M2AN. MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS - MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

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M2AN. Mathematical modelling and numerical analysis - Modélisation mathématique et analyse numérique, tome 25, nº 6 (1991), p. 783-807

<http://www.numdam.org/item?id=M2AN_1991__25_6_783_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

(Vol. 25, n° 6, 1991, p. 783 à 807)

THE *h-p* VERSION OF THE BOUNDARY ELEMENT METHOD ON POLYGONAL DOMAINS WITH QUASIUNIFORM MESHES (*)

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Communicated by J. DOUGLAS

Abstract. — *We investigate the h-p version of the boundary element method for integral equation formulations for PDEs over polygonal domains where both the mesh size h and the polynomial degree p are changed to improve accuracy. Under the assumption of quasiuniform meshes, we obtain estimates for the rate of convergence which show the effects on the error of changing h and p either separately or together. Using precise results for the singular behaviour of the solution near corners of the domain, it is shown that the rate of convergence for the p-version (h fixed) is twice that of the h version (p fixed) for most problems, which agrees with computational results reported in the literature. Interior estimates are also derived.*

Résumé. — *Nous étudions la version h-p des éléments finis pour la méthode intégrale appliquée à la résolution d'équations aux dérivées partielles sur des domaines polygonaux. Le pas de maillage h ainsi que le degré des polynômes p sont modifiés afin d'améliorer la précision. Pour des maillages quasi uniformes, nous obtenons des estimations de convergence qui montrent l'incidence sur l'erreur si on change h et p en même temps ou séparément.*

En utilisant des résultats précis sur la singularité de la solution près des coins du domaine, on montre que la vitesse de convergence de la version p (pour h fixé) est deux fois supérieure à celle de la version (pour p fixé) pour la plupart des problèmes, ce qui confirme les résultats numériques rencontrés dans la littérature. On en déduit des estimations pour l'intérieur du domaine.

(*) Received February 1990.

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(**) The first author was supported by the National Science Foundation under Grants DMS-8603954 and DMS-8704463 and the second author was supported by the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under Grant AFOSR-85-0322. This paper was partly written while the first author was visiting at IPST, University of Maryland, College Park, Maryland.

1. INTRODUCTION

The boundary element method (BEM) has received a tremendous amount of attention by researchers in terms of both its theoretical aspects and applications. Most research so far has been carried out in the framework of the h-version, in which accuracy is achieved by decreasing the mesh size h of elements on the boundary while keeping the degree p of piecewise polynomials used fixed (usually at a low level, $p = 1$ or 2). For this method, several detailed results, including asymptotic rates of convergence for both first-kind and second-kind integral equations are well-known (see [14], [24] for example).

The basic idea of the above convergence proofs is the observation that for strongly elliptic pseudodifferential operators one obtains quasioptimal convergence in the energy norm for any Galerkin scheme with conforming boundary elements (see [21]). This result can also be used to analyze the recently introduced p- and h-p versions for BEM (see [1], [2], [3], [27], [20], [21]). In the p-version, a fixed mesh with constant h is used and accuracy is achieved by increasing the degrees p of the polynomials used. The h-p version combines the two approaches. These two extension processes were first analyzed theoretically for the finite element method in [8] and [6] respectively, in 1981. Since then, their basic properties have been rigorously established for finite element methods (see [4] for a survey) and they have been translated into industrial codes like MSC\PROBE.

In [21], we presented the p-version of BEM for some first-kind integral equations arising from two-dimensional screen Neumann and Dirichlet problems in acoustics. For these problems, sharp regularity results from [23], [24] showed that near 0, an end of the obstacle, the solutions behaved, respectively, as $r^{1/2}$ and $r^{-1/2}$ where r was the distance from 0. Using this knowledge, it was shown that the rate of convergence using the p-version in the $\tilde{H}^{1/2}$ and $\tilde{H}^{-1/2}$ norms respectively was twice that of the corresponding h-version.

