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ANALYSIS OF DOMAIN DECOMPOSITION FOR NON SYMMETRIC PROBLEMS : APPLICATION TO THE NAVIER-STOKES EQUATIONS (*)

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Résumé — *Cet article est consacré aux fondements mathématiques d'un algorithme de gradient conjugué pour la solution de problèmes non symétriques par décomposition de domaine, proposé par Sonké* et al [5] *et appliqué à la simulation d'écoulements laminaires dans une cavité en L*

L'analyse est effectuée pour un probleme modele de Dirichlet associé a un opérateur linéaire général du second ordre, non symétrique

En utilisant une décomposition de domaine sans recouvrement, nous prouvons l'existence et l'unicité de la solution du probleme de décomposition-coordination équivalent, au moyen de l'opérateur de Steklov-Poincaré Une technique de symétrisation est ensuite appliquée au problème de décomposition coordination et permet de construire un algorithme de gradient-conjugué pour le calcul de la solution

L'application de la méthode a une forme linéarisée des équations de Navier-Stokes est expliquée à la fin de l'article

D'un point de vue mathématique, les conditions de cette application sont assez restrictives Cependant, les résultats numériques présentés dans [5] suggerent que ces conditions ne sont pas optimales Elles sont donc susceptibles d'amélioration

Abstract — *A Dirichlet problem with a general second order and non symmetric linear operator is solved* via *a domain decomposition method without overlapping existence and uniqueness of solution for the equivalent decomposition-coordination problem is proved, using Steklov-Poincaré operator A symmetrization technique is applied to obtain a conjugate gradient algorithm for computation of solution Application to a linearized form of the Navier Stokes equations is explained*

Key Words — Domain decomposition , Steklov-Poincaré operator , symmetrization technique , conjugate gradient , Navier-Stokes equations

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1. INTRODUCTION

During the last decade, scientific calculation has witnessed an increasing interest in domain decomposition methods. Development of parallel calculation has favoured numerical experiments in this scope of science. Besides these experiments, numerous theoretical investigations have been devoted to domain decomposition methods without overlapping. It can be refered to Glowinski [1] as one of pioneer works accomplished in domain decomposition without overlapping. Among recent studies, reader can consult Glowinski and Wheeler [2], Chan [3], Chan and Resasco [4], Sonké *et al.* [5]. Except [5], these investigations have been restricted to symmetric problems. Iterative methods based on the classical conjugate gradient method are often used. Most of non symmetric problems are solved by numerical analogs to the Schwarz alternating method [6]. In [1], a method based on a combination of optimal control and domain decomposition for symmetric problems is proposed for the study of nonlinear problems. The same idea can be applied to non symmetric problems and utilization of preconditionners [3], [4] can increase performance of such procedure. Nevertheless, this method is expensive, for each iteration of the least squares algorithm [1] needs several domain decomposition procedures. The first conjugate gradient algorithm for domain decomposition methods for non symmetric problems was proposed in [5] and applied to the study of fluid motiòn in an L-shapped cavity.

This paper is devoted to the mathematical background of the non symmetric method mentioned above. Using domain decomposition without overlapping, we prove the existence and uniqueness of solution for a Dirichlet problem with a linear and non symmetric general second order operator. For computation of this solution, we propose a conjugate gradient algorithm based on a « symmetrization » technique introduced by Sonké [7].

2. MATHEMATICAL ANALYSIS

2.1. Model problem

Let Ω be an open bounded set of R^N, with regular boundary Γ. Denote A, B and D operators defined by

$$Au = \alpha_0 u - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(\alpha_{ij}\frac{\partial u}{\partial x_j}\right)$$

$$Bu = \sum_{i=1}^{N} \beta_i \frac{\partial u}{\partial x_i} \qquad Du = Au + Bu$$

where α_0 belongs to $L^\infty(\Omega)$, the α_{ij}'s $1 \leqslant i, j \leqslant N$ belong to $L^\infty(\Omega)$ and the β_i's belong to $L^\infty(\Omega)$. Suppose f belongs to $L^2(\Omega)$, take g in $H^{1/2}(\Gamma)$ and consider the following global problem

$$\begin{cases} Du = f & \text{in} \quad \Omega \\ u = g & \text{on} \quad \Gamma . \end{cases} \tag{1}$$

Problem (1) is non symmetric, because of operator B, and also because we do not suppose the usual symmetry conditions $\alpha_{ij} = \alpha_{ji}$.

Denote $\beta = (\beta_1, \beta_2, \ldots, \beta_N)^T$. Under the following conditions

$$\operatorname{div} \beta = 0 \tag{2}$$

$$\exists \tau > 0 \text{ such that } \forall \xi \in R^N , \quad \sum_{i,j=1}^{N} \alpha_{ij}(x)\, \xi_i\, \xi_j \geqslant \tau\, |\xi|^2 \text{ a.e. in } \Omega , \tag{3}$$

$$\alpha_0(x) \geqslant \tau_0 \text{ a.e. in } \Omega , \tag{4}$$

with $\tau_0 > -\tau/C_\Omega^2$, where C_Ω is the Poincaré constant of Ω.

Sonké [7] proved the existence and uniqueness of solution for problem (1) using the Lax-Milgram theorem [8] and the following lemma of Temam [9].

LEMMA 1 : *Suppose u, v and w are vector functins with $\nabla . u = 0$. Operator $b(.,.,.)$ defined by*

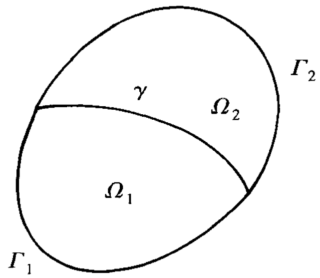
$$b(u, v, w) = \sum_{i,j=1}^{N} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \, dx$$

is a continuous trilinear form on $H_0^1(\Omega) \times H_0^1(\Omega) \times (H_0^1(\Omega) \cap L^N(\Omega))$ and one has the following estimate

$$b(u, v, w) \leqslant c(N) \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)} \|w\|_{H_0^1(\Omega) \cap L^N(\Omega)} .$$

Now consider a family of non-intersecting open sets Ω_k, $n = 1, 2, \ldots, n$ $(n \geqslant 2)$, which is a partition of Ω. For simplicity, we limit our study to the case $n = 2$. Ω is the union of Ω_1, Ω_2 and γ, Ω_1 and Ω_2 are non-intersecting domains and γ is the intersection of their closures.

$$\Omega = \Omega_1 \cup \Omega_2$$
$$\Omega_1 \cap \Omega_2 = \varnothing .$$



Denote

$$\Gamma_k = \Gamma \cap \partial\Omega_k \quad k = 1, 2$$
$$\gamma = \partial\Omega_1 \cap \partial\Omega_2 .$$

