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MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

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ON CONVEX BÉZIER TRIANGLES

by H. PRAUTZSCH ([1])

Abstract. — *Goodman [8] showed that uniform subdivision of triangular Bézier nets preserves convexity. Here, a very short proof of this fact is given which applies even to box spline surfaces and degree elevation instead of subdivision.*

Secondly, it is shown that every Bézier net of a quadratic convex Bézier triangle can be subdivided such that the net becomes convex.

Keywords : *Convexity, Bernstein polynomials, Bézier triangles, subdivision, degree elevation, box splines.*

Résumé. — Sur les triangles de Bézier convexes. *Goodman [8] a montré que la sous-division uniforme de réseaux de Bézier triangulaires conserve la convexité. Nous donnons ici une preuve très courte de cette propriété, preuve qui s'applique aussi aux surfaces spline tensorielles et à l'augmentation du degré.*

Deuxièmement, on montre que tout réseau de Bézier d'un triangle de Bézier quadratique et convexe peut être sous-divisé de façon à ce que le réseau résultant devienne convexe.

1. INTRODUCTION

Grandine observed that subdivision of triangular Bézier nets does not always preserve convexity [9]. However, convexity is preserved by uniform subdivision as Goodman found out recently [8]. We will show that Goodman's result is a simple consequence of the fact that subdivision and differentiation commute.

Before we embark on this topic let us recall the definitions of the terms already used and yet to come. Δ denotes some triangle in $\mathbb{R}^2$ and d_0, d_1, d_2 its vertices. It serves as coordinate frame for the barycentric coordinates which are used throughout this paper. Thus, a tuple

$$(\alpha_0, \alpha_1, \alpha_2)\,, \quad a_0 + \alpha_1 + \alpha_2 = 1\,,$$

represents the point

$$\alpha_0 d_0 + \alpha_1 d_1 + \alpha_2 d_2\,.$$

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It is well-known that every polynomial p of degree $\leq n$ has a unique **n-th degree Bézier representation** over Δ of the form

$$(1.1) \qquad p(\alpha_0, \alpha_1, \alpha_2) = \sum_{\substack{i+j+k=n \\ i,j,k \geq 0}} b_{i,j,k} \left(\frac{n!}{i!j!k!} \alpha_0! \, \alpha_1! \, \alpha_2! \right) .$$

This representation is associated with the so-called Bézier net. The Bézier net is related to a uniform subdivision, i.e., a triangulation, of Δ generated by the gridlines

$$\alpha_0 = \mu/n, \quad \alpha_1 = \mu/n, \quad \alpha_2 = \mu/n, \quad \mu = 0, 1, ..., n \text{ ; see figure 1 .}$$

Δ_n will denote the set of all subtriangles of Δ belonging to this triangulation.

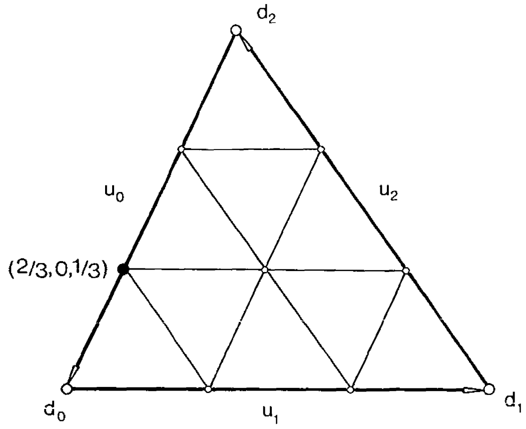

Figure 1. — The uniform subdivision Δ_3 of Δ.

The n-th Bézier net of p can now be defined as the piecewise linear function b over Δ which interpolates the **Bézier ordinates** $b_{i,j,k}$ of p at the abscissas $(i, j, k)/n$ and is linear over all triangles in Δ_n.

The Bézièr net is a useful approximation to p, even geometrically. In particular, it is well-known that p is convex if b is convex, see [1]. However, the converse is not true in general. This observation becomes interesting as one can « refine » the Bézier net to an arbitrarily close approximation of p. This paper investigates certain refinement methods with regard to convexity. First, in section 3 a short and general proof is given that uniform refinement methods preserve the convexity of Bézier nets. In section 4 it is shown that the proof applies even to box spline surfaces. In section 5 iterated uniform refinement is considered and two examples are presented in section 6. Finally, in section 7 it is shown that the Bézier net of any convex quadratic polynomial p over Δ can be refined to a convex Bézier net.

