setminus F} j[\mathbf{e}(\mathbf{r}(x)), \boldsymbol{\rho}(\boldsymbol{\varphi}(x)), \boldsymbol{\chi}^i + \boldsymbol{\kappa}_y(\mathbf{v}^i(y))] \, dy \right\} dx . \quad (4.16)$$

Taking now the infimum in the r.h.s. of the last inequality, when $\mathbf{v}^i$ runs over the set K_{YF}, we arrive at the relation

$$G^s(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}) \leq \limsup_{\varepsilon \to 0} G^\varepsilon(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}^\varepsilon) \leq$$

$$\leq \int_\Omega j_0[\mathbf{e}(\mathbf{r}(x)), \boldsymbol{\rho}(\boldsymbol{\varphi}(x))] \, dx + \sum_{i \in J} W_1[\mathbf{e}(\mathbf{r}(x)), \boldsymbol{\chi}_i] \, dx =$$

$$= \int_\Omega j_0[\mathbf{e}(\mathbf{r}(x)), \boldsymbol{\rho}(\boldsymbol{\varphi}(x))] \, dx + \int_\Omega W_1[\mathbf{e}(\mathbf{r}(x)), \boldsymbol{\kappa}(\boldsymbol{\psi}(x))] \, dx . \quad (4.17)$$

Step 2.

Let us put $\mathbf{v} = 0$ in (4.3). Then one obtains

$$j_0(\mathbf{E}, \boldsymbol{\theta}) + W_1(\mathbf{E}, \boldsymbol{\chi}) \leqslant j_0(\mathbf{E}, \boldsymbol{\theta}) + \frac{1}{|Y|} \int_{Y \setminus F} j_1(\mathbf{E}, \boldsymbol{\chi})\, dy =$$

$$= j_0(\mathbf{E}, \boldsymbol{\theta}) + j_1(\mathbf{E}, \boldsymbol{\chi}) \leqslant \lambda_1(|\mathbf{E}|^2 + |\boldsymbol{\theta}|^2 + |\boldsymbol{\chi}|^2). \quad (4.18)$$

From the property (ii) of epi-limit (Section 2) it follows that the convexity of $G^{\varepsilon}(\mathbf{r}, \boldsymbol{\varphi}, \,.\,)$ is preserved by the epi-limit superior $G^s(\mathbf{r}, \boldsymbol{\varphi}, \,.\,)$. Moreover, (4.18) yields

$$G^s(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}) \leqslant \lambda_1 \int_{\Omega} (|\mathbf{e}(\mathbf{r}(x))|^2 + |\boldsymbol{\rho}(\boldsymbol{\varphi}(x))|^2 + |\boldsymbol{\kappa}(\boldsymbol{\psi}(x))|^2)\, dx\,, \quad (4.19)$$

for each $\boldsymbol{\psi} \in H^1(\Omega, \mathbb{R}^2)$. Being convex and finite, the functional $G^s(\mathbf{r}, \boldsymbol{\varphi}, \,.\,)$ is continuous on $H^1(\Omega, \mathbb{R}^2)$. Consequently, due to the density of piecewise affine continuous functions in $H^1(\Omega)$ [14], the inequality $G^s(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}) \leqslant G(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi})$ can be extended to the whole space $H^1(\Omega, \mathbb{R}^2)$. Moreover, the functional $G(\mathbf{r}, \boldsymbol{\varphi}, \,.\,)$ is convex and continuous on this space. More precisely, for each $\boldsymbol{\psi} \in H^1(\Omega, \mathbb{R}^2)$ there exists a sequence $\{\boldsymbol{\psi}^k\}_{k \in \mathbb{N}}$ of piecewise affine continuous functions, such that $\boldsymbol{\psi}^k \underset{k \to \infty}{\to} \boldsymbol{\psi}$ in the strong topology of $H^1(\Omega, \mathbb{R}^2)$. Here $\mathbb{N}$ stands for the set of natural numbers. Consequently, one has

$$G(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}^k) \underset{k \to \infty}{\to} G(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}). \quad (4.20)$$

From the previous steps it results that there exists a sequence $\{\boldsymbol{\psi}^{k,\varepsilon}\}_{\varepsilon > 0}$ converging strongly to $\boldsymbol{\psi}^k$ in $L^2(\Omega, \mathbb{R}^2)$ as $\varepsilon \to 0$ and such that

$$\limsup_{\varepsilon \to 0} G^{\varepsilon}(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}^{k,\varepsilon}) \leqslant G(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}^k). \quad (4.21)$$

Thus (4.20) and (4.21) result in

$$\limsup_{k \to \infty} \limsup_{\varepsilon \to 0} G^{\varepsilon}(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}^{k,\varepsilon}) \leqslant G(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}),$$

$$\limsup_{k \to \infty} \limsup_{\varepsilon \to 0} \|\boldsymbol{\psi}^{k,\varepsilon} - \boldsymbol{\psi}\|_{L^2(\Omega, \mathbb{R}^2)} = 0.$$

Using now Lemma 4.1 we deduce that there exists a mapping $\varepsilon \to k(\varepsilon)$ with

$k(\varepsilon) \to \infty$ such that, setting $\boldsymbol{\psi}^{\varepsilon} = \boldsymbol{\psi}^{k(\varepsilon), \varepsilon}$ we eventually obtain

$$G^{s}(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}) \leq \limsup_{\varepsilon \to 0} G^{\varepsilon}(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}^{\varepsilon}) \leq G(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}),$$

for each $\boldsymbol{\psi} \in H^{1}(\Omega, \mathbb{R}^{2})$. In this way (4.7) has been proved.

We pass now to proving the inequality (4.8).

Step 3.

Firstly, let us prove that if $\boldsymbol{\psi}$ is an affine function

$$\psi_{\alpha}(x) = \chi_{\alpha\beta}\, x_{\beta} + z_{\alpha}\,; \quad \boldsymbol{\chi} \in \mathbb{E}_{s}^{2}, \quad \mathbf{z} \in \mathbb{R}^{2}, \tag{4.22}$$

then

$$\lim_{\varepsilon \to 0} G^{\varepsilon}(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}^{\varepsilon}) = G(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}), \tag{4.23}$$

where

$$\boldsymbol{\psi}^{\varepsilon}(x) = \boldsymbol{\psi}(x) + \varepsilon \psi_{(\mathbf{E}, \boldsymbol{\chi})}\left(\frac{x}{\varepsilon}\right). \tag{4.24}$$

Here $\boldsymbol{\psi}_{(\mathbf{E}, \boldsymbol{\chi})}$ is a solution to the following local problem

$$\inf_{\mathbf{v} \in K_{YF}} \frac{1}{Y} \int_{Y \setminus F} [j_{1}(\mathbf{E}, \boldsymbol{\kappa}_{y}(\mathbf{v}) + \boldsymbol{\chi})]\, dy\,. \tag{4.25}$$

Passing with ε to zero in the functional $G^{\varepsilon}(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}^{\varepsilon})$ we obtain, (*cf.* Refs. [18, 19])

$$\lim_{\varepsilon \to 0} G^{\varepsilon}(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}^{\varepsilon}) = \int_{\Omega} j_{0}[\mathbf{e}(\mathbf{r}(x)), \boldsymbol{\rho}(\boldsymbol{\varphi}(x))]\, dx +$$

$$+ \int_{\Omega} \frac{1}{|Y|} \int_{Y \setminus F} \{j_{1}[\mathbf{e}(\mathbf{r}(x)), \boldsymbol{\chi} + \boldsymbol{\kappa}_{y}(\boldsymbol{\psi}_{(\mathbf{e}(\mathbf{r}(x)), \boldsymbol{\chi}))}(y))]\, dy\}\, dx =$$

$$= \int_{\Omega} \{j_{0}[\mathbf{e}(\mathbf{r}(x)), \boldsymbol{\rho}(\boldsymbol{\varphi}(x))] + W_{1}(\mathbf{e}(\mathbf{r}(x)), \boldsymbol{\chi})\}\, dx\,.$$

Equation (4.22) gives $\boldsymbol{\kappa}(\boldsymbol{\psi}) = \boldsymbol{\chi}$. Consequently we may write

$$\lim_{\varepsilon \to 0} G^{\varepsilon}(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}^{\varepsilon}) =$$

$$= \int_{\Omega} \{j_{0}[\mathbf{e}(\mathbf{r}(x)), \boldsymbol{\rho}(\boldsymbol{\varphi}(x))]\, dx + W_{1}[\mathbf{e}(\mathbf{r}(x)), \boldsymbol{\kappa}(\boldsymbol{\psi}(x))]\}$$

$$= G(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi})\,.$$

Step 4.

Let $\{\boldsymbol{\psi}^{\varepsilon}\}_{\varepsilon>0} \subset K^{\varepsilon}$ be a sequence of functions converging strongly to a certain $\boldsymbol{\psi} \in L^2(\Omega, \mathbb{R}^2)$. We take $\mathbf{q}$, a continuous piecewise affine function ; hence

$$q_\alpha(x) = \chi^i_{\alpha\beta}\, x_\beta + z^i_\alpha\,, \quad x \in \Omega_i\,, \tag{4.26}$$

$$\boldsymbol{\kappa}(\mathbf{q}) = \boldsymbol{\chi}^i\,, \quad \text{on} \quad \Omega_i\,, \tag{4.27}$$

where $\boldsymbol{\chi}^i \in \mathbb{E}^2_s$, $\mathbf{z}^i \in \mathbb{R}^2$, $i \in J$.

Let us denote by $\tilde{\mathbf{v}}^i\,(i \in J)$ a solution to the following local problem

$$\inf_{\mathbf{v} \in K_{YF}} \frac{1}{Y} \int_{Y \setminus F} [j_1(\mathbf{E}, \boldsymbol{\kappa}_y(\mathbf{v}) + \boldsymbol{\chi}^i)]\, dy\,. \tag{4.28}$$

Obviously, $\tilde{\mathbf{v}}^i$ depends on $\mathbf{E}$ and $\boldsymbol{\chi}^i$. We set

$$\mathbf{q}^{\varepsilon, i}(x) = \mathbf{q}(x) + \varepsilon \tilde{\mathbf{v}}^i\left(\frac{x}{\varepsilon}\right), \quad i \in J\,. \tag{4.29}$$

Clearly, for each $i \in J$ we have $\mathbf{q}^{\varepsilon, i} \underset{\varepsilon \to 0}{\to} \mathbf{q}$ strongly in $L^2(\Omega, \mathbb{R}^2)$.

