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SOME OPTIMAL CONTROL PROBLEMS OF MULTISTATE EQUATIONS APPEARING IN FLUID MECHANICS

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Abstract — *This work deals with two optimal control problems associated to the steady state Navier Stokes equations The state of the system is the velocity of the fluid and the controls are the body forces or the heat flux on the boundary In the second case the Navier-Stokes equations are coupled with the stationary heat equation The control problems consist in minimizing a cost functional involving the turbulence Some control constraints can be added to the problem Existence of an optimal control is proved and some optimality conditions are derived In both problems the relation control → state is multi-valued and therefore the derivation of the optimality conditions is not obvious To overcome this difficulty, we introduce an approximate family of optimal control problems governed by a well posed linear elliptic system, we obtain the optimality conditions for these problems and then we pass to the limit The approach followed in this study can be used in the numerical resolution of the optimal control problem*

Résumé — *Nous étudions deux problemes de contrôle optimal se rapportant aux équations de Navier-Stokes stationnaires L'état du système est le champ de vitesses dans le fluide, et les contrôles sont, soit les forces volumiques, soit le flux de chaleur au bord , dans le second cas, les équations de Navier-Stokes sont couplees avec l'équation de convection-diffusion pour la température, dans l'approximation de Boussinesq On cherche a minimiser une fonctionnelle caractérisant l'état de la turbulence à l'intérieur du fluide, éventuellement sous certaines contraintes portant sur les contrôles Nous prouvons l existence d'un contrôle optimal, et donnons les conditions d'optimalité qui le caractérisent Dans les deux cas, la relation contrôle-état est multivaluée , nous surmontons les difficultés que cela entraîne en utilisant une famille de problemes approchés, qui suggèrent par la même occasion un algorithme numérique adapté à la résolution de ces problemes*

Keywords — Optimal control, multistate elliptic systems, Navier-Stokes equations, optimality conditions

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1. INTRODUCTION

In this paper we study two optimal control problems that consist in minimizing a cost functional involving the vorticity in the fluid. The controls are the body forces or the heat flux on the boundary. The state is the velocity of the fluid and the equations relating the control and state are the Navier-Stokes equations. If the control is the temperature then the heat equation must be added to the previous ones.

This type of problems have been studied by Abergel and Temam ; the time-dependent two-dimensional case can be studied directly [1], whereas, for three-dimensional evolutionary flows, they obtain partial results [2], which comes from the fact that the Navier-Stokes equations are not known to be well posed. For time-dependent problems, one should also mention recent results by Choi *et al.* [4], which pertains rather to the problem of characterizing a feedback control operator.

When the stationary equations are considered, the nonuniqueness of solution occurs in dimensions two and three. Hereinafter we will deal with this situation : stationary equations. In order to simplify the exposition we will only consider the three-dimensional case, however the results and methods are the same for the two-dimensional flows. The control of the stationary Navier-Stokes equations has been investigated by Gunzburger *et al.* [6], [7]. They derived the optimality conditions for these problems by using a theorem of Ioffe and Tikhomorov [8] and assuming a property, called property C, on the feasible control set. We will follow a different approach which allows us to deduce some optimality conditions of Fritz John type for any convex feasible control set and derive these conditions in a qualified form when the property C is assumed. Our approach provides a numerical method to deal with these multistate equations and solve the control problems.

In [2], the authors use a method similar to ours, in order to deal with the optimal control of the high frequencies for the stationary Navier-Stokes equations.

In the next section we formulate a distributed control problem that corresponds to the control by the body forces. We prove the existence of a solution for this problem and derive some optimality conditions satisfied by the optimal controls. To obtain these conditions we introduce a family of control problems that approximate the initial problem and that are associated with linear and well-posed state equations. We deduce the optimality conditions for these problems and then we pass to the limit and derive the desired conditions for our control problem. In Section 3 this scheme of work is repeated for a boundary control problem, the control being the temperature. For a precise account of the methods and results of the optimal control problems governed by partial differential equations, the reader is referred to Lions [9].

Before finishing this section let us introduce some notation. The fluid is supposed to occupy a physical domain $\Omega \subset R^3$. We assume that Ω is bounded and its boundary Γ is Lipschitz, $\vec{n}(x)$ denoting the outward unit normal vector to Γ at the point x; see Nečas [11]. $L^2(\Omega)$ is the space of square integrable functions and $H^1(\Omega)$ is the Sobolev space formed by the real-valued functions which, together with all their partial distributional derivatives of first order, belong to $L^2(\Omega)$. $H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$ constituted by the functions of null trace and $H^{-1}(\Omega)$ is its dual. We set

$$Y = \{\vec{y} \in (H^1(\Omega))^3 : \operatorname{div} \vec{y} = 0\} \quad \text{and} \quad Y_0 = \{\vec{y} \in (H_0^1(\Omega))^3 : \operatorname{div} \vec{y} = 0\}.$$

It is well known that Y and Y_0 are separable Hilbert spaces ; see Temam [1]. Finally $\vec{\phi}_\Gamma$ will denote a fixed element of $(H^{1/2}(\Gamma))^3$, $H^{1/2}(\Gamma)$ being the trace space of $H^1(\Omega)$, such that

$$\vec{\phi}_\Gamma \cdot \vec{n} = 0 \quad \text{on } \Gamma. \tag{1.1}$$

In Appendix we will prove that for every $\mu > 0$ we can find an element $\vec{\phi} \in Y$ such that the trace of $\vec{\phi}$ on Γ is $\vec{\phi}_\Gamma$ and

$$\left| \sum_{i,j=1}^{3} \int_\Omega y_i \, \partial_{x_i} z_j \, \phi_j \, dx \right| \leq \mu \, \|\vec{y}\|_{(H^1(\Omega))^3} \, \|\vec{z}\|_{(H^1(\Omega))^3} \quad \forall \vec{y}, \vec{z} \in Y_0. \tag{1.2}$$

2. A DISTRIBUTED CONTROL PROBLEM

Let us consider a stationary viscous incompressible flow in Ω; the equations of motion are

$$\begin{cases} -\nu \Delta \vec{y} + (\vec{y} \cdot \nabla)\vec{y} + \nabla \pi = \vec{f} + Bu \text{ in } \Omega, \\ \operatorname{div} \vec{y} = 0 \text{ in } \Omega, \quad \vec{y} = \vec{\phi}_\Gamma \text{ on } \Gamma, \end{cases} \tag{2.1}$$

where $\nu > 0$, $\vec{f} \in (H^{-1}(\Omega))^3$, $B \in \mathcal{L}(U, (H^{-1}(\Omega))^3)$, $u \in U$, U being a Hilbert space. $\vec{y}$ is the velocity, π the pressure, $\vec{f}$ the body forces and u is the control that can act over all domain Ω or only over a part of Ω or even only in a given direction of the space. All these possibilities can be treated by choosing a suitable space U and the corresponding linear mapping B.

It is well known that (2.1) has at least one solution $(\vec{y}, \pi) \in (H_0^1(\Omega))^3 \times L^2(\Omega)$; see for example Lions [10] or Temam [13]. However there is not, in general, a unique solution.

Now we define the functional $J : (H^1(\Omega))^3 \times U \to R$ by

$$J(\vec{y}, u) = \frac{1}{2} \int_\Omega |\nabla \times \vec{y}|^2 \, dx + \frac{N}{2} \|u\|_U^2 ,$$

with $N \geq 0$ and

$$\nabla \times \vec{y} = (\partial_{x_2} y_3 - \partial_{x_3} y_2, \partial_{x_3} y_1 - \partial_{x_1} y_3, \partial_{x_1} y_2 - \partial_{x_2} y_1)$$

denoting the vorticity of the flow. The physically relevant term in J is of course

$$\frac{1}{2} \int_\Omega |\nabla \times \vec{y}|^2 \, dx ,$$

which provides an estimate of the level of turbulence within the flow.

Given a nonempty convex closed subset K of U, we formulate the optimal control problem as follows :

$$\text{(P1)} \begin{cases} \text{Minimize } J(\vec{y}, u) \\ (\vec{y}, u) \in (H^1(\Omega))^3 \times K \text{ and } (\vec{y}, u) \text{ satisfies (2.1) for some } \pi \in L^2(\Omega). \end{cases}$$

The first thing to study is the existence of a solution of (P1).

THEOREM 2 : *Assumed that $N > 0$ or K is bounded in U, then* (P1) *has at least one solution.*

Proof : Let $\{(\vec{y}_k, u_k)\}_{k=1}^{\infty} \subset (H^1(\Omega))^3 \times K$ be a minimizing sequence. From the definition of J and the assumption of the theorem it follows that $\{u_k\}_{k=1}^{\infty}$ is a bounded sequence in U. Now using (2.1) we deduce the estimate

$$\|\vec{y}_k\|_{(H^1(\Omega))^3} \leq C_1 \|\vec{f} + Bu_k\|_{(H^{-1}(\Omega))^3} + C_2 \leq$$
$$\leq C_1 \left(\|\vec{f}\|_{(H^{-1}(\Omega))^3} + \|B\| \, \|u_k\|_U \right) + C_2 \leq C ,$$

where C_2 depends on $\vec{\phi}_\Gamma$.

Thus we can take a subsequence, denoted in the same way, and an element $(\vec{y}_0, u_0) \in (H^1(\Omega))^3 \times U$ such that $(\vec{y}_k, u_k) \to (\vec{y}_0, u_0)$ in $(H^1(\Omega))^3 \times U$ weakly. Using the compactness of the inclusion $H^1(\Omega) \subset L^6(\Omega)$ it is easy to pass to the limit in the state equation and verify that $(\vec{y}, u_0)$ satisfies (2.1) for some pressure π_0. Since K is convex and closed, we deduce that

$u_0 \in K$. Finally, from the convexity and continuity of J it follows the lower semicontinuity of J in the weak topology, which allows us to conclude that

$$J(\vec{y}_0, u_0) \leq \lim_{k \to \infty} \inf J(\vec{y}_k, u_k) = \inf (\mathrm{P1}),$$

and thus $(\vec{y}_0, u_0)$ is a solution of (P1). □

We now state the optimality system for Problem (P1).