2. THE FUNDAMENTAL FACTS

First, we introduce operators to discuss subdivision, degree elevation and differentiation. These operators are defined on the set of all Bézier nets. Still, p denotes some bivariate polynomial and b its n-th degree Bézier net over Δ.

- The **uniform subdivision operator** U_m is defined such that $U_m b$ is a piecewise linear function over Δ where $[U_m b]|_{\Delta'}$, is the n-th degree Bézier net of p over Δ' for all $\Delta' \in \Delta_m$. Note that $U_m b$ is continuous.
- The **degree elevation operator** E_m is defined by $E_m b := (n+m)$-th degree Bézier net of p.
- The **differentiation operator** $D(u)$ is defined by $[D(u)\,b](x) := (n-1)$-th degree Bézier net of $\frac{d}{dt} p(x+tu)|_{t=0}$.

The operator D is mainly used in connection with the three directions of the edges of Δ

$$u_0 := d_0 - d_2, \quad u_1 := d_1 - d_0, \quad u_2 := d_2 - d_1 .$$

Also, the abbreviation

$$D_{i,j} := D(u_i) \circ D(u_j)$$

is used. For later reference we mention that, *cf.* e.g. [7],

$$(2.1) \qquad [D(u_1)\,b]((i,j,k)/(n-1)) = b_{i,j+1,k} - b_{i+1,j,k} .$$

An analogous identity holds for $D(u_0)$ and $D(u_2)$.

The operator D facilitates the test whether b is a convex function over Δ:

(2.2) LEMMA [1, 10] : *b is convex if and only if $D_{0,1}\, b \leq 0$, $D_{1,2}\, b \leq 0$ and $D_{2,0}\, b \leq 0$ over Δ.*

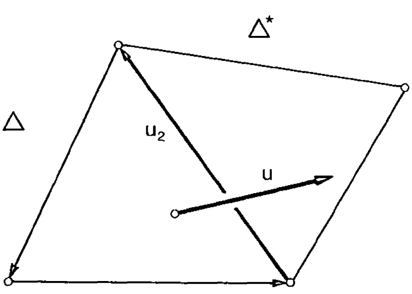

Figure 2. — The triangles Δ and Δ^*.

Of course, $D_{i,j}\, b$ is only defined for $n \geqslant 2$. In case $n = 0$ or $1, b$ is constant or linear and trivially convex.

This Lemma can be generalized to composite Bézier nets. Let Δ^* be a triangle adjacent to Δ such that $\Gamma := \Delta \cap \Delta^*$ is a common edge of Δ and Δ^*. Without loss of generality u_2 is assumed to be the direction of Γ, see figure 2.

Further, let b^* be a Bézier net over Δ^* of the same degree as b such that the composite function over $\Delta \cup \Delta^*$

$$b_{\cup}(x) := \begin{cases} b(x) & \text{if} \quad x \in \Delta \\ b^*(x) & \text{if} \quad x \in \Delta^* \end{cases}$$

is continuous. We observe that $b_{\cup}$ is convex if and only if both b and b^* are convex and for some direction $u = \alpha_1 u_1 + \alpha_2 u_2 \neq 0$, with $\alpha_1 > 0$

$$[D(u)\, b](x) \leqslant [D(u)\, b^*](x)\,, \quad x \in \Gamma\,. \tag{2.3}$$

(This condition appears in [8] for u being the diagonal of the quadrilateral $\Delta \cup \Delta^*$ across Γ.)

Besides (2.2) and the uniqueness of the Bézier net there is another property crucial for the analysis here. Namely, $R = U, E$ preserves positivity, more general

$$\min b \leqslant \min R_m\, b \quad \text{and} \quad \max R_m\, b \leqslant \max b\,. \tag{2.4}$$

Inequalities (2.4) are due to the fact that the Bézier ordinates interpolated by $R_m\, b$ lie in the convex hull of all Bézier ordinates $b_{i,j,k}$, *cf.* [4, 7].