According to distribution theory [10], the global problem (1) is equivalent to the following set of problems

$$\begin{cases} Du_k = f_k & \text{in} \quad \Omega_k \\ u_k = g_k & \text{on} \quad \Gamma_k \end{cases} \tag{5}$$

$$\begin{gathered} u_1 = u_2 \quad \text{on} \quad \gamma \\ \frac{\partial u_1}{\partial \nu_A^{(1)}} + \frac{\partial u_2}{\partial \nu_A^{(2)}} - (\beta^{(1)} \cdot \nu^{(1)})\, u_1 - (\beta^{(2)} \cdot \nu^{(2)})\, u_2 = 0 \text{ on } \gamma \end{gathered} \tag{6}$$

$\beta^{(k)}$, u_k, f_k(resp. g_k) denote the restriction of β, u, f (resp. g) on Ω_k (resp. Γ_k). $\nu^{(k)}$ is the unit normal vector directed out of Ω_k.

$$\frac{\partial u}{\partial \nu_A^{(k)}} = \sum_{i,j=1}^{N} \alpha_{ij} \frac{\partial u}{\partial x_j} \nu_i^{(k)}$$

is the conormal derivative associated to operator A, directed out of Ω_k. $(\beta^{(k)} \cdot \nu^{(k)})$ denotes the inner product of $\beta^{(k)}$ and $\nu^{(k)}$ in R^N.

From the coordination problem (6), we can rewrite problem (5)-(6) in the following primal approach

$$\begin{gathered} \text{Find } \lambda \text{ defined on } \gamma \text{ such that} \\ \frac{\partial u_1(\lambda)}{\partial \nu_A^{(1)}} + \frac{\partial u_2(\lambda)}{\partial \nu_A^{(2)}} - (\beta^{(1)} \cdot \nu^{(1)})\, u_1(\lambda) - (\beta^{(2)} \cdot \nu^{(2)})\, u_2(\lambda) = 0 \text{ on } \gamma \end{gathered} \tag{7}$$

where $u_k(\lambda)$ is the solution of (5) with

$$u_k(\lambda)\big|_\gamma = \lambda\,. \tag{8}$$

We are now going to prove the existence and uniqueness of solution for problem (7)-(8).

2.2. Solution of problem (7)-(8)

Define the following spaces

$$\Lambda_0 = \{\mu \in L^2(\gamma) \text{ such that } \mu = \tilde{\mu} \mid \gamma \text{ where } \tilde{\mu} \in H_0^1(\Omega)\}$$

$$\Lambda_g = \\ = \{\mu \in L^2(\gamma) \text{ such that } \mu = \tilde{\mu} \mid \gamma \text{ where } \tilde{\mu} \in H^1(\Omega) \text{ and } \tilde{\mu} = g \text{ on } \Gamma\}\,.$$

For $\mu \in \Lambda_0$, we need $\tilde{\mu}_k$, $k = 1, 2$, which is an extension of μ in

$$H_k = \{u_k \in H^1(\Omega_k) \text{ such that } u_k = 0 \text{ on } \Gamma_k\}\,.$$

This extension can be defined as the unique element $\tilde{\mu}_k \in K_k$ satisfying

$$\tilde{\mu}_k\big|_\gamma = \mu$$

with

$$H_k = H_0^1(\Omega) \oplus K_k\,, \quad k = 1, 2$$

Λ_0 is a Hilbert space with norm $\|\mu\|_{\Lambda_0} = \|\tilde{\mu}\|_{H_0^1}$ and the induced inner product. We now give a weak formulation of problem (7)-(8).

Define functions $u_k(\lambda)$, u_k^λ and u_k^0, $k = 1, 2$ by

$$u_k(\lambda) = u_k^\lambda + u_k^0\,, \quad k = 1, 2$$

where $u_k(\lambda)$ is the solution of problem (5) with boundary conditions (8) u_k^λ is the solution of the following problem.

Find $u_k^\lambda \in H_k$, $u_k^\lambda = \lambda$ on γ

$$\int_{\Omega_k} \left(\sum_{i,j=1}^{N} \alpha_{ij} \frac{\partial u_k^\lambda}{\partial x_j} \frac{\partial v_k}{\partial x_i} \right) dx + \int_{\Omega_k} \alpha_0\, u_k^\lambda\, v_k\, dx + \\ + \sum_{i=1}^{N} \int_{\Omega_k} \beta_i \frac{\partial u_k^\lambda}{\partial x_i} v_k\, dx = 0 \quad \forall v_k \in H_0^1(\Omega_k) \tag{9}$$

u_k^0 is the solution of the following problem.

Find $u_k^0 \in H^1(\Omega_k)$, $u_k^0 = g_k$ on Γ_k, $u_k^0 = 0$ on γ

$$\int_{\Omega_k} \left(\sum_{i,j=1}^{N} \alpha_{ij} \frac{\partial u_k^0}{\partial x_j} \frac{\partial v_k}{\partial x_i} \right) dx + \int_{\Omega_k} \alpha_0\, u_k^0\, v_k\, dx + \\ + \sum_{i=1}^{N} \int_{\Omega_k} \beta_i \frac{\partial u_k^0}{\partial x_i} v_k\, dx = \int_{\Omega_k} f_k\, v_k\, dx \quad \forall v_k \in H_0^1(\Omega_k)\,. \tag{10}$$

Define the non symmetric Steklov-Poincaré operator T [11] on Λ_0 by

$$T\lambda = \frac{\partial u_1^\lambda}{\partial \nu_A^{(1)}} + \frac{\partial u_2^\lambda}{\partial \nu_A^{(2)}} - (\beta^{(1)} \cdot \nu^{(1)})\, u_1^\lambda - (\beta^{(2)} \cdot \nu^{(2)})\, u_2^\lambda\,. \tag{11}$$

For $\mu \in \Lambda_0$, we have

$$\int_\gamma T\lambda \cdot \mu\, d\gamma + \int_\gamma \left(\frac{\partial u_1^0}{\partial \nu_A^{(1)}} + \frac{\partial u_2^0}{\partial \nu_A^{(2)}} \right) - (\beta^{(1)} \cdot \nu^{(1)})\, u_1^0 - \\ - ((\beta^{(2)} \cdot \nu^{(2)})\, u_2^0)\, \mu\, d\gamma = 0\,.$$

We have then the following variational formulation for problem (7)-(8)

$$\begin{cases} \text{Find } \lambda \in \Lambda_0 \text{ such that} \\ t(\lambda, \mu) + l(\mu) = 0 \quad \forall \mu \in \Lambda_0 \end{cases} \tag{12}$$

where $t(.,.)$ is the bilinear form defined on Λ_0 by

$$t(.,.) : \Lambda_0 \times \Lambda_0 \to R$$

$$(\lambda, \mu) \to t(\lambda, \mu) = \int_\gamma T\lambda \, . \, \mu \, d\gamma$$

and $l(.)$ is the linear form defined on Λ_0 by

$$l(.) : \Lambda_0 \to R$$

$\mu \to l(\mu) =$

$$= \int_\gamma \left(\frac{\partial u_1^0}{\partial \nu_A^{(1)}} + \frac{\partial u_2^0}{\partial \nu_A^{(2)}} - (\beta^{(1)} . \nu^{(1)}) u_1^0 - (\beta^{(2)} . \nu^{(2)}) u_2^0 \right) \mu \, d\gamma \, .$$

We have the following result.