THEOREM 2.2 : *If* $(\vec{y}_0, u_0) \in (H^1(\Omega))^3 \times U$ *is a solution of* (P1), *then there exist a number* $\alpha \geq 0$ *and some elements* $\vec{p}_0 \in (H^1(\Omega))^3$ *and* $\pi_0, \lambda_0 \in L^2(\Omega)$ *verifying*

$$\alpha + \|\vec{p}_0\|_{(H^1(\Omega))^3} > 0, \tag{2.2}$$

$$\begin{cases} -\nu \Delta \vec{y}_0 + (\vec{y}_0 \cdot \nabla) \vec{y}_0 + \nabla \pi_0 = \vec{f} + B u_0 \ in\ \Omega \\ \operatorname{div} \vec{y}_0 = 0 \ in\ \Omega, \quad \vec{y}_0 = \vec{\phi}_\Gamma \ on\ \Gamma, \end{cases} \tag{2.3}$$

$$\begin{cases} -\nu \Delta \vec{p}_0 - (\vec{y}_0 \cdot \nabla) \vec{p}_0 + (\nabla \vec{y}_0)^T \vec{p}_0 + \nabla \lambda_0 = \alpha \nabla \times (\nabla \times \vec{y}_0) \ in\ \Omega \\ \operatorname{div} \vec{p}_0 = 0 \ in\ \Omega, \quad \vec{p}_0 = 0 \ on\ \Gamma, \end{cases} \tag{2.4}$$

$$(B^* \vec{p}_0 + \alpha N u_0, u - u_0)_U \geq 0 \quad \forall u \in K. \tag{2.5}$$

Before proving this theorem let us remark that sometimes it is possible to get (2.3)-(2.5) with $\alpha = 1$. Following Gunzburger *et al.* [7] we say that the control set K has property C at $(\vec{y}_0, u_0)$ if for any nonzero solution $(\vec{p}, \pi) \in (H^1(\Omega))^3 \times L^2(\Omega)$ of the system

$$\begin{cases} -\nu \Delta \vec{p} - (\vec{y}_0 \cdot \nabla) \vec{p} + (\nabla \vec{y}_0)^T \vec{p} + \nabla \lambda = 0 \text{ in } \Omega \\ \operatorname{div} \vec{p} = 0 \text{ in } \Omega, \quad \vec{p} = 0 \text{ on } \Gamma, \end{cases} \tag{2.6}$$

we can find $u \in K$ such that

$$(B^* \vec{p}, u - u_0) < 0. \tag{2.7}$$

Convention will have it that property C is to hold vacuously if there are no nonzero solutions of (2.6).

COROLLARY 2.3 : *If* K *has property* C *at* $(\vec{y}_0, u_0)$, *then there exist* $\vec{p}_0 \in (H^1(\Omega))^3$ *and* $\pi_0, \lambda_0 \in L^2(\Omega)$ *verifying* (2.3)-(2.5) *with* $\alpha = 1$.

Proof: It is enough to remark that (2.6) and (2.7) implies that $\alpha \neq 0$ in (2.3)-(2.5). Then we can replace $\vec{p}_0$ by $\vec{p}_0/\alpha$ and so deduce the desired result. □

Remark 1 : It is obvious that if $U = K = (L^2(\Omega))^3$ and B = inclusion operator from $(L^2(\Omega))^3$ into $(H^{-1}(\Omega))^3$, then K has property C at $(\vec{y}_0, u_0)$.

The rest of this section is devoted to the proof of the optimality conditions exhibited in Theorem 2.2.

2.1. The problems $(\mathrm{P1}_\varepsilon)$

In order to prove Theorem 2.2 we are going to introduce a family of problems $(\mathrm{P1}_\varepsilon)$, whose solutions converge towards a solution $(\vec{y}_0, u_0)$, then we will derive the optimality conditions for these problems and finally we will pass to the limit in these optimality conditions.

First let us introduce some notations. We will denote by

$$a : (H^1(\Omega))^3 \times (H^1(\Omega))^3 \to R$$

and

$$b : (H^1(\Omega))^3 \times (H^1(\Omega))^3 \times (H^1(\Omega))^3 \to R$$

the bilinear and trilinear forms defined by

$$a(\vec{y}, \vec{z}) = \sum_{j=1}^{3} \int_\Omega \nabla y_j \,.\, \nabla z_j \, dx$$

and

$$b(\vec{y}, \vec{z}, \vec{w}) = \sum_{i,j=1}^{3} \int_\Omega y_i \, \partial_{x_i} z_j \, w_j \, dx = \int_\Omega (\vec{y} \,.\, \nabla) \vec{z} \,.\, \vec{w} \, dx \,.$$

Concerning the trilinear form b, the following properties can be easily proved for every $(\vec{y}, \vec{z}, \vec{w}) \in Y \times (H^1(\Omega))^3 \times (H^1(\Omega))^3$:

1) $b(\vec{y}, \vec{z}, \vec{w}) = -b(\vec{y}, \vec{w}, \vec{z})$ if $\vec{y} \,.\, \vec{n} = 0$ on Γ.

2) $b(\vec{y}, \vec{z}, \vec{z}) = 0$ if $\vec{y} \,.\, \vec{n} = 0$ on Γ.

3) $|b(\vec{y}, \vec{z}, \vec{w})| \leq \|\vec{y}\|_{(L^4(\Omega))^3} \|\vec{z}\|_{(H^1(\Omega))^3} \|\vec{w}\|_{(L^4(\Omega))^3}$.

On the other hand it is well known that $\vec{y} \in (H^1(\Omega))^3$ is a solution of the problem

$$\begin{cases} \text{Find } \vec{y} \in Y \text{ such that} \\ \vec{y} = \vec{\phi}_\Gamma \text{ on } \Gamma \text{ and} \\ \nu a(\vec{y}, \vec{z}) + b(\vec{y}, \vec{y}, \vec{z}) = \langle \vec{f} + Bu, \vec{z} \rangle \quad \forall \vec{z} \in Y_0 \end{cases} \tag{2.8}$$

if and only if there exists an element (unique up to the addition of a constant) $\pi \in L^2(\Omega)$ such that $(\vec{y}, \pi)$ satisfies (2.1) ; see Temam [13]. Problem (2.8) is the variational formulation of (2.1).

Let us fix a solution $(\vec{y}_0, u_0)$ of (P1). For every $\varepsilon > 0$ we define the functional $J_\varepsilon : \{\vec{w} \in Y : \vec{w} = \vec{\phi}_\Gamma \text{ on } \Gamma\} \times U \to R$ by

$$J_\varepsilon(\vec{w}, u) = J(\vec{y}(\vec{w}, u), u) + \frac{1}{2\varepsilon} \sum_{j=1}^{3} \int_\Omega |\nabla y_j(\vec{w}, u) - \nabla w_j|^2 \, dx + \\ + \frac{1}{2} \sum_{j=1}^{3} \int_\Omega |y_j - y_{0j}|^2 \, dx + \frac{1}{2} \|u - u_0\|_U^2 ,$$

where $\vec{y}(\vec{w}, u)$ is the unique solution of the variational problem

$$\begin{cases} \text{Find } \vec{y} \in Y \text{ such that} \\ \vec{y} = \vec{\phi}_\Gamma \text{ on } \Gamma \text{ and} \\ \nu a(\vec{y}, \vec{z}) + b(\vec{w}, \vec{y}, \vec{z}) = \langle \vec{f} + Bu, \vec{z} \rangle \quad \forall \vec{z} \in Y_0 . \end{cases} \tag{2.9}$$

The existence and uniqueness of solution of (2.9) is a direct consequence of the Lax-Milgram theorem and the second property of b stated above.

Now we formulate the problem $(\mathrm{P1}_\varepsilon)$ in the following way

$$(\mathrm{P1}_\varepsilon) \quad \begin{cases} \text{Minimize } J_\varepsilon(\vec{w}, u) \\ (\vec{w}, u) \in Y \times K \quad \text{and} \quad \vec{w} = \vec{\phi}_\Gamma \quad \text{on} \quad \Gamma . \end{cases}$$

We prove that each problem $(\mathrm{P1}_\varepsilon)$ has at least one solution and that they form an approximating family for (P1) in a sense that we make precise.

PROPOSITION 2.4 : *For every $\varepsilon > 0$ there exists at least one solution $(\vec{w}_\varepsilon, u_\varepsilon)$ of $(\mathrm{P1}_\varepsilon)$. Moreover if we denote by $\vec{y}_\varepsilon$ the solution of* (2.9) *corresponding to $(\vec{w}_\varepsilon, u_\varepsilon)$, then we have*

$$\lim_{\varepsilon \to 0} \|u_\varepsilon - u_0\|_U = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \sum_{j=1}^{3} \int_\Omega |\nabla y_{\varepsilon j} - \nabla w_{\varepsilon j}|^2 \, dx = 0 , \tag{2.10}$$

$$\vec{w}_\varepsilon \to \vec{y}_0 \quad \textit{weakly in } Y , \tag{2.11}$$

$$\vec{y}_\varepsilon \to \vec{y}_0 \quad \textit{weakly in } Y , \tag{2.12}$$

$$\lim_{\varepsilon \to 0} J_\varepsilon(\vec{w}_\varepsilon, u_\varepsilon) = J(\vec{y}_0, u_0) . \tag{2.13}$$

Proof : The existence of a solution can be proved as in Theorem 2.1. Let us prove the second part of the theorem. Let $\{(\vec{w}_\varepsilon, u_\varepsilon)\}_{\varepsilon > 0}$ be solutions of $(P1_\varepsilon)$. Since $\vec{y}(\vec{y}_0, u_0) = \vec{y}_0$, we have

$$J_\varepsilon(\vec{w}_\varepsilon, u_\varepsilon) \leq J_\varepsilon(\vec{y}_0, u_0) = J(\vec{y}_0, u_0),$$

from where it follows

$$\|u_\varepsilon - u_0\|_U^2 \leq 2\,J(\vec{y}_0, u_0) \Rightarrow \|u_\varepsilon\|_U^2 \leq 4\,J(\vec{y}_0, u_0) + 2\|u_0\|_U^2 \quad (2.14)$$

and

$$\int_\Omega |\nabla y_{\varepsilon j} - \nabla w_{\varepsilon j}|^2\, dx \leq 2\,\varepsilon J(\vec{y}_0, u_0) \to 0 \quad \text{when} \quad \varepsilon \to 0\,. \quad (2.15)$$

From here and the identity $\vec{y}_\varepsilon = \vec{w}_\varepsilon = \vec{\phi}_\Gamma$ on Γ we deduce the convergence $(\vec{y}_\varepsilon - \vec{w}_\varepsilon) \to \vec{0}$ strongly in $(H^1(\Omega))^3$. Thus there exists $\varepsilon_0 > 0$ such that

$$\|\vec{w}_\varepsilon - \vec{y}_\varepsilon\|_{(H^1(\Omega))^3} \leq 1 \quad \forall\, \varepsilon < \varepsilon_0\,. \quad (2.16)$$

Now let us take $\vec{\phi} \in Y$ verifying that $\vec{\phi} = \vec{\phi}_\Gamma$ on Γ and (1.2). Let us denote $\vec{z}_{0\,\varepsilon} = \vec{y}_\varepsilon - \vec{\phi} \in Y_0$. Then we get from (2.9) with $\vec{z} = \vec{z}_{0\varepsilon}$

$$\nu a(\vec{z}_{0\,\varepsilon}, \vec{z}_{0\,\varepsilon}) = \langle \vec{f} + Bu_\varepsilon, \vec{z}_{0\,\varepsilon}\rangle - \nu a(\vec{\phi}, \vec{z}_{0\,\varepsilon}) - b(\vec{w}_\varepsilon, \vec{\phi}, \vec{z}_{0\,\varepsilon}),$$

so with (2.14)

$$\|\vec{z}_{0\,\varepsilon}\|^2_{(H^1(\Omega))^3} \leq C_1 \|\vec{z}_{0\,\varepsilon}\|_{(H^1(\Omega))^3} + C_2 |b(\vec{w}_\varepsilon, \vec{\phi}, \vec{z}_{0\,\varepsilon})|\,.$$

