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MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
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UNIFORM CONVERGENCE OF MIXED INTERPOLATED ELEMENTS FOR REISSNER-MINDLIN PLATES (*)

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Abstract — *The mixed-interpolated elements of Bathe and Dvorkin [4] and Bathe, Brezzi, and Cho [3] are analyzed It is shown that convergence is uniform in the thickness parameter when the Mindlin-Reissner plate is treated To this end a discrete analog of the Helmholtz decomposition of* L_2 *is introduced*

Résumé — *On considère les éléments de Bathe, Dvorkin [4] et Bathe, Brezzi, Cho [3] utilisant une interpolation composée pour la plaque de Mindlin-Reissner On démontre que la convergence est uniforme par rapport au paramètre d'épaisseur La démonstration est basée sur une décomposition discrète de type Helmholtz*

1. INTRODUCTION

When the Mindlin plate is treated by finite elements, some extra devices are necessary to get convergence uniformly with respect to the thickness parameter. In particular, selected reduced integration or a mixed method with a penalty term is often applied. In 1986 Brezzi and Fortin [8] showed that a Helmholtz decomposition of L_2 is an efficient tool in the analysis. Later Arnold and Falk [1] detected that a discrete version exists for a certain pairing of finite elements.

In the last few years the MITCn elements ($n = 4$, 7, 8 and 9) of Bathe and Dvorkin [4] and Bathe *et al.* [3] have attracted much attention. The analysis of these mixed elements was done for the limit case in which the thickness parameter t is zero.

In this paper we will extend the analysis to positive thickness. To this end we will use a discrete Helmholtz decomposition. For this, the decomposition

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cannot be done with the standard operators grad and curl. Instead we will replace the latter by a discrete operator which will be called curl_h. The properties of this operator may be derived from the axioms of Brezzi *et al.* [5].

Unless otherwise stated we will adopt the notation of the paper just cited.

2. THE PLATE MODEL

The energy functional of the Mindlin-Reissner plate can be written as

$$\frac{1}{2}t^3 a(\theta,\ \theta) + \frac{1}{2}\lambda t \|\nabla w - \theta\|_0^2 - t^3(f,\ w)\,, \tag{2.1}$$

where $\theta = (\theta_1,\ \theta_2)$ denotes the rotation, w the transverse displacement and t the thickness of the plate. We assume that the plate is clamped so that $\theta \in [H_0^1(\Omega)]^2$ and $w \in H_0^1(\Omega)$. The terms in (2.1) are the bending energy, the shear energy and the energy induced by the load $t^3 f$, resp. As usual $(\cdot,\cdot)$ refers to the inner product in $L_2(\Omega)$ and $\|\cdot\|_s$ is the norm in the Sobolev spaces $H^s(\Omega)$ and $H_0^s(\Omega)$. Furthermore the bilinear form a is given by

$$a(\theta,\ \psi) := \frac{E}{12(1-\nu^2)}\int_{\Omega}\left[\sum_{i,j}(1-\nu)\,\varepsilon_{ij}(\theta)\,\varepsilon_{ij}(\psi) + \nu \operatorname{div}\theta \operatorname{div}\psi\right]dx\,,$$

where $\varepsilon_{ij}(\theta) = \frac{1}{2}(\partial_i\theta_j + \partial_j\theta_i)$ is the linear strain tensor, E and ν are Young's modulus and Poisson's ratio, resp. By Korn's inequality, a is an inner product on $[H_0^1(\Omega)]^2$ that is equivalent to the usual one.

In the numerical solution the Sobolev spaces are replaced by finite element subspaces $\Theta_h \subset [H_0^1(\Omega)]^2$ and $W_h \subset H_0^1(\Omega)$, where h is a mesh parameter. Furthermore, often some reduced integration is used when evaluating the shear energy in order to prevent the scheme from locking. For that purpose a linear reduction operator

$$R : [H_0^1(\Omega)]^2 \to \Gamma_h$$

is introduced in [5]. It makes the shear terms belong to a third finite element space Γ_h. The discretized problem then has the form

$$\frac{1}{2}a(\theta_h,\ \theta_h) + \frac{1}{2}\lambda t^{-2}\|\nabla w_h - R\theta_h\|_0^2 - (f,\ w_h) \to \min_{\substack{w_h \in W_h \\ \theta_h \in \Theta_h}}\ ! \tag{2.2}$$

Obviously, after adjusting the thickness parameter we may assume that $\lambda = 1$.

An essential step in the development of stable elements for the discretization of (2.1) was done when Brezzi and Fortin [8] used a Helmholtz decomposition to represent the Mindlin-Reissner plate model as two Poisson equations and one Stokes-like problem. Let

$$\gamma := t^{-2}(\nabla w - \theta)$$

denote the shear strain vector. Then the solution of the variational problem associated to (2.1) satisfies

$$\begin{aligned} & a(\theta, \psi) + (\gamma, \nabla v - \psi) = (f, v) \quad \forall \psi \in [H_0^1(\Omega)]^2, \quad v \in H_0^1(\Omega), \\ & (\nabla w - \theta, \eta) - t^2(\gamma, \eta) = 0 \quad \forall \eta \in L_2(\Omega)^2 . \end{aligned} \tag{2.3}$$

Using the Helmholtz Theorem [13]

$$[L_2(\Omega)]^2 = \nabla H_0^1(\Omega) \oplus \text{curl } (H^1(\Omega)/\mathbb{R})$$

the shear strains γ and η are decomposed as

$$\gamma = \nabla r + \text{curl}\, p \quad \text{and} \quad \eta = \nabla z + \text{curl}\, q . \tag{2.4}$$

Substituting (2.4) in (2.3) Brezzi and Fortin [8] obtained the following system of equations for $(\theta, w, r, p) \in [H_0^1(\Omega)]^2 \times H_0^1(\Omega) \times H^1(\Omega)/\mathbb{R}$:

$$\begin{aligned} (\nabla r, \nabla v) &= (f, v) & \forall v &\in H_0^1(\Omega) & (2.5)\\ a(\theta, \psi) - (p, \text{rot}\, \psi) &= (\nabla r, \psi) & \forall \psi &\in [H_0^1(\Omega)]^2 & (2.6a)\\ -(\text{rot}\, \theta, q) - t^2(\text{curl}\, p, \text{curl}\, q) &= 0 & \forall q &\in H^1(\Omega)/\mathbb{R} & (2.6b)\\ (\nabla w, \nabla z) &= (\theta, \nabla z) + t^2(f, z) & \forall z &\in H_0^1(\Omega) . & (2.7) \end{aligned}$$

Note that (2.5) is a simple Poisson equation, which is decoupled from the other equations. Furthermore, let $\perp$ denote the isometry in two space : $(x, y)^\perp := (y, -x)$. Then

$$\text{rot}\, \theta := \frac{\partial \theta_2}{\partial x} - \frac{\partial \theta_1}{\partial y} = \text{div}\, \theta^\perp ,$$

$$\text{curl}\, q := \left(\frac{\partial q}{\partial y}, -\frac{\partial q}{\partial x} \right) = (\nabla q)^\perp .$$

Thus (2.6) is related to a Stokes problem with penalty term $\frac{1}{2} t^2 \| \nabla p \|_0^2$. Finally, (2.7) is again a Poisson equation.