THEOREM 1 : *Suppose conditions* (2), (3) *and* (4) *are verified. Therefore, problem* (12) *has a unique solution.*

Demonstration of theorem 1 : Denote

$$| \| \alpha \| | = \max \left\{ \| \alpha_{ij} \|_\infty, \, 1 \le i, j \le N \right\}$$

where

$$\| \alpha_{ij} \|_\infty = \text{sup ess } \{ \alpha_{ij}(x), \, x \in \Omega \}$$

i) $t(.,.)$ is continuous on $\Lambda_0 \times \Lambda_0$.

Let $(\lambda, \mu) \in \Lambda_0 \times \Lambda_0$. Using Green formulas, we obtain

$$t(\lambda, \mu) = \sum_{k=1}^{2} \left(\int_{\Omega_k} \left(\sum_{i,j=1}^{N} \alpha_{ij} \frac{\partial u_k^\lambda}{\partial x_j} \frac{\partial \tilde{\mu}_k}{\partial x_i} \right) dx + \int_{\Omega_k} \alpha_0 u_k^\lambda \tilde{\mu}_k \, dx - \int_{\Omega} \sum_{i=1}^{N} \beta_i \frac{\partial \tilde{\mu}_k}{\partial x_i} u_k^\lambda \, dx \right)$$

$$= \sum_{k=1}^{2} \left(\int_{\Omega_k} \left(\sum_{i,j=1}^{N} \alpha_{ij} \frac{\partial u_k^\lambda}{\partial x_j} \frac{\partial u_k^\mu}{\partial x_i} \right) dx + \int_{\Omega_k} \alpha_0 u_k^\lambda u_k^\mu \, dx - \int_{\Omega} \sum_{i=1}^{N} \beta_i \frac{\partial u_k^\mu}{\partial x_i} u_k^\lambda \, dx \right) .$$

Therefore

$$|t(\lambda, \mu)| \leq \left| \sum_{k=1}^{2} \left(\int_{\Omega_k} \left(\sum_{i,j=1}^{N} \alpha_{ij} \frac{\partial u_k^\lambda}{\partial x_j} \frac{\partial u_k^\mu}{\partial x_i} \right) dx + \int_{\Omega_k} \alpha_0 \, u_k^\lambda \, u_k^\mu \, dx \right) \right| + \left| \sum_{k=1}^{2} \int_{\Omega} \sum_{i=1}^{N} \beta_i \frac{\partial u_k^\mu}{\partial x_i} u_k^\lambda \, dx \right| .$$

From Lemma 1, we obtain

$$|t(\lambda, \mu)| \leq \left(N \,.\, \mathrm{Cte} \; | \, \| \alpha \| \, | \int_{\Omega_k} \sum_{i=1}^{N} \left| \frac{\partial u_k^\lambda}{\partial x_i} \right| \left| \frac{\partial u_k^\mu}{\partial x_i} \right| dx + \| \alpha_0 \|_\infty \int_{\Omega_k} |u_k^\lambda| \, |u_k^\mu| \, dx \right)$$

$$+ \sum_{k=1}^{2} c(N) \,.\, \| \beta \|_\infty \, \| u_k^\lambda \|_{H_0^1} \, \| u_k^\mu \|_{H_0^1}$$

$$\leq \max \left(N \,.\, \mathrm{Cte} \; | \, \| \alpha \| \, | , \; \| \alpha_0 \|_\infty , \; \| \beta \|_\infty \right) \times \left(\sum_{k=1}^{2} \| u_k^\lambda \|_{H^1}^2 \right)^{1/2} \left(\sum_{k=1}^{2} \| u_k^\mu \|_{H^1}^2 \right)^{1/2}$$

thus $t(\,.,\,.\,)$ is continuous.

ii) $t(\,.,\,.\,)$ is Λ_0-elliptic.

Let $\lambda \in \Lambda_0$. We have

$$\int_{\Omega} \sum_{i=1}^{N} \beta_i \frac{\partial u_k^\lambda}{\partial x_i} u_k^\lambda \, dx = 0 \, .$$

We then obtain from Green formulas

$$t(\lambda, \lambda) = \sum_{k=2}^{N} \int_{\Omega_k} \left(\sum_{i,j=1}^{N} \alpha_{ij} \frac{\partial u_k^\lambda}{\partial x_j} \frac{\partial u_k^\lambda}{\partial x_i} + \alpha_0 (u_k^\lambda)^2 \right) dx \, .$$

Using conditions (3)-(4), we obtain

$$t(\lambda, \lambda) \geq \sum_{k=2}^{N} \left(\tau \, \| \nabla u_k^\lambda \|_{L^2}^2 + \tau_0 \| u_k^\lambda \|_{L^2}^2 \right)$$

$$\geq \min (\tau, \tau_0) \sum_{k=1}^{N} \| u_k^\lambda \|_{H^1}^2$$

$t(\,.,\,.\,)$ is therefore Λ_0-elliptic.

The linear form $\ell(\,.\,)$ beeing continuous, we conclude that problem (12) has a unique solution by the Lax-Milgram theorem.

3. ALGORITHMIC APPROACH

Our objective in this section is the construction of an algorithm for the solution of problem (12). Recall the following result.