Taking μ in (1.2) such that $C_2\,\mu < 1/2$ and using (2.16) we obtain

$$\|\vec{z}_{0\,\varepsilon}\|^2_{(H^1(\Omega))^3} \leq C_1 \|\vec{z}_{0\,\varepsilon}\|_{(H^1(\Omega))^3} + C_3 \|\vec{z}_{0\,\varepsilon}\|_{(H^1(\Omega))^3} + C_2\,\mu\, \|\vec{z}_{0\,\varepsilon}\|^2_{(H^1(\Omega))^3},$$

then

$$\|\vec{z}_{0\,\varepsilon}\|_{(H^1(\Omega))^3} \leq C_4\,,$$

therefore

$$\|\vec{y}_\varepsilon\| \leq \|\vec{\phi}\|_{(H^1(\Omega))^3} + \|\vec{z}_{0\,\varepsilon}\|_{(H^1(\Omega))^3} \leq C_5\,. \quad (2.17)$$

Then we can extract subsequences and elements $(u, \vec{y}) \in K \times Y$ satisfying

$$\begin{aligned} u_{\varepsilon(k)} &\to u \quad \text{in } U \text{ weakly}\,, \\ \vec{y}_{\varepsilon(k)} &\to \vec{y} \quad \text{in } (H^1(\Omega))^3 \text{ weakly and} \\ \vec{w}_{\varepsilon(k)} &\to \vec{y} \quad \text{in } (H^1(\Omega))^3 \text{ weakly}\,, \end{aligned} \quad (2.18)$$

with $\varepsilon(k) \to 0$. From these convergences and using the Rellich's theorem, we can pass to the limit in

$$\nu a(\vec{y}_{\varepsilon(k)}, \vec{z}) + b(\vec{w}_{\varepsilon(k)}, \vec{y}_{\varepsilon(k)}, \vec{z}) = \langle \vec{f} + Bu_{\varepsilon(k)}, \vec{z} \rangle$$

and obtain

$$\nu a(\vec{y}, \vec{z}) + b(\vec{y}, \vec{y}, \vec{z}) = \langle \vec{f} + Bu, \vec{z} \rangle \quad \forall \vec{z} \in Y_0 . \tag{2.19}$$

From (2.19) we deduce the existence of an element $\pi \in L^2(\Omega)$ such that $(\vec{y}, \pi)$ and u satisfy (2.1), therefore $(\vec{y}, u)$ is a feasible point for $(P1)$. On the other hand, since $(\vec{y}_0, u_0)$ is a solution of $(P1)$, we have

$$\begin{aligned} J(\vec{y}, u) \leq J(\vec{y}, u) + \frac{1}{2} \sum_{j=1}^{3} \int_\Omega |y_j - y_{0j}|^2 \, dx + \frac{1}{2} \|u - u_0\|_U^2 \leq \\ \leq \liminf_{k \to \infty} J_{\varepsilon(k)}(\vec{w}_{\varepsilon(k)}, u_{\varepsilon(k)}) \leq J(\vec{y}_0, u_0) \leq J(\vec{y}, u) , \end{aligned}$$

which implies that $\vec{y} = \vec{y}_0$ and $u = u_0$. Thus the whole family $\{(\vec{w}_\varepsilon, \vec{y}_\varepsilon, u_\varepsilon)\}_{\varepsilon > 0}$ converges to $(\vec{y}_0, \vec{y}_0, u_0)$ weakly in $(H^1(\Omega))^3 \times (H^1(\Omega))^3 \times U$. Now (2.13) is deduced in the following way

$$J(\vec{y}_0, u_0) \leq \liminf_{\varepsilon \to 0} J_\varepsilon(\vec{w}_\varepsilon, u_\varepsilon) \leq \limsup_{\varepsilon \to 0} J_\varepsilon(\vec{w}_\varepsilon, u_\varepsilon) \leq J(\vec{y}_0, u_0) .$$

Finally (2.10) is proved

$$\begin{aligned} \limsup_{\varepsilon \to 0} \left(\frac{1}{2\varepsilon} \sum_{j=1}^{3} \int_\Omega |\nabla y_{\varepsilon j} - \nabla w_{\varepsilon j}|^2 \, dx + \frac{1}{2} \|u_0 - u_\varepsilon\|_U^2 \right) \leq \\ \leq \limsup_{\varepsilon \to 0} \, [J_\varepsilon(\vec{w}_\varepsilon, u_\varepsilon) - J(\vec{y}_\varepsilon, u_\varepsilon)] \\ \leq \limsup_{\varepsilon \to 0} J_\varepsilon(\vec{w}_\varepsilon, u_\varepsilon) - \liminf_{\varepsilon \to 0} J(\vec{y}_\varepsilon, u_\varepsilon) \leq 0 . \end{aligned}$$ □

The following theorem states the optimality conditions for $(P1_\varepsilon)$.

PROPOSITION 2.5 : *Let us suppose that* $(\vec{w}_\varepsilon, u_\varepsilon)$ *is a solution of* $(P1_\varepsilon)$, *then there exist two elements* $\vec{y}_\varepsilon \in Y$, *with* $\vec{y}_\varepsilon = \vec{\phi}_\Gamma$ *on* Γ, *and* $\vec{p}_\varepsilon \in Y_0$ *such that the following system is satisfied*

$$\nu a(\vec{y}_\varepsilon, \vec{z}) + b(\vec{w}_\varepsilon, \vec{y}_\varepsilon, \vec{z}) = \langle \vec{f} + Bu_\varepsilon \rangle \quad \forall \vec{z} \in Y_0 , \tag{2.20}$$

$$\nu a(\vec{p}_\varepsilon, \vec{z}) - b(\vec{w}_\varepsilon, \vec{p}_\varepsilon, \vec{z}) - b(\vec{z}, \vec{p}_\varepsilon, \vec{y}_\varepsilon) = = \int_\Omega (\nabla \times \vec{y}_\varepsilon) \cdot (\nabla \times \vec{z})\, dx + \sum_{j=1}^{3} \int_\Omega (y_{\varepsilon j} - y_{0j})\, z_j\, dx \quad \forall \vec{z} \in Y_0, \tag{2.21}$$

$$(B^* \vec{p}_\varepsilon + N u_\varepsilon + u_\varepsilon - u_0, u - u_\varepsilon)_U \geqslant 0 \quad \forall u \in K. \tag{2.22}$$

Proof: Let $\vec{y}_\varepsilon \in Y$ be the solution of (2.20) and let us take $\vec{p}_\varepsilon \in Y_0$ satisfying

$$\nu a(\vec{p}_\varepsilon, \vec{z}) - b(\vec{w}_\varepsilon, \vec{p}_\varepsilon, \vec{z}) = = \int_\Omega (\nabla \times \vec{y}_\varepsilon) \cdot (\nabla \times \vec{z})\, dx + \sum_{j=1}^{3} \int_\Omega (y_{\varepsilon j} - y_{0j})\, z_j\, dx + + \frac{1}{\varepsilon} \sum_{j=1}^{3} \int_\Omega (\nabla y_{\varepsilon j} - \nabla w_{\varepsilon j})\, \nabla z_j\, dx \quad \forall \vec{z} \in Y_0. \tag{2.23}$$

Now, given $\vec{w} \in Y_0$ and $u \in U$ we denote by $\vec{z}_{\vec{w}}$ and $\vec{z}_u$ the elements of Y_0 verifying

$$\nu a(\vec{z}_{\vec{w}}, \vec{z}) + b(\vec{w}_\varepsilon, \vec{z}_{\vec{w}}, \vec{z}) + b(\vec{w}, \vec{y}_\varepsilon, \vec{z}) = 0 \quad \forall \vec{z} \in Y_0 \tag{2.24}$$

and

$$\nu a(\vec{z}_u, \vec{z}) + b(\vec{w}_\varepsilon, \vec{z}_u, \vec{z}) = \langle Bu, \vec{z} \rangle \quad \forall \vec{z} \in Y_0 \tag{2.25}$$

respectively.

It is easy to verify that J_ε is of class C^1 and we obtain with (2.23)-(2.25) for every $(\vec{w}, u) \in Y_0 \times U$

$$\begin{aligned}
\frac{\partial J_\varepsilon}{\partial \vec{w}} (\vec{w}_\varepsilon, u_\varepsilon) \cdot \vec{w} &= \int_\Omega (\nabla \times \vec{y}_\varepsilon) \cdot (\nabla \times \vec{z}_{\vec{w}})\, dx \\
&+ \frac{1}{\varepsilon} \sum_{j=1}^{3} \int_\Omega (\nabla y_{\varepsilon j} - \nabla w_{\varepsilon j}) \cdot (\nabla z_{\vec{w} j} - \nabla w_j)\, dx + \sum_{j=1}^{3} \int_\Omega (y_{\varepsilon j} - y_{0j})\, z_{\vec{w} j}\, dx \\
&= \nu a(\vec{p}_\varepsilon, \vec{z}_{\vec{w}}) - b(\vec{w}_\varepsilon, \vec{p}_\varepsilon, \vec{z}_{\vec{w}}) - \frac{1}{\varepsilon} \sum_{j=1}^{3} \int_\Omega (\nabla y_{\varepsilon j} - \nabla w_{\varepsilon j})\, \nabla w_j\, dx \\
&= \nu a(\vec{z}_{\vec{w}}, \vec{p}_\varepsilon) + b(\vec{w}_\varepsilon, \vec{z}_{\vec{w}}, \vec{p}_\varepsilon) - \frac{1}{\varepsilon} \sum_{j=1}^{3} \int_\Omega (\nabla y_{\varepsilon j} - \nabla w_{\varepsilon j})\, \nabla w_j\, dx \\
&= b(\vec{w}, \vec{p}_\varepsilon, \vec{y}_\varepsilon) - \frac{1}{\varepsilon} \sum_{j=1}^{3} \int_\Omega (\nabla y_{\varepsilon j} - \nabla w_{\varepsilon j})\, \nabla w_j\, dx
\end{aligned} \tag{2.26}$$

and

$$\frac{\partial J_\varepsilon}{\partial u}(\vec{w}_\varepsilon, u_\varepsilon) . u = \int_\Omega (\nabla \times \vec{y}_\varepsilon) . (\nabla \times \vec{z}_u)\, dx + N(u_\varepsilon, u)_U$$
$$+ \frac{1}{\varepsilon}\sum_{j=1}^{3}\int_\Omega (\nabla y_{\varepsilon j} - \nabla w_{\varepsilon j}) . \nabla z_{uj}\, dx$$
$$+ \sum_{j=1}^{3}\int_\Omega (y_{\varepsilon j} - y_{0j})\, z_{uj}\, dx + (u_\varepsilon - u_0, u)_U$$
$$= \nu a(\vec{p}_\varepsilon, \vec{z}_u) - b(\vec{w}_\varepsilon, \vec{p}_\varepsilon, \vec{z}_u) + N(u_\varepsilon, u)_U + (u_\varepsilon - u_0, u)_U$$
$$= \nu a(\vec{z}_u, \vec{p}_\varepsilon) b + (\vec{w}_\varepsilon, \vec{z}_u, \vec{p}_\varepsilon) + (N u_\varepsilon + u_\varepsilon - u_0, u)_U$$
$$= \langle Bu, \vec{p}_\varepsilon \rangle + (N u_\varepsilon + u_\varepsilon - u_0, u)_U = (B^* \vec{p}_\varepsilon + N u_\varepsilon + u_\varepsilon - u_0, u)_U . \tag{2.27}$$