It turns out that for integral equation formulations for PDEs over polygonal domains, the solution over any piece of the boundary can be shown to behave like r^{α}, where $\alpha > 1/2$ for the Neumann problem and $\alpha > -1/2$ for the Dirichlet problem (see [12], [14]). In this paper, our first goal is to extend the results from [21] (which were restricted to the special cases $\alpha = 1/2$ and $\alpha = -1/2$) to the case for general α. We show in Section 3 that the p-version once again yields a theoretical rate of convergence which is twice that of the h-version. Recently some computations have been reported by E. Rank on the p-version for a model integral equation problem ([20]). In Section 3, we discuss some of his computational results in terms of our theoretical work.

In [21], we dealt only with the p-version. Our second goal here is to analyze the more general h-p version and derive estimates for the rate of convergence when arbitrary combinations of both p and h are used, under the assumption that a quasiuniform mesh is used. From these estimates, the rates for the p and the h versions can be obtained by fixing either h or p to be a constant. Using the estimates for the solution of the integral equation on the boundary, we will also obtain asymptotic error rates for the solution in the interior of the domain.

Finally, in [21], we dealt only with the pure Neumann and pure Dirichlet problems. We analyze here the more general mixed problem from which these follow as special cases.

2. PRELIMINARIES

We consider the mixed boundary value problem for the Laplacian : *Find* $u \in H^1(\Omega)$ *satisfying*

$$\begin{aligned} \Delta u &= 0 \; in \; \Omega \\ u &= g_1 \; on \; \Gamma_1 \\ \frac{\partial u}{\partial n} &= g_2 \; on \; \Gamma_2 \end{aligned} \tag{2.1}$$

where $g_1 \in H^{1/2}(\Gamma_1)$, $g_2 \in H^{-1/2}(\Gamma_2)$.

We assume Ω to be a bounded plane domain with a polygonal boundary $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 = \bigcup_{j=1}^{J} \bar{\Gamma}^j$, Γ^j being open straight line segments. In the following we always assume $\Gamma_1 \neq 0$. By t_j ($j = 0, \dots, J$) we denote the corner points where Γ^j and Γ^{j+1} meet ($t_J = t_0$). The interior angle at t_j is denoted by ω_j. We assume $\omega_j \neq 0$ or 2π. Let D, N, and M be the subsets of $\{1, \dots, J\}$ for which $t_j \in \Gamma_1$, $t_j \in \Gamma_2$, or $t_j \in \bar{\Gamma}_1 \cap \bar{\Gamma}_2$, respectively. $\frac{\partial u}{\partial n}$ means the normal derivative with respect to the outer normal $\vec{n}$, which exists outside the corners. The definition of Sobolev spaces is as usual [18] :

$$H^s(\Omega) = \{u|_\Omega : u \in H^s(\mathbb{R}^2)\} \quad (s \in \mathbb{R}) . \tag{2.2}$$

$$H^s(\Gamma) = \begin{cases} \{u|_\Gamma : u \in H^{s+1/2}(\mathbb{R}^2)\} & (s > 0) \\ L^2(\Gamma) & (s = 0) \\ (H^{-s}(\Gamma))' \text{ (dual space)} & (s < 0) \end{cases} \tag{2.3}$$

$$H^s(\Gamma_j) = \{u|_{\Gamma_j} : u \in H^s(\Gamma)\} \quad (s \geq 0) \quad (j = 1, 2 \text{ ; similarly for } \Gamma^j) \tag{2.4}$$

$$\tilde{H}^s(\Gamma_j) = \{u \in H^s(\Gamma_j) : u^* \in H^s(\Gamma)\} \quad (s \geq 0) . \tag{2.5}$$

Here $u^* = \begin{cases} u \text{ on } \Gamma_j \\ 0 \text{ on } \Gamma\backslash\bar{\Gamma}_j \end{cases}$ means the continuation of u by 0 outside Γ_j.

Finally,

$$H^s(\Gamma_j) = (\tilde{H}^{-s}(\Gamma_j))' \quad (s < 0) \tag{2.6}$$

$$\tilde{H}^s(\Gamma_j) = (H^{-s}(\Gamma_j))' \quad (s < 0) . \tag{2.7}$$

In [13] the boundary value problem (2.1) has been converted by the direct method to a system of boundary integral equations on Γ for the unknown Cauchy data $v := u$ on Γ_2 and $\psi := \frac{\partial u}{\partial n}$ on Γ_1. In the direct method one uses the representation formula for the solution of (2.1) arising from Green's formula which contains the partly given and partly unknown Cauchy data on the boundary. By taking boundary values in this representation, one finds a relation between the Cauchy data which is a system of integral (or rather pseudodifferential) equations on the boundary. This is then used to determine the unknown Cauchy data from the prescribed data. For the solution of (2.1) the representation formula is