Consider the following problem

$$\begin{cases} \textit{Find } u \in V & \textit{such that} \\ a(u, v) = L(v)\,, & \forall v \in V \end{cases} \tag{13}$$

where

i) *V is a Hilbert space for $(\,.,\,.\,)$ and $\|\,.\,\|$,*
ii) *$a : V \times V \to R$ is bilinear, continuous, V-elliptic,*
iii) *$L : V \to R$ is continuous.*

Problem (13) has a unique solution. More, if $a(\,.,\,.\,)$ is *symmetric*, therefore (13) is equivalent to the following minimization problem

$$\begin{cases} \text{Find } u \in V & \text{such that} \\ J(u) \leqslant J(v)\,, & \forall v \in V \end{cases} \tag{14}$$

where
$$J(v) = \frac{1}{2} a(v, v) - L(v)\,.$$

A conjugate gradient algorithm for solution of (14) is given in [2], but we cannot use this algorithm to compute the solution of (12), since this problem is not symmetric. In order to find an algorithm for solution of (12), we are going to apply the symmetrization technique described in [7].

3.1. « Symmetrization »

The principle of the symmetrization technique is described below.

Given a non symmetric linear problem (P), which solution belongs to a Hilbert space V, we construct in $\mathbf{V} = V \times V$, a symmetric positive definite problem (P') which contains problem (P). Iterative methods of the conjugate gradient type can then be used to solve (P'). Solution of (P) is deduced from the solution of (P').

Now define operators $\bar{A}$, $\bar{B}$ and $\bar{D}$ by

$$\bar{A}\psi = \alpha_0 \psi - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(\bar{\alpha}_{ij} \frac{\partial \psi}{\partial x_j}\right) \quad \text{where} \quad \bar{\alpha}_{ij} = \alpha_{ji}$$

$$\bar{B}\psi = \sum_{i=1}^{N} \bar{\beta}_i \frac{\partial \psi}{\partial x_i} \quad \text{where} \quad \bar{\beta}_i = -\beta_i \quad \text{and} \quad \bar{D}\psi = \bar{A}\psi + \bar{B}\psi\,.$$

We use large characters to denote product spaces, for example $\mathbf{H}^1(\Omega) = (H^1(\Omega))^2$.

Define $F = (f, f)^T$, $G = (g, g)^T$ and operator $P : \mathbf{H}^1(\Omega) \to \mathbf{H}^{-1}(\Omega)$ by

$$\Phi = (\phi, \psi)^T \to P\Phi = (\bar{D}\psi, D\phi)^T .$$

Consider the following problem, which is a juxtaposition of two independant problems similar to (1).

$$\begin{cases} P\Phi = F & \text{in } \Omega \\ \Phi = G & \text{on } \Gamma . \end{cases} \tag{15}$$

The following result is proved in [7].

PROPOSITION 1 : *Denote* $\beta = (\beta_1, \beta_2, \dots, \beta_N)^T$, $U = (u_1, u_2)^T$ *and* $V = (v_1, v_2)^T$. *Define operator* $\sigma(.,.)$ on $\mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega)$ *by*

$$\sigma(U, V) = a(U, V) + b(U, V)$$

where

$$a(U, V) = \int_\Omega \left(\sum_{i,j=1}^N \left(\alpha_{ij} \frac{\partial u_1}{\partial x_j} \frac{\partial v_2}{\partial x_i} + \bar{\alpha}_{ij} \frac{\partial u_2}{\partial x_j} \frac{\partial v_1}{\partial x_i} \right) \right) dx + \\ + \int_\Omega \alpha_0 (u_1 v_2 + u_2 v_1)\, dx$$

$$b(U, V) = b_\beta(u_1, v_2) + b_{\bar{\beta}}(u_2, v_1)\; b_\beta(u, v) = \sum_{i=1}^N \int_\Omega \beta_i \frac{\partial u}{\partial x_i} v \, dx .$$

We suppose hypothesis (2).

Therefore $\sigma(.,.)$ *is a bilinear symmetric form on* $\mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega)$.

One can also prove (see [7]) that $\sigma(.,.)$ is the bilinear form of the variational formulation of (15), $\sigma(.,.)$ is continuous on $\mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega)$ and one has the following estimate

$$|\sigma(U, V)| \leqslant 4 . \max (N . \mathrm{Cte}\ \|\alpha\|, \|\alpha_0\|_\infty, \|\beta\|_\infty)\ \|U\|_{\mathbf{H}^1} \|V\|_{\mathbf{H}^1} . \tag{16}$$

In the following, *we suppose* that $\sigma(.,.)$ is coercive.

Global problem (15) is equivalent to the family of problems

$$\begin{cases} P\Phi_k = F_k & \text{in } \Omega \\ \Phi_k = G_k & \text{on } \Gamma \end{cases} \tag{17}$$

with the coordination problem

$$\left\{\begin{array}{l} \Phi_1 = \Phi_2 \quad \text{on} \quad \gamma \\ \dfrac{D\Phi_1}{D\nu^{(1)}} + \dfrac{D\Phi_2}{D\nu^{(2)}} - \nu_\beta^{(1)} \cdot \Phi_1 - \nu_\beta^{(2)} \cdot \Phi_2 = 0 \quad \text{on} \quad \gamma \end{array}\right. \tag{18}$$

where for $k = 1, 2$.

Φ_k and F_k (resp. G_k) are restrictions of Φ and F (resp. G) to Ω_k (resp. Γ_k), $\dfrac{D}{D\nu^{(k)}}$ is defined by

$$\Phi = (\phi, \psi)^T \rightarrow \frac{D\Phi}{D\nu^{(k)}} = \left(\frac{\partial \psi}{\partial \nu_A^{(k)}}, \frac{\partial \phi}{\partial \nu_A^{(k)}} \right)^T$$

and $$\nu_\beta^{(k)} \cdot \Phi = ((\bar{\beta}^{(k)} \cdot \nu^{(k)})\, \psi, (\beta^{(k)} \cdot \nu^{(k)})\, \phi)^T .$$

We have the following formulation of (17)-(18) in the primal approach.

$$\left\{\begin{array}{l} \text{Find } M \text{ such that} \\ \dfrac{D\Phi_1(M)}{D\nu^{(1)}} + \dfrac{D\Phi_2(M)}{D\nu^{(2)}} - \nu_\beta^{(1)} \cdot \Phi_1(M) - \nu_\beta^{(2)} \cdot \Phi_2(M) = 0 \text{ on } \gamma \end{array}\right. \tag{19}$$

where

$$\Phi_k(M)|_\gamma = M , \qquad k = 1, 2 . \tag{20}$$

3.2. Solution of problem (19)-(20)

Define the following spaces

$$\Lambda_0 = \left\{ M = (\mu, \mu)^T \in L^2(\gamma) \times L^2(\gamma) \quad \text{such that} \quad M = \tilde{M}|_\gamma , \quad \tilde{M} \in \mathbf{H}_0^1(\Omega) \right\}$$

$$\Lambda_g = \left\{ M \in L^2(\gamma) \times L^2(\gamma) \quad \text{such that} \quad M = \tilde{M}|_\gamma , \quad \tilde{M} \in \mathbf{H}^1(\Omega) \quad \text{and} \quad \tilde{M} = G \quad \text{on} \quad \Gamma \right\}$$