3. SUBDIVISION AND DEGREE ELEVATION PRESERVE CONVEXITY AND MONOTONICITY

With the prerequisites of section 2 there is a quick proof of

(3.1) THEOREM : *Let b be a convex n-th degree Bézier net. Then $U_m\, b$ is convex, for all $m \in \mathbb{N}$ (See* [8] *for $m = 2^{\nu}$).*

Proof : If b is linear, $U_m\, b$ is again linear. Hence, suppose b is non linear. Then by (2.2)

$$D_{0,1}\, b \leqslant 0\,, \quad D_{1,2}\, b \leqslant 0\,, \quad D_{2,0}\, b \leqslant 0$$

and since U preserves negativity, (2.3),

$$U_m\, D_{0,1}\, b \leqslant 0\,, \quad U_m\, D_{1,2}\, b \leqslant 0\,, \quad U_m\, D_{2,0}\, b \leqslant 0\,.$$

U and D commute because every polynomial has a unique Bézier net. Thus

$$D_{0,1}\, U_m\, b \leqslant 0\,, \quad D_{1,2}\, U_m\, b \leqslant 0\,, \quad D_{2,0}\, U_m\, b \leqslant 0\,,$$

i.e., $U_m\, b$ is convex for each $\Delta' \in \Delta_m$.

Note that $D_{i,j}\, U_m\, b$ is to be understood piecewise for each $\Delta' \in \Delta_m$, i.e.,

$$(D_{i,j}\, U_m\, b)\big|_{\Delta'} = D_{i,j}[(U_m\, b)\big|_{\Delta'}] .$$

As mentioned in section 2, $D(u)\, U_m\, b = U_m\, D(u)\, b$ is continuous, $u \in \mathbb{R}^2$ arbitrary. Thus (2.3) holds with equality and implies that $U_m\, b$ is convex over its entire domain Δ. ∎

This proof works in the same way if U is replaced by E, i.e., degree elevation, too, preserves convexity. This was first observed by Chang and Feng [2] who proved this fact without the use of operators.

Moreover, convexity can be replaced by monotonicity in the direction of some $u = \alpha_1 u_1 + \alpha_2 u_2$. The polynomial p is monotonic in the direction u if

$$D(u)\, b = \alpha_1 D(u_1)\, b + \alpha_2 D(u_2)\, b \geqslant 0 . \tag{3.2}$$

(Because of (2.1) this condition means that b is increasing over the triangles in Δ_n parallel to Δ but not necessarily over the other triangles in Δ_n.) Since U and E preserve property (3.2), one has

(3.3) COROLLARY : *Let p be a polynomial whose Bézier net b satisfies* (3.2). *Then $U_m\, b$ and $E_m\, b$ satisfy* (3.2), *too.* ((3.3) *appears in* [8] *for U_m, $m = 2^\nu$.*)

The proof of (3.1) can also be carried over to composite Bézier nets. Rewritting (2.3) gives

$$D(u)[b^* - b](x) \geqslant 0 , \quad \text{for all} \quad x \in \Gamma , \tag{3.4}$$

where the notation of section 2 is used. Then an immediate consequence is

(3.5) COROLLARY (*see also* [8]) : *If b_u is convex, then so also is $R_m\, b_u$, $R = U$, E where*

$$[R_m\, b_u]\,(x) := \begin{cases} [R_m\, b]\,(x) & if \quad x \in \Delta \\ [R_m\, b^*]\,(x) & if \quad x \in \Delta^* . \end{cases}$$

Proof: The convexity of b and b^* is preserved under R because of (3.1). Analogously (3.4) is preserved by R. ∎

4. BOX SPLINE SURFACES

This section shows that Theorem (3.1) is even valid for box spline surfaces. Following the notation of [6] we introduce the subset

$$X_0 := \{e_1, \ldots, e_s, e\} \quad \text{of} \quad R^s$$

where $e_i := (\delta_{ij})_{j=1}^{s}$, $e := e_1 + \cdots + e_s$. Furthermore,

$$X := \{x_1, ..., x_n\}$$

refers to a set of not necessarily distinct vectors in $\mathbb{Z}^s$ and also to the matrix whose columns are $x_1, ..., x_n$. The box spline is defined by the requirement that

$$\int_{R^s} f(x)\, B(x|X)\, dx = \int_{[0,1]} f(Xu)\, du$$

holds for any continuous function f on $\mathbb{R}^s$. For more information confer [5].

A spline function

$$s(x|X) = \sum_{\alpha \in \mathbb{Z}^s} c_\alpha\, B(x - \alpha\,|X)$$

has similar properties as a polynomial with its Bézier representation :

- The hyperplanes $\alpha + \operatorname{span} V$, $\alpha \in Z^s$, $V \subset X_0$, $|V| = s - 1$, form a triangulation of $\mathbb{R}^s$ such that $\mathbb{Z}^s$ forms the set of its vertices. Let T be this triangulation. The piecewise linear function c which is linear over all simplices of T and which has the interpolatory property

$$c(\alpha) = c_\alpha \quad \text{for all} \quad \alpha \in \mathbb{Z}^s$$

is the so-called control net of s.