Next, let us introduce, for each $i \in J$, a function $\varphi_i \in \mathcal{D}(\Omega_i)$ such that $0 \leqslant \varphi_i(x) \leqslant 1$, $x \in \Omega_i$. The fact that the function j is positive implies

$$\int_\Omega j_0(\mathbf{e}(\mathbf{r}), \boldsymbol{\rho}(\boldsymbol{\varphi}))\, dx + \int_{\Omega^\varepsilon} j_1(\mathbf{e}(\mathbf{r}), \boldsymbol{\kappa}(\boldsymbol{\psi}^\varepsilon))\, dx \geqslant$$

$$\geqslant \sum_{i \in J} \int_{\Omega_i} \varphi_i(x)\, j_0(\mathbf{e}(\mathbf{r}), \boldsymbol{\rho}(\boldsymbol{\varphi}))\, dx + \sum_{i \in J} \int_{\omega_i^\varepsilon} \varphi_i\, j_1(\mathbf{e}(\mathbf{r}), \boldsymbol{\kappa}(\boldsymbol{\psi}^\varepsilon))\, dx\,, \tag{4.30}$$

where $\omega_i^\varepsilon = \Omega^\varepsilon \cap \Omega_i$. Subdifferential inequality for the subdifferential at the point $\boldsymbol{\kappa}(\mathbf{q}^{\varepsilon, i})$ gives

$$\sum_{i \in J} \int_{\Omega_i} \varphi_i\, j_0(\mathbf{e}(\mathbf{r}), \boldsymbol{\rho}(\boldsymbol{\varphi}))\, dx + \sum_{i \in J} \int_{\omega_i^\varepsilon} \varphi_i\, j_1(\mathbf{e}(\mathbf{r}), \boldsymbol{\kappa}(\boldsymbol{\psi}^\varepsilon))\, dx -$$

$$- \sum_{i \in J} \int_{\Omega_i} \varphi_i\, j_0(\mathbf{e}(\mathbf{r}), \boldsymbol{\rho}(\boldsymbol{\varphi}))\, dx - \sum_{i \in J} \int_{\omega_i^\varepsilon} \varphi_i\, j_1(\mathbf{e}(\mathbf{r}), \boldsymbol{\kappa}(\mathbf{q}^{\varepsilon, i}))\, dx \geqslant$$

$$\geqslant \sum_{i \in J} \int_{\Omega_i} \varphi_i\, D_{\alpha\beta}[(\mathbf{e}(\mathbf{r}), \boldsymbol{\kappa}(\mathbf{q}^{\varepsilon, i}))]\, \kappa_{\alpha\beta}(\boldsymbol{\psi}^\varepsilon - \mathbf{q}^{\varepsilon, i})\, dx\,, \tag{4.31}$$

where

$$D_{\alpha\beta}[\mathbf{e}(\mathbf{r}), \boldsymbol{\kappa}(\mathbf{q}^{\varepsilon, i})] = [D_2 j_1(\mathbf{e}(\mathbf{r}), \boldsymbol{\kappa}(\mathbf{q}^{\varepsilon, i}))]_{\alpha\beta} =$$

$$= \frac{b^3}{3} C_{\alpha\beta\lambda\mu}\, \kappa_{\lambda\mu}(\mathbf{q}^{\varepsilon, i}) + \frac{b^2}{2} C_{\alpha\beta\mu}\, e_{\lambda\mu}(\mathbf{r})$$

Here $D_2 j_1(\mathbf{e}(\mathbf{r}), \,.\,)$. D is the gradient of the function $j_1(\mathbf{e}(\mathbf{r}), \,.)$

By using (4.23) we arrive at

$$\lim_{\varepsilon \to 0} \left[\int_{\Omega_i} \varphi_i j_0(\mathbf{e}(\mathbf{r}), \boldsymbol{\rho}(\boldsymbol{\varphi}))\, dx + \int_{\omega_i^\varepsilon} \varphi_i j_1(\mathbf{e}(\mathbf{r}), \boldsymbol{\kappa}(\mathbf{q}^{\varepsilon, i}))\, dx \right] =$$

$$= \int_{\Omega_i} \varphi_i [j_0(\mathbf{e}(\mathbf{r}), \boldsymbol{\rho}(\boldsymbol{\varphi})) + W_1(\mathbf{e}(\mathbf{r}), \boldsymbol{\kappa}(\mathbf{q}))]\, dx\,, \quad i \in J\,. \quad (4.32)$$

Integrating by parts the last term of the inequality (4.31) we readily obtain

$$\int_{\omega_i^\varepsilon} \varphi_i D_{\alpha\beta}(\mathbf{e}(\mathbf{r}), \boldsymbol{\kappa}(\mathbf{q}^{\varepsilon, i}))\, \kappa_{\alpha\beta}(\boldsymbol{\psi}^\varepsilon - \mathbf{q}^{\varepsilon, i})\, dx = A_i^\varepsilon + B_i^\varepsilon + C_i^\varepsilon\,, \quad (4.33)$$

where (no summation over i)

$$A_i^\varepsilon = - \int_{\omega_i^\varepsilon} \varphi_i [D_{\alpha\beta}(\mathbf{e}(\mathbf{r}), \boldsymbol{\kappa}(\mathbf{q}^{\varepsilon, i}))]_{,\beta}\, (\psi_\alpha^\varepsilon - q_\alpha^{\varepsilon, i})\, dx\,, \quad (4.34)$$

$$B_i^\varepsilon = - \int_{\omega_i^\varepsilon} D_{\alpha\beta}(\mathbf{e}(\mathbf{r}), \boldsymbol{\kappa}(\mathbf{q}^{\varepsilon, i}))\, \varphi_{i,\beta}\, (\psi_\alpha^\varepsilon - q_\alpha^{\varepsilon, i})\, dx\,, \quad (4.35)$$

$$C_i^\varepsilon = \int_{F_i^\varepsilon} \varphi_i [(\psi_\alpha^\varepsilon - q_\alpha^{\varepsilon, i})\big|_1\, (D_{\alpha\beta}(\mathbf{e}(\mathbf{r}), \boldsymbol{\kappa}(\mathbf{q}^{\varepsilon, i}))\, n_\beta)\big|_1 -$$

$$- (\psi_\alpha^\varepsilon - q_\alpha^{\varepsilon, i})\big|_2\, (D_{\alpha\beta}(\mathbf{e}(\mathbf{r}), \boldsymbol{\kappa}(\mathbf{q}^{\varepsilon, i})\, n_\beta))\big|\,]_2\, dS\,. \quad (4.36)$$

Here $F_i^\varepsilon = F^\varepsilon \cap \Omega_i$.

Before proceeding further, some localization considerations are indispensable. The local minimization problem (4.28) is equivalent to the following variational inequality [1] :

$$\left| \begin{array}{l} \text{find } \tilde{\mathbf{v}}^i \in K_{\mathrm{YF}} \text{ such that} \\ \displaystyle\int_{Y \setminus F} D_{\alpha\beta}(\mathbf{E}, \boldsymbol{\kappa}_y(\tilde{\mathbf{v}}^i) + \boldsymbol{\chi}_i)\, \kappa_{y\alpha\beta}(\mathbf{v} - \tilde{\mathbf{v}}^i)\, dy \geq 0\,, \quad \forall \mathbf{v} \in K_{YF}\,, \end{array} \right. \quad (4.37)$$

where $\mathbf{E} = \mathbf{e}(\mathbf{r})$. Let us now take $\mathbf{v} = \tilde{\mathbf{v}}^i + \boldsymbol{\varphi}$, $\boldsymbol{\varphi} \in K_{YF}$. Then (4.37) gives

$$\int_{Y \setminus F} D_{\alpha\beta}(\mathbf{E}, \boldsymbol{\kappa}_y(\tilde{\mathbf{v}}^i) + \boldsymbol{\chi}_i)\, \kappa_{y\alpha\beta}(\boldsymbol{\varphi})\, dy \geq 0\,, \quad \forall \boldsymbol{\varphi} \in K_{YF}\,. \quad (4.38)$$

Particularly, one can take $\boldsymbol{\varphi} \in \mathcal{D}(Y \setminus F)$, that is $\boldsymbol{\varphi}$ equals (0, 0) in a neighbourhood of $\partial(Y \setminus F) = \partial Y \cup F$. In such a case from (4.38) one obtains

$$- \frac{\partial}{\partial y_\beta} D_{\alpha\beta}(\mathbf{E}, \boldsymbol{\kappa}_y(\tilde{\mathbf{v}}^i) + \boldsymbol{\chi}^i) = 0\,, \qquad \text{in} \qquad \mathcal{D}'(Y \setminus F, \mathbb{R}^2)\,. \quad (4.39)$$

Let us consider (4.38) once again and take $\boldsymbol{\varphi} \in K_{YF}$ such that $\boldsymbol{\varphi} = (0, 0)$ in a neighbourhood of F. Integrating by parts and taking account of equation (4.39) we deduce that

$$\text{vectors } [D_{\alpha\beta}(\mathbf{E}, \boldsymbol{\kappa}_y(\tilde{\mathbf{v}}^i) + \boldsymbol{\chi}_i) N_\beta] \text{ are opposite at the opposite sides of } Y. \tag{4.40}$$

Therefore equation (4.39) can be extended to $\mathbb{R}^2 \setminus \cup (F + \mathfrak{N}_1 y_1, \mathfrak{N}_2 y_2)$, that is

$$-\frac{\partial}{\partial y_\beta} D_{\alpha\beta}(\mathbf{E}, \boldsymbol{\kappa}_y(\tilde{\mathbf{v}}^i) + \boldsymbol{\chi}^i) = 0\,, \quad \text{in} \quad \mathscr{D}'(\mathbb{R}^2 \setminus \cup (F + \mathfrak{N}_1, \mathfrak{N}_2 y_2))\,, \tag{4.41}$$

where $\mathfrak{N}_1, \mathfrak{N}_2 \in \mathfrak{Z}$, and $\mathfrak{Z}$ stands for the set of integers.