Since $(\vec{w}_\varepsilon, u_\varepsilon)$ is a solution of $(\mathrm{P1}_\varepsilon)$ there holds

$$\frac{\partial J_\varepsilon}{\partial \vec{w}}(\vec{w}_\varepsilon, u_\varepsilon) . \vec{w} = 0 \quad \forall \vec{w} \in Y_0 \quad \text{and} \quad \frac{\partial J_\varepsilon}{\partial u}(\vec{w}_\varepsilon, u_\varepsilon) . (u - u_\varepsilon) \geqslant 0 \quad \forall u \in K .$$

These relations together with (2.26) and (2.27) allow us to obtain (2.22) and

$$b(\vec{w}, \vec{p}_\varepsilon, \vec{y}_\varepsilon) = \frac{1}{\varepsilon}\sum_{j=1}^{3}\int_\Omega (\nabla y_{\varepsilon j} - \nabla w_{\varepsilon j}) \nabla w_j\, dx \quad \forall \vec{w} \in Y_0 . \tag{2.28}$$

Finally (2.21) follows from (2.23) and (2.28). □

It is obvious that the optimality conditions (2.20)-(2.22) can be written in the following way.

COROLLARY 2.6 : *If $(\vec{w}_\varepsilon, u_\varepsilon)$ is a solution of $(\mathrm{P1}_\varepsilon)$, then there exist elements $\vec{y}_\varepsilon, \vec{p}_\varepsilon \in (H^1(\Omega))^3$ and $\lambda_\varepsilon, \pi_\varepsilon \in L^2(\Omega)$ such that*

$$\begin{cases} -\nu \Delta \vec{y}_\varepsilon + (\vec{w}_\varepsilon . \nabla)\vec{y}_\varepsilon + \nabla \pi_\varepsilon = \vec{f} + B u_\varepsilon \ in\ \Omega \\ \operatorname{div} \vec{y}_\varepsilon = 0 \quad in\ \Omega , \quad \vec{y}_\varepsilon = \vec{\phi}_\Gamma \quad on\ \Gamma , \end{cases} \tag{2.29}$$

$$\begin{cases} -\nu \Delta \vec{p}_\varepsilon - (\vec{w}_\varepsilon . \nabla)\vec{p}_\varepsilon + (\nabla \vec{y}_\varepsilon)^T \vec{p}_\varepsilon + \nabla \lambda_\varepsilon = \nabla \times (\nabla \times \vec{y}_\varepsilon) + \vec{y}_\varepsilon - \vec{y}_0 \ in\ \Omega \\ \operatorname{div} \vec{p}_\varepsilon = 0 \quad in\ \Omega , \quad \vec{p}_\varepsilon = 0 \quad on\ \Gamma , \end{cases} \tag{2.30}$$

$$(B^* \vec{p}_\varepsilon + N u_\varepsilon + u_\varepsilon - u_0, u - u_\varepsilon)_U \geqslant 0 \quad \forall u \in K . \tag{2.31}$$

Remark 2 : The method described in this section provides an efficient numerical scheme to solve Problem (P1) ; obviously, the functional

J_ε should be modified by removing the last two terms. Proposition 2.4 may then fail to be true, but, under the assumptions of Theorem 2.1, it is still possible to prove that $\{u_\varepsilon\}_{\varepsilon > 0}$ is a bounded sequence in U and every weak limit point, when $\varepsilon \to 0$, is a solution of (P1). In fact these subsequences converge strongly in U if $N > 0$. Furthermore $\vec{y}_\varepsilon \to \vec{y}_0$ weakly in $(H^1(\Omega))^3$ and $\inf (\mathrm{P1}_\varepsilon) \to \inf (\mathrm{P1})$.

2.2. Proof of Theorem 2.2

We are going to pass to the limit in the system (2.20)-(2.22) with the help of Proposition 2.4. In this process the essential point is the boundedness of $\{\vec{p}_\varepsilon\}_{\varepsilon > 0}$ in $(H^1(\Omega))^3$. First let us assume that $\{\vec{p}_\varepsilon\}_{\varepsilon > 0}$ is bounded in $(L^2(\Omega))^3$. Choosing in (2.21) $\vec{z} = \vec{p}_\varepsilon$ and remembering the properties of the trilinear form b we get

$$\nu a(\vec{p}_\varepsilon, \vec{p}_\varepsilon) + b(\vec{p}_\varepsilon, \vec{y}_\varepsilon, \vec{p}_\varepsilon) =$$

$$= \int_\Omega (\nabla \times \vec{y}_\varepsilon) \cdot (\nabla \times \vec{p}_\varepsilon)\, dx + \sum_{j=1}^{3} \int_\Omega (y_{\varepsilon j} - y_{0j})\, p_{\varepsilon j}\, dx ,$$

therefore

$$\|\vec{p}_\varepsilon\|^2_{(H^1(\Omega))^3} \leq C_1 \left(|b(\vec{p}_\varepsilon, \vec{y}_\varepsilon, \vec{p}_\varepsilon)| + \|\vec{p}_\varepsilon\|_{(H^1(\Omega))^3} \right) \leq$$

$$\leq C_2 \|\vec{p}_\varepsilon\|^2_{(L^4(\Omega))^3} + C_1 \|\vec{p}_\varepsilon\|_{(H^1(\Omega))^3} . \tag{2.32}$$

From the inequality (Temam [13, page 296])

$$\|\vec{p}_\varepsilon\|_{(L^4(\Omega))^3} \leq \sqrt{2}\, \|\vec{p}_\varepsilon\|^{1/4}_{(L^2(\Omega))^3} \|\vec{p}_\varepsilon\|^{3/4}_{(H^1(\Omega))^3} \tag{2.33}$$

and (2.32), we obtain

$$\|\vec{p}_\varepsilon\|^2_{(H^1(\Omega))^3} \leq C_3 \|\vec{p}_\varepsilon\|^{3/2}_{(H^1(\Omega))^3} + C_1 \|\vec{p}_\varepsilon\|_{(H^1(\Omega))^3} ,$$

which proves the boundedness of $\{\vec{p}_\varepsilon\}_{\varepsilon > 0}$ in $(H^1(\Omega))^3$. Then we extract a subsequence, denoted in the same way, and an element $\vec{p}_0 \in (H^1(\Omega))^3$ such that $\vec{p}_\varepsilon \to \vec{p}_0$ weakly in $(H^1(\Omega))^3$. Now it is easy to pass to the limit in (2.21), using (2.10) and (2.12), and obtain for every $\vec{z} \in Y_0$:

$$\nu a(\vec{p}_0, \vec{z}) - b(\vec{y}_0, \vec{p}_0, \vec{z}) - b(\vec{z}, \vec{p}_0, \vec{y}_0) = \int_\Omega (\nabla \times \vec{y}_0) \cdot (\nabla \times \vec{z})\, dx .$$

From here follows the existence of $\lambda_0 \in L^2(\Omega)$ verifying (2.4) with $\alpha = 1$. Analogously, we can pass to the limit in (2.20) and derive (2.3). Finally (2.5) is easily deduced from (2.22).

If $\{\vec{p}_\varepsilon\}_{\varepsilon > 0}$ is not bounded in $(L^2(\Omega))^3$ we set

$$\alpha_\varepsilon = \frac{1}{\|\vec{p}_\varepsilon\|_{(L^2(\Omega))^3}} \to 0 \quad \text{when} \quad \varepsilon \to \infty$$

and again we denote $\alpha_\varepsilon \vec{p}_\varepsilon$ by $\vec{p}_\varepsilon$. Then (2.21) and (2.22) turn into

$$\nu a(\vec{p}_\varepsilon, \vec{z}) - b(\vec{w}_\varepsilon, \vec{p}_\varepsilon, \vec{z}) - b(\vec{z}, \vec{p}_\varepsilon, \vec{y}_\varepsilon) =$$
$$= \alpha_\varepsilon \int_\Omega (\nabla \times \vec{y}_\varepsilon) \cdot (\nabla \times \vec{z})\, dx + \alpha_\varepsilon \sum_{j=1}^{3} \int_\Omega (y_{\varepsilon j} - y_{0j})\, z_j\, dx \quad \forall \vec{z} \in Y_0,$$

and

$$(B^* \vec{p}_\varepsilon + \alpha_\varepsilon N u_\varepsilon, u - u_\varepsilon)_U \geqslant 0 \quad \forall u \in K$$

respectively. Now repeating the previous argument, we derive (2.3)-(2.5) with $\alpha = 0$. It remains to prove (2.2) or equivalently that $\vec{p}_0 \neq 0$. From the weak convergence $\vec{p}_\varepsilon \to \vec{p}_0$ in $(H^1(\Omega))^3$ and Rellich's theorem, follows the strong convergence of $\{\vec{p}_\varepsilon\}_{\varepsilon > 0}$ to $\vec{p}_0$ in $(L^2(\Omega))^3$, which proves, remembering the redefinition of $\vec{p}_\varepsilon$, that

$$\|\vec{p}_0\|_{(L^2(\Omega))^3} = \lim_{\varepsilon \to 0} \|\vec{p}_\varepsilon\|_{(L^2(\Omega))^3} = 1 . \quad \square$$

3. A BOUNDARY CONTROL PROBLEM

It is very important to consider the applicability of the method we present here to more realistic problems. In this section, the issue of controlling the turbulence caused by heat convection is considered. We study a boundary control problem, and the state of the system solves the equations of

$$\begin{cases} -\nu\, \Delta \vec{y} + (\vec{y} \cdot \nabla)\, \vec{y} + \nabla \pi = \vec{f} + \vec{\beta}\, \tau \text{ in } \Omega, \\ -\kappa\, \Delta \tau + \vec{y} \cdot \nabla \tau = g \text{ in } \Omega, \\ \operatorname{div} \vec{y} = 0 \text{ in } \Omega, \quad \vec{y} = \vec{\phi}_\Gamma \text{ on } \Gamma, \\ \tau = h \text{ on } \Gamma_0, \quad \partial_n \tau = u \text{ on } \Gamma_1, \end{cases} \tag{3.1}$$

where $\nu, \kappa > 0$, $\vec{f} \in (H^{-1}(\Omega))^3$, $\vec{\beta} \in (L^\infty(\Omega))^3$, $g \in L^{6/5}(\Omega)$, $h \in H^{1/2}(\Gamma_0)$, $u \in L^2(\Gamma_1)$, $\Gamma = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \varnothing$ and $\sigma(\Gamma_0)$, $\sigma(\Gamma_1) > 0$. The reader is referred to [1] for an application of Problem (P2) to the case of a fluid in a dirven cavity. Here $\vec{y}$, π and $\vec{f}$ are the same things as in (2.1), τ is the temperature inside the fluid and u is the heat flux through the boundary. The control problem is formulated in the following way

$$(\text{P2}) \quad \begin{cases} \text{Minimize } J(\vec{y}, u) \\ (\vec{y}, u) \in (H^1(\Omega))^3 \times K \text{ and } (\vec{y}, u) \text{ satisfies (3.1) for some } (\pi, \tau), \end{cases}$$

with $J : (H^1(\Omega))^3 \times L^2(\Gamma_1) \to R$ being defined by

$$J(\vec{y}, u) = \frac{1}{2} \int_\Omega |\nabla \times \vec{y}|^2 \, dx + \frac{N}{2} \int_{\Gamma_1} |u|^2 \, d\sigma ,$$

$N \geq 0$ and $K \subset L^2(\Gamma_1)$ nonempty, convex and closed. In this problem the role of the control is to cool suitably the fluid from a part of the boundary in order to minimize the turbulence inside the flow.