- As for Bézier nets we introduce a **difference operator** $D(u)$ for $u \in X_0$. Here, D is defined on the space of all box spline control nets by the rule

$$[D(u)\, c]\, (\alpha) = c_\alpha - c_{\alpha - u}\,, \quad \alpha \in \mathbb{Z}^s\,.$$

If $u \in X$ and $s \in C^1(R^s)$, one has

$$\frac{d}{dt} s(x - tu\,|X)\bigg|_{t=0} = \sum_{\alpha \in \mathbb{Z}^s} [D(u)\, c]\, (\alpha)\, B(x - \alpha\,|X - \{u\})\,.$$

The operator D can be used as in (2.2).

(4.1) LEMMA [6] : *Suppose $X_0 \subset X$. Then $c(x)$ is convex if and only if $D(u)\, D(v)\, c \geq 0$ for all u, $v \in X_0$, $u \neq v$.*

- Subdividing box splines means to present $s(x|X)$ over the finer grid $h\mathbb{Z}^s$, $h^{-1} \in \mathbb{N}$ by translates of the scaled box spline $B(x|hX)$, i.e., subdivision means to produce a control net $U(c|X, h)$ such that

$$s(x) = \sum_{\alpha \in hZ^s} U(c|X, h)(\alpha) B(x - \alpha | hX).$$

Note that U has a different meaning in this section.

Let us recall [13, 5] how to generate $U(c|X, h)(\alpha)$, $\alpha \in hZ^s$. The method is best described algorithmically ; here in a slightly different form than elsewhere :

1. Set $d(\alpha) := 0$ for all $\alpha \in h\mathbb{Z}^s$.
2. Set $d(\alpha) := h^{-s} c(\alpha)$ for all $\alpha \in \mathbb{Z}^s$.
3. For $u = x_1, ..., x_n$ set $U(c|X, h)(\alpha) := h \sum_{i=0}^{h^{-1}-1} c(\alpha - ihu)$ for all $\alpha \in h\mathbb{Z}^s$.

The subdivision operator U and the difference operator D commute in the following sense :

(4.2) LEMMA : *Suppose $u \in X$. Then one has*

$$U(D(u) c | X - \{u\}, h) = h^{-1} D(hu) U(c|X, h).$$

The proof is not difficult and omitted.

Lemmata (4.1) and (4.2) and the fact that U preserves positivity establish the analogy to Theorem (3.1) :

(4.3) THEOREM : *Suppose that $X_0 \subset X$. If c is a convex box spline control net, then $U(c|X, h)$ is also convex for all $h^{-1} \in \mathbb{N}$.*

5. ITERATED REFINEMENT

On returning to the notations of sections 1 through 3 there are two powerful and intriguing properties :

(5.1) $U_m b$ converges uniformly to p over Δ as $m \to \infty$, see, e.g., [12].

(5.2) $E_m b$ converges also uniformly to p over Δ as $m \to 0$, see, e.g., [3] for a proof and further references.

With the abbreviations

$$p_u(.) := \frac{d}{dt} p(. + tu)|_{t=0}$$

$$p_{i,j} := (p_{u_i})_{u_j}, \qquad i, j = 0, 1, 2,$$

for the directional derivatives one gets

(5.3) THEOREM : *Suppose* $p_{0,1} < 0$, $p_{1,2} < 0$, *and* $p_{2,0} < 0$. *Then there is some* $m \in \mathbb{N}$ *such that for all* $\mu > m$ $U_\mu b$ and $E_\mu b$ *are convex.* (*In* [8] *this result is proved for* U_μ, $\mu = 2^\nu$.)

Proof: The definition of D, (5.1) and the assumption of (5.3) imply that there exists an $m \in \mathbb{N}$ such that for all $\mu > m$

$$U_\mu D_{i,j} b = D_{i,j} U_\mu b < 0 .$$

i.e., $U_\mu b$ is convex for all $\mu > m$. Similarly $E_\mu b$ is convex for sufficiently large μ. ■

We like to mention that Chang and Feng [2] used that degree elevation preserves convexity and (5.2) to prove that p is convex whenever b is convex. This proof is somewhat involved because of (5.2). The original proof in [1] and other proofs [7, 10] show that the Hessian of p is positive definite. Here we present yet another and even more elementary proof.