Let us return to the inequality (4.37). Integrating by parts and taking account of (4.39) and (4.40) we arrive at the following relation

$$\int_F \Big\{ (v_\alpha - \tilde{\mathbf{v}}^i_\alpha)\big|_1 [D_{\alpha\beta}(\mathbf{E}, \boldsymbol{\kappa}_y(\tilde{\mathbf{v}}^i) + \boldsymbol{\chi}^i)]\big|_1 N_\beta - - (v_\alpha - \tilde{\mathbf{v}}^i_\alpha)\big|_2 [D_{\alpha\beta}(\mathbf{E}, \boldsymbol{\kappa}_y(\tilde{\mathbf{v}}^i) + \boldsymbol{\chi}^i)]\big|_2 N_\beta \Big\}\, ds \geqslant 0\,, \quad \forall \mathbf{v} \in K_{YF}\,. \tag{4.42}$$

Now we replace $\mathbf{v}$ by $\mathbf{t} = (1 - \varphi)(\tilde{\mathbf{v}}^i + \boldsymbol{\chi}^i y) + \varphi \boldsymbol{\xi} - \boldsymbol{\chi}^i y$, where $\boldsymbol{\xi} \in K_{YF}$ and $0 \leqslant \varphi \leqslant 1$, $\varphi \in \mathscr{D}(Y)$. Obviously, we have $[\![\mathbf{t}]\!] \geqslant 0$, on F. In such a case the inequality (4.42) yields

$$\int_F \varphi \Big\{ [\xi_\alpha - \tilde{\mathbf{v}}^i_\alpha + \chi^i_{\alpha\gamma} y_\gamma]\big|_1 [D_{\alpha\beta}(\mathbf{E}, \boldsymbol{\kappa}_y(\tilde{\mathbf{v}}^i) + \boldsymbol{\chi}^i)]\big|_1 N_\beta - - [\xi_\alpha - (\tilde{\mathbf{v}}^i_\alpha + \chi^i_{\alpha\gamma} y_\gamma)\big|_2 [D_{\alpha\beta}(\mathbf{E}, \boldsymbol{\kappa}_y(\tilde{\mathbf{v}}^i) + \boldsymbol{\chi}^i)]\big|_2 N_\beta \Big\}\, ds \geqslant 0\,, \tag{4.43}$$

$$\forall \boldsymbol{\xi} \in K_{YF}\,, \qquad \varphi \in \mathscr{D}^+(Y)\,.$$

By changing scale $\left(y \to \dfrac{x}{\varepsilon}\right)$, from (4.41) and (4.43), we obtain

$$\{D_{\alpha\beta}[\mathbf{e}(\mathbf{r}(\mathbf{x})), \boldsymbol{\kappa}(\mathbf{q}^{\varepsilon,i}(x))]\}_{,\beta} = 0\,, \qquad \text{in} \qquad \mathscr{D}'(\Omega^\varepsilon)\,, \tag{4.44}$$

and

$$\int_{F^\varepsilon_i} \varphi_i \Big\{ (\psi^\varepsilon_\alpha - q^{\varepsilon,i}_\alpha)\big|_1 [D_{\alpha\beta}(\mathbf{e}(\mathbf{r}), \boldsymbol{\kappa}(\mathbf{q}^{\varepsilon,i}))]\big|_1 n_\beta - - (\psi^\varepsilon_\alpha - q^{\varepsilon,i}_\alpha)\big|_2 [D_{\alpha\beta}(\mathbf{e}(\mathbf{r}), \boldsymbol{\kappa}(\mathbf{q}^{\varepsilon,i}))]\big|_2 n_\beta \Big\}\, ds \geqslant 0\,, \tag{4.45}$$

respectively, since $\boldsymbol{\psi}^\varepsilon \in K^\varepsilon$ and $\varphi_i \in \mathscr{D}^+(\omega^\varepsilon_i)$. Now, taking account of

(4.44) and (4.45) from (4.30), (4.31) and (4.33)-(4.36) we derive the following relation

$$G^{\varepsilon}(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}^{\varepsilon}) = \int_{\Omega} j_0(\mathbf{e}(\mathbf{r}), \boldsymbol{\rho}(\boldsymbol{\varphi}))\, dx + \int_{\Omega^{\varepsilon}} j_1(\mathbf{e}(\mathbf{r}), \boldsymbol{\kappa}(\boldsymbol{\psi}^{\varepsilon}))\, dx \geqslant$$

$$\geqslant \sum_{i \in J} \int_{\Omega_i} \varphi_i\, j_0(\mathbf{e}(\mathbf{r}), \boldsymbol{\rho}(\boldsymbol{\varphi}))\, dx + \sum_{i \in J} \int_{\omega_i^{\varepsilon}} \varphi_i\, j_1(\mathbf{e}(\mathbf{r}), \boldsymbol{\kappa}(\mathbf{q}^{\varepsilon, i}))\, dx -$$

$$- \sum_{i \in J} \int_{\omega_i^{\varepsilon}} D_{\alpha\beta}(\mathbf{e}(\mathbf{r}), \boldsymbol{\kappa}(\mathbf{q}^{\varepsilon, i}))\, \varphi_{i,\beta}(\psi_{\alpha}^{\varepsilon} - q_{\alpha}^{\varepsilon, i})\, dx\,. \tag{4.46}$$

Before passing to the limit ($\varepsilon \to 0$) in (4.46) we shall demonstrate the following property

$$D_2 W_1(\mathbf{E}, \boldsymbol{\chi}) = \frac{1}{|Y|} \int_{Y \setminus F} D_2 j_1(\mathbf{E}, \boldsymbol{\kappa}_y(\tilde{\mathbf{v}}) + \boldsymbol{\chi})\, dy\,, \tag{4.47}$$

where $\tilde{\mathbf{v}} \in K_{YF}$ realizes the minimum in (4.3). Obviously we have

$$[D_2 j_1(\mathbf{E}, \boldsymbol{\kappa}_y(\tilde{\mathbf{v}}) + \boldsymbol{\chi})]_{\alpha\beta} = D_{\alpha\beta}(\mathbf{E}, \boldsymbol{\kappa}_y(\tilde{\mathbf{v}}) + \boldsymbol{\chi}) =$$

$$= \frac{b^3}{3} C_{\alpha\beta\lambda\mu}(\boldsymbol{\kappa}_{y\lambda\mu}(\tilde{\mathbf{v}}) + \chi_{\alpha\beta}) + \frac{b^2}{2} C_{\alpha\beta\lambda\mu} E_{\lambda\mu}\,. \tag{4.48}$$

Since $j_1(\mathbf{E}, .\,)$ is convex and finite, therefore we have, *cf.* [21]

$$D_2 j_1(\mathbf{E}, .\,) = \partial_2 j_1(\mathbf{E}, .\,)\,. \tag{4.49}$$

Here $\partial_2 j_1$ is the subdifferential of the function $j_1(\mathbf{E}, .\,)$. Similarly we can write

$$D_2 W_1(\mathbf{E}, \boldsymbol{\chi}) = \partial_2 W_1(\mathbf{E}, \boldsymbol{\chi})\,. \tag{4.50}$$

Let us denote by $(m_{\alpha\beta}(y))$ ($y \in Y \setminus F$) the microscopic moment tensor corresponding to $\tilde{\mathbf{v}}(y)$, that is, *cf.* (4.37)

$$m_{\alpha\beta}(y) = D_{\alpha\beta}(\mathbf{E}, \boldsymbol{\kappa}_y(\tilde{\mathbf{v}}) + \boldsymbol{\chi})\,,$$

and

$$\int_{Y \setminus F} m_{\alpha\beta}(y)\, \kappa_{y\alpha\beta}(\mathbf{v} - \tilde{\mathbf{v}})\, dy \geqslant 0\,, \qquad \forall \mathbf{v} \in K_{YF}\,. \tag{4.51}$$

By using (4.49) and (4.50) we have the following subdifferential inequality

$$W_1(\mathbf{E}, \boldsymbol{\chi}^{(1)}) - W_1(\mathbf{E}, \boldsymbol{\chi}) =$$

$$= \frac{1}{|Y|} \int_{Y \setminus F} D_2 j_1(\mathbf{E}, \boldsymbol{\kappa}_y(\tilde{\mathbf{v}}^{(1)}) + \boldsymbol{\chi}^{(1)}) - D_2 j_1(\mathbf{E}, \boldsymbol{\kappa}_y(\tilde{\mathbf{v}}) + \boldsymbol{\chi})] \, dy \geqslant$$

$$\geqslant \frac{1}{|Y|} \int_{Y \setminus F} m_{\alpha\beta}(y) \, [\kappa_{y\alpha\beta}(\tilde{\mathbf{v}}^{(1)}) + \chi_{\alpha\beta}^{(1)} - (\kappa_{y\alpha\beta}(\tilde{\mathbf{v}}) + \chi_{\alpha\beta})] \, dy \, .$$

Hence, taking account of (4.51) we obtain

$$W_1(\mathbf{E}, \boldsymbol{\chi}^{(1)}) - W_1(\mathbf{E}, \boldsymbol{\chi}) \geqslant \left\{ \frac{1}{|Y|} \int_{Y \setminus F} m_{\alpha\beta}(y) \, dy \right\} (\chi_{\alpha\beta}^{(1)} - \chi_{\alpha\beta}) =$$

$$= L_{\alpha\beta}^h (\chi_{\alpha\beta}^{(1)} - \chi_{\alpha\beta}) \, , \quad (4.52)$$

where

$$L_{\alpha\beta}^h = \frac{1}{|Y|} \int_{Y \setminus F} m_{\alpha\beta}(y) \, dy \, , \quad (4.53)$$

are macroscopic moments in the lower layer of the homogenized plate.

Now we can pass to the limit in (4.46) as $\varepsilon \to 0$. First, we notice that

$$\boldsymbol{\psi}^{\varepsilon} - \mathbf{q}^{\varepsilon, i} \underset{\varepsilon \to 0}{\to} \boldsymbol{\psi} - \mathbf{q} \quad \text{strongly in} \quad L^2(\Omega, \mathbb{R}^2), \quad (4.54)$$

$$D_{\alpha\beta}(\mathbf{e}(\mathbf{r})), \boldsymbol{\kappa}(\mathbf{q}^{\varepsilon, i})) =$$

$$= D_{\alpha\beta}\left[\mathbf{e}(\mathbf{r}), \boldsymbol{\chi}^i + \boldsymbol{\kappa}(\tilde{\mathbf{v}}^i)\left(\frac{x}{\varepsilon}\right)\right] \underset{\varepsilon \to 0}{\to} \frac{1}{|Y|} \int_{Y \setminus F} D_{\alpha\beta}[\mathbf{e}(\mathbf{r}), \boldsymbol{\chi}^i + \boldsymbol{\kappa}_y(\tilde{\mathbf{v}}^i(y))] \, dy \, , \quad (4.55)$$

weakly in $L^2(\Omega, \mathbb{E}_s^2)$ (*cf.* Refs. [18, 19]). Taking account of equation (4.47) we see that the limit in (4.55) is equal just to $D_2 W_1(\mathbf{e}(\mathbf{r}), \boldsymbol{\chi}^i)$. Thus we obtain

$$\liminf_{\varepsilon \to 0} G^{\varepsilon}(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}^{\varepsilon}) \geqslant \sum_{i \in J} \int_{\Omega_i} \varphi_i(x) \, [j_0(\mathbf{e}(\mathbf{r}), \boldsymbol{\rho}(\boldsymbol{\varphi})) + W_1(\mathbf{e}(\mathbf{r}), \boldsymbol{\kappa}(\mathbf{q}))] \, dx -$$

$$- \sum_{i \in J} \int_{\Omega_i} [D_2 W_1(\mathbf{e}(\mathbf{r}), \boldsymbol{\chi}^i)]_{\alpha\beta} \, \varphi_{i,\beta}(\psi_\alpha - q_\alpha) \, dx \, . \quad (4.56)$$

According to theorem 2.2 of the paper [20], there exists an extension operator

$$\mathbb{Q}^{\varepsilon} : H^1(\Omega^{\varepsilon}, \mathbb{R}^2) \to H^1(\Omega, \mathbb{R}^2)$$

such that the sequence $\{\mathbb{Q}^{\varepsilon} \mathbf{v}^{\varepsilon}\}_{\varepsilon \to 0}$ is bounded and $\|\mathbb{Q}^{\varepsilon} \mathbf{v}^{\varepsilon} - \mathbf{v}^{\varepsilon}\|_{L^2(\Omega, \mathbb{R}^2)} \to 0$ as $\varepsilon \to 0$, provided that a sequence $\{\mathbf{v}^{\varepsilon}\}_{\varepsilon \to 0}$ is bounded in $H^1(\Omega^{\varepsilon}, \mathbb{R}^2)$.