The Stokes-like problem (2.6) can be written in a more compact form by introducing the bilinear form

$$\begin{aligned} A_t((\theta, p), (\psi, q)) := a(\theta, \psi) - (p, \text{rot}\, \psi) - \\ - (q, \text{rot}\, \theta) - t^2(\text{curl}\, p, \text{curl}\, q) . \end{aligned} \tag{2.8}$$

The following regularity result can be found in [1, 8]

THEOREM 2 1 *Let Ω be a convex polygonal or a smoothly bounded domain in the plane For any $t > 0$ and $g \in [H^{-1}(\Omega)]^2$ there is a unique solution $\chi \in [H_0^1(\Omega)]^2$ and $p \in H^1(\Omega)/\mathbb{R}$ of*

$$A_t((\chi, \rho), (\psi, q)) = (g, \psi) \quad \forall (\psi, q) \in [H_0^1(\Omega)]^2 \times H^1(\Omega)/\mathbb{R} \tag{2 9}$$

Moreover, if $g \in [L_2(\Omega)]^2$, then $\chi \in [H^2(\Omega)]^2$, $\rho \in H^2(\Omega)$ and there exists a constant C independant of t and g, such that

$$\|\chi\|_2 + \|\rho\|_1 + t\|\rho\|_2 \leq C\|g\|_0 \tag{2 10}$$

The analysis of a special finite element approximation of the scheme (2 2) was simplified by the existence of a discrete version of the Helmholtz Theorem [1] Although a decomposition is not always given in this strong sense, a certain step in this direction was done when Brezzi *et al* [5] introduced the following five properties

Assume that besides

$$W_h \subset H_0^1(\Omega) \qquad \text{(transverse displacement)} ,$$
$$\Theta_h \subset [H_0^1(\Omega)]^2 \quad \text{(rotations)}$$

two additional spaces

$$\Gamma_h \subset H_0(\mathrm{rot}, \Omega) \quad \text{(shear strains)} ,$$
$$Q_h \subset L_2(\Omega)/\mathbb{R} \qquad \text{(pressure)}$$

and a reduction operator

$$R \quad [H_0^1(\Omega)]^2 \to \Gamma_h$$

are given such that the following properties hold

P_1) $\nabla W_h \subset \Gamma_h$, i e the discrete shear

$$\gamma_h = t^{-2}(\nabla w_h - R\theta_h)$$

belongs to Γ_h

P_2) rot $\Gamma_h \subset Q_h$

P_3) The pair of spaces (Θ_h, Q_h) satisfies the inf-sup condition

$$\lim_{q_l \in Q_l} \sup_{\psi_h \in \Theta_h} \frac{(\mathrm{rot}\ \psi_h, q_h)}{\|\psi_h\|_1 \ \|q_h\|_0} = \beta > 0$$

with β being independent of the mesh size h

P_4) *Commuting diagram property* Let P_h be the L_2-projection onto Q_h Then

$$\operatorname{rot} R\eta = P_h \operatorname{rot} \eta \quad \forall \eta \in [H_0^1(\Omega)]^2 ,$$

i e the following diagram commutes

$$\begin{array}{ccc} [H_0^1(\Omega)]^2 & \xrightarrow{\operatorname{rot}} & L_2(\Omega) \\ R \downarrow & & \downarrow P_h \\ \Gamma_h & \xrightarrow{\operatorname{rot}} & Q_h \end{array}$$

P_5) *Completeness of decomposition* If $\delta_h \in \Gamma_h$ and $\operatorname{rot} \delta_h = 0$ then $\delta_h \in \nabla W_h$

Recall that P_3 essentially states that the pair (Θ_h, Q_h) is « good » for the Stokes problem [13], whereas P_4 states that the pair $(\Gamma_h^{\perp}, Q_h)$ is « good » for the mixed variable formulation of an elliptic second order equation [6, 7, 11, 15]

Brezzi, Bathe and Fortin [5] used these properties for the analysis of the plate in the limit case $t = 0$ It is our aim to show that the properties P_1 to P_5 also imply convergence which is uniformly good for every $t > 0$ To this end we will extend the Helmholtz decomposition to the finite element spaces

3 THE HELMHOLTZ DECOMPOSITION OF Γ_h

DEFINITION 3 1 *Let the operator*

$$\operatorname{curl}_h \; Q_h \to \Gamma_h$$

be defined by

$$(\operatorname{curl}_h q_h, \eta) = (q_h, \operatorname{rot} \eta) \quad \textit{for all} \quad \eta \in \Gamma_h \tag{3 1}$$

We recall that $\Gamma_h \subset H_0(\operatorname{rot}, \Omega)$, where $H_0(\operatorname{rot}, \Omega) = \{\eta \in [L_2(\Omega)]^2, \operatorname{rot} \eta \in L_2(\Omega), \eta\tau = 0 \text{ on } \partial\Omega\}$, τ being the unit tangent vector Therefore, the functional $\eta \mapsto (q_h, \operatorname{rot} \eta)$ is well defined and $\operatorname{curl}_h q_h$ is uniquely determined by (3 1) — Formally, curl_h is introduced like a distributional derivative on a finite dimensional space

THEOREM 3 2 *Assume that the properties P_1, P_2 and P_5 hold Then an L_2-orthogonal decomposition is given by*

$$\Gamma_h = \nabla W_h \oplus \operatorname{curl}_h Q_h \tag{3 2}$$

Proof (1) From Definition 3 1 and P_1 it follows that

$$\nabla W_h \oplus \operatorname{curl}_h Q_h \subset \Gamma_h$$

Furthermore, given $q_h \in Q_h$ and $w_h \in W_h$ it follows that

$$(\mathrm{curl}_h\, q_h, \nabla w_h) = (q_h, \mathrm{rot}\, \nabla w_h) = 0\,.$$

Therefore, the functions $\mathrm{curl}_h\, q_h$ and ∇w_h are L_2-orthogonal.