Let us remark that the hypothesis $g \in L^{6/5}(\Omega)$ is made to give a sense to the Neumann boundary condition of (3.1). Thanks to this assumption the term $\partial_n \tau$ is well defined and the usual variational formulation of this problem is equivalent to (3.1) ; see Casas and Fernández [3]. Now we will analyze the state equation. We will state the existence of a solution of the coupled system (3.1) without any restriction on the size of the viscosity ν and the diffusion coefficient κ ; see for example Gaultier and Lezaun [5]. However we can not hope, in general, to have uniqueness, therefore we are dealing with a multistate equation ; see Rabinowitz [12].

THEOREM 3.1 : *Assuming the above conditions, the system* (3.1) *has at least one solution* $(\vec{y}, \pi, \tau) \in (H^1(\Omega))^3 \times L^2(\Omega) \times H^1(\Omega)$. *Furthermore there exist constants* M_1, $M_2 > 0$ *such that*

$$\|\tau\|_{H^1(\Omega)} + \|\vec{y}\|_{(H^1(\Omega))^3} \leq$$
$$\leq M_1 \left(\|\vec{f}\|_{(H^{-1}(\Omega))^3} + \|g\|_{L^{6/5}(\Omega)} + \|u\|_{L^2(\Gamma_1)} \right) + M_2 , \quad (3.2)$$

where M_2 *depends on* $\vec{\phi}_\Gamma$ *and* h, *being zero when these functions are zero.*

Proof : Let us set

$$T = \left\{ \tau \in H^1(\Omega) : \tau|_{\Gamma_0} = 0 \right\} .$$

Together with the bilinear and trilinear forms a and b defined in Section 2.1, here we will need $a_0 : H^1(\Omega) \times H^1(\Omega) \to R$ and

$$b_0 : (H^1(\Omega))^3 \times H^1(\Omega) \times H^1(\Omega) \to R$$

given by

$$a_0(\tau, \zeta) = \int_\Omega \nabla\tau \,.\, \nabla\zeta \, dx$$

and

$$b_0(\vec{y}, \tau, \zeta) = \int_\Omega (\vec{y} \,.\, \nabla\tau)\, \zeta \, dx \,.$$

To prove the existence of a solution of (3.1) it is enough to state that there exists an element $(\vec{y}, \tau) \in Y \times H^1(\Omega)$ satisfying

$$\begin{cases} \vec{y} = \vec{\phi}_\Gamma \text{ on } \Gamma \,, \quad \tau = h \text{ on } \Gamma_0 \\ \nu a(\vec{y}, \vec{z}) + b(\vec{y}, \vec{y}, \vec{z}) = \langle \vec{f}, \vec{z} \rangle + \int_\Omega (\vec{\beta} \,.\, \vec{z})\, \tau \, dx \quad \forall \vec{z} \in Y_0 \,, \\ \kappa a_0(\tau, \zeta) + b_0(\vec{y}, \tau, \zeta) = \langle g, \zeta \rangle + \int_{\Gamma_1} u\zeta \, d\sigma \quad \forall \zeta \in T \,. \end{cases} \tag{3.3}$$

Since Y_0 is a separable space, there exists a sequence $\{\vec{w}_k\}_{k=1}^{\infty}$ which is a Hilbertian basis of Y_0. Let Y_m be the space generated by the functions $\{\vec{w}_1, \ldots, \vec{w}_m\}$ and let $\vec{\phi} \in Y$ with trace equal to $\vec{\phi}_\Gamma$ and verifying (1.2). For each fixed integer $m \geq 1$ we will prove the existence of an approximate solution $(\vec{y}_m, \tau_m) \in Y_m \times T$ of (3.3)

$$\begin{cases} \vec{y}_m = \vec{\phi} + \sum_{k=1}^{m} \xi_{k,m} \vec{w}_k \,, \quad \xi_{k,m} \in R \,, \quad \tau_m = h \quad \text{on } \Gamma_0 \\ \nu a(\vec{y}_m, \vec{w}_k) + b(\vec{y}_m, \vec{y}_m, \vec{w}_k) = \langle \vec{f}, \vec{w}_k \rangle + \int_\Omega (\vec{\beta} \,.\, \vec{w}_k)\, \tau_m \, dx \,, \quad 1 \leq k \leq m \,, \\ \kappa a_0(\tau_m, \zeta) + b_0(\vec{y}_m, \tau_m, \zeta) = \int_\Omega g\zeta \, dx + \int_{\Gamma_1} u\zeta \, d\sigma \quad \forall \zeta \in T \,. \end{cases} \tag{3.4}$$

Let us prove that (3.4) has at least one solution. First we define the mapping $F : Y_m \to Y_m$ in such a way that $F(\vec{w})$ is the unique element $\vec{z}_m$ that, together with $\tau_m \in H^1(\Omega)$ verifying $\tau_m = h$ on Γ_0, satisfies

$$\begin{cases} \nu a(\vec{\phi} + \vec{z}_m, \vec{w}_k) + b(\vec{\phi} + \vec{w}, \vec{\phi} + \vec{z}_m, \vec{w}_k) = \langle \vec{f}, \vec{w}_k \rangle + \int_\Omega (\vec{\beta} \,.\, \vec{w}_k)\, \tau_m \, dx, \\ \kappa a_0(\tau_m, \zeta) + b_0(\vec{\phi} + \vec{w}, \tau_m, \zeta) = \int_\Omega g\zeta \, dx + \int_{\Gamma_1} u\zeta \, d\sigma \,, \\ \forall \zeta \in T \quad \text{and} \quad 1 \leq k \leq m \,. \end{cases} \tag{3.5}$$

It is an immediate consequence of the Lax-Milgram theorem that this problem has a unique solution $(\vec{z}_m, \tau_m) \in Y_m \times H^1(\Omega)$, remark that we can find firstly τ_m and then $\vec{z}_m \in Y_m$. Let us take $\psi \in h^1(\Omega)$ such that $\psi = h$ on Γ_0 and $\partial_n \psi = 0$ on Γ_1, for example ψ could be the solution of

$$\begin{cases} -\Delta\psi = 0 & \text{in } \Omega \\ \psi = h & \text{on } \Gamma_0 \\ \partial_n \psi = 0 & \text{on } \Gamma_1 . \end{cases}$$

Let now $\rho_\varepsilon \in D(R^3)$ verifying

$$\rho_\varepsilon(x) = \begin{cases} 1 & \text{if} \quad d(x, \Gamma) \leq \varepsilon/2 \\ 0 & \text{if} \quad d(x, \Gamma) \geq \varepsilon . \end{cases}$$

Given $\delta > 0$, redefining ψ as $\rho_\varepsilon\, \psi$ and taken ε small enough, we can suppose that

$$\|\psi\|_{L^4(\Omega)} \leq \delta . \tag{3.6}$$

Taking $\theta_m = \tau_m - \psi \in T$ and setting $\zeta = \theta_m$ in the second equation of (3.5), we deduce with the aid of (3.6) and the identities

$$b_0(\vec{y}, \theta, \theta) = 0 \;\; \forall \theta \in H^1(\Omega) \text{ and } \forall \vec{y} \in Y \text{ such that } \vec{y} \,.\, n = 0 \text{ on } \Gamma ,$$

$$b_0(\vec{y}, \theta, \zeta) = - b_0(\vec{y}, \zeta, \theta) \;\; \forall \theta, \zeta \in H^1(\Omega) \text{ and } \forall \vec{y} \in Y$$
$$\text{such that } \vec{y} \,.\, n = 0 \text{ on } \Gamma ,$$

that

$$\kappa a_0(\theta_m, \theta_m) = \\ = \int_\Omega g\,\theta_m\, dx + \int_{\Gamma_1} u\,\theta_m\, d\sigma - \kappa a_0(\psi, \theta_m) - b_0(\vec{\phi} + \vec{w}, \psi, \theta_m) ,$$

therefore

$$\|\theta_m\|_{H^1(\Omega)} \leq C_1 \left(\|g\|_{L^{6/5}(\Omega)} + \|u\|_{L^2(\Gamma_1)} + \|\psi\|_{H^1(\Omega)} + \delta \left\|\vec{\phi} + \vec{w}\right\|_{(H^1(\Omega))^3} \right) ,$$

and

$$\|\tau_m\|_{H^1(\Omega)} \leq C_1 \left(\|g\|_{L^{6/5}(\Omega)} + \|u\|_{L^2(\Gamma_1)} + \delta \left\|\vec{\phi} + \vec{w}\right\|_{(H^1(\Omega))^3} \right) + \\ + (1 + C_1)\|\psi\|_{H^1(\Omega)} .$$

Analogously from the first equation of (3.5) we get

$$\nu a(\vec{z}_m, \vec{z}_m) = \\ = \left\langle \vec{f}, \vec{z}_m \right\rangle + \int_\Omega (\vec{\beta} \,.\, \vec{z}_m)\, \tau_m\, dx - \nu a_0(\vec{\phi}, \vec{z}_m) - b(\vec{\phi} + \vec{w}, \vec{\phi}, \vec{z}_m) .$$

From (1.2) and the estimates for τ_m we obtain

$$\|\vec{z}_m\|_{(H^1(\Omega))^3} \leq C_2 \Big(\|\vec{f}\|_{(H^{-1}(\Omega))^3} + \|g\|_{L^{6/5}(\Omega)} + \|u\|_{L^2(\Gamma_1)} + + (\delta + \mu)\|\vec{w}\|_{(H^1(\Omega))^3} + \|\psi\|_{H^1(\Omega)} + \|\vec{\phi}\|_{(H^1(\Omega))^3} \Big) ,$$

then

$$\|\vec{y}_m\|_{(H^1(\Omega))^3} \leq C_3 + C_2(\delta + \mu)\|\vec{w}\|_{(H^1(\Omega))^3} ,$$

where C_3 is independent of $\vec{w}$.