By using this result and knowing that $\boldsymbol{\psi}^{\varepsilon} \underset{\varepsilon \to 0}{\to} \boldsymbol{\psi}$ in $L^2(\Omega, \mathbb{R}^2)$ we infer that $\psi \in H^1(\Omega, \mathbb{R}^2)$. Performing integration by parts in the last term of the inequality (4.56) we obtain

$$\liminf_{\varepsilon \to 0} G^{\varepsilon}(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}^{\varepsilon}) \geq \sum_{i \in J} \int_{\Omega_i} \varphi_i(x) \, [j_0(\mathbf{e}(\mathbf{r}), \boldsymbol{\rho}(\boldsymbol{\varphi})) +$$

$$+ W_1(\mathbf{e}(\mathbf{r}), \boldsymbol{\kappa}(\mathbf{q}))] \, dx + \sum_{i \in J} \int_{\Omega_i} \varphi_i(x) \, [D_2 W_1(\mathbf{e}(\mathbf{r}), \boldsymbol{\kappa}(\mathbf{q}))]_{\alpha\beta,\,\beta} (\psi_{\alpha} - q_{\alpha}) \, dx +$$

$$+ \sum_{i \in J} \int_{\Omega_i} \varphi_i(x) \, [D_2 W_1(\mathbf{e}(\mathbf{r}), \boldsymbol{\kappa}(\mathbf{q}))]_{\alpha\beta} \, \kappa_{\alpha\beta} \, (\boldsymbol{\psi} - \mathbf{q}) \, dx \,, \tag{4.57}$$

since $\boldsymbol{\kappa}(\mathbf{q}) = \boldsymbol{\chi}^i$ on Ω_i. Let now φ_i converges to 1 for $i \in J$. Hence we can write

$$\liminf_{\varepsilon \to 0} G^{\varepsilon}(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}^{\varepsilon}) \geq \int_{\Omega} [j_0(\mathbf{e}(\mathbf{r}), \boldsymbol{\rho}(\boldsymbol{\varphi})) + W_1(\mathbf{e}(\mathbf{r}), \boldsymbol{\kappa}(\mathbf{q}))] \, dx +$$

$$+ \int_{\Omega} [D_2 W_1(\mathbf{e}(\mathbf{r}), \boldsymbol{\kappa}(\mathbf{q}))]_{\alpha\beta,\,\beta} (\psi_{\alpha} - q_{\alpha}) \, dx +$$

$$+ \int_{\Omega} [D_2 W_1(\mathbf{e}(\mathbf{r}), \boldsymbol{\kappa}(\mathbf{q}))]_{\alpha\beta} \, \kappa_{\alpha\beta} (\boldsymbol{\psi} - \mathbf{q}) \, dx \,. \tag{4.58}$$

In the last inequality the function $\mathbf{q}$ is continuous and piecewise affine. Thus one can use an argumentation based on the density of such functions in the space $H^1(\Omega, \mathbb{R}^2)$. Because the functional $G(\mathbf{r}, \boldsymbol{\varphi}, \,.\,)$ is convex and continuous on the space $H^1(\Omega, \mathbb{R}^2)$, the inequality (4.58) can be extended to an arbitrary $\mathbf{q} \in H^1(\Omega, \mathbb{R}^2)$.

Setting now $\boldsymbol{\psi} = \mathbf{q}$ we finally arrive at the following inequality

$$\liminf_{\varepsilon \to 0} \; G^{\varepsilon}(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}^{\varepsilon}) \geq G(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}) \,,$$

what proves (4.8) ■

Remark 4.1 : From the above considerations we conclude that

$$J^{\varepsilon}(\mathbf{r}^{\varepsilon}, \boldsymbol{\varphi}^{\varepsilon}, \boldsymbol{\psi}^{\varepsilon}, w^{\varepsilon}) \to J^h(\mathbf{r}^h, \boldsymbol{\varphi}^h, \boldsymbol{\psi}^h, w^h) \quad \text{as} \quad \varepsilon \to 0 \,, \tag{4.59}$$

where (*cf.* Ref. [1])

$$(P_{\varepsilon})\, J^{\varepsilon}(\mathbf{r}^{\varepsilon}, \boldsymbol{\varphi}^{\varepsilon}, \boldsymbol{\psi}^{\varepsilon}, w^{\varepsilon}) =$$

$$= \inf \left\{J^{\varepsilon}(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}, w) \mid \mathbf{r}, \boldsymbol{\varphi} \in H_0^1(\Omega, \mathbb{R}^2), \boldsymbol{\psi} \in K_{\varepsilon}, w \in H_0^1(\Omega)\right\},$$

$$(P_h)\, J^h(\mathbf{r}^h, \boldsymbol{\varphi}^h, \boldsymbol{\psi}^h, w^h) =$$

$$= \inf \left\{J^h(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}, w) \mid \mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi} \in H_0^1(\Omega, \mathbb{R}^2), w \in H_0^1(\Omega)\right\}.$$

Indeed, according to our theorem 4.1 and comments preceding it we infer that for $(\mathbf{r}^h, \boldsymbol{\varphi}^h, \boldsymbol{\psi}^h, w^h) \in [H_0^1(\Omega, \mathbb{R}^2)]^3 \times H_0^1(\Omega)$ there exists a sequence $\left\{\tilde{\mathbf{r}}^{\varepsilon}, \tilde{\boldsymbol{\varphi}}^{\varepsilon}, \tilde{\boldsymbol{\psi}}^{\varepsilon}, \tilde{\mathbf{w}}^{\varepsilon}\right\}_{\varepsilon > 0} \subset [H_0^1(\Omega, \mathbb{R}^2)]^2 \times K_{\varepsilon} \times H_0^1(\Omega)$ strongly convergent in $[L^2(\Omega, \mathbb{R}^2)]^3 \times L^2(\Omega)$ and such that

$$J^h(\mathbf{r}^h, \boldsymbol{\varphi}^h, \boldsymbol{\psi}^h, w^h) \geq \liminf_{\varepsilon \to 0} J^{\varepsilon}\left\{\tilde{\mathbf{r}}^{\varepsilon}, \tilde{\boldsymbol{\varphi}}^{\varepsilon}, \tilde{\boldsymbol{\psi}}^{\varepsilon}, \tilde{\mathbf{w}}^{\varepsilon}\right\}, \tag{4.60}$$

see the formula (4.7). Moreover, taking account of (4.8) we can write

$$\liminf_{\varepsilon \to 0} J^{\varepsilon}(\mathbf{r}^{\varepsilon}, \boldsymbol{\varphi}^{\varepsilon}, \boldsymbol{\psi}^{\varepsilon}, w^{\varepsilon}) \geq J^h(\mathbf{r}^h, \boldsymbol{\varphi}^h, \boldsymbol{\psi}^h, w^h). \tag{4.61}$$

Since $(\mathbf{r}^{\varepsilon}, \boldsymbol{\varphi}^{\varepsilon}, \boldsymbol{\psi}^{\varepsilon}, w^{\varepsilon})$ is a minimizer of the problem (P_{ε}), therefore we have

$$J^{\varepsilon}(\mathbf{r}^{\varepsilon}, \boldsymbol{\varphi}^{\varepsilon}, \boldsymbol{\psi}^{\varepsilon}, w^{\varepsilon}) \leq J^{\varepsilon}\left\{\tilde{\mathbf{r}}^{\varepsilon}, \tilde{\boldsymbol{\varphi}}^{\varepsilon}, \tilde{\boldsymbol{\psi}}^{\varepsilon}, \tilde{\mathbf{w}}^{\varepsilon}\right\}. \tag{4.62}$$

The relations (4.60)-(4.62) demonstrate that the total potential energy of the fissured plate converges to the total potential energy of the homogenized plate as $\varepsilon \to 0$. We also note that such a convergence holds true for more general boundary conditions like mixed ones.

5. DUAL HOMOGENIZATION

The direct homogenization of fissured plates performed both in [1] and in the previous section involves generalized displacements only. Now we shall study the problem of the dual homogenization involving generalized stresses.

As we know, the matrices $\mathbf{C}$ and $\mathbf{D}^{(1)}$ given by (I.2.4) and (I.2.27), respectively, are non-singular. Hence we can write

$$\mathbf{D}^{-1} = [\mathbf{D}^{(1)}]^{-1} \otimes \mathbf{B}, \qquad \mathbf{B} = \mathbf{C}^{-1}, \tag{5.1}$$

where

$$[\mathbf{D}^{(1)}]^{-1} = \begin{bmatrix} \dfrac{4}{h} & \dfrac{6}{ah} & -\dfrac{6}{bh} \\ \dfrac{6}{ah} & \dfrac{12\,a + 3\,b}{a^3\,h} & -\dfrac{9}{abh} \\ -\dfrac{6}{bh} & -\dfrac{9}{abh} & \dfrac{3\,a + 12\,b}{b^3\,h} \end{bmatrix}. \tag{5.2}$$

Thus the inverse constitutive relations have the following form

$$\begin{bmatrix} \mathbf{e} \\ \boldsymbol{\varphi} \\ \boldsymbol{\kappa} \end{bmatrix} = \mathbf{D}^{-1} \begin{bmatrix} \mathbf{N} \\ \mathbf{M} \\ \mathbf{L} \end{bmatrix}, \tag{5.3}$$

$$g_\alpha = a^{-1} Z_{\alpha\beta} Q_\beta , \quad d_\alpha = b^{-1} Z_{\alpha\beta} T_\beta , \quad \mathbf{Z} = \mathbf{H}^{-1} . \tag{5.4}$$