Suppose b is convex, i.e., $D_{i,j} b \leqslant 0$, $i \neq j$, see (2.2). Thus we get by the convex hull property, see, e.g., [7], $p_{i,j} \leqslant 0$ and therefore

$$p_{i,i} = -p_{i,j} - p_{i,k} \geqslant 0 , \quad \{i, j, k\} = \{0, 1, 2\} .$$

Every vector $u \in \mathbb{R}^2$ can be written as

$$u = \alpha_i u_i + \alpha_j u_j , \quad i, j \in \{0, 1, 2\} ,$$

where $\alpha_i \alpha_j \leqslant 0$, see figure 3.

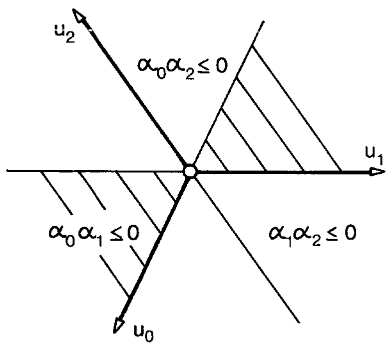

Figure 3. — A partition of $\mathbb{R}^2$.

Thus

$$p_{uu} = \alpha_i^2 p_{i,i} + 2 \alpha_i \alpha_j p_{i,j} + \alpha_j^2 p_{j,j} \geqslant 0 ,$$

i.e., p is convex because u is arbitrary.

6. TWO EXAMPLES

Unfortunately, iterated refinement by U_m or E_m does not always provide a method to obtain a convex approximation of a convex polynomial p.

Consider, e.g., the quadratic polynomial p whose Bézier net b is given by

$$\begin{matrix} & & b_{002} & & \\ & b_{101} & & b_{011} & \\ b_{200} & & b_{110} & & b_{020} \end{matrix} \quad = \quad \begin{matrix} & & 2 & & \\ & -1 & & 0 & \\ 2 & & 0 & & 0 \end{matrix} .$$

On using (2.1) and (1.1) one obtains for $D_{ij}\, b = p_{ij}$

$$p_{01} = -6, \quad p_{12} = 2, \quad p_{20} = -6,$$

which entails by (2.2) and (2.4) that all $U_m\, b$ and $E_m\, b$ are non-convex. On the other hand, p is convex. Namely, every $u \in \mathbb{R}^2 \setminus \{0\}$ can be written as $u = \alpha_1 u_1 + \alpha_2 u_2$. Then

$$p_{uu} = \alpha_1^2 p_{1,1} + 2\,\alpha_1\,\alpha_2 p_{1,2} + \alpha_2^2 p_{2,2} = 2(\alpha_1^2 + \alpha_2^2 + (\alpha_1 + \alpha_2)^2) > 0 .$$

Similar examples exist even for Bézier curves as is shown in the sequel. We assume familiarity with Bézier curves and refer, e.g., to [4] for details. Consider the univariate polynomial

$$q(x) = 1/2 + 1/2(2\,x - 1)^4 = (1 - x)^4 + 6\,x^2(1 - x)^2 + x^4, \quad x \in [0, 1],$$

whose Bézier polygon is non-convex, see figure 4.

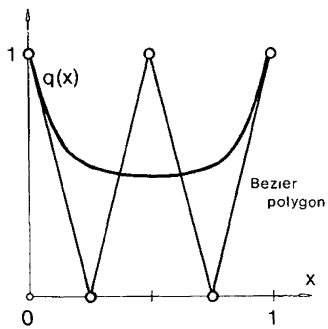

Figure 4. — The Bézier curve $q(x)$.

On the other hand, $q(x)$ is convex since $q''(x) \geq 0$, see figure 5.

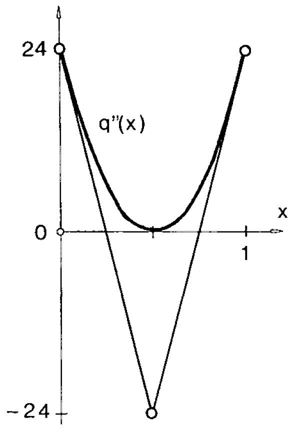

Figure 5. — The Bézier curve $q''(x)$.