(2) Given $\gamma_h \in \Gamma_h$, let η_h be the L_2-projection onto $\mathrm{curl}_h\, Q_h$. Then η_h is characterized by

$$(\gamma_h - \eta_h, \mathrm{curl}_h\, q_h) = 0 \quad \text{for all} \quad q_h \in Q_h\,.$$

From Definition 3.1 we conclude that $(\mathrm{rot}\,(\gamma_h - \eta_h), q_h) = 0\ \forall q_h \in Q_h$, and P_2 implies that

$$\mathrm{rot}\,(\gamma_h - \eta_h) = 0\,.$$

Therefore, P_5 asserts that $\gamma_h - \eta_h \in \nabla W_h$, so that, by construction, $\gamma_h \in \nabla W_h \oplus \mathrm{curl}_h\, Q_h$. □

Using the decomposition (3.2) we will immediately obtain a representation of the approximation scheme (2.2), which is analogous to (2.5)-(2.7). Obviously, when $\lambda = 1$, the solution of the variational problem (2.2) is characterized by

$$\begin{aligned} a(\theta_h, \psi_h) + (\gamma_h, \nabla v_h - R\psi_h) &= (f, v_h) \quad \forall \psi_h \in \Theta_h, v_h \in W_h\,, \\ (\nabla w_h - R\theta_h, \eta_h) - t^2(\gamma_h, \eta_h) &= 0 \qquad\quad \forall \eta_h \in \Gamma_h\,. \end{aligned} \tag{3.3}$$

Inserting the L_2-orthogonal decompositions

$$\gamma_h = \nabla r_h + \mathrm{curl}_h\, p_h \quad \text{and} \quad \eta_h = \nabla z_h + \mathrm{curl}_h\, q_h$$

into (3.3), and observing that by Definition 3.1 and P_4

$$(\mathrm{curl}_h\, q_h, R\theta_h) = (q_h, \mathrm{rot}\, R\theta_h) = (q_h, P_h\, \mathrm{rot}\, \theta_h) = (q_h, \mathrm{rot}\, \theta_h)\,,$$

we obtain the discrete version of the decomposition (2.5)-(2.7) :

$$(\nabla r_h, \nabla v_h) = (f, v_h) \quad \forall v_h \in W_h\,, \tag{3.4}$$

$$\begin{aligned} a(\theta_h, \psi_h) - (p_h, \mathrm{rot}\, \psi_h) &= (\nabla r_h, R\psi_h) \quad \forall \psi_h \in \Theta_h\,, \\ -(\mathrm{rot}\, \theta_h, q_h) - t^2(\mathrm{curl}_h\, p_h, \mathrm{curl}_h\, q_h) &= 0 \quad \forall q_h \in Q_h\,, \end{aligned} \tag{3.5}$$

$$(\nabla w_h, \nabla z_h) = (R\theta_h, \nabla z_h) + t^2(f, z_h) \quad \forall z_h \in W_h\,. \tag{3.6}$$

For abbreviation we introduce the discrete bilinear form

$$A_t^h((\theta_h, p_h), (\psi_h, q_h)) := \\ := a(\theta_h, \psi_h) - (p_h, \mathrm{rot}\, \psi_h) - (q_h, \mathrm{rot}\, \theta_h) - t^2(\mathrm{curl}_h\, p_h, \mathrm{curl}_h\, q_h)\,. \tag{3.7}$$

As a consequence of property P_3 the following stability estimate for the discrete Stokes like problem (3.5) is valid, *cf.* [8, 14].

LEMMA 3.3 (Stability) : *Assume that the pair of spaces* (Θ_h, Q_h) *satisfies* P_3. *Then there exists a positive constant* α *which is independent of the parameters* t *and* h, *such that for all* $(\theta_h, q_h) \in \Theta_h \times Q_h$

$$\sup_{\substack{\psi_h \in \Theta_h \\ q_h \in Q_h}} \frac{A_t^h((\theta_h, p_h), (\psi_h, q_h))}{\|\psi_h\|_1 + \|q_h\|_0 + t\|\mathrm{curl}_h\, q_h\|_0} \geqslant \alpha\left(\|\theta_h\|_1 + \|p_h\|_0 + t\|\mathrm{curl}_h\, p_h\|_0\right). \tag{3.8}$$

We conclude this section with an observation concerning the discrete operator curl_h.

PROPOSITION 3.4 : *Let* P_h *and* Π_h *denote the* L_2*-projection onto* Q_h *and* Γ_h, *resp. Then*

$$\mathrm{curl}_h P_h \rho = \Pi_h \,\mathrm{curl}\, \rho \quad \textit{for all} \quad \rho \in H^1(\Omega),$$

i.e. the following diagram commutes :

$$\begin{array}{ccc} H^1(\Omega) & \xrightarrow{\mathrm{curl}} & L_2(\Omega)^2 \\ {\scriptstyle P_h}\downarrow & & \downarrow{\scriptstyle \Pi_h} \\ Q_h & \xrightarrow{\mathrm{curl}_h} & \Gamma_h . \end{array}$$

Proof: Given $\rho \in H^1(\Omega)$ and $\gamma_h \in \Gamma_h \subset H_0(\mathrm{rot}\,\Omega)$ we use Definition 3.1, P_2 and integration by parts to obtain

$$\begin{aligned}(\mathrm{curl}_h P_h \rho, \gamma_h) = (P_h \rho, \mathrm{rot}\,\gamma_h) = (\rho, \mathrm{rot}\,\gamma_h) &= \\ = (\mathrm{curl}\,\rho, \gamma_h) = (\Pi_h\, \mathrm{curl}\,\rho, \gamma_h)\,. \quad \square\end{aligned}$$

4. ERROR ANALYSIS

In this section we will derive abstract error estimates assuming the properties P_1 to P_5. Specifically, we will establish H^1-estimates on the rotation vector and the transverse displacement, whereas L_2-estimates are postponed to the next section.

The most difficult part is the derivation of error bounds for the Stokes-like problem. Since the discrete bilinear form A_t^h differs from A_t, the approximation scheme is nonconforming and we have to deal with consistency errors. We assume throughout the remainder of this paper that Ω is a convex polygon or smoothly bounded domain in the plane, so that the regularity result of Theorem 2.1 is valid.