To derive the dual problem (P_ε^*) we apply the Rockafellar theory of duality in the form presented by Ekeland and Temam [14]. Let us define Λ, a linear and continuous operator, in the following way

$$\begin{aligned} \Lambda(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}, w) &= (\Lambda_1 \mathbf{r}, \Lambda_2 \boldsymbol{\varphi} . \Lambda_3 \boldsymbol{\psi}, \Lambda_4(w, \boldsymbol{\varphi}), \Lambda_5(w, \boldsymbol{\psi})) \\ &= (\mathbf{e}(\mathbf{r}), \boldsymbol{\rho}(\boldsymbol{\varphi}), \boldsymbol{\kappa}(\boldsymbol{\psi}), \mathbf{g}(w, \boldsymbol{\varphi}), \mathbf{d}(w, \boldsymbol{\psi})), \end{aligned}$$

$$\begin{aligned} \Lambda : [H_0^1(\Omega, \mathbb{R}^2)]^2 \times H_1^1(\Omega^\varepsilon, \mathbb{R}^2) \times H_0^1(\Omega) \to \\ \to [L^2(\Omega, \mathbb{E}_s^2)]^2 \times L^2(\Omega^\varepsilon, \mathbb{E}_s^2) \times [L^2(\Omega, \mathbb{R}^2)]^2 , \end{aligned}$$

where

$$H_1^1(\Omega^\varepsilon, \mathbb{R}^2) = \left\{ \mathbf{v} = (v_\alpha) \middle| v_\alpha \in H^1(\Omega^\varepsilon), \ \mathbf{v} = 0 \text{ on } \partial\Omega \right\} . \tag{5.5}$$

To find the conjugate operator Λ^* we have to give explicit expressions for the operators $\Lambda_1^*, \dots, \Lambda_5^*$. Simple calculations yield

$$\begin{aligned} \langle \mathbf{N}, \Lambda_1 \mathbf{r} \rangle_{L^2 \times L^2} = \int_\Omega N_{\alpha\beta}\, e_{\alpha\beta}(\mathbf{r})\, dx = - \int_\Omega N_{\alpha\beta,\beta}\, r_\alpha\, dx = \\ = \langle \Lambda_1^* \mathbf{N}, \mathbf{r} \rangle_{H^{-1} \times H_0^1}, \end{aligned} \tag{5.6}$$

provided that $\mathbf{r} \in H_0^1(\Omega, \mathbb{R}^2)$. In a similar manner we obtain

$$\Lambda_2^* \mathbf{M} = (-M_{\alpha\beta,\beta}) , \quad \text{in } \Omega ; \tag{5.7}$$

$$\Lambda_3^* \mathbf{L} = \begin{cases} -L_{\alpha\beta,\beta} , & \text{in } \ \Omega^\varepsilon , \\ -L_n , & \text{on } \ F^\varepsilon ; \end{cases} \tag{5.8}$$

$$\Lambda_4^* \mathbf{Q} = \begin{cases} -Q_{\alpha,\alpha} , & \text{in } \ \Omega , \quad (w) \\ -\mathbf{Q} , & \text{in } \ \Omega , \quad (\varphi) \end{cases} \tag{5.9}$$

$$\Lambda_5^* \mathbf{T} = \begin{cases} -T_{\alpha,\alpha} , & \text{in } \ \Omega , \quad (w) \\ \ \ \mathbf{T} , & \text{in } \ \Omega , \quad (\psi) . \end{cases} \tag{5.10}$$

Further, we set

$$\mathscr{L}_\varepsilon(v, \boldsymbol{\psi}) = \Phi_2(v) + I_{K_\varepsilon}(\boldsymbol{\psi}) ,$$

where I_{K_ε} stands for the indicator function of the set K_ε. To formulate the dual problem P_ε^* we first calculate

$$\mathscr{L}_\varepsilon^*(-\Lambda^*(\mathbf{N}, \mathbf{M}, \mathbf{L}, \mathbf{Q}, \mathbf{T})) =$$

$$= \sup\left\{\int_\Omega (N_{\alpha\beta,\beta}\, r_\alpha + M_{\alpha\beta,\beta}\, \varphi_\alpha + Q_{\alpha,\alpha}\, w - Q_\alpha\, \varphi_\alpha + \right.$$

$$+ T_{\alpha,\alpha}\, w - T_\alpha\, \psi_\alpha + pw)\, dx + \int_{\Omega^\varepsilon} L_{\alpha\beta,\beta}\, \psi_\alpha\, dx + \int_{F^\varepsilon} L_n[\psi_n]\, ds$$

$$\left. - I_{K_\varepsilon}(\boldsymbol{\psi}) \,|\, \mathbf{r}, \boldsymbol{\varphi} \in H_0^1(\Omega, \mathbb{R}^2),\ \boldsymbol{\psi} \in H_1^1(\Omega^\varepsilon, \mathbb{R}^2),\ w \in H_0^1(\Omega)\right\}$$

$$= \begin{cases} 0, \text{ if} & N_{\alpha\beta,\beta} = 0,\ -M_{\alpha\beta,\beta} + Q_\alpha = 0, \text{ in } \Omega\,, \\ & (Q_\alpha + T_\alpha)_{,\alpha} + p = 0, \text{ in } \Omega\,, \\ & -L_{\alpha\beta,\beta} + T_\alpha = 0, \text{ in } \Omega^\varepsilon, \\ & L_n \leq 0, \text{ on } F^\varepsilon\,; \\ +\infty, \text{ otherwise}\,. \end{cases} \tag{5.11}$$

We set

$$G_2^\varepsilon(\Lambda(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}, w)) = \int_{\Omega^\varepsilon} j\,[\mathbf{e}(\mathbf{r}), \boldsymbol{\rho}(\boldsymbol{\varphi}), \boldsymbol{\kappa}(\boldsymbol{\varphi})]\, dx + + \int_\Omega j_2[\mathbf{g}(w, \boldsymbol{\varphi}), \mathbf{d}(w, \boldsymbol{\psi})]\, dx\,, \tag{5.12}$$

where

$$j_2[\mathbf{g}(w, \boldsymbol{\varphi}), \mathbf{d}(w, \boldsymbol{\psi})] = = \frac{1}{2}\,[Q_\alpha(w, \boldsymbol{\varphi})\, g_\alpha(w, \boldsymbol{\varphi}) + T_\alpha(w, \boldsymbol{\psi})\, d_\alpha(w, \boldsymbol{\psi})]\,. \tag{5.13}$$

In our case we can write, *cf.* reference [14, Chapter 9, Th. 2.1]

$$(G_2^\varepsilon)^*\,(\mathbf{N}, \mathbf{M}, \mathbf{L}, \mathbf{Q}, \mathbf{T}) = \int_{\Omega^\varepsilon} j^*[\mathbf{N}(x), \mathbf{M}(x), \mathbf{L}(x)]\, dx + + \int_\Omega j_2^*[\mathbf{Q}(x), \mathbf{T}(x)]\, dx\,, \tag{5.14}$$

where

$$j_2^*(\mathbf{Q}, \mathbf{T}) = \sup\left\{\mathbf{Q}\cdot\mathbf{g} + \mathbf{T}\cdot\mathbf{d} - j_2(\mathbf{g}, \mathbf{d}) \,|\, \mathbf{g}, \mathbf{d} \in \mathbb{R}^2\right\} = = \frac{1}{2}\,(a^{-1} Z_{\alpha\beta}\, Q_\alpha\, Q_\beta + b^{-1} Z_{\alpha\beta}\, T_\alpha\, T_\beta)\,, \tag{5.15}$$

$$j^*(\mathbf{N}, \mathbf{M}, \mathbf{L}) = \sup \{N_{\alpha\beta} e_{\alpha\beta} + M_{\alpha\beta} \rho_{\alpha\beta} + L_{\alpha\beta} \kappa_{\alpha\beta} - j(\mathbf{e}, \boldsymbol{\rho}, \boldsymbol{\kappa}) | \mathbf{e}, \boldsymbol{\rho}, \boldsymbol{\kappa} \in \mathbb{E}_s^2\}$$

$$= \frac{1}{2} [\mathbf{N}, \mathbf{M}, \mathbf{L}] \mathbf{D}^{-1} [\mathbf{N}, \mathbf{M}, \mathbf{L}]^t$$

$$= B_{\alpha\beta\lambda\mu} \left(\frac{2}{h} N_{\alpha\beta} N_{\lambda\mu} + \frac{6}{ah} M_{\alpha\beta} N_{\lambda\mu} - \frac{6}{bh} N_{\alpha\beta} L_{\lambda\mu} + \frac{3(4a+b)}{2a^3 h} M_{\alpha\beta} M_{\lambda\mu} + \frac{3(4b+a)}{2b^3 h} L_{\alpha\beta} L_{\lambda\mu} - \frac{9}{abh} L_{\alpha\beta} M_{\lambda\mu} \right), \quad (5.16)$$

where the superscript « t » denotes transposition.

For a fixed $\varepsilon > 0$ the set $\mathscr{C}_s^\varepsilon$ of statically admissible generalized stresses is defined by

$$\mathscr{C}_s^\varepsilon = \{\mathbf{N} \in L^2(\Omega, \mathbb{E}_s^2), \mathbf{M} \in L^2(\Omega, \mathbb{E}_s^2), \mathbf{L} \in L^2(\Omega^\varepsilon, \mathbb{E}_s^2), \mathbf{Q} \in L^2(\Omega, \mathbb{R}^2), \mathbf{T} \in L^2(\Omega, \mathbb{E}^2) | N_{\alpha\beta,\beta} = 0, \text{ in } \Omega ; - M_{\alpha\beta,\beta} + Q_\alpha = 0, \text{ in } \Omega ; (Q_\alpha + T_\alpha)_{,\alpha} + p = 0, \text{ in } \Omega ; - L_{\alpha\beta,\beta} + T_\alpha = 0, \text{ in } \Omega^\varepsilon ; L_n \leq 0, \text{ on } F^\varepsilon\} . \quad (5.17)$$

By using the aforementioned theory of duality [14] we can formulate the dual problem or the complementary energy principle for a fixed $\varepsilon > 0$.

Problem $P_\varepsilon^*(\varepsilon > 0)$

Find

$$\sup \left\{ - \int_\Omega [j^*(\mathbf{N}(x), \mathbf{M}(x), \mathbf{L}(x)) + j_2^*(\mathbf{Q}(x), \mathbf{T}(x))] \, dx \,|\, (\mathbf{N}, \mathbf{M}, \mathbf{L}, \mathbf{Q}, \mathbf{T}) \in \mathscr{C}_s^\varepsilon \right\} . \quad (5.18)$$

Introducing the indicator function of the set $\mathscr{C}_s^\varepsilon$ we can formulate the above problem in an equivalent way.