$$2\,u(z) = -\int_\Gamma u(\zeta)\frac{\partial}{\partial n_\zeta} G(z,\zeta)\,ds_\zeta + \int_\Gamma \frac{\partial u}{\partial n}(\zeta)\,G(z,\zeta)\,ds_\zeta \quad \text{for} \quad z \in \Omega\,, \tag{2.8}$$

where $\frac{1}{2} G(z,\zeta)$ is the fundamental solution $-\frac{1}{2\pi}\log|z-\zeta|$ of the Laplace equation.

Taking the limit of $u(z)$ for $z \in \Gamma_2$ and the normal derivative $\frac{\partial u}{\partial n}(z)$ for $z \in \Gamma_1$ in this formula and using the jump relations, one finds the system

$$\begin{bmatrix} D_{22} & K'_{12} \\ -K_{21} & V_{11} \end{bmatrix}\begin{bmatrix} v \\ \psi \end{bmatrix} = \begin{bmatrix} -D_{12} & 1-K'_{22} \\ 1+K_{11} & -V_{21} \end{bmatrix}\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \tag{2.9}$$

where the subscripts in D_{jk}, etc., mean integration over Γ_j and evaluation on Γ_k. The integral operators arising in (2.9) are given by :

$$Vf(z) = \int_\Gamma f(\zeta)\,G(z,\zeta)\,ds_\zeta\,, \quad Kf(z) = \int_\Gamma f(\zeta)\frac{\partial}{\partial n_\zeta}G(z,\zeta)\,ds_\zeta$$

$$K'f(z) = \int_\Gamma f(\zeta)\frac{\partial}{\partial n_z}G(z,\zeta)\,ds_\zeta\,,$$

$$Df(z) = -\frac{\partial}{\partial n_z}\int_\Gamma f(\zeta)\frac{\partial}{\partial n_\zeta}G(z,\zeta)\,ds_\zeta\,.$$

The system (2.9) is a system of first kind integral equations for v and ψ. As shown in [13] this system satisfies a Gårding inequality. The natural bilinear form associated with (2.9) is equivalent to the « energy norm », i.e., the natural norm for the Cauchy data of the weak (or variational) solution of problem (2.1). Making use of localization and the Mellin transformation one obtains in [13] the following results :

THEOREM 2.1 : *Let* $s \in \left(-\frac{1}{2}, \frac{3}{2}\right)$. *The operator*

$$A = \begin{bmatrix} D_{22} & K'_{12} \\ -K_{21} & V_{11} \end{bmatrix} \text{ maps } \begin{matrix} \tilde{H}^s(\Gamma_2) \\ \times \\ \tilde{H}^{s-1}(\Gamma_1) \end{matrix} \text{ continuously into } \begin{matrix} H^{s-1}(\Gamma_2) \\ \times \\ H^s(\Gamma_1) \end{matrix} . \tag{2.10}$$

In order to use Theorem 2.1 to obtain information on the solvability of the system (2.9) we rewrite (2.9) by substituting $v = v^* + \ell g_1$, $\psi = \psi^* + \ell g_2$ with arbitrary extensions $\ell g_1 \in H^{1/2}(\Gamma)$ and $\ell g_2 \in H^{-1/2}(\Gamma)$. Thus (2.9) becomes

$$A \begin{bmatrix} v^* \\ \psi^* \end{bmatrix} = \begin{bmatrix} -D_{\Gamma 2} & (1 - K'_{\Gamma 2}) \\ (1 + K_{\Gamma 1}) & -V_{\Gamma 1} \end{bmatrix} \begin{bmatrix} \ell g_1 \\ \ell g_2 \end{bmatrix} =: \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \tag{2.11}$$

where $D_{\Gamma 2}$ for example denotes integration on Γ and evaluation on Γ_2. From [13], [14] follows $f_1 \in H^{s-1}(\Gamma_2)$, $f_2 \in H^s(\Gamma_1)$ for $g_1 \in H^s(\Gamma_1)$ and $g_2 \in H^{s-1}(\Gamma_2)$ for any $s \in (-1/2, 3/2)$.