Remark 4.1 : Let (χ, ρ) be the unique solution of the Stokes-like problem (2.9) with $t > 0$. Setting $\psi = 0$ in (2.9) we have that

$$t^2(\operatorname{curl} \rho, \operatorname{curl} p) = (\operatorname{rot} \psi, q) \quad \forall q \in H^1(\Omega)/\mathbb{R} .$$

From this we conclude that

$$\begin{aligned} t^2 \operatorname{rot} \operatorname{curl} \rho &= \operatorname{rot} \psi && \in L_2(\Omega) \\ \operatorname{curl} \rho \cdot \tau &= \frac{\partial \rho}{\partial n} = 0 && \text{on } \partial\Omega . \end{aligned}$$

Specifically,

$$(\operatorname{curl} \rho, \operatorname{curl} q) = (\operatorname{rot} \operatorname{curl} \rho, q) \quad \forall q \in H^1(\Omega) . \tag{4.1}$$

Therefore, by a density argument test functions $q \in L_2(\Omega)/\mathbb{R}$ are also permitted in (2.9).

We now present the energy estimate for the Stokes-like problem.

THEOREM 4.2 : *Assume that the properties P_1 to P_5 hold. Let (θ, p) and (θ_h, p_h) be the solutions of* (2.6) *and* (3.5), *resp. Then the following error bound*

$$\begin{aligned} &\|\theta - \theta_h\|_1 + \|p - p_h\|_0 + t\|\operatorname{curl} p - \operatorname{curl}_h p_h\|_0 \leq \\ &\leq C \left\{ \inf_{\psi_h \in \Theta_h} \|\theta - \psi_h\|_1 + \inf_{q_h \in Q_h} \|p - q_h\|_0 + t\|R \operatorname{curl} p - \operatorname{curl} p\|_0 \right. \\ &\quad \left. + \|r - r_h\|_0 + \sup_{\psi_h \in \Theta_h} \frac{|(\nabla r_h, R\psi_h - \psi_h)|}{\|\psi_h\|_1} \right\} \end{aligned} \tag{4.2}$$

holds with C being independent of the mesh size h and of the parameter t.

Remark : The first two terms on the right hand side of (4.2) are the standard terms for the approximation from the subspaces Θ_h and Q_h. By Remark 4.1, the (nonconforming) difference of A_t and A_t^h arises only from the difference between curl and curl_h. This effect is in principle treated by deriving bounds for

$$\frac{|A_t^h((\theta, P_h p), (\chi_h, q_h)) - (\nabla r, \chi_h)|}{\|\chi_h\|_1 + \|q_h\|_0 + t\|\operatorname{curl}_h q_h\|}$$

see e.g. the proofs of Strang's lemmas in [10]. The result is the third term in (4.2). Obviously, the fourth term shows that the error in the solution of the Poisson equation (2.5) is inherited to the Stokes problem. The last term represents the additional consistency error induced by the operator R on the right hand side of (3.5).

Proof : (1) Let $\tilde{\theta}_h \in \Theta_h$ be an approximation to θ satisfying

$$\left\| \theta - \tilde{\theta}_h \right\|_1 = \inf_{\psi_h \in \Theta_h} \| \theta - \psi_h \|_1 .$$

Using the relation $\operatorname{curl}_h P_h p = \Pi_h \operatorname{curl} p$ given in Proposition 3.4, we conclude that

$$\begin{aligned} &\| \theta - \theta_h \|_1 + \| p - p_h \|_0 + t \| \operatorname{curl} p - \operatorname{curl}_h p_h \|_0 \\ &\quad \leq \left(\left\| \theta - \tilde{\theta}_h \right\|_1 + \| p - P_h p \|_0 + t \| \operatorname{curl} p - \Pi_h \operatorname{curl} p \|_0 \right) \\ &\quad + \left(\left\| \tilde{\theta}_h - \theta_h \right\|_1 + \| P_h p - p_h \|_0 + t \| \operatorname{curl}_h (P_h p - p_h) \|_0 \right) . \end{aligned} \tag{4.3}$$

Since Π_h is the L_2-projection onto Γ_h, $\| \operatorname{curl} p - \Pi_h \operatorname{curl} p \|_0 = \inf_{\eta \in \Gamma_h} \| \operatorname{curl} p - \eta \|_0 \leq \| \operatorname{curl} p - R \operatorname{curl} p \|_0$. Therefore, the first three terms are bounded by the right hand side of (4.2). Lemma 3.3 ensures that there exists a pair $(\chi_h, q_h) \in \Theta_h \times Q_h$ such that

$$\| \chi_h \|_1 + \| q_h \|_0 + t \| \operatorname{curl}_h q_h \|_0 = 1 \tag{4.4}$$

and

$$\begin{aligned} &\alpha \left(\left\| \theta_h - \tilde{\theta}_h \right\|_1 + \| p_h - P_h p \|_0 + t \| \operatorname{curl}_h (p_h - P_h p) \|_0 \right) \\ &\qquad \leq A_t^h ((\theta_h - \tilde{\theta}_h, p_h - P_h p), (\chi_h, q_h)) . \end{aligned} \tag{4.5}$$

Using (3.5) the right hand side of (4.5) can be rewritten as

$$\begin{aligned} A_t^h ((\theta_h - \tilde{\theta}_h, p_h - P_h p), (\chi_h, q_h)) = \\ &= (\nabla r_h, R \chi_h) - A_t^h ((\tilde{\theta}_h, P_h p), (\chi_h, q_h)) \\ &= [(\nabla r, \chi_h) - A_t^h ((\theta, P_h p), (\chi_h, q_h))] \\ &\quad + A_t^h ((\theta - \tilde{\theta}_h, 0), (\chi_h, q_h)) \\ &\quad + (\nabla r_h - \nabla r, \chi_h) + (\nabla r_h, R \chi_h - \chi_h) \\ &:= E_1 + E_2 + E_3 . \end{aligned} \tag{4.6}$$

We will establish bounds for the three terms separately.