Choosing δ and μ in such a way that

$$C_2(\delta + \mu) \leq 1/2$$

and setting $r = 2\, C_3$ we deduce that F applies the ball $\bar{B}_r(0)$ of Y_m into itself, moreover it is easy to verify that F is continuous. Then by applying the Brouwer's theorem we deduce the existence of a fixed-point of $F\vec{z}_{0m}$ and an associated temperature τ_{0m} in such a way that $(\vec{\phi} + \vec{z}_{0m}, \tau_{0m})$ is obviously a solution of (3.4). Now we can pass to the limit in a similar way as in Teman [13 Theorem 1.2] and deduce the convergence of a subsequence towards a solution of (3.3).

In order to prove (3.2) we will write $\vec{y} = \vec{\phi} + \vec{z}_0$ and $\tau = \psi + \theta_0$, where $\vec{\phi}$ and ψ are defined as above and $\vec{z}_0 \in Y_0$ and $\theta_0 \in T$. Now putting $\vec{z} = \vec{z}_0$ and $\zeta = \theta_0$ in (3.3) and using the properties of the trilinear forms b and b_0 we obtain

$$\|\theta_0\| \leq C_4 \Big(\|g\|_{L^{6/5}(\Omega)} + \|u\|_{L^2(\Gamma_1)} + \|\psi\|_{H^1(\Omega)} + \delta \|\vec{\phi} + \vec{z}_0\|_{(H^1(\Omega))^3} \Big)$$

and

$$\|\vec{z}_0\|_{(H^1(\Omega))^3} \leq C_5 \Big(\|\vec{f}\|_{(H^{-1}(\Omega))^3} + \|g\|_{L^{6/5}(\Omega)} + \|u\|_{L^2(\Gamma_1)} + + (\delta + \mu)\|\vec{z}_0\|_{(H^1(\Omega))^3} + \|\psi\|_{H^1(\Omega)} + \|\vec{\phi}\|_{(H^1(\Omega))^3} \Big) ,$$

which allows us to conclude (3.2) by choosing δ and μ in such a way that $C_5(\delta + \mu) \leq 1/2$. □

We can proceed as in Theorem 2.1, using the estimates (3.2), and deduce the following theorem.

THEOREM 3.2 : *If $N > 0$ or K is bounded in $L^2(\Gamma_1)$, then* (P2) *has at least one solution.*

Next we state the optimality conditions of problem (P2).

THEOREM 3.3 : *Let* $(\vec{y}_0, u_0)$ *be a solution of* (P2), *then there exist a constant* $\alpha \geq 0$ *and elements* $\vec{p}_0 \in (H^1(\Omega))^3$, τ_0, $q_0 \in H^1(\Omega)$ *and* π_0, $\lambda_0 \in L^2(\Omega)$ *such that*

$$\alpha + \|q_0\|_{H^1(\Omega)} > 0 \tag{3.7}$$

$$\begin{cases} -\nu\,\Delta\vec{y}_0 + (\vec{y}_0 \cdot \nabla)\,\vec{y}_0 + \nabla\pi_0 = \vec{f} + \vec{\beta}\,\tau_0 & in\ \Omega\,, \\ -\kappa\,\Delta\tau_0 + \vec{y}_0 \cdot \nabla\tau_0 = g \quad in\ \Omega\,, \\ \operatorname{div}\vec{y}_0 = 0 \quad in\ \Omega\,, \quad \vec{y}_0 = \vec{\phi}_\Gamma \quad on\ \Gamma\,, \\ \tau_0 = h \quad on\ \Gamma_0\,, \quad \partial_n\tau_0 = u_0 \quad on\ \Gamma_1\,, \end{cases} \tag{3.8}$$

$$\begin{cases} -\nu\,\Delta\vec{p}_0 - (\vec{y}_0 \cdot \nabla)\,\vec{p}_0 + (\nabla\vec{y}_0)^T\,\vec{p}_0 + \nabla\lambda_0 = \tau_0\,\nabla q_0 + \alpha\nabla\times(\nabla\times\vec{y}_0) \ in\ \Omega, \\ -\kappa\,\Delta q_0 - \vec{y}_0 \cdot \nabla q_0 = \vec{\beta}\vec{p}_0 \quad in\ \Omega\,, \\ \operatorname{div}\vec{p}_0 = 0 \quad in\ \Omega\,, \quad \vec{p}_0 = 0 \quad on\ \Gamma\,, \\ q_0 = 0 \quad on\ \Gamma_0\,, \quad \partial_n q_0 = 0 \quad on\ \Gamma_1\,, \end{cases} \tag{3.9}$$

$$\int_{\Gamma_1} (q_0 + \alpha N u_0)(u - u_0)\,d\sigma \geq 0 \quad \forall u \in K\,. \tag{3.10}$$

Similarly to Theorem 2.2 here we could formulate a statement analogous to that of Corollary 2.3, which would allow to conclude (3.8)-(3.10) with $\alpha = 1$ if K had property C at $(\vec{y}_0, \vec{u}_0)$. The proof of this theorem follows the same steps as that of Theorem 2.2.

3.1. The problems $(\mathbf{P2}_\varepsilon)$

Let $(\vec{y}_0, u_0)$ be a fixed solution of (P2) and for each $\varepsilon > 0$ let us define the functionals $J_\varepsilon : \{\vec{w} \in Y : \vec{w} = \vec{\phi}_\Gamma \text{ on } \Gamma\} \times L^2(\Gamma_1) \to R$ by

$$J_\varepsilon(\vec{w}, u) = J(\vec{y}(\vec{w}, u), u) + \frac{1}{2\,\varepsilon}\sum_{j=1}^{3}\int_\Omega |\nabla y_j(\vec{w}, u) - \nabla w_j|^2\,dx + \\ + \frac{1}{2}\sum_{j=1}^{3}\int_\Omega |y_j - y_{0j}|^2\,dx + \frac{1}{2}\int_{\Gamma_1} |u - u_0|^2\,d\sigma\,,$$

where $\vec{y}(\vec{w}, u)$ is the unique element of Y satisfying together with an element $\tau \in H^1(\Omega)$

$$\begin{cases} \vec{y} = \vec{\phi}_\Gamma \quad \text{on } \Gamma \quad \text{and} \quad \tau = h \quad \text{on } \Gamma_0\,, \\ \nu a(\vec{y}, \vec{z}) + b(\vec{w}, \vec{y}, \vec{z}) = \langle \vec{f}, \vec{z} \rangle + \int_\Omega (\vec{\beta} \cdot \vec{z})\,\tau\,dx \quad \forall \vec{z} \in Y_0\,, \\ \kappa a_0(\tau, \zeta) + b_0(\vec{w}, \tau, \zeta) = \langle g, \zeta \rangle + \int_{\Gamma_1} u\zeta\,d\sigma \quad \forall \zeta \in T\,. \end{cases} \tag{3.11}$$

Now we formulate the approximate control problem as in Section 2.1

$$(\mathrm{P2}_\varepsilon) \quad \begin{cases} \text{Minimise } J_\varepsilon(\vec{w}, u) \\ (\vec{w}, u) \in Y \times K \quad \text{and} \quad \vec{w} = \vec{\phi}_\Gamma \quad \text{on } \Gamma . \end{cases}$$

The proof of Proposition 2.4, with the obvious modifications, can be repeated to derive the following result.

PROPOSITION 3.4 : *For every* $\varepsilon > 0$ *there exists at least one solution* $(\vec{w}_\varepsilon, u_\varepsilon)$ *of* $(\mathrm{P2}_\varepsilon)$. *Moreover, if we denote by* $(\vec{y}_\varepsilon, \tau_\varepsilon)$ *the solution of* (3.11) *corresponding to* $(\vec{w}_\varepsilon, u_\varepsilon)$, *then we have*

$$\lim_{\varepsilon \to 0} \|u_\varepsilon - u_0\|_{L^2(\Gamma_1)} = \lim_{\varepsilon \to 0} \frac{1}{2\,\varepsilon} \sum_{j=1}^{3} \int_\Omega |\nabla y_{\varepsilon j} - \nabla w_{\varepsilon j}|^2 \, dx = 0 \,, \quad (3.12)$$

$$\vec{w}_\varepsilon \to \vec{y}_0 \quad \textit{wheakly in } Y \,, \quad (3.13)$$

$$\vec{y}_\varepsilon \to \vec{y}_0 \quad \textit{weakly in } Y \,, \quad (3.14)$$

$$\lim_{\varepsilon \to 0} \|\tau_\varepsilon - \tau_0\|_{H^1(\Omega)} = 0 \,, \quad (3.15)$$

$$\lim_{\varepsilon \to 0} J_\varepsilon(\vec{w}_\varepsilon, u_\varepsilon) = J(\vec{y}_0, u_0) \,. \quad (3.16)$$

The next step consists in deriving the optimality conditions satisfied by $(\vec{w}_\varepsilon, u_\varepsilon)$.

PROPOSITION 3.5 : *Let us suppose that* $(\vec{w}_\varepsilon, u_\varepsilon)$ *is a solution of* $(\mathrm{P2}_\varepsilon)$, *then there exist elements* $\vec{y}_\varepsilon \in Y$, $\tau_\varepsilon \in H^1(\Omega)$, $\vec{p}_\varepsilon \in Y_0$ *and* $q_\varepsilon \in T$ *such that the following system is verified*

$$\begin{cases} \vec{y}_\varepsilon = \vec{\phi}_\Gamma \quad \text{on } \Gamma \quad \text{and} \quad \tau_\varepsilon = h \quad \text{on } \Gamma_0 \,, \\ \nu a(\vec{y}_\varepsilon, \vec{z}) + b(\vec{w}_\varepsilon, \vec{y}_\varepsilon, \vec{z}) = \langle \vec{f}, \vec{z} \rangle + \int_\Omega (\vec{\beta} \,.\, \vec{z})\, \tau_\varepsilon \, dx \quad \forall \vec{z} \in Y_0 \,, \\ \kappa a_0(\tau_\varepsilon, \zeta) + b_0(\vec{w}_\varepsilon, \tau_\varepsilon, \zeta) = \langle g, \zeta \rangle + \int_{\Gamma_1} u_\varepsilon \, \zeta \, d\sigma \quad \forall \zeta \in T \,, \end{cases} \quad (3.17)$$

$$\begin{cases} \nu a(\vec{p}_\varepsilon, \vec{z}) - b(\vec{w}_\varepsilon, \vec{p}_\varepsilon, \vec{z}) - b(\vec{z}, \vec{p}_\varepsilon, \vec{y}_\varepsilon) = b_0(\vec{z}, q_\varepsilon, \tau_\varepsilon) + \\ \quad + \int_\Omega (\nabla \times \vec{y}_\varepsilon) \,.\, (\nabla \times \vec{z}) \, dx + \sum_{j=1}^{3} \int_\Omega (y_{\varepsilon j} - y_{0 j})\, z_j \, dx \quad \forall \vec{z} \in Y_0 \,, \\ \kappa a_0(q_\varepsilon, \zeta) - b_0(\vec{w}_\varepsilon, q_\varepsilon, \zeta) = \int_\Omega (\vec{\beta} \,.\, \vec{p}_\varepsilon)\, \zeta \, dx \quad \forall \zeta \in T \,, \end{cases} \quad (3.18)$$

$$\int_{\Gamma_1} (q_\varepsilon + N u_\varepsilon + u_\varepsilon - u_0)(u - u_\varepsilon)\, d\sigma \geqslant 0 \quad \forall u \in K\,. \tag{3.19}$$