Let

$$q(x) = \sum_{i=0}^{n} b_i \binom{n}{i} x^i (1-x)^{n-i}$$

be a degree elevated representation of $q(x)$. Then

$$q''(x) = n(n-1) \sum_{j=0}^{n-2} \Delta^2 b_j \binom{n-2}{j} x^j (1-x)^{n-2-j} ,$$

$$\Delta^2 b_j = b_{j+2} - 2\, b_{j+1} + b_j ,$$

is a degree elevated representation of $q''(x)$. In analogy to (2.2) the Bézier polygon $(b_i, i/u)$, $i = 0, 1, ..., n$ is convex if and only if all $\Delta^2 b_j \geq 0$.

Since $q''(1/2) = 0$ and because of the convex hull property some $\Delta^2 b_j$ must be non-positive. Moreover, degree elevation is a corner cutting procedure, i.e., in order to « degree elevate » a Bézier polygon one has to cut all of its corners. As a consequence, some $\Delta^2 b_j$ must even be negative. Otherwise further degree elevation would yield strictly positive Bézier ordinates. Hence, the Bézier polygon $(b_i, i/n)$, $i = 0, ..., n$, cannot be convex.

Similar examples can be given with subdivision instead of degree elevation. Obviously, the Bézier polygon of a convex polynomial stays non-convex under degree elevation or subdivision only if p'' has a zero in (0, 1).

7. CONSTRUCTING A CONVEX BÉZIER NET FOR A CONVEX QUADRATIC

In this section p denotes a convex quadratic, i.e., we may assume without loss of generality

$$0 \leqslant p_{00} \leqslant p_{11} \leqslant p_{22}. \tag{7.1}$$

Then, because of

$$p_{00} = p_{11} + 2\, p_{12} + p_{22} \tag{7.2}$$

and

$$p_{11} = p_{00} + 2\, p_{02} + p_{22}$$

one has

$$p_{12} \leqslant 0 \quad \text{and} \quad p_{02} \leqslant 0\,, \tag{7.3}$$

Figure 6 shows this situation. There $p(d_0) = p_0(d_0) = p_1(d_0) = 0$ is assumed, but this does not mean a loss of generality since only second derivatives matter here.

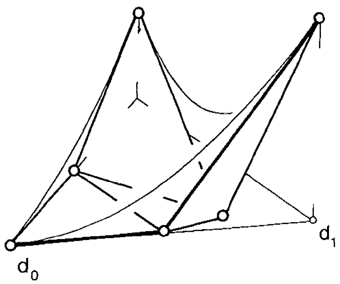

Figure 6. — A convex quadratic with a non-convex Bézier net.

As was shown, the Bézier net b of p and all nets $U_m b$, $m \in \mathbb{N}$, are not convex if $p_{01} > 0$. Nevertheless, one can subdivide b such that a convex composite Bézier net is obtained for p over Δ.

So, suppose $p_{01} > 0$. As a consequence

$$p_{12} < 0 \quad \text{and} \quad p_{02} < 0 \tag{7.4}$$

since otherwise one could slightly perturb Δ into a new triangle Δ^* such that « $p_{12} > 0$ » (or « $p_{02} > 0$ » respectively) and still have « $p_{0i} > 0$ » which contradicts (7.3).

On introducing the notation $B(p \mid T)$ for the Bézier net of p with respect to the triangle T and $\langle r_1, \ldots, r_m \rangle$ for the convex hull of the points $r_1, \ldots, r_m$ one has

(7.5) THEOREM : *There is a unique point* $c \in (d_1, d_2)$ *such that* $B(p \mid \langle d_0, d_2, c \rangle)$ *and* $B(p \mid \langle d_0, d_1, c \rangle)$ *are convex, see figure* 7.

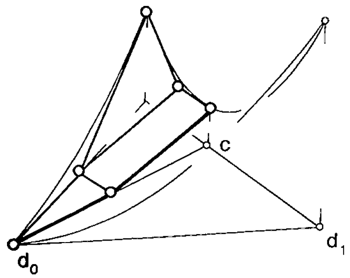

Figure 7. — *b* after the subdivision along $[c, d_0]$.