(2) First, we treat the most difficult term E_1. From Remark 4.1 it follows that $A_t((\theta, p), (\chi_h, q_h)) = (\nabla r, \chi_h)$. Hence,

$$\begin{aligned} E_1 &= - A_t^h ((\theta, P_h p), (\chi_h, q_h)) + A_t ((\theta, p), (\chi_h, q_h)) \\ &= (P_h p - p, \operatorname{rot} \chi_h) - t^2 [(\operatorname{rot} \operatorname{curl} p, q_h) - (\operatorname{curl}_h P_h p, \operatorname{curl}_h q_h)] . \end{aligned} \tag{4.7}$$

Next, the commuting diagram property P_4 and $q_h \in Q_h$ imply that

$$(\operatorname{rot} \operatorname{curl} p, q_h) = (P_h \operatorname{rot} (\operatorname{curl} p), q_h) = (\operatorname{rot} R \operatorname{curl} p, q_h) = \\ = (R \operatorname{curl} p, \operatorname{curl}_h q_h) .$$

Furthermore, by applying Proposition 3.4 we obtain

$$(\operatorname{curl}_h P_h p, \operatorname{curl}_h q_h) = (\Pi_h \operatorname{curl} p, \operatorname{curl}_h q_h) = (\operatorname{curl} p, \operatorname{curl}_h q_h) .$$

Subtraction of the last two equations yields

$$(\operatorname{rot} \operatorname{curl} p, q_h) - (\operatorname{curl}_h P_h p, \operatorname{curl}_h q_h) = \\ = (R \operatorname{curl} p - \operatorname{curl} p, \operatorname{curl}_h q_h) . \quad (4.8)$$

Inserting this into (4.7) and recalling the normalization (4.4) we have

$$\begin{aligned} E_1 &\leq \|p - P_h p\|_0 \, \|\chi_h\|_1 + (t\|R \operatorname{curl} p - \operatorname{curl} p\|)(t\|\operatorname{curl}_h q_h\|) \\ &\leq \|p - P_h p\|_0 + t\|R \operatorname{curl} p - \operatorname{curl} p\| . \end{aligned} \quad (4.9)$$

(3) A bound for E_2 is easily obtained by recalling the definition of A_t^h and the normalization (4.4)

$$\begin{aligned} E_2 &= a(\theta - \tilde{\theta}_h, \psi_h) - (\operatorname{rot} (\theta - \tilde{\theta}_h), q_h) \\ &\leq C \left\| \theta - \tilde{\theta}_h \right\|_1 \cdot (\|\psi_h\|_1 + \|q_h\|_0) \\ &\leq C \left\| \theta - \tilde{\theta}_h \right\|_1 . \end{aligned}$$

(4) Finally, we consider the remaining term. Since $\|\chi_h\|_1 \leq 1$, we have

$$\begin{aligned} E_3 &= (\nabla r_h - \nabla r, \chi_h) + (\nabla r_h, R\chi_h - \chi_h) \\ &= (r_h - r, \operatorname{div} \chi_h) + (\nabla r_h, R\chi_h - \chi_h) \\ &\leq \|r_h - r\|_0 + \sup_{\psi_h \in \Theta_h} \frac{(\nabla r_h, R\psi_h - \psi_h)}{\|\psi_h\|_1} . \end{aligned}$$

Collecting the terms from the estimates of E_1, E_2 and E_3, we have a bound for the right hand side of (4.5) and the proof of the theorem is complete. □

The following lemma completes the estimates by those for the remaining variables. The proof is standard and can be found e.g. in [10].

LEMMA 4.3 : *Let r, w and r_h, w_h be the solutions of* (2.5), (2.7) *and* (3.4), (3.5), *resp. Then the following error bounds hold*

$$\|\nabla(r - r_h)\|_0 = \inf_{v_h \in W_h} \|\nabla(r - v_h)\|_0 , \tag{4.10}$$

$$\|\nabla(w - w_h)\|_0 \leq \inf_{v_h \in W_h} \|\nabla(w - v_h)\|_0 + \|R\theta_h - \theta\|_0 . \tag{4.11}$$

5. ERROR ANALYSIS CONTINUED : L_2 ESTIMATES

We continue our abstract error estimates based on the properties P_1 to P_5 to obtain bounds on the L_2-norm of the errors $\theta - \theta_h$ and $w - w_h$.

THEOREM 5.1 : *Assume that the properties P_1 to P_5 hold. Let (θ, p) and (θ_h, p_h) be the solutions of* (2.6) *and* (3.5), *resp. Then there exists a constant C independant of the thickness parameter t and the mesh size h such that*

$$\begin{aligned}
\|\theta - \theta_h\|_0 \leq \sup_{g \in L_2(\Omega)^2} \frac{1}{\|g\|_0} \inf_{\chi_h \in \Theta_h} \Big\{ & M(\|\theta - \theta_h\|_1 + \|(I - P_h)\operatorname{rot}\theta\|_0 \\
& + \|p - p_h\|_0 + t\|\operatorname{curl} p - \operatorname{curl}_h p_h\|_0) \\
& \times (\|\chi_g - \chi_h\|_1 + \|(I - P_h)\operatorname{rot}\chi_g\|_0 + \|(\rho_g - P_h\rho_g)\|_0 \\
& + t\|\operatorname{curl}\rho_g - R\operatorname{curl}\rho_g\|_0) \\
& + (\nabla r, \chi_h) - (\nabla r_h, R\chi_h)\Big\} ,
\end{aligned} \tag{5.1}$$

where for each $g \in L_2(\Omega)^2$ the pair (χ_g, ρ_g) is the unique solution of the Stokes-like problem (2.9) *and P_h is the L_2-projection onto Q_h.*

Proof : Given $g \in L_2(\Omega)^2$, let $(\chi, \rho) := (\chi_g, \rho_g)$ be the solution of (2.9). Since θ and θ_h are solutions of the mixed problems (2.6) and (3.5), resp., we obtain

$$\begin{aligned}
(g, \theta - \theta_h) &= (g, \theta - \theta_h) - [A_t((\theta, p), (\chi, \rho)) - (\nabla r, \chi)] \\
&\quad + [A_t^h((\theta_h, p_h), (\chi_h, P_h\rho)) - (\nabla r_h, R\chi_h)] \\
&= [(g, \theta - \theta_h) - A_t((\chi, \rho), (\theta - \theta_h, p - p_h))] \\
&\quad + [(\nabla r, \chi - \chi_h) - A_t((\theta, p), (\chi - \chi_h, \rho - P_h\rho))] \\
&\quad + A_t((\theta, p), (\chi - \chi_h, \rho - P_h\rho)) - A_t((\chi, \rho), (\theta_h, p_h)) \\
&\quad + A_t^h((\chi_h, P_h\rho), (\theta_h, p_h)) \\
&\quad + (\nabla r, \chi_h) - (\nabla r_h, R\chi_h) .
\end{aligned} \tag{5.2}$$