Λ_0 is a Hilbert space with norm $\|M\|_{\Lambda_0} = \|\tilde{M}\|_{\mathbf{H}_0^1}$, where $\tilde{M}$ is the unique element of

$$\mathbf{H}_k = \{ U_k \in \mathbf{H}^1(\Omega_k) \quad \text{such that} \quad U_k = 0 \quad \text{on} \quad \Gamma_k \}$$

defined by its restrictions $\tilde{M}_k = \tilde{M}|_{\Omega_k} \in \mathbf{K}_k$ such that

$$\tilde{M}_k|_\gamma = M \quad \text{with} \quad \mathbf{H}_k = \mathbf{H}_0^1(\Omega) \oplus \mathbf{K}_k , \quad k = 1, 2 .$$

Consider the following decomposition

$$\Phi_k(M) = \Phi_k^M + \Phi_k^0 \tag{21}$$

where $\Phi_k(M)$ is the solution of problem (17) with boundary conditions (20), Φ_k^M is the solution of the following variational problem.

Find $\Phi_k^M = (\phi_k^\mu, \psi_k^\mu)^T \in \mathbf{H}_k$, $\Phi_k^M = M = (\mu, \mu)^T$ on γ

$$\int_{\Omega_k} \left(\sum_{i,j=1}^N \left(\alpha_{ij} \frac{\partial \phi_k^\mu}{\partial x_j} \frac{\partial v_k}{\partial x_i} + \bar{\alpha}_{ij} \frac{\partial \psi_k^\mu}{\partial x_j} \frac{\partial u_k}{\partial x_i} \right) \right) dx + \int_{\Omega_k} \alpha_0 (\phi_k^\mu v_k + \psi_k^\mu u_k)\, dx$$

$$\sum_{i=1}^N \int_{\Omega_k} \beta_i \frac{\partial \phi_k^\mu}{\partial x_i} v_k\, dx + \sum_{i=1}^N \int_{\Omega_k} \bar{\beta}_i \frac{\partial \psi_k^\mu}{\partial x_i} u_k\, dx$$

$$= 0 \quad \forall U_k = (u_k, v_k)^T \in \mathbf{H}_0^1(\Omega_k) \tag{22}$$

Φ_k^0 is the solution of the following variational problem.

Find $\Phi_k^0 = (\phi_k^0, \psi_k^0) \in \mathbf{H}^1(\Omega)$, $\Phi_k^0 = G_k$ on Γ_k, $\Phi_k^0 = 0$ on γ

$$\int_{\Omega_k} \left(\sum_{i,j=1}^N \left(\alpha_{ij} \frac{\partial \phi_k^0}{\partial x_j} \frac{\partial v_k}{\partial x_i} + \bar{\alpha}_{ij} \frac{\partial \psi_k^0}{\partial x_j} \frac{\partial u_k}{\partial x_i} \right) \right) dx + \int_{\Omega_k} \alpha_0 (\phi_k^0 v_k + \psi_k^0 u_k)\, dx$$

$$\sum_{i=1}^N \int_{\Omega_k} \beta_i \frac{\partial \phi_k^0}{\partial x_i} v_k dx + \sum_{i=1}^N \int_{\Omega_k} \bar{\beta}_i \frac{\partial \psi_k^0}{\partial x_i} u_k dx$$

$$= \int_{\Omega_k} (f u_k + f v_k)\, dx \quad \forall U_k = (u_k, v_k)^T \in \mathbf{H}_0^1(\Omega_k)\,. \tag{23}$$

Define the following Steklov-Poincaré operator Ξ

$$\Xi M = \frac{D\Phi_1^M}{D\nu^{(1)}} + \frac{D\Phi_2^M}{D\nu^{(2)}} - \nu_\beta^{(1)} \cdot \Phi_1^M - \nu_\beta^{(2)} \cdot \Phi_2^M\,. \tag{24}$$

Define the linear form L by

$$L: \quad \Lambda_0 \to R$$

$$M \to L(M) = \int_\gamma \left(\frac{D\Phi_1^0}{D\nu^{(1)}} + \frac{D\Phi_2^0}{D\nu^{(2)}} - \nu_\beta^{(1)} \cdot \Phi_1^0 - \nu_\beta^{(2)} \cdot \Phi_2^0 \right) . M\, d\gamma\,.$$

Define the bilinear form $\zeta(.,.)$ by

$$\zeta: \quad \Lambda_0 \times \Lambda_0 \to R$$

$$(M, M') \to \zeta(M, M') = \int_\gamma (\Xi M) . M'\, d\gamma\,.$$

We obtain the following variational formulation for problem (19)-(20)

$$\begin{cases} \text{Find } M = (\lambda, \lambda)^T \in \Lambda_0 \quad \text{such that} \\ \zeta(M, M') + L(M') = 0 \quad \forall M' = (\mu, \mu)^T \in \Lambda_0 \end{cases} \tag{25}$$

PROPOSITION 2 : *The bilinear form* $\zeta(.,.)$ *is symmetric*

Demonstration of proposition 2 : Take a couple $(M, M')^T = ((\lambda, \lambda)^T, (\mu, \mu)^T)$ in Λ_0. We have

$$\begin{aligned} \zeta(M, M') &= \int_\gamma (\Xi M) . M' \, d\gamma \\ &= \sum_{k=1}^{2} \int_\gamma \left(\frac{D\Phi_k^M}{D\nu^{(k)}} - \nu_\beta^{(k)} . \Phi_k^M \right) . M' \, d\gamma \\ &= \sum_{k=1}^{2} \int_\gamma \left(\frac{\partial \psi_k^\lambda}{\partial \nu_{\bar{A}}} \mu + \frac{\partial \phi_k^\lambda}{\partial \nu_A} \mu \right) d\gamma \\ &\quad - \sum_{k=1}^{2} ((\bar{\beta}_k . \nu^{(k)}) \psi_k^\lambda \mu + \beta_k . \nu^{(k)} \phi_k^\lambda \mu) \, d\gamma . \end{aligned}$$