Problem R_ε^*

Find

$$\inf \left\{ \int_\Omega [j^*(\mathbf{N}(x), \mathbf{M}(x), \mathbf{L}(x)) + j_2^*(\mathbf{Q}(x), \mathbf{T}(x))] \, dx + I_{\mathscr{C}_s^\varepsilon}(\mathbf{N}, \mathbf{M}, \mathbf{L}, \mathbf{Q}, \mathbf{T}) \,|\, (\mathbf{N}, \mathbf{M}, \mathbf{L}, \mathbf{Q}, \mathbf{T}) \in \mathfrak{H} \right\} , \quad (5.19)$$

where

$$\mathfrak{H} = [L^2(\Omega, \mathbb{E}_s^2)]^3 \times [L^2(\Omega, \mathbb{R}^2)]^2 . \tag{5.20}$$

Obviously we have

$$\sup P_\varepsilon^* = -\inf R_\varepsilon^* .$$

By using theorem 4.2 of Ekeland and Temam [14, Chap. 3] we infer that a solution $(\mathbf{N}^\varepsilon, \mathbf{M}^\varepsilon, \mathbf{L}^\varepsilon, \mathbf{Q}^\varepsilon, \mathbf{T}^\varepsilon) \in \mathscr{C}_s^\varepsilon$ to the problem P_ε^* exists and is unique. We observe that the complementary potential $j^* + j_2^*$ is strictly convex.

Let us set

$$\mathfrak{S}^\varepsilon(\mathbf{N}, \mathbf{M}, \mathbf{L}, \mathbf{Q}, \mathbf{T}) = \int_\Omega [j^*(\mathbf{N}, \mathbf{M}, \mathbf{L}) + j_2^*(\mathbf{Q}, \mathbf{T})]\, dx + + I_{\mathscr{C}_s^\varepsilon}(\mathbf{N}, \mathbf{M}, \mathbf{L}, \mathbf{Q}, \mathbf{T}) . \tag{5.21}$$

Before discussing the problem of epi-convergence of the sequence of functionals $\{\mathfrak{S}^\varepsilon\}_{\varepsilon > 0}$ we shall derive the dual macroscopic potential W^*, where W is given by (I.4.35).

Towards this end we follow an approach primarily used by Telega [2] for a three-dimensional fissured solid. By using Fenchel transformation we write

$$\begin{aligned} W^*(\mathbf{N}, \mathbf{M}, \mathbf{L}, \mathbf{Q}, \mathbf{T}) &= \\ &= \sup \{N_{\alpha\beta} E_{\alpha\beta} + M_{\alpha\beta}\, \theta_{\alpha\beta} + L_{\alpha\beta}\, \chi_{\alpha\beta} + Q_\alpha\, g_\alpha + T_\alpha\, d_\alpha - \\ &\quad - W(\mathbf{E}, \boldsymbol{\theta}, \boldsymbol{\chi}, \mathbf{g}, \mathbf{d}) \,|\, \mathbf{E}, \boldsymbol{\theta}, \boldsymbol{\chi} \in \mathbb{E}_2^2 ;\, \mathbf{g}, \mathbf{d} \in \mathbb{R}^2\} \\ &= (j_0 + W_1)^*(\mathbf{N}, \mathbf{M}, \mathbf{L}) + j_2^*(\mathbf{Q}, \mathbf{T}) , \end{aligned} \tag{5.22}$$

where $\mathbf{N}, \mathbf{M}, \mathbf{L} \in \mathbb{E}_s^\varepsilon$ and $\mathbf{Q}, \mathbf{T} \in \mathbb{R}^2$; moreover

$$(j_0 + W_1)^*(\mathbf{N}, \mathbf{M}, \mathbf{L}) = \sup_{\mathbf{E}, \boldsymbol{\theta}, \boldsymbol{\chi} \in \mathbb{E}_s^2} \Big\{N_{\alpha\beta} E_{\alpha\beta} + M_{\alpha\beta}\, \theta_{\alpha\beta} + + L_{\alpha\beta}\, \chi_{\alpha\beta} - j_0(\mathbf{E}, \boldsymbol{\theta}) - \frac{1}{|Y|} \inf_{\mathbf{v} \in K_{YF}} \int_{Y \setminus F} j_1(\mathbf{E}, \boldsymbol{\kappa}_y(\mathbf{v}) + \boldsymbol{\chi})\, dy\Big\} . \tag{5.23}$$

To find the explicit form of the function $(j_0 + W_1)^*$ we consider the following convex optimization problem :

$$\begin{aligned} (P_{(\mathbf{E}, \boldsymbol{\chi})}) \inf \left\{\int_{Y \setminus F} j_1(\mathbf{E}, \boldsymbol{\kappa}_y(\mathbf{v}) + \boldsymbol{\chi})\, dy \,|\, \mathbf{v} \in K_{YF}\right\} &= \\ = \inf \left\{\int_{Y \setminus F} j_1(\mathbf{E}, \boldsymbol{\kappa}_y(\mathbf{v}) + \boldsymbol{\chi})\, dy + I_{K_{YF}}(\mathbf{v}) \,|\, \mathbf{v} \in H_{\text{per}}(Y \setminus F, \mathbb{R}^2)\right\} . \end{aligned}$$

Let us set $\Lambda \mathbf{v} = \boldsymbol{\kappa}_y(\mathbf{v})$ and

$$G_{(\mathbf{E}, \boldsymbol{\chi})}(\Lambda \mathbf{v}) = \int_{Y \setminus F} j_1(\mathbf{E}, \Lambda \mathbf{v} + \boldsymbol{\chi})\, dy, \ \mathbf{v} \in H^1(Y \setminus F, \mathbb{R}^2),$$

$$\mathcal{F}(\mathbf{v}) = I_{K_{YF}}(\mathbf{v}), \ \mathbf{v} \in H_{\mathrm{per}}(Y \setminus F, \mathbb{R}^2).$$

We see that $\Lambda : H^1(Y \setminus F, \mathbb{R}^2) \to L^2(Y \setminus F, \mathbb{E}_s^2)$. One readily finds the conjugate functional $G^*_{(\mathbf{E}, \boldsymbol{\chi})}$. It has the following form

$$G^*_{(\mathbf{E}, \boldsymbol{\chi})}(\mathbf{p}^*) = \int_{Y \setminus F} \tilde{j}_1(\mathbf{p}^*(y), \mathbf{E}, \boldsymbol{\chi})\, dy, \tag{5.24}$$

where

$$\tilde{j}_1(\mathbf{p}^*, \mathbf{E}, \boldsymbol{\chi}) = \sup \left\{ p^*_{\alpha\beta}\, p_{\alpha\beta} - j_1(\mathbf{E}, \mathbf{p} + \boldsymbol{\chi}) \mid \mathbf{p} \in \mathbb{E}_s^2 \right\} =$$
$$= \frac{3}{2\, b^3} B_{\alpha\beta\lambda\mu}\, p^*_{\alpha\beta}\, p^*_{\lambda\mu} - \frac{3}{2\, b} p^*_{\alpha\beta}\, E_{\alpha\beta} - p^*_{\alpha\beta}\, \chi_{\alpha\beta} - \frac{3\, b}{8} C_{\alpha\beta\lambda\mu}\, E_{\alpha\beta}\, E_{\lambda\mu}\,.$$

Further, we calculate

$$\langle \Lambda \mathbf{v}, \mathbf{p}^* \rangle_{L^2 \times L^2} = \int_{Y \setminus F} \kappa_{y\alpha\beta}(\mathbf{v})\, p^*_{\alpha\beta}\, dy = - \int_{Y \setminus F} p^*_{\alpha\beta,\, \beta}\, v_\alpha\, dy +$$
$$+ \int_{\partial Y} p^*_{\alpha\beta}\, n_\beta\, v_\alpha\, ds - \int_F p^*_N [\![v_N]\!]\, ds$$
$$- \int_F \mathbf{p}^*_T \cdot [\![\mathbf{v}_T]\!]\, ds = \langle \Lambda^* \mathbf{p}^*, \mathbf{v} \rangle_{H^{-1} \times H^1}\,.$$

Thus we can write

$$\Lambda^* \mathbf{p}^* = \begin{cases} - \operatorname{div}_y \mathbf{p}^*, & \text{in } \ Y \setminus F\,, \\ (p^*_{\alpha\beta}\, n_\beta)\,, & \text{on } \ \partial Y\,, \\ - p^*_N\,, & \text{on } \ F\,, \\ - \mathbf{p}^*_T\,, & \text{on } \ F\,. \end{cases} \tag{5.25}$$

Taking account of (5.25) we obtain

$$\mathcal{F}^*(- \Lambda^* \mathbf{p}^*) = \sup \left\{ \langle - \Lambda^* \mathbf{p}^*, \mathbf{v} \rangle - \mathcal{F}(\mathbf{v}) \mid \mathbf{v} \in H_{\mathrm{per}}(Y \setminus F, \mathbb{R}^2) \right\} =$$
$$= \begin{cases} 0 & \text{if } \ \mathbf{p}^* \in S^c_{\mathrm{per}}\,, \\ + \infty, & \text{otherwise}\,, \end{cases} \tag{5.26}$$

where

$$S^c_{\text{per}} = \{\mathbf{p}^* \in L^2(Y\backslash F, \mathbb{E}^2_s) | \operatorname{div}_y \mathbf{p}^* = 0, \quad \text{in } Y\backslash F ; \mathbf{p}^*_T = 0,$$

$$p^*_N \leqslant 0, \text{ on } F ; \mathbf{p}^* \cdot \mathbf{n} \text{ takes opposite values at opposite sides of } Y \} . \quad (5.27)$$

It is worth noting that

$$S^c_{\text{per}} = [\boldsymbol{\kappa}_y(K_{YF})]^* , \quad (5.28)$$

or S^c_{per} is the polar set of $\boldsymbol{\kappa}_y(K_{YF})$ (in $L^2(Y\backslash F, \mathbb{E}^2_s)$). To corroborate this statement we have to find the set $[\boldsymbol{\kappa}_y(K_{YF})]^*$. By definition, we have, *cf.* Refs. [10, 14, 21]

$$[\boldsymbol{\kappa}_y(K_{YF})]^* = \left\{ \mathbf{p}^* \in L^2(Y\backslash F, \mathbb{E}^2_s) \left| \int_{Y\backslash F} p^*_{\alpha\beta}\, \kappa_{y\alpha\beta}(\mathbf{v})\, dy \geqslant 0 \quad \forall \mathbf{v} \in K_{YF} \right. \right\} . \quad (5.29)$$

Performing the integration by parts and using (5.25) we arrive at (5.28).