Using Remark 4.1 and recalling (2.6) and (2.9), we find that the terms in brackets in the first two rows on the right hand side of (5.2) vanish. We separate the t-dependent terms and note that $A_0 = A_0^h$:

$$\begin{aligned}(g, \theta - \theta_h) &= A_0((\chi - \chi_h, \rho - P_h \rho), (\theta - \theta_h, p - p_h)) - \\ &\quad - t^2\{(\operatorname{rot}\operatorname{curl} p, \rho - P_h \rho) - (\operatorname{rot}\operatorname{curl}\rho, p_h) + (\operatorname{curl}_h P_h \rho, \operatorname{curl}_h p_h)\} \\ &\quad + (\nabla r, \chi_h) - (\nabla r_h, R\chi_h) \\ &=: E_1 + E_2 + E_3 . \end{aligned} \tag{5.3}$$

Here E_i refers to the term of the i-th row on the right hand side of (5.3). Obviously,

$$E_1 \leqslant C\,(\|\theta - \theta_h\|_1 + \|p - p_h\|_0)(\|\chi - \chi_h\|_1 + \|\rho - P_h \rho\|_0), \tag{5.4}$$

while E_3 is found directly from (5.1).

This leaves the estimate of E_2. From Remark 4.1 we know that (2.6b) may be read as $-t^2 \operatorname{rot}\operatorname{curl} p = \operatorname{rot}\theta$, and Theorem 2.1 provides us with an H^1-estimate of $\operatorname{rot}\theta$. Since P_h is a projection, we obtain from (2.6b) with $q = \rho - P_h \rho$ that

$$\begin{aligned} -t^2(\operatorname{rot}\operatorname{curl} p, \rho - P_h \rho) &= (\operatorname{rot}\theta, \rho - P_h\rho) \\ &= ((I - P_h)\operatorname{rot}\theta, \rho - P_h\rho) \\ &\leqslant \|(I - P_h)\operatorname{rot}\theta\|_0\, \|\rho - P_h\rho\|_0 . \end{aligned} \tag{5.5}$$

Next we use (4.8) with ρ instead of p and p_h instead of q_h to obtain

$$\begin{aligned} (\operatorname{rot}\operatorname{curl}\rho, p_h) - (\operatorname{curl}_h P_h\rho, \operatorname{curl}_h p_h) &= \\ &= (R\operatorname{curl}\rho - \operatorname{curl}\rho, \operatorname{curl}_h p_h) \\ &= (R\operatorname{curl}\rho - \operatorname{curl}\rho, \operatorname{curl}_h p_h - \operatorname{curl} p) \\ &\quad + (R\operatorname{curl}\rho - \operatorname{curl}\rho, \operatorname{curl} p) . \end{aligned} \tag{5.6}$$

Furthermore, P_4 implies that

$$\begin{aligned} ((R - I)\operatorname{curl}\rho, \operatorname{curl} p) &= (\operatorname{rot}\,(R - I)\operatorname{curl}\rho, p) \\ &= ((P_h - I)\operatorname{rot}\operatorname{curl}\rho, p) \\ &= (\operatorname{rot}\operatorname{curl}\rho, P_h p - p) . \end{aligned} \tag{5.7}$$

We proceed analogously to (5.5) and use (2.9) with $q = p - P_h p$, $\psi = 0$ to obtain

$$\begin{aligned} -t^2(\operatorname{rot}\operatorname{curl}\rho, p - P_h p) &= (\operatorname{rot}\chi, p - P_h p) \\ &= ((I - P_h)\operatorname{rot}\chi, p - P_h p) \\ &\leqslant \|(I - P_h)\operatorname{rot}\chi\|_0\, \|p - p_h\|_0 . \end{aligned} \tag{5.8}$$

Substituting (5.8) in (5.7) and (5.7) in (5.6) and recalling (5.5) we obtain

$$E_2 \leqslant (\|(I - P_h) \operatorname{rot} \theta\|_0 + \|p - p_h\|_0 + t\|\operatorname{curl} p - \operatorname{curl}_h p_h\|_0)$$
$$\times (\|(I - P_h) \operatorname{rot} \chi_g\|_0 + \|\rho - P_h \rho\|_0 + t\|\operatorname{curl} \rho - R \operatorname{curl} \rho\|_0) . \quad (5.9)$$

Combining (5.9), (5.4) and (5.3) yields the estimate (5.1). □

The L_2-estimate for the displacement error is standard, see e.g. [10, p. 203].

THEOREM 5.2 : *Let w and w_h denote the solutions of* (2.7) *and* (3.6), *resp. Then*

$$\|w - w_h\|_0 \leqslant \sup_{g \in L_2(\Omega)} \frac{1}{\|g\|_0} \inf_{\xi_h \in W_h}$$
$$\left\{ \|w - w_h\|_1 \, \|\xi_g - \xi_h\|_1 + |(\theta - R\theta_h, \nabla \xi_h)| \right\} , \quad (5.10)$$

where for each $g \in L_2(\Omega)$ the function $\xi_g \in H_0^1(\Omega)$ is the unique solution of the variational problem

$$(\nabla \xi_g, \nabla v) = (g, v) \quad \forall v \in H_0^1(\Omega) .$$

6. APPLICATIONS

Numerous examples of finite element spaces satisfying the five properties P_1 to P_5 can be found in [5], where a systematic study of the MITCn elements is given. We will apply our abstract error estimates to those elements.

We treat the triangular elements first. Let $\mathscr{T}_h$ be a regular triangulation of the convex, polygonal domain Ω into triangles T of diameter less than h.