Using Green formulas, we obtain

$$\begin{aligned} \zeta(M, M') &= \sum_{k=1}^{2} \left(\int_{\Omega_k} \left(\sum_{i,j=1}^{N} \bar{\alpha}_{ij} \frac{\partial \psi_k^\lambda}{\partial x_j} \frac{\partial \hat{\mu}}{\partial x_i} \right) dx \right. \\ &\quad \left. + \int_{\Omega_k} \alpha_0 \psi_k^\lambda \hat{\mu} \, dx - \sum_{i=1}^{N} \int_{\Omega_k} \bar{\beta}_i \frac{\partial \hat{\mu}}{\partial x_i} \psi_k^\lambda \, dx \right) \\ &\quad + \sum_{k=1}^{2} \left(\int_{\Omega_k} \left(\sum_{i,j=1}^{N} \alpha_{ij} \frac{\partial \phi_k^\lambda}{\partial x_j} \frac{\partial \hat{\mu}}{\partial x_i} \right) dx \right. \\ &\quad \left. + \int_{\Omega_k} \alpha_0 \phi_k^\lambda \hat{\mu} \, dx - \sum_{i=1}^{N} \int_{\Omega_k} \beta_i \frac{\partial \hat{\mu}}{\partial x_i} \phi_k^\lambda \, dx \right) \\ &= \sum_{k=1}^{2} \left(\int_{\Omega_k} \left(\sum_{i,j=1}^{N} \alpha_{ij} \frac{\partial \phi_k^\lambda}{\partial x_j} \frac{\partial \psi_k^\mu}{\partial x_i} + \bar{\alpha}_{ij} \frac{\partial \psi_k^\lambda}{\partial x_j} \frac{\partial \phi_k^\mu}{\partial x_i} \right) dx \right. \\ &\quad \left. + \int_{\Omega_k} \alpha_0 (\phi_k^\lambda \psi_k^\mu + \psi_k^\lambda \phi_k^\mu) \, dx \right) \\ &\quad - \sum_{k=1}^{2} \left(\sum_{i=1}^{N} \int_{\Omega_k} \beta_i \frac{\partial \psi_k^\mu}{\partial x_i} \phi_k^\lambda \, dx \sum_{i=1}^{N} \int_{\Omega_k} \bar{\beta}_i \frac{\partial \phi_k^\mu}{\partial x_i} \psi k^\lambda \, dx \right) \end{aligned}$$

prolonging functions ϕ_k^λ, ψ_k^λ, Φ_k^M, and $\Phi_k^{M'}$ on $\Omega - \Omega_k$ by 0, we can extend integrals on Ω. We obtain

$$\begin{aligned}\zeta(M, M') = \sum_{k=1}^{2} &\left(\sum_{i=1}^{N} \int_\Omega \beta_i \frac{\partial \phi_k^\lambda}{\partial x_i} \psi_k^\mu \, dx \sum_{i=1}^{N} \int_\Omega \bar{\beta}_i \frac{\partial \psi_k^\lambda}{\partial x_i} \phi_k^\mu \, dx \right) \\ &+ \sum_{k=1}^{2} \left(\left(\int_\Omega \left(\sum_{i,j=1}^{N} \alpha_{ij} \frac{\partial \phi_k^\lambda}{\partial x_j} \frac{\partial \psi_k^\mu}{\partial x_i} + \bar{\alpha}_{ij} \frac{\partial \psi_k^\lambda}{\partial x_j} \frac{\partial \phi_k^\mu}{\partial x_i} \right) dx \right.\right. \\ &+ \left. \int_\Omega \alpha_0 (\phi_k^\lambda \psi_k^\mu + \psi_k^\lambda \phi_k^\mu) \, dx \right) = \sum_{k=1}^{2} \sigma(\Phi_k^M, \Phi_k^{M'})\end{aligned}$$

where $\sigma(.,.)$ is the symmetric bilinear form defined in proposition 1. We conclude that $\zeta(.,.)$ is symmetric.

THEOREM 2 : *Under hypothesis* (2), (3), (4), *with* $\tau_0 > 0$, *and the following additional condition*

$$\exists \tau_1 > 0 \,, \quad |\alpha_0| \leq \tau_1 \quad and \quad \tau + (\tau_0 - 2\,\tau_1)\, C_\Omega^2 \geq \alpha > 0 \,, \qquad (26)$$

solution of problem (12) *can be computed by means of a conjugate gradient algorithm.*

Démonstration of Theorem 2 : Since problem (25) contains problem (12), we just have to prove that one can compute solution of (25) using a conjugate gradient algorithm.

i) $\zeta(.,.)$ is continuous.

From the proof of proposition 2, we have

$$\zeta(M, M') = \sigma(\Phi_1^M, \Phi_1^{M'}) + \sigma(\Phi_2^M, \Phi_2^{M'}) \,.$$

We have then

$$|\zeta(M, M')| \leq |\sigma(\Phi_1^M, \Phi_1^{M'})| + |\sigma(\Phi_2^M, \Phi_2^{M'})| \,.$$

From estimate (16) and Cauchy-Schwarz inequalities, we obtain

$$\begin{aligned}|\zeta(M, M')| &\leq 4 \,.\, \max\,(N \,.\, \mathrm{Cte}\ |||\alpha|||,\ \|\alpha_0\|_\infty,\ \|\beta\|_\infty) \times \\ &\qquad \times \sum_{k=1}^{2} \|\Phi_k^M\|_{H^1} \|\Phi_k^{M'}\|_{H^1} \\ &\leq 4 \,.\, \max\,(N \,.\, \mathrm{Cte}\ |||\alpha|||,\ \|\alpha_0\|_\infty,\ \|\beta\|_\infty) \times \\ &\qquad \times \left(\sum_{k=1}^{2} \|\Phi_k^M\|_{H^1}^2 \right)^{1/2} \left(\sum_{k=1}^{2} \|\Phi_k^{M'}\|_{H^1}^2 \right)^{1/2}\end{aligned}$$

$\zeta(.,.)$ is thus continuous.

ii) Coercivity.

We suppose that $\sigma(.,.)$ is coercive. Actually, this can be proved under condition (26).

Then, relation

$$\zeta(M, M) = \sigma(\Phi_1^M, \Phi_1^M) + \sigma(\Phi_2^M, \Phi_2^M)$$

and the fact that the Λ_0-norm is induced by the $\mathbf{H}^1$-norm prove that $\zeta(.,.)$ is coercive. Proposition 2 completes this demonstration.

Remark : One can prove that $\zeta(.,.)$ is coercive, under the more precise following condition

$$\min\left(\tau + (\tau_0 - 2\tau_1) C^2_{\Omega_1}, \tau + (\tau_0 - 2\tau_1) C^2_{\Omega_2}\right) \geqslant \alpha > 0, \tag{27}$$

where $\Omega_{i'} = \Omega_i \cup \gamma \cup W_{ij}$, $W_{ij} = \Omega_{i'} \cap \Omega_j$ is a very thin open set.