Proof : The proof follows the same steps as that of Proposition 2.5. First we take $\vec{p}_\varepsilon \in Y_0$ and $q_\varepsilon \in T$ satisfying :

$$\begin{cases} \nu a(\vec{p}_\varepsilon, \vec{z}) - b(\vec{w}_\varepsilon, \vec{p}_\varepsilon, \vec{z}) = \displaystyle\int_\Omega (\nabla \times \vec{y}_\varepsilon) \cdot (\nabla \times \vec{z})\, dx + \\ \quad + \displaystyle\sum_{j=1}^{3} \int_\Omega (y_{\varepsilon j} - y_{0j}) z_j\, dx + \frac{1}{\varepsilon} \sum_{j=1}^{3} \int_\Omega (\nabla y_{\varepsilon j} - \nabla w_{\varepsilon j}) \nabla z_j\, dx \quad \forall \vec{z} \in Y_0\,, \\ \kappa a_0(q_\varepsilon, \zeta) - b_0(\vec{w}_\varepsilon, q_\varepsilon, \zeta) = \displaystyle\int_\Omega (\vec{\beta} \cdot \vec{p}_\varepsilon)\, \zeta\, dx \quad \forall \zeta \in T\,. \end{cases} \tag{3.20}$$

Now, given $\vec{w} \in Y_0$ and $u \in L^2(\Gamma_1)$ we denote by $(\vec{z}_{\vec{w}}, \theta_{\vec{w}})$ and $(\vec{z}_u, \theta_u)$ the elements of $Y_0 \times T$ verifying

$$\begin{cases} \nu a(\vec{z}_{\vec{w}}, \vec{z}) + b(\vec{w}_\varepsilon, \vec{z}_{\vec{w}}, \vec{z}) + b(\vec{w}, \vec{y}_\varepsilon, \vec{z}) = \displaystyle\int_\Omega (\vec{\beta} \cdot \vec{z})\, \theta_{\vec{w}}\, dx \quad \forall \vec{z} \in Y_0 \\ \kappa a_0(\theta_{\vec{w}}, \zeta) + b_0(\vec{w}_\varepsilon, \theta_{\vec{w}}, \zeta) + b_0(\vec{w}, \tau_\varepsilon, \zeta) = 0 \quad \forall \zeta \in T\,, \end{cases} \tag{3.21}$$

and

$$\begin{cases} \nu a(\vec{z}_u, \vec{z}) + b(\vec{w}_\varepsilon, \vec{z}_u, \vec{z}) = \displaystyle\int_\Omega (\vec{\beta} \cdot \vec{z})\, \theta_u\, dx \quad \forall \vec{z} \in Y_0 \\ \kappa a_0(\theta_u, \zeta) + b_0(\vec{w}_\varepsilon, \theta_u, \zeta) = \displaystyle\int_{\Gamma_1} u \zeta\, d\sigma \quad \forall \zeta \in T\,, \end{cases} \tag{3.22}$$

respectively.

With the aid of (3.20) and (3.21) we get

$$\begin{aligned} \frac{\partial J_\varepsilon}{\partial \vec{w}}(\vec{w}_\varepsilon, u_\varepsilon) \cdot \vec{w} &= \int_\Omega (\nabla \times \vec{y}_\varepsilon) \cdot (\nabla \times \vec{z}_{\vec{w}})\, dx \\ &\quad + \frac{1}{\varepsilon} \sum_{j=1}^{3} \int_\Omega (\nabla y_{\varepsilon j} - \nabla w_{\varepsilon j}) \cdot (\nabla z_{\vec{w} j} - \nabla w_j)\, dx + \sum_{j=1}^{3} \int_\Omega (y_{\varepsilon j} - y_{0j})\, z_{\vec{w} j}\, dx \\ &= \nu a(\vec{p}_\varepsilon, \vec{z}_{\vec{w}}) - b(\vec{w}_\varepsilon, \vec{p}_\varepsilon, \vec{z}_{\vec{w}}) - \frac{1}{\varepsilon} \sum_{j=1}^{3} \int_\Omega (\nabla y_{\varepsilon j} - \nabla w_{\varepsilon j}) \nabla w_j\, dx \\ &= \nu a(\vec{z}_{\vec{w}}, \vec{p}_\varepsilon) + b(\vec{w}_\varepsilon, \vec{z}_{\vec{w}}, \vec{p}_\varepsilon) - \frac{1}{\varepsilon} \sum_{j=1}^{3} \int_\Omega (\nabla y_{\varepsilon j} - \nabla w_{\varepsilon j}) \nabla w_j\, dx \end{aligned}$$

$$= b(\vec{w}, \vec{p}_\varepsilon, \vec{y}_\varepsilon) + \kappa a_0(q_\varepsilon, \theta_{\vec{w}}) - b_0(\vec{w}_\varepsilon, q_\varepsilon, \theta_{\vec{w}})$$
$$- \frac{1}{\varepsilon} \sum_{j=1}^{3} \int_\Omega (\nabla y_{\varepsilon j} - \nabla w_{\varepsilon j}) \nabla w_j \, dx$$

$$= b(\vec{w}, \vec{p}_\varepsilon, \vec{y}_\varepsilon) + \kappa a_0(\theta_{\vec{w}}, q_\varepsilon) + b_0(\vec{w}_\varepsilon, \theta_{\vec{w}}, q_\varepsilon)$$
$$- \frac{1}{\varepsilon} \sum_{j=1}^{3} \int_\Omega (\nabla y_{\varepsilon j} - \nabla w_{\varepsilon j}) \nabla w_j \, dx$$

$$= b(\vec{w}, \vec{p}_\varepsilon, \vec{y}_\varepsilon) + \kappa a_0(\theta_{\vec{w}}, q_\varepsilon) + b_0(\vec{w}_\varepsilon, \theta_{\vec{w}}, q_\varepsilon)$$
$$- \frac{1}{\varepsilon} \sum_{j=1}^{3} \int_\Omega (\nabla y_{\varepsilon j} - \nabla w_{\varepsilon j}) \nabla w_j \, dx$$

$$= b(\vec{w}, \vec{p}_\varepsilon, \vec{y}_\varepsilon) + b_0(\vec{w}, q_\varepsilon, \tau_\varepsilon) - \frac{1}{\varepsilon} \sum_{j=1}^{3} \int_\Omega (\nabla y_{\varepsilon j} - \nabla w_{\varepsilon j}) \nabla w_j \, dx \,. \tag{3.23}$$

Using now (3.20) and (3.22) we obtain

$$\frac{\partial J_\varepsilon}{\partial u} (\vec{w}_\varepsilon, u_\varepsilon) \,.\, u = \int_\Omega (\nabla \times \vec{y}_\varepsilon) \,.\, (\nabla \times \vec{z}_u) \, dx + N \int_{\Gamma_1} u_\varepsilon \, u \, d\sigma +$$
$$+ \frac{1}{\varepsilon} \sum_{j=1}^{3} \int_\Omega (\nabla y_{\varepsilon j} - \nabla w_{\varepsilon j}) \,.\, \nabla z_{uj} \, dx$$
$$+ \sum_{j=1}^{3} \int_\Omega (y_{\varepsilon j} - y_{0j}) \, z_{uj} \, dx + \int_{\Gamma_1} (u_\varepsilon - u_0) \, u \, d\sigma$$

$$= \nu a(\vec{p}_\varepsilon, \vec{z}_u) - b(\vec{w}_\varepsilon, \vec{p}_\varepsilon, \vec{z}_u) + N \int_{\Gamma_1} u_\varepsilon \, u \, d\sigma + \int_{\Gamma_1} (u_\varepsilon - u_0) \, u \, d\sigma$$

$$= \nu a(\vec{z}_u, \vec{p}_\varepsilon) + b(\vec{w}_\varepsilon, \vec{z}_u, \vec{p}_\varepsilon) + \int_{\Gamma_1} (N u_\varepsilon + u_\varepsilon - u_0) \, u \, d\sigma$$

$$= \int_\Omega (\vec{\beta} \,.\, \vec{p}_\varepsilon) \, \theta_u \, dx + \int_{\Gamma_1} (N u_\varepsilon + u_\varepsilon - u_0) \, u \, d\sigma$$

$$= \kappa a_0(q_\varepsilon, \theta_u) - b_0(\vec{w}_\varepsilon, q_\varepsilon, \theta_u) + \int_{\Gamma_1} (N u_\varepsilon + u_\varepsilon - u_0) \, u \, d\sigma$$

$$= \kappa a_0(\theta_u, q_\varepsilon) + b_0(\vec{w}_\varepsilon, \theta_u, q_\varepsilon) + \int_{\Gamma_1} (N u_\varepsilon + u_\varepsilon - u_0) \, u \, d\sigma$$

$$= \int_{\Gamma_1} (q_\varepsilon + N u_\varepsilon + u_\varepsilon - u_0) \, u \, d\sigma \,. \tag{3.24}$$

We finish the proof as in Proposition 2.5, taking into account this time the relations (3.20), (3.23) and (3.24). □

COROLLARY 3.6 : *If* $(\vec{w}_\varepsilon, u_\varepsilon)$ *is a solution of* $(P2_\varepsilon)$, *then there exist elements* $\vec{y}_\varepsilon, \vec{p}_\varepsilon \in (H^1(\Omega))^3$, $\tau_\varepsilon, q_\varepsilon \in H^1(\Omega)$ *and* $\lambda_\varepsilon, \pi_\varepsilon \in L^2(\Omega)$ *such that*

$$\begin{cases} -\nu\,\Delta\vec{y}_\varepsilon + (\vec{w}_\varepsilon\,.\,\nabla)\,\vec{y}_\varepsilon + \nabla\pi_\varepsilon = \vec{f} + \vec{\beta}\,\tau_\varepsilon \quad in\ \Omega\,, \\ -\kappa\,\Delta\tau_\varepsilon + \vec{w}_\varepsilon\,.\,\nabla\tau_\varepsilon = g \quad in\ \Omega\,, \\ \operatorname{div}\vec{y}_\varepsilon = 0 \quad in\ \Omega\,, \quad \vec{y}_\varepsilon = \vec{\phi}_\Gamma \quad on\ \Gamma\,, \\ \tau_\varepsilon = h \quad on\ \Gamma_0\,, \quad \partial_n\tau_\varepsilon = u_\varepsilon \quad on\ \Gamma_1\,, \end{cases} \tag{3.25}$$

$$\begin{cases} -\nu\,\Delta\vec{p}_\varepsilon - (\vec{w}_\varepsilon\,.\,\nabla)\,\vec{p}_\varepsilon + (\nabla\vec{y}_\varepsilon)^T\,\vec{p}_\varepsilon + \nabla\lambda_\varepsilon = \\ \quad = \tau_\varepsilon\,\nabla q_\varepsilon + \nabla\times(\nabla\times\vec{y}_\varepsilon) + \vec{y}_\varepsilon - \vec{y}_0 \quad in\ \Omega\,, \\ -\kappa\,\Delta q_\varepsilon - \vec{w}_\varepsilon\,.\,\nabla q_\varepsilon = \vec{\beta}\vec{p}_\varepsilon \quad in\ \Omega\,, \\ \operatorname{div}\vec{p}_\varepsilon = 0 \quad in\ \Omega\,, \quad \vec{p}_\varepsilon = 0 \quad on\ \Gamma, \\ q_\varepsilon = 0 \quad on\ \Gamma_0\,, \quad \partial_n q_\varepsilon = 0 \quad on\ \Gamma_1\,, \end{cases} \tag{3.26}$$

$$\int_{\Gamma_1} (q_\varepsilon + Nu_\varepsilon + u_\varepsilon - u_0)(u - u_\varepsilon)\,d\sigma \geqslant 0 \quad \forall u \in K\,. \tag{3.27}$$