Now taking account of the relations (5.24) and (5.26) we obtain the dual problem of $(P_{(\mathbf{E}, \chi)})$

$$(P^*_{(\mathbf{E}, \chi)}) \qquad - \inf \left\{ \int_{Y\backslash F} \tilde{j}_1(\mathbf{p}^*(y), \mathbf{E}, \boldsymbol{\chi})\, dy \,|\, \mathbf{p}^* \in S^c_{\text{per}} \right\} .$$

By applying the convex duality theorem [14] we write

$$\inf P_{(\mathbf{E}, \boldsymbol{\chi})} = - \inf P^*_{(\mathbf{E}, \boldsymbol{\chi})} . \quad (5.30)$$

Substituting (5.30) into (5.23) we obtain

$$(j_0 + W_1)^* (\mathbf{N}, \mathbf{M}, \mathbf{L}) = \sup_{\mathbf{E}, \boldsymbol{\theta}, \boldsymbol{\chi} \in \mathbb{E}^2_s} \{ N_{\alpha\beta} E_{\alpha\beta} + M_{\alpha\beta}\, \theta_{\alpha\beta} +$$

$$+ L_{\alpha\beta}\, \chi_{\alpha\beta} - j_0(\mathbf{E}, \boldsymbol{\theta}) + \frac{3b}{8} C_{\alpha\beta\lambda\mu} E_{\alpha\beta} E_{\lambda\mu}$$

$$+ \frac{1}{|Y|} \inf_{\mathbf{p}^* \in S^c_{\text{per}}} \int_{Y\backslash F} \left(\frac{3}{2b^3} B_{\alpha\beta\lambda\mu}\, p^*_{\alpha\beta}\, p^*_{\lambda\mu} \right.$$

$$\left. \left. - \frac{3}{2b} p^*_{\alpha\beta} E_{\alpha\beta} - p^*_{\alpha\beta}\, \chi_{\alpha\beta} \right) dy \right\} . \quad (5.31)$$

Let us set

$$j_3(\mathbf{p}^*, \mathbf{E}) = \frac{3}{2b^3} B_{\alpha\beta\lambda\mu}\, p^*_{\alpha\beta}\, p^*_{\lambda\mu} - \frac{3}{2b} p^*_{\alpha\beta} E_{\alpha\beta} . \quad (5.32)$$

Calculating the supremum with respect to $\boldsymbol{\theta}$ in (5.31) and taking account of (5.32) we obtain

$$(j_0 + W_1)^* (\mathbf{N}, \mathbf{M}, \mathbf{L}) = \frac{3}{2a^3} B_{\alpha\beta\lambda\mu} M_{\alpha\beta} M_{\lambda\mu} +$$

$$+ \sup_{\mathbf{E}, \boldsymbol{\chi} \in \mathbb{E}_s^2} \frac{1}{|Y|} \left\{ \int_{Y \setminus F} \left(N_{\alpha\beta} E_{\alpha\beta} - \frac{h}{8} C_{\alpha\beta\lambda\mu} E_{\alpha\beta} E_{\lambda\mu} + \frac{3}{2a} M_{\alpha\beta} E_{\alpha\beta} \right) dy \right.$$

$$\left. + \inf_{\mathbf{p}^* \in S_{\text{per}}^c} \left[\int_{Y \setminus F} j_3(\mathbf{p}^*(y), \mathbf{E})\, dy + \int_{Y \setminus F} (L_{\alpha\beta} - p_{\alpha\beta}^*(y))\, \chi_{\alpha\beta}\, dy \right] \right\}. \quad (5.33)$$

We shall now transform the last infimum by using the notion of inf-convolution, see [10, 21]. In our case we have

$$\inf_{\mathbf{p}^* \in S_{\text{per}}^c} \left[\int_{Y \setminus F} j_3(\mathbf{p}^*(y), \mathbf{E})\, dy + \int_{Y \setminus F} (L_{\alpha\beta} - p_{\alpha\beta}^*(y))\, \chi_{\alpha\beta}\, dy \right] =$$

$$= \inf \left[\int_{Y \setminus F} j_3(\mathbf{p}^*(y), \mathbf{E})\, dy + I_{S_{\text{per}}^c}(\mathbf{p}^*) \right.$$

$$\left. + \int_{Y \setminus F} (L_{\alpha\beta} - p_{\alpha\beta}^*(y))\, \chi_{\alpha\beta}\, dy \,|\, \mathbf{p}^* \in L^2(Y \setminus F, \mathbb{E}_s^2) \right]$$

$$= \inf \left[\int_{Y \setminus F} j_3(\mathbf{L} - \mathbf{p}^*(y), \mathbf{E})\, dy + I_{S_{\text{per}}^c}(\mathbf{L} - \mathbf{p}^*) \right.$$

$$\left. + \int_{Y \setminus F} p_{\alpha\beta}^*(y)\, \chi_{\alpha\beta}\, dy \,|\, \mathbf{p}^* \in L^2(Y \setminus F, \mathbb{E}_s^2) \right]. \quad (5.34)$$

Substituting (5.34) into (5.33) we conclude that the supremum with respect to $\boldsymbol{\chi}$ is finite, provided that

$$\int_{Y \setminus F} \mathbf{p}^*(y)\, dy = 0. \quad (5.35)$$

Of interest are thus only those local fields $\mathbf{p}^* \in L^2(Y \setminus F, \mathbb{E}_s^2)$ satisfying the following condition

$$\mathbf{p}^* \in (\mathbf{L} - S_{\text{per}}^c) \cap \left\{ \mathbf{q}^* \in L^2(Y \setminus F, \mathbb{E}_s^2) \,|\, \int_{Y \setminus F} \mathbf{q}^*(y)\, dy = 0 \right\} =$$

$$= (\mathbf{L} - S_{\text{per}}^c) \cap (\mathbb{E}_s^2)^\perp. \quad (5.36)$$

Here $(\mathbb{E}_s^2)^\perp$ denotes the complement of $\mathbb{E}_s^2$ in the space $L^2(Y \setminus F, \mathbb{E}_s^2)$. Simple calculation yields

$$(\mathbb{E}_s^2)^\perp = \left\{ \mathbf{q}^* \in L^2(Y \backslash F, \mathbb{E}_s^2) \middle| \int_{Y \backslash F} \mathbf{q}^*_{\alpha\beta}(y) E_{\alpha\beta} \, dy = 0, \forall \mathbf{E} \in \mathbb{E}_s^2 \right\} =$$

$$= \left\{ \mathbf{q}^* \in L^2(Y \backslash F, \mathbb{E}_s^2) \middle| \int_{Y \backslash F} \mathbf{q}^*(y) \, dy = 0 \right\} . \quad (5.37)$$

Let us set

$$S^0_{\text{per}}(\mathbf{L}) = (\mathbf{L} - S^c_{\text{per}}) \cap (\mathbb{E}_s^2)^\perp . \quad (5.38)$$

Thus we can write

$$(j_0 + W_1)^* (\mathbf{N}, \mathbf{M}, \mathbf{L}) = \frac{3}{2a^3} B_{\alpha\beta\lambda\mu} M_{\alpha\beta} M_{\alpha\beta} +$$

$$+ \sup_{\mathbf{E} \in \mathbb{E}_s^2} \inf_{\mathbf{q}^* \in S^0_{\text{per}}(\mathbf{L})} \left\{ \frac{1}{|Y|} \int_{Y \backslash F} \left[N_{\alpha\beta} E_{\alpha\beta} - \frac{h}{8} C_{\alpha\beta\lambda\mu} E_{\alpha\beta} E_{\lambda\mu} \right. \right.$$

$$+ \frac{3}{2a} M_{\alpha\beta} E_{\alpha\beta} - \frac{3}{2b} L_{\alpha\beta} E_{\alpha\beta}$$

$$\left. \left. + \frac{3}{2b} B_{\alpha\beta\lambda\mu} (L_{\alpha\beta} - q_{\alpha\beta})(L_{\lambda\mu} - q_{\lambda\mu}) \right] dy \right\} .$$

Calculating now the supremum over $\mathbf{E} \in \mathbb{E}_s^2$ we finally obtain

$$(j_0 + W_1)^* (\mathbf{N}, \mathbf{M}, \mathbf{L}) = B_{\alpha\beta\lambda\mu} \left(\frac{2}{h} N_{\alpha\beta} N_{\lambda\mu} + \frac{6}{ah} N_{\alpha\beta} M_{\lambda\mu} - \right.$$

$$- \frac{6}{bh} N_{\alpha\beta} L_{\lambda\mu} + \frac{3(4a + b)}{2a^3 h} M_{\alpha\beta} M_{\lambda\mu}$$

$$- \frac{9}{abh} M_{\alpha\beta} L_{\lambda\mu} + \frac{9}{2b^2 h} L_{\alpha\beta} L_{\lambda\mu}$$

$$+ \inf \left\{ \frac{1}{|Y|} \int_{Y \backslash F} \frac{3}{2b^3} B_{\alpha\beta\lambda\mu} (L_{\alpha\beta} - q_{\alpha\beta})(L_{\lambda\mu} - q_{\lambda\mu}) \, dy \,|\, \mathbf{q} \in S^0_{\text{per}}(\mathbf{L}) \right\} , \quad (5.39)$$

where $\mathbf{N}, \mathbf{M}, \mathbf{L} \in \mathbb{E}_s^2$.

The complete form of the complementary macroscopic potential W^* is found by using (5.22), where $(j_0 + W_1)^*$ is given by the formula (5.39). It is worth noting that in order to calculate W^* we have to solve solely the following local problem :

$$\left|\begin{array}{l}\text{for a given } \mathbf{L} \in \mathbb{E}_s^2 \text{ find} \\ \inf \left\{ \dfrac{1}{|Y|} \displaystyle\int_{Y \setminus F} \dfrac{3}{2\, b^3} B_{\alpha\beta\lambda\mu} (L_{\alpha\beta} - q_{\alpha\beta}(y)) \times \right. \\ \qquad\qquad \left. \times (L_{\alpha\beta} - q_{\alpha\beta}(y))\, dy \,|\, \mathbf{q} \in S_{\text{per}}^0(\mathbf{L}) \right\} . \end{array}\right. \quad (5.40)$$

The infimum over $\mathbf{q} \in S_{\text{per}}^0(\mathbf{L})$ can be written in an equivalent manner. Namely we observe that for a $\mathbf{q} \in S_{\text{per}}^0(\mathbf{L})$ we have

$$\mathbf{q} = \mathbf{L} - \mathbf{p}^*, \ \langle \mathbf{q} \rangle = \frac{1}{|Y|} \int_{Y \setminus F} \mathbf{q}(y)\, dy = 0, \ \mathbf{p}^* \in S_{\text{per}}^c, \ \langle \mathbf{p}^* \rangle = \mathbf{L} .$$

Hence the last local problem can be reformulated in the following way

$$\left|\begin{array}{l}\text{for a given } \mathbf{L} \in \mathbb{E}_s^2 \text{ find} \\ \inf \left\{ \dfrac{1}{|Y|} \displaystyle\int_{Y \setminus F} \dfrac{3}{2\, b^3} B_{\alpha\beta\lambda\mu}\, p_{\alpha\beta}^* \, p_{\lambda\mu}^* \, dy \,|\, \mathbf{p}^* \in S_{\text{per}}^c, \ \langle \mathbf{p}^* \rangle = \mathbf{L} \right\} . \end{array}\right. \quad (5.41)$$