3.3. Algorithm

We apply an algorithm of minimization for the solution of problem (25), see Glowinski [12]. From this algorithm, we obtain the following conjugate gradient algorithm for the solution of (12).

Initialization.

Choose λ^0 in Λ_g compute ρ^0.

For $k = 1, 2$.

Find $u_k^0 \in H^1(\Omega_k)$, $u_k^0 = g_k$ on Γ_k, $u_k^0 = \lambda^0$ on γ such that

$$\int_{\Omega_k} \left(\sum_{i,j=1}^N \alpha_{ij} \frac{\partial u_k^0}{\partial x_j} \frac{\partial v_k}{\partial x_i}\right) dx + \int_{\Omega_k} \alpha_0 u_k^0 v_k \, dx + \\ + \sum_{k=1}^N \int_{\Omega_k} \beta_i \frac{\partial u_k^0}{\partial x_i} v_k \, dx = \int_{\Omega_k} f_k v_k \, dx \; \forall v_k \in H_0^1(\Omega_k). \tag{28}$$

Find $\rho^0 \in \Lambda_0$ such that

$$\sum_{k=1}^2 \left(\int_{\Omega_k} \left(\nabla \tilde{\rho}_k^0 \nabla \tilde{\mu}_k \, dx + \int_{\Omega_k} \tilde{\rho}_k^0 \tilde{\mu}_k \, dx\right)\right) = \\ = \sum_{k=1}^2 \left(\int_{\Omega_k} \left(\sum_{i,j=1}^N \alpha_{ij} \frac{\partial u_k^0}{\partial x_j} \frac{\partial \tilde{\mu}_k}{\partial x_i}\right) dx \right. \\ + \int_{\Omega_k} \alpha_0 u_k^0 \tilde{\mu}_x \, dx + \sum_{i=1}^N \int_{\Omega_k} \beta_i \frac{\partial u_k^0}{\partial x_i} \tilde{\mu}_k \, dx \\ \left. - \int_{\Omega_k} f_k \tilde{\mu}_k \, dx\right) \quad \forall \mu \in \Lambda_0. \tag{29}$$

Take

$$\bar{\delta}^0 = \delta^0 = \bar{\rho}^0 = \rho^0 . \tag{30}$$

Iterations.

$n \geq 0$, λ^n, ρ^n, $\bar{\rho}^n$, δ^n et $\bar{\delta}^n$ known, compute λ^{n+1}, ρ^{n+1}, $\bar{\rho}^{n+1}$, δ^{n+1} et $\bar{\delta}^{n+1}$.

For $k = 1, 2$.

Find $v_k^n \in H_k$, $v_k^n = \delta^n$ on γ such that

$$\int_{\Omega_k} \left(\sum_{i,j=1}^N \alpha_{ij} \frac{\partial v_k^n}{\partial x_j} \frac{\partial z_k}{\partial x_i} \right) dx + \int_{\Omega_k} \alpha_0 v_k^n z_k \, dx + \\ + \sum_{i=1}^N \int_{\Omega_k} \beta_i \frac{\partial v_k^n}{\partial x_i} z_k \, dx = 0 \quad \forall z_k \in H_0^1(\Omega_k) . \tag{31}$$

Find $w_k^n \in H_k$, $w_k^n = \bar{\delta}^n$ on γ such that

$$\int_{\Omega_k} \left(\sum_{i,j=1}^N \bar{\alpha}_{ij} \frac{\partial w_k^n}{\partial x_j} \frac{\partial y_k}{\partial x_i} \right) dx + \int_{\Omega_k} \alpha_0 w_k^n y_k \, dx + \\ + \sum_{i=1}^N \int_{\Omega_k} \beta_i \frac{\partial w_k^n}{\partial x_i} y_k \, dx = 0 \quad \forall y_k \in H_0^1(\Omega_k) . \tag{32}$$

Compute η^n and $\bar{\eta}^n$ as solutions of the following variational problems.

Find $\eta^n \in \Lambda_0$ such that

$$\sum_{k=1}^2 \left(\int_{\Omega_k} \nabla \tilde{\eta}_k^n \cdot \nabla \tilde{\zeta}_k \, dx + \int_{\Omega_k} \tilde{\eta}_k^n \tilde{\zeta}_k \, dx \right) = \\ = \sum_{k=1}^2 \left(\int_{\Omega_k} \left(\sum_{i,j=1}^N \tilde{\alpha}_{ij} \frac{\partial w_k^n}{\partial x_j} \frac{\partial \tilde{\zeta}_k}{\partial x_i} \right) dx + \int_{\Omega_k} \alpha_0 w_k^n \tilde{\zeta}_k \, dx \right. \\ \left. + \sum_{i=1}^N \int_{\Omega_k} \beta_i \frac{\partial w_k^n}{\partial x_i} \tilde{\zeta}_k \, dx \quad \forall \zeta \in \Lambda_0 . \right. \tag{33}$$

Find $\bar{\eta}^n \in \Lambda_0$ such that

$$\sum_{k=1}^2 \left(\int_{\Omega_k} \nabla \tilde{\bar{\eta}}_k^n \cdot \nabla \tilde{\zeta}_k \, dx + \int_{\Omega_k} \tilde{\bar{\eta}}_k^n \tilde{\zeta}_k \, dx \right) = \\ = \sum_{k=1}^2 \left(\int_{\Omega_k} \left(\sum_{i,j=1}^N \alpha_{ij} \frac{\partial v_k^n}{\partial x_j} \frac{\partial \tilde{\zeta}_k}{\partial x_i} \right) dx + \int_{\Omega_k} \alpha_0 v_k^n \tilde{\zeta}_k \, dx \right. \\ \left. + \sum_{i=1}^N \int_{\Omega_k} \beta_i \frac{\partial v_k^n}{\partial x_i} \tilde{\zeta}_k \, dx \quad \forall \zeta \in \Lambda_0 \right. \tag{34}$$

$$\varphi_n = \frac{\sum_{k=1}^{2} \int_{\Omega_k} (|\nabla \tilde{\rho}_k^n|^2 + |\tilde{\rho}_k^n|^2)\, dx + \sum_{k=1}^{2} \int_{\Omega_k} (|\nabla \tilde{\bar{\rho}}_k^n|^2 + |\tilde{\bar{\rho}}_k^n|^2)\, dx}{\sum_{k=1}^{2} \int_{\Omega_k} (\nabla \tilde{\eta}_k^n \,.\, \nabla v_k^n + \tilde{\eta}_k^n v_k^n)\, dx + \sum_{k=1}^{2} \int_{\Omega_k} (\nabla \tilde{\bar{\eta}}_k^n \,.\, \nabla v_k^n + \tilde{\bar{\eta}}_k^n v_k^n)\, dx} \tag{35}$$