6.1. Triangular elements of order $k \geqslant 2$

For each integer $k \geqslant 0$ we denote by $P_k(T)$ the space of polynomials of degree $\leqslant k$ on T. Let $k \geqslant 2$. Then the following system $(W_h, \Theta_h, Q_h, \Gamma_h, R)$ satisfies P_1 to P_5 [5] :

$$W_h := \{w \in H_0^1(\Omega) ; w|_T \in P_k(T) \quad \forall T \in \mathscr{T}_h\} ;$$
$$\Theta_h := \{\psi \in [H_0^1(\Omega)]^2 ; \psi|_T \in P_k(T)^2 \oplus B_{k+1}(T)^2 \quad \forall T \in \mathscr{T}_h\} ,$$

where

$$B_{k+1}(T) := \{\lambda_1 \lambda_2 \lambda_3 p ; p \in P_{k-2}(T)\}$$

and $\lambda_1 \lambda_2 \lambda_3$ is the cubic bubble function on T vanishing on ∂T;

$$Q_h := \{q \in L_2(\Omega)/\mathbb{R} \,;\, q|_T \in P_{k-1}(T) \quad \forall T \in \mathcal{T}_h\} \,;$$
$$\Gamma_h := \{\gamma \in H_0(\mathrm{rot}, \Omega) \,;\, \gamma|_T \in RT_{k-1} \quad \forall T \in \mathcal{T}_h\} \,,$$

where

$$RT_{k-1} := \left\{ \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + p_3 \begin{pmatrix} y \\ -x \end{pmatrix} \,;\quad p_i \in P_{k-1} \right\}$$

is a rotated Raviart-Thomas-space of order $k-1$ [15]. Furthermore, the operator R is defined by

$$\int_e (\gamma - R\gamma)\, \tau p_{k-1}\, ds = 0 \quad \text{for each edge } e \text{ and } p_{k-1} \in P_{k-1}(e), \tag{6.1}$$

$$\int_T (\gamma - R\gamma)\, p_{k-2}\, dx = 0 \quad \text{for each } T \in \mathcal{T}_h \text{ and } p_{k-2} \in P_{k-2}(T)\,, \tag{6.2}$$

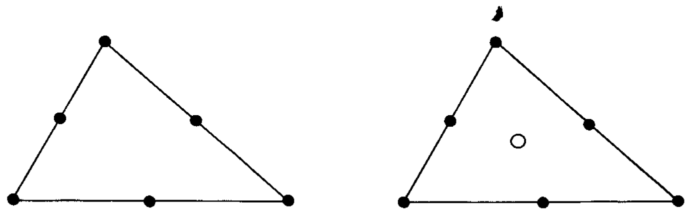
where τ denotes the unit tangent vector. In particular, the estimate

$$\| \eta - R\eta \|_0 \leq c h^s \| \eta \|_s\,, \quad 1 \leq s \leq k \tag{6.3}$$

can be found in [15].

We will use the orthogonality relation (6.2) in the derivation of the error estimates. For this purpose we denote by Π_{k-2} the L_2-projection onto the space of piecewise polynomials of order $k-2$. Moreover, we recall that a piecewise polynomial $\gamma = (\gamma_1, \gamma_2)$ belongs to $H_0(\mathrm{rot}\,\Omega)$ if and only if $\gamma \,.\, \tau$ is continuous at the interelement boundaries and vanishes on $\partial\Omega$ [15].

An important element of this class is the MITC7 element, which is obtained for $k = 2$. If $k \geq 4$, then the pairs with polynomials of order k for the velocities and $k-1$ for the pressure are stable elements for the Stokes-problem [13, 16], and the space of bubble functions B_{k+1} may be dropped in the definition of Θ_h. Formally, the definitions can also be extended to



Triangular elements : MITC6 and MITC7
(• rotation and displacement, ○ rotation only).

$k = 1$. However, in this case some extra devices are necessary to ensure stability, see e.g. [12].

Now we deduce from the abstract error estimates.

THEOREM 6.2 : *Consider the elements of order $k \geq 2$ defined in* 6.1. *Then for $\ell = 0, 1$ and $1 \leq s \leq k$ the following error estimates*

$$\begin{aligned}
\| \theta - \theta_h \|_\ell &\leq ch^{s+1-\ell}(\| \theta \|_{s+1} + \|p\|_s + t\|p\|_{s+1} + \|r\|_{s+1-\ell}) \\
\|w - w_h\|_\ell &\leq ch^{s+1-\ell}(\|w\|_{s+1} + \| \theta \|_{s+1-\ell} + \|p\|_{s-\ell} \\
&\quad + t\|p\|_{s+1-\ell} + \|r\|_{s+1-\ell})
\end{aligned} \tag{6.4}$$

hold with c being independant of t and h.

Proof : Let $1 \leq s \leq k$. The estimate for the solution of the first Poisson equation is standard [10]

$$\|r - r_h\|_0 + h\|r - r_h\|_1 \leq ch^{s+1}\|r\|_{s+1} . \tag{6.5}$$

To estimate the last term from (4.2) we note that in each triangle T the function $\Pi_{k-2}(\nabla r)$ is orthogonal to $\psi_h - R\psi_h$. Moreover, $(I - \Pi_{k-1})(\nabla r)$ may be estimated by standard approximation arguments. Therefore, (6.3) and (6.5) yield

$$\begin{aligned}
|(\nabla r_h, \psi_h - R\psi_h)| &= |(\nabla(r_h - r), \psi_h - R\psi_h) + (\nabla r - \Pi_{k-2}\nabla r, \psi_h - R\psi_h)| \\
&\leq ch\| \psi_h\|_1\{ \|r_h - r\|_1 + \|(I - \Pi_{k-2}) \nabla r\|_0\} \\
&\leq ch^s\| \psi_h\|_1 \|r\|_s .
\end{aligned} \tag{6.6}$$

Substituting (6.3), (6.5) and (6.6) in (4.2) and using the well-known approximation properties of piecewise polynomials of degree $\leq k$, we have

$$\begin{aligned}
&\| \theta - \theta_h\|_1 + \|p - p_h\|_0 + t\|\operatorname{curl} p - \operatorname{curl}_h p_h\|_0 \leq \\
&\quad \leq ch^s (\| \theta \|_{s+1} + \|p\|_s + t\|p\|_{s+1} + \|r\|_s) , \quad 1 \leq s \leq k .
\end{aligned} \tag{6.7}$$

This gives the estimate as stated in (6.4*a*) with $\ell = 1$.