Let us pass now to the problem of epi-convergence of the sequence of functionals $\{\mathfrak{S}^\varepsilon\}_{\varepsilon > 0}$, where $\mathfrak{S}^\varepsilon$ is given by the formula (5.21). We set

$$J_p^\varepsilon(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}, w\, ; \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_5) = \\ = \int_{\Omega^\varepsilon} \{ j\, [\mathbf{e}(\mathbf{r}) + \mathbf{p}_1, \boldsymbol{\rho}(\boldsymbol{\varphi}) + \mathbf{p}_2, \boldsymbol{\kappa}(\boldsymbol{\psi}) + \mathbf{p}_3] + \\ + j_2 [\mathbf{g}(w, \boldsymbol{\varphi}) + \mathbf{p}_4, \mathbf{d}(w, \boldsymbol{\psi}) + \mathbf{p}_5] \}\, dx , \quad (5.42)$$

where $\mathbf{p}_\alpha \in L^2(\Omega, \mathbb{E}_s^2)$, $\mathbf{p}_3 \in L^2(\Omega^\varepsilon, \mathbb{E}_s^2)$, $\mathbf{p}_4 \in L^2(\Omega, \mathbb{R}^2)$, $\mathbf{p}_5 \in L^2(\Omega, \mathbb{R}^2)$. The epi-convergence of the sequence $\{J^\varepsilon\}_{\varepsilon > 0}$ to the functional J^h implies the epi-convergence of $\{J_p^\varepsilon\}_{\varepsilon > 0}$, in the strong topology of the space $[L^2(\Omega, \mathbb{R}^2)]^3 \times L^2(\Omega) \times \mathfrak{H}$, to the following functional

$$J_p^h(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}, w\, ; \mathbf{p}_I) = \int_\Omega \{ W[\mathbf{e}(\mathbf{r}) + \mathbf{p}_1, \boldsymbol{\rho}(\boldsymbol{\psi}) + \mathbf{p}_2, \boldsymbol{\kappa}(\boldsymbol{\psi}) + \\ + \mathbf{p}_3, \mathbf{g}(w, \boldsymbol{\varphi}) + \mathbf{p}_4, \mathbf{d}(w, \boldsymbol{\psi}) + \mathbf{p}_5] \} . \quad (5.43)$$

where $I = 1, 2, 3, 4, 5$. Strictly speaking, to prove the aforementioned epi-convergence of the sequence $\{J_p^\varepsilon\}_{\varepsilon > 0}$ one can exploit the approach proposed by Azé [12]. In our case we use the fact that J^h is the epi-limit of $\{J^\varepsilon\}_{\varepsilon > 0}$. Moreover, one has to choose approximating sequences for $\mathbf{p}_I (I = 1, 2, \ldots, 5)$. Towards this end we first consider the case when the functions $\mathbf{p}_I$ are constant on the sets Ω_i constituting a partition of Ω^ε. For such functions $\mathbf{p}_I$ one proves the following inequalities

$$J_p^h(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}, w ; \mathbf{p}_I) \leqslant \liminf_{\varepsilon \to 0} J_p^\varepsilon(\mathbf{r}^\varepsilon, \boldsymbol{\varphi}^\varepsilon, \boldsymbol{\psi}^\varepsilon, w^\varepsilon ; \mathbf{p}_I), \tag{5.44}$$

$$J_p^h(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}, w ; \mathbf{p}_I) \geqslant \limsup_{\varepsilon \to 0} J_p^\varepsilon(\mathbf{r}^\varepsilon, \boldsymbol{\varphi}^\varepsilon, \boldsymbol{\psi}^\varepsilon, w^\varepsilon ; \mathbf{p}_I), \tag{5.45}$$

according to the property (iv) of the epi-limit (*cf.* Section 2) of the present paper. In the second stage arbitrary elements $\mathbf{p}_I$ ($I = 1, 2, \ldots, 5$) from the corresponding spaces L^2 are approximated by piecewise constant functions. Next, inequalities like (5.44) and (5.45) are proved for such a general case.

To apply theorem 2.4 one has to verify the coercivity condition (2.11). To prove it let us set

$$\mathfrak{S}_p^\varepsilon(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}, w ; \mathbf{p}_I) = J_p^\varepsilon(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}, w ; \mathbf{p}_I) + \Phi_2(w). \tag{5.46}$$

By using the relations (3.13), (3.24) and the Korn inequality for the two-dimensional domain Ω^ε [20] we arrive at the following inequality

$$\mathfrak{S}_p^\varepsilon(\mathbf{r}, \boldsymbol{\varphi}, \boldsymbol{\psi}, w ; \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}) \geqslant$$
$$\geqslant c(\|\mathbf{r}\|_{1,\Omega}^2 + \|\boldsymbol{\varphi}\|_{1,\Omega}^2 + \|\boldsymbol{\psi}\|_{1,\Omega^\varepsilon} + \|w\|_{1,\Omega}^2 - \|p\|_{0,\Omega}\, \|w\|_{1,\Omega}),$$

where $c > 0$ is a constant. The assumption ($\mathscr{X}$) of the theorem 2.4 is obviously verified for each $r > 0$. Thus we may write

$$\mathfrak{S}^h = [w - \mathfrak{H}] - \lim_{\varepsilon \to 0} \mathfrak{S}^\varepsilon, \tag{5.47}$$

where

$$\mathfrak{S}^h(\mathbf{N}, \mathbf{M}, \mathbf{L}, \mathbf{Q}, \mathbf{T}) = \int_\Omega W^*[\mathbf{N}(x), \mathbf{M}(x), \mathbf{L}(x), \mathbf{Q}(x), \mathbf{T}(x)] +$$
$$+ I_{\mathscr{C}_s}(\mathbf{N}, \mathbf{M}, \mathbf{L}, \mathbf{Q}, \mathbf{T}), \tag{5.48}$$

and

$$\mathscr{C}_s = \{\mathbf{N}, \mathbf{M}, \mathbf{L} \in L^2(\Omega, \mathbb{E}_s^2) ; \mathbf{Q}, \mathbf{T} \in L^2(\Omega, \mathbb{R}^2) |\, N_{\alpha\beta,\beta} = 0,$$
$$- M_{\alpha\beta,\beta} + Q_\alpha = 0, - L_{\alpha\beta,\beta} + T_\alpha = 0, (Q_\alpha + T_\alpha)_{,\alpha} + p = 0, \text{ in } \Omega\}. \tag{5.49}$$

We recall that w-$\mathfrak{H}$ stands for the weak topology of the space $\mathfrak{H}$.

Remark 5.1 : The study of duality performed in Section 5 suggests that the formulas given in the paper [2] for the complementary macroscopic potentials are not the general ones. After [4] we shall now present general results. We preserve notations of our paper [2], where three-dimensional problems were investigated.

(i) **Frictionless case**

The general form of the formula (5.15) of reference [2] is

$$W^*(\mathbf{T}) = \inf \left\{ \frac{1}{|Y|} \int_{Y \setminus F} J^*(y, \mathbf{T} - \boldsymbol{\tau})\, dy \mid \boldsymbol{\tau} \in S^0_{\text{per}}(\mathbf{T}) \right\},$$

where $\mathbf{T} \in \mathbb{E}^3_s$ is a macroscopic stress tensor and

$$S^0_{\text{per}}(\mathbf{T}) = (\mathbf{T} - S^c_{\text{per}}) \cap \left\{ \boldsymbol{\tau} \mid \int_{Y \setminus F} \boldsymbol{\tau}(y)\, dy = 0 \right\}.$$

The cone S^c_{per} is still given by the relation (5.11) of reference [2].

(ii) **Frictional case — $\boldsymbol{\sigma}_N$ prescribed**

Now the general form of the formula (5.25) of reference [2] is

$$W^*_{\sigma_N}(\mathbf{T}, \boldsymbol{\alpha}^*) = \inf \left\{ \frac{1}{|Y|} \int_{Y \setminus F} J^*(y, \mathbf{T} - \boldsymbol{\tau})\, dy \mid \boldsymbol{\tau} \in \mathscr{C}_{\text{per}}(\sigma_N, \mathbf{T}) \right\},$$

where

$$\mathscr{C}_{\text{per}}(\sigma_N, \mathbf{T}) = [\mathbf{T} - \mathscr{K}(\boldsymbol{\alpha}^*)] \cap \left\{ \boldsymbol{\tau} \mid \int_{Y \setminus F} \boldsymbol{\tau}(y)\, dy = 0 \right\},$$

and

$\mathscr{K}(\boldsymbol{\alpha}^*) = \{ \boldsymbol{\tau} \in L^2(Y \setminus F, \mathbb{E}^2_s) \mid \text{div}_y\, \boldsymbol{\tau} = 0, \text{ in } Y \setminus F$; $\boldsymbol{\tau}(y)\, \mathbf{n}(y)$ takes opposite values on opposite sides of Y ; $\tau_N \leq 0$, $\boldsymbol{\tau}_T \in C(\sigma_N)$, $\boldsymbol{\tau}_T + |Y|\, \boldsymbol{\alpha}^* = 0$, on $F \}$.

(iii) **Frictional case — $\boldsymbol{\sigma}_N$ unspresribed**

The complementary macroscopic potential $W^*_{\sigma_N}$ has now the following form

$$W^*_{\sigma_N}(\mathbf{T}, \mathbf{A}) = \frac{1}{|Y|} \int_{Y \setminus F} J^*(y, \mathbf{T} - \boldsymbol{\sigma}(\mathbf{T}, \mathbf{A}))\, dy,$$

and replaces the one given by equation (5.31) of reference [2]. Here $\boldsymbol{\sigma}(\mathbf{T}, \mathbf{A})$ is a solution to the following local problem :

$$\left| \begin{array}{l} \text{for } \mathbf{T} \in \mathbb{E}^3_s \text{ and } \mathbf{A} \text{ given, find } \boldsymbol{\sigma} \in \mathscr{C}_{\text{per}}(\sigma_N, \mathbf{T}) \text{ such that} \\ \displaystyle\int_{Y \setminus F} \langle \partial J^*(y, \mathbf{T} - \boldsymbol{\sigma}), \boldsymbol{\tau} - \boldsymbol{\sigma} \rangle\, dy \geq 0 \quad \forall \boldsymbol{\tau} \in \mathscr{C}_{\text{per}}(\sigma_N, \mathbf{T}). \end{array} \right.$$

ACKNOWLEDGEMENTS

This paper was partially supported by the State Committee for Scientific Research under grant No. 3 001991 01.

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