3.2 Proof of Theorem 3.3

In order to pass to the limit in the system (3.17)-(3.19) we will proceed as in Section 2.2. First, from the last relation of (3.18) we obtain

$$\|q_\varepsilon\|_{H^1(\Omega)} \leqslant C_1\,\|\vec{p}_\varepsilon\|_{(H^1(\Omega))^3}\,. \tag{3.28}$$

Now, from the first relation of (3.18) we derive with the aid of (3.28) that

$$\begin{aligned} \|\vec{p}_\varepsilon\|^2_{(H^1(\Omega))^3} &\leqslant C_2(|b(\vec{p}_\varepsilon, \vec{p}_\varepsilon, \vec{y}_\varepsilon)| + |b_0(\vec{p}_\varepsilon, q_\varepsilon, \tau_\varepsilon)| + \|\vec{p}_\varepsilon\|_{(H^1(\Omega))^3}) \leqslant \\ &\leqslant C_3(\|\vec{p}_\varepsilon\|^2_{(L^4(\Omega))^3} + \|\vec{p}_\varepsilon\|_{(L^4(\Omega))^3}\|\vec{p}_\varepsilon\|_{(H^1(\Omega))^3} + \|\vec{p}_\varepsilon\|_{(H^1(\Omega))^3})\,. \end{aligned} \tag{3.29}$$

From (2.33), (3.28) and (3.29), we deduce that the boundedness of $\{\vec{p}_\varepsilon\}_{\varepsilon>0}$ in $(L^2(\Omega))^3$ implies the boundedness of $\{q_\varepsilon\}_{\varepsilon>0}$ and $\{\vec{p}_\varepsilon\}_{\varepsilon>0}$ in the spaces $H^1(\Omega)$ and $(H^1(\Omega))^3$ respectively. Therefore we can argue as in the proof of Theorem 2.2 distinguishing two situations, according to whether $\{\vec{p}_\varepsilon\}_{\varepsilon>0}$ is bounded in $(L^2(\Omega))^3$ or not. In this way we can pass to the limit and derive (3.8)-(3.10) with $\alpha = 1$ or $\alpha = 0$ and

$$\|\vec{p}_0\|_{(L^2(\Omega))^3} = 1\,.$$

In this second case, we deduce from (3.9) that q_0 can not be null, which proves (3.7). □

APPENDIX

The aim of this appendix is to prove that for every $\vec{\phi}_\Gamma \in (H^{1/2}(\Gamma))^3$ satisfying (1.1) there exists at least one element $\vec{\phi} \in Y$ with trace equal to $\vec{\phi}_\Gamma$ and verifying (1.2). Here we will follow Temam [13 Lemma 1.8 and Appendix I]. Whereas the proof of Temam assumes Γ of class C^2, here we only will assume that Γ is Lipschitz. However our proof uses the hypothesis (1.1) in a essential way, which is not necessary in [13].

Before stating the main result we need to prove a lemma.

LEMMA A.1 : *There exists an element $\vec{\varphi} \in Y \cap H^2(\Omega)$ such that the trace of $\vec{\phi} = \operatorname{curl} \vec{\varphi} \in Y$ on Γ is $\vec{\phi}_\Gamma$.*

Proof : Since $(H^{1/2}(\Gamma))^3$ is the trace space of $(H^1(\Omega))^3$, there exists an element $\vec{\Phi}_1 \in (H^1(\Omega))^3$ such taht $\gamma(\vec{\Phi}_1) = \vec{\phi}_\Gamma$. From (1.1) and the formula of integration by parts, Nečas [11] we obtain

$$\int_\Omega \operatorname{div} \vec{\Phi}_1(x)\, dx = \int_\Gamma \vec{\Phi}_1(x) \cdot \vec{n}(x)\, d\sigma(x) = 0 .$$

Then Lemma 2.4 of [13] proves the existence of a function $\vec{\Phi}_2 \in (H_0^1(\Omega))^3$ such that $\operatorname{div} \vec{\Phi}_2 = -\operatorname{div} \vec{\Phi}_1$. Therefore $\vec{\Phi}_3 = \vec{\Phi}_1 + \vec{\Phi}_2 \in Y$ and its trace is $\vec{\phi}_\Gamma$.

Let $B_r(0)$ be a ball of R^3 such that $\bar{\Omega} \subset B_r(0)$ and take $\vec{\Phi}_r : B_r(0) \to R$ defined by

$$\vec{\Phi}_r(x) = \begin{cases} \vec{\Phi}_3(x) & \text{if} \quad x \in \Omega \\ 0 & \text{if} \quad x \notin \Omega . \end{cases}$$

Then $\vec{\Phi}_r \in (L^2(\Omega))^3$ and $\forall \psi \in D(B_r(0))$ we deduce from (1.1)

$$\begin{aligned} \left\langle \operatorname{div} \vec{\Phi}_r, \psi \right\rangle_{D'(B_r(0))\, D(B_r(0))} &= - \left\langle \vec{\Phi}_r, \nabla\psi \right\rangle_{D'(B_r(0))\, D(B_r(0))} \\ &= - \int_\Omega \vec{\Phi}_3(x)\, \nabla\psi(x)\, dx \\ &= \int_\Omega \operatorname{div} \vec{\Phi}_3(x)\, \psi(x)\, dx - \int_\Gamma \vec{\Phi}_3(x) \cdot \vec{n}(x)\, \psi(x)\, d\sigma(x) = 0 , \end{aligned}$$

which proves that div $\vec{\Phi}_r = 0$. Then Proposition 1.3 and Lemma 1.5 of Appendix I of [13] allows us to deduce the existence of an element $\vec{\varphi}_r \in Y \cap H^2(\Omega)$ such that $\vec{\Phi}_r = \text{curl } \vec{\varphi}_r$. Then the restriction $\vec{\varphi}$ of $\vec{\varphi}_r$ to Ω verifies the statement of lemma. □

PROPOSITION A.2 : *For any $\mu > 0$ there exists an element $\vec{\phi} \in Y$ whose trace is $\vec{\phi}_\Gamma$ and that verifies* (1.2).

Proof : Taking into account that $\rho(x) = d(x, \Gamma)$ is a Lipschitz function in Ω, we can follow the approach of Lemma 1.9 of [1] to deduce the existence of a function $\theta_\varepsilon \in C^2(\bar{\Omega})$ such that

$$\theta_\varepsilon = 1 \text{ in some neighbourhood of } \Gamma ,$$

$$\theta_\varepsilon = 0 \quad \text{if} \quad \rho(x) \geqslant 2\,\delta(\varepsilon), \quad \delta(\varepsilon) = \exp(-1/\varepsilon),$$

$$\left|\partial_{x_j}\theta_\varepsilon(x)\right| \leqslant \frac{\varepsilon}{\rho(x)} \quad \text{if} \quad \rho(x) \leqslant 2\,\delta(\varepsilon), \quad 1 \leqslant j \leqslant n .$$

Now we can take $\vec{\phi} = \text{curl}\,(\theta_\varepsilon\,\vec{\varphi})$, $\vec{\varphi}$ satisfying the conditions of Lemma A.1. Thus we have

$$\left|\sum_{i,j=1}^{3} \int_\Omega y_i\, \partial_{x_i} z_j\, \phi_j\, dx\right| \leqslant \sum_{j=1}^{3} \|y_j\, \phi_j\|_{(L^2(\Omega))^3} \|\vec{z}\|_{(H^1(\Omega))^3} \quad \forall \vec{y}, \vec{z} \in Y_0$$

and the proof can be continued as in Lemma 1.8 of [13]. □

REFERENCES

[1] ABERGEL and R. TEMAM, 1990, On some control problems in fluid mechanics, *Theoret. Comput. Fluid Dynamics,* **1,** 303-325.

[2] F. ABERGEL and R. TEMAM, 1992, Optimal control of turbulent flows, *in Optimal control of viscous flows, S. S. Sritharan ed., Frontiers in Applied Mathematics Series, SIAM,* Philadelphia.

[3] E. CASAS and L. FERNANDEZ, 1989, A Green's formula for quasilinear elliptic operators, *J. of Math. Anal. & Appl.,* **142,** 62-72.

[4] H. CHOI, J. KIM, P. MOIN, R. TEMAM, à paraître, *Methods of feedback control for distributed systems and applications to Burgers equations.*

[5] M. GAULTIER and M. LEZAUN, 1989, Equations de Navier-Stokes couplées à des équations de la chaleur : résolution par une méthode de point fixe en dimension infinie, *Ann. Sc. Math.* Québec, **13,** 1-17.

[6] M. GUNZBURGER, L. HOU and T. SVOBODNY, 1991, Analysis and finite element approximations of optimal control problems for the stationary Navier-Stokes equations with Dirichlet conditions, *M²AN,* **25,** 711-748.

[7] M. GUNZBURGER, L. HOU and T. SVOBODNY, 1991, Boundary velocity control of incompressible flow with an application to viscous drag reduction, *SIAM J. on Control & Optimization.*

[8] A. IOFFE and V. TIKHOMOROV, 1979, *Extremal Problems,* North-Holland, Amsterdam.

[9] J. LIONS, 1968, *Contrôle de Systèmes Gouvernés par des Equations aux Dérivées Partielles,* Dunod, Paris.

[10] J. LIONS, 1969, *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires,* Dunod, Paris.

[11] J. NEČAS, 1967, *Les Méthodes Directes en Théorie des Equations Elliptiques,* Editeurs Academia, Prague.

[12] P. RABINOWITZ, 1968, Existence and nonuniqueness of rectangular solutions of the Bénard problem, *Arch. Rational Mech. Anal.,* **29**, 32-57.

[13] R. TEMAM, 1979, *Navier-Stokes Equations,* North-Holland, Amsterdam.