Proof : For better readibility we introduce the notation

$$D(u)\,p := p_u\,.$$

Note that the operator D is now used in two different contexts, for Bézier nets and also for polynomials. Then let

$$f(\lambda) = D(u_2)\,D(u_1 + \lambda u_2)\,p = p_{21} + \lambda p_{22}, \quad \text{i.e.},$$

$$f(0) = p_{21} < 0 \quad \text{and} \quad f(1) = -p_{20} > 0\,.$$

Hence, the linear function $f(\lambda)$ has a zero λ_0 in (0, 1) which corresponds to the point $c = d_1 + \lambda_0 u_2$. For the triangle $\langle d_2, d_0, c \rangle$ one gets $p_{20} \leq 0$, $D(u_2)\ D(c - d_0)\,p = 0$. Which implies by the contraposition of (7.4) $D(u_0)\,D(c - d_0)\,p \leq 0$. Thus $B(p \mid \langle d_2, d_0, c \rangle)$ is convex and similarly is $B(p \mid \langle d_0, d_1, c \rangle)$. ∎

We will call the Bézier net b over Δ strictly convex if the strict inequalities

$$D_{01}\,b < 0\,, \quad D_{12}\,b < 0\,, \quad D_{02}\,b < 0$$

hold. For example, the two Bézier nets produced in Theorem (7.5) are not strictly convex. Hence, it might be impossible to conclude numerically that these two nets are convex. Fortunately, one can overcome this difficulty. First we observe from (7.5) by some continuity arguements.

(7.6) COROLLARY : *Suppose* $D(c - d_0)\,D(c - d_0)\,p > 0$. *Then there exists a* $c' \in (c, d_1)$ *such that for all* $d \in (c, c')$ *the net* $B(p \mid \langle d, d_0, d_2 \rangle)$ *is convex. (The net* $B(p \mid \langle d, d_1, d_0 \rangle)$ *is not convex.)*

One can continue subdividing $B(p \mid \langle d, d_0, d_1 \rangle)$ as indicated by (7.6) thereby producing more and more strictly convex Bézier nets for p over a region which will fill out Δ in the limit.

Figure 8 depicts the triangulation of Δ associated with such an iterated subdivision. The triangulation shown in figure 8 is quite simple to compute because the edges $\langle d_0\, d_2\rangle$, $\langle d, d'\rangle$, $\langle d'', d'''\rangle$, ... are all parallel and also the edges $\langle d_0\, d\rangle$, $\langle d', d''\rangle$, ...

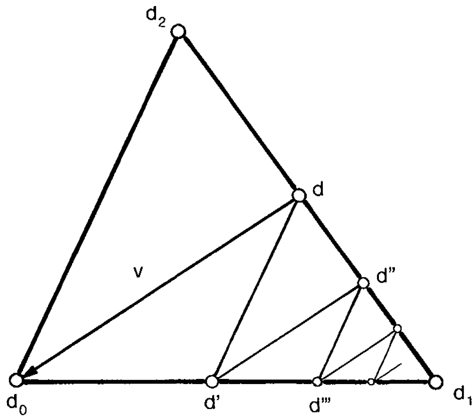

Figure 8. — A subdivision of Δ produced by iterated applications of (7.6).

The following theorem verifies that one can always choose such a simple triangulation.

(7.7) THEOREM : *Let* $d \in (d_1, d_2)$, $v := d_0 - d$, *and* p *such that* $B(p \mid \langle d_0, d, d_2\rangle)$ *is strictly convex, i.e.,* $D(u_0)\, D(-v)\, p$, $D(-v)\, D(u_2)\, p$, *and* $p_{0,2} < 0$. *Also, let* $p_{0,1} > 0$ *and* $p_{1,2} < 0$. *If* d', d'', ... *are as in figure* 8, *then* $B(p \mid \langle d_0, d, d'\rangle)$ *is also strictly convex.*

Because of parallelism all nets $B(p \mid \langle d^{(m)}, d^{(m+1)}, d^{(m+2)}\rangle)$, $m \in \mathbb{N} \cup \{0\}$, $d = d^{(0)}$, are also convex.

Proof: The three relevant derivatives are, *cf.* (2.2) :

(i) $D(v)\, D(u_1)\, p$ which is negative : Since $D(u_2)\, D(v)\, p > 0$ and $p_{1,2} < 0$ by assumption, one can use (7.4) with respect to $\langle d_0, d_1, d\rangle$ and gets $D(v)\, D(u_1)\, p < 0$.

(ii) $D(u_1)\, D(-u_0)\, p = -p_{0,1} < 0$.

(iii) $D(-u_0)\, D(v)\, p = D(u_0)\, D(-v)\, p < 0$. ■

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