Next we apply the abstract error estimate (5.1) to derive an L_2-estimate of the rotation vector. Let χ_h be the $\| \cdot \|_1$-nearest element to χ_g in Θ_h :

$$\|\chi_g - \chi_h\|_1 = \lim_{\psi_h \in \Theta_h} \|\chi_g - \psi_h\|_1 .$$

Inserting (6.7), (6.3), and the regularity estimate (2.10) into (5.1) we get

$$\begin{aligned}
\| \theta - \theta_h\|_0 &\leq ch^{s+1}(\| \theta \|_{s+1} + \|p\|_s + t\|p\|_{s+1} + \|r\|_s) + \\
&\quad + \frac{1}{\|g\|_0} |(\nabla r, \chi_h) - (\nabla r_h, R\chi_h)| .
\end{aligned} \tag{6.8}$$

To treat the last term we use similar arguments as in the derivation of (6 6)

$$\begin{aligned}|(\nabla r, \chi_h) - (\nabla r_h, R\chi_h)| &\leq |(\nabla(r - r_h), \chi_h)| + |(\nabla r_h, (I - R)\chi_h)| \\ &\leq \|r - r_h\|_0 \|\chi_h\|_1 + |(\nabla(r_h - r), (I - R)\chi_h)| \\ &\quad + |(\nabla r, (I - R)(\chi_h - \chi_g))| + |(\nabla r, (I - R)\chi_g)| \\ &\leq c(h^{s+1}\|r\|_{s+1} \|g\|_0 + \|(I - \Pi_{k-2})\nabla r\|_0 h^2 \|g\|_0) \\ &\leq ch^{s+1}\|r\|_{s+1} \|g\|_0, \quad 1 \leq s \leq k. \end{aligned} \tag{6.9}$$

Combining (6.8) and (6.9) yields the estimate as stated in (6.4a) with $\ell = 0$.

Finally, we will establish bounds of the error of the transversal displacement w. First we consider $\ell = 1$. From (4.11) it follows that

$$\|w - w_h\|_1 \leq ch^s\|w\|_{s+1} + \|\theta - R\theta_h\|_0 . \tag{6.10}$$

Using (6.3) we get

$$\begin{aligned}\|\theta - R\theta_h\|_0 &\leq \|(I - R)\theta\|_0 + \|(I - R)(\theta_h - \theta)\|_0 + \|\theta - \theta_h\|_0 \\ &\leq c(h^s\|\theta\|_s + h\|\theta - \theta_h\|_1 + \|\theta - \theta_h\|_0) .\end{aligned}$$

The estimates of $\|\theta - \theta_h\|_\ell$ from (6 4a) may now be applied to obtain

$$\|\theta - R\theta_h\|_0 \leq ch^s(\|\theta\|_s + \|p\|_{s-1} + t\|p\|_s + \|r\|_s) .$$

Inserting this estimate into (6.10) we have the estimate as stated in (6.4b) for $\ell = 1$.

Now we turn to the case $\ell = 0$ and recall Theorem 5.2. Given $g \in L_2(\Omega)$, let $\xi_g \in H_0^1(\Omega)$ denote the solution of (5.11). By standard results on conforming methods there is a $\xi_h \in W_h$ such that

$$\|\xi_g - \xi_h\|_1 \leq ch\|\xi_g\|_2 \leq ch\|g\|_0 .$$

We combine this fact with the defining equations (6 2) and (6 3) for R Also we may use the bounds of $\|\theta - \theta_h\|$ from (6 4a) to obtain

$$\begin{aligned}|(\theta - R\theta_h, \nabla\xi_h)| &= \\ &= |(\theta - R\theta, \nabla(\xi_h - \xi_g)) + (\theta - R\theta, \nabla\xi_g) + (R(\theta - \theta_h), \partial\xi_h)| \\ &\leq \|\theta - R\theta\|_0 \|\nabla(\xi_h - \xi_g)\|_0 + |(\theta - R\theta, (I - \Pi_{k-2})\nabla\xi_g)| \\ &\quad + \|\nabla\xi_h\|_0 (\|(I - R)(\theta - \theta_h)\|_0 + \|\theta - \theta_h\|_0)\end{aligned}$$

$$\leq c\|g\|_0 (h\|\theta - R\theta\|_0 + h\|\theta - \theta_h\|_1 + \|\theta - \theta_h\|_0)$$
$$\leq ch^{s+1}\|g\|_0 (\|\theta\|_{s+1} + \|p\|_s + t\|p\|_{s+1} + \|r\|_{s+1}) . \tag{6.11}$$

Inserting (6.11) into (5.10) completes the proof. □

According to Theorem 6.2 the finite elements defined in 6.1 provide an approximation scheme of arbitrary order of accuracy uniformly in a fixed range $0 \leq t \leq C$. In particular, the limit case $t = 0$, i.e. the Kirchhoff plate model, is included.

Even if the boundary and the load f are smooth, due to the boundary layer in case $t > 0$ neither $\|\theta\|_3$ nor $\|p\|_2$ may be bounded independently of the thickness parameter [1, 8]. Therefore a high order approximation is not always advantageous. For this reason the following MITC6 element also seems to be attractive : Choose Θ_h and W_h such that each contains continuous piecewise quadratics and combine these two spaces with a piecewise constant pressure space Q_h and $\Gamma_h := \{\gamma \in H_0(\mathrm{rot}, \Omega) ; \ \gamma|_T$ linear $\forall T\}$. The latter is the rotated Brezzi-Douglas-Marini space of lowest order [7]. The operator R is defined by

$$\int_e (\gamma - R\gamma) p_1 \, ds = 0 \quad \text{for each edge } e \text{ of } T , \quad p_1 \in P_1(e) ,$$

i.e. the tensorial component is interpolated at the two Gauß points of each edge. Here, the Stokes-like problem is discretized using the $P_2 - P_0$-element. Therefore, only $k = 1$ holds in (6.4a). On the other hand, the displacement error estimate (6.4b) holds with $\ell = 1$ and $k = 2$.

PROPOSITION 6.3 : *Let* (θ_h, w_h) *be the solution of* (2.2) *for the MITC6-element. Assume in addition to* (2.10) *that the solution* w *of the Poisson equation* (2.7) *is* H^3*-regular. Then the following error estimate*

$$\begin{aligned} \|\theta - \theta_h\|_0 &\leq ch^2\|f\|_0 \\ \|w - w_h\|_1 &\leq ch^2(\|f\|_0 + t^2\|f\|_1) \end{aligned} \tag{6.12}$$

is valid.

Furthermore, in [5] rectangular elements are also presented. In particular, the MITC4 and the MITC9 rectangular elements are described in detail [2, 3, 4]. The analysis of the MITC9-element follows the same lines as in the proof of Theorem 6.2 and the error estimate (6.4) holds with $k = 2$.

After completing this manuscript we have heard that Brezzi and Stenberg are preparing a similar theory and that in a forthcoming book by Brezzi and Fortin [17] *also some equivalent ideas can be found.*

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