$$\lambda^{n+1} = \lambda^n + \varphi_n \delta^n \tag{36}$$

$$\rho^{n+1} = \rho^n - \varphi_n \eta^n \tag{37}$$

$$\bar{\rho}^{n+1} = \bar{\rho}^n - \varphi_n \bar{\eta}^n \tag{38}$$

$$\rho_n = \frac{\sum_{k=1}^{2} \int_{\Omega_k} (|\nabla \tilde{\rho}_k^{n+1}|^2 + |\tilde{\rho}_k^{n+1}|^2)\, dx + \sum_{k=1}^{2} \int_{\Omega_k} (|\nabla \tilde{\bar{\rho}}_k^{n+1}|^2 + |\tilde{\bar{\rho}}_k^{n+1}|^2)\, dx}{\sum_{k=1}^{2} \int_{\Omega_k} (|\nabla \tilde{\rho}_k^n|^2 + |\tilde{\rho}_k^n|^2)\, dx + \sum_{k=1}^{2} \int_{\Omega_k} (|\nabla \tilde{\bar{\rho}}_k^n|^2 + |\tilde{\bar{\rho}}_k^n|^2)\, dx} \tag{39}$$

$$\delta^{n+1} = \rho^{n+1} + \rho_n \delta^n \tag{40}$$

$$\bar{\delta}^{n+1} = \bar{\rho}^{n+1} + \rho_n \bar{\delta}^n . \tag{41}$$

Do $n = n + 1$, go to (31).

4. APPLICATION TO THE 2D NAVIER-STOKES EQUATIONS

The $2D$ Navier-Stokes equations for laminar unsteady flow of an incompressible fluid are considered in the velocity-vorticity formulation [13]. We have the following system

$$\frac{\partial \omega}{\partial t} + u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y} - \frac{1}{\mathrm{Re}} \nabla^2 \omega = 0 \tag{42}$$

$$\nabla^2 u = \frac{\partial \omega}{\partial y} \tag{43}$$

$$\nabla^2 v = - \frac{\partial \omega}{\partial x} \tag{44}$$

plus boundary and initial conditions.
Where Re is the Reynolds number, u and v are velocity components and vorticity is defined by

$$\omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} .$$

Continuity equation $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ is implicitly taken into account, see Fasel [14].

Denote $U = (u, v)^T$ the velocity vector. At each time step, velocity components can be computed by a conjugate gradient algorithm for symmetric problems with domain decomposition, proposed by Sonké *et al.* [15]. The same algorithm can be applied for vorticity computation, *via* the following time discretization

$$\frac{3\,\omega^{n+1} - 4\,\omega^n + \omega^{n-1}}{2\,\Delta t} + 2(U\,.\,\nabla\omega\,)^n - (U\,.\,\nabla\omega\,)^{n-1} = \frac{1}{\mathrm{Re}}\,\nabla^2\omega^{n+1} \tag{45}$$

which is obtained using an explicit scheme of Adams-Bashforth type for non linear term [16]. We have then to solve an Helmholtz problem at each time step.

For reasons mentioned in [5], we avoid scheme (45). Using a semi-implicit two-steps second order accurate θ-scheme [17] for vorticity, we obtain at each time step the following problem

$$\begin{cases} \left(\frac{I}{\Delta t} + \theta\left[(\tilde{U}^{n+1}\,.\,\nabla\,) - \frac{1}{\mathrm{Re}}\,\nabla^2\right]\right)\,\omega^{n+1} = \\ \quad = \left(\frac{I}{\Delta t} - (1-\theta)\left[(U^n\,.\,\nabla\,) - \frac{1}{\mathrm{Re}}\,\nabla^2\right]\right)\,\omega^n \quad \text{in} \quad \Omega \\ \omega^{n+1} = \frac{\partial\tilde{u}^{n+1}}{\partial y} - \frac{\partial\tilde{v}^{n+1}}{\partial x} \quad \text{on } \Gamma\,, \quad 0 \leq \theta \leq 1\,. \end{cases} \tag{46}$$

This problem is similar to the model problem (1), operator A is replaced by

$$\frac{I}{\Delta t} - \frac{\theta}{\mathrm{Re}}\,\nabla^2$$

where I is the identity operator, that is

$$\alpha_0 = \frac{1}{\Delta t} \quad \text{and} \quad \alpha_{ij} = \frac{\theta}{\mathrm{Re}}\,\delta_{ij}$$

δ_{ij} is the Kronecker symbol, operator B is replaced by $\theta\,(\tilde{U}^{n+1}\,.\,\nabla\,)$, where $\tilde{U}^{n+1}$ is a second order prediction of U^{n+1}.

Evolution of fluid motion can then be studied by solving system (43), (44), (46).

Condition (2) is satisfied, since continuity equation is implicitly verified [14].

Conditions (3) and (4) are satisfied with $\tau = \frac{\theta}{\mathrm{Re}}$ and $\tau_0 = \frac{1}{\Delta t}$.

Now, takıng $\tau_1 = \frac{1}{\Delta t}$, condıtıon (27) can be expressed ın the followıng form

$$\min \left(\frac{\theta}{\mathrm{Re}} - \frac{C_{\Omega_1}^2}{\Delta t}, \frac{\theta}{\mathrm{Re}} - \frac{C_{\Omega_2}^2}{\Delta t} \right) \geqslant \alpha > 0$$

that ıs

$$\mathrm{Re} < \frac{\theta \, . \, \Delta t}{\max \left(C_{\Omega_1}^2, C_{\Omega_2}^2 \right)} \tag{47}$$

Recall that ıf l ıs the thıckness of a bounded set O, then one has $C_O^2 \simeq \frac{1}{2} l^2$, see Ravıart and Thomas [18]

Scheme (46) ıs uncondıtıonally stable for $1/2 \leqslant \theta \leqslant 1$ and one can choose quıte large tıme steps, e g $\Delta t = 1/10$

Suppose Ω ıs decomposed ınto subdomaıns of thıckness $\leqslant 1/10$ Then for a fully ımplıcıt scheme, that ıs $\theta = 1$, condıtıon (47) ıs satısfıed for $\mathrm{Re} \leqslant 20$

The dıscrete versıon of algorıthm presented ın thıs paper and numerıcal results of ıts applıcatıon to problem (46) can be seen ın [5] Thıs applıcatıon shows that condıtıon (27) ıs restrıctıve, sınce succesful results have been obtaıned ın sımulatıon of fluıd motıon for Re = 100, computatıonal domaın beıng decomposed ınto subdomaıns of thıckness $l = 1/2$

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