THEOREM 2.2 : *There exists a constant* $\gamma > 0$ *and a compact operator* $C : \tilde{H}^{1/2}(\Gamma_2) \times \tilde{H}^{-1/2}(\Gamma_1) \to H^{-1/2}(\Gamma_2) \times H^{1/2}(\Gamma_1)$ *such that for all* $U = (v, \psi) \in \tilde{H}^{1/2}(\Gamma_2) \times \tilde{H}^{-1/2}(\Gamma_1)$ *there holds*

$$\operatorname{Re} \langle (A + C)\, U, \bar{U} \rangle \geq \gamma \left(\| v \|^2_{\tilde{H}^{1/2}(\Gamma_2)} + \| \psi \|^2_{\tilde{H}^{-1/2}(\Gamma_1)} \right) . \tag{2.12}$$

Here $\langle \cdot, \cdot \rangle$ means the natural duality between $\tilde{H}^{1/2}(\Gamma_2) \times \tilde{H}^{-1/2}(\Gamma_1)$ and $H^{-1/2}(\Gamma_2) \times H^{1/2}(\Gamma_1)$, which is, for smooth functions $U = (v, \psi)$ and $W = (f_1, f_2)$, given by

$$\langle W, U \rangle := \int_{\Gamma_2} f_1 \, v \, ds + \int_{\Gamma_1} f_2 \, \psi \, ds .$$

Since the problem (2.1) has no eigensolutions we obtain from Theorems 2.1 and 2.2 the existence of the solutions of (2.11) and (2.9).

THEOREM 2.3 : *Let* $g_1 \in H^{1/2}(\Gamma_1)$, $g_2 \in H^{-1/2}(\Gamma_2)$ *with arbitrary extensions* ℓg_j *of* g_j $(j = 1, 2)$. *Then there exists exactly one solution* $v^* \in \tilde{H}^{1/2}(\Gamma_2)$, $\psi^* \in \tilde{H}^{-1/2}(\Gamma_1)$ *of* (2.11). *Furthermore, let* $v := v^* + \ell g_1|_{\Gamma_2}$ *and* $\psi := \psi^* + \ell g_2|_{\Gamma_1}$, *then* $v \in H^{1/2}(\Gamma_2)$, $\psi \in H^{-1/2}(\Gamma_1)$ *solve* (2.9).

It is well-known that for a polygonal domain Ω the solution u of the mixed b.v.p. (2.1) has unbounded gradients near the corners where the boundary conditions change — even for smooth boundaries. The same phenomenon holds for the layer (v, ψ) of our system of integral equations (2.9). In order to obtain improved error estimates for the Galerkin's method we need more information about the regularity of the solution, for example its local expansion in terms of singularity functions near corners. As indicated in [12], [13], [14] the Mellin transform together with the Cauchy integral theorem for analytic functions gives an expansion of (v, ψ) in terms of singularity functions which are explicitly given below. By shifting the path of integration for the inverse Mellin transform and computing the residues at the zeros of det $\hat{A}(\lambda)$ we obtain the following regularity result. For its formulation we introduce some notation.

For localization we need a partition of unity $(\chi_1, \ldots, \chi_J)$ with the following properties :

$$\chi_j \text{ is the restriction of a } C_0^\infty(\mathbb{R}^2)\text{-function to } \Gamma$$

$$\chi_j \equiv 1 \text{ in a neighborhood of the vertex } t_j, \text{ and supp } \chi_j \subset \Gamma^j \cup \{t_j\} \cup \Gamma^{j+1}.$$

For every function on Γ we then have $u = \sum_{j=1}^{J} \chi_j u$ so that $\chi_j u$ is the « local representation » of u at t_j. Each $\chi_j u$ has its support on the set $S_j := \Gamma^j \cup \{t_j\} \cup \Gamma^{j+1}$. By means of an affine transformation of variables S_j can be considered as a part of the set $\Gamma^{\omega_j} := \Gamma_- \cup \{0\} \cup \Gamma_+$ where $\Gamma_- = e^{i\omega_j}\mathbb{R}_+$ corresponds to Γ^j and $\Gamma_+ = \mathbb{R}_+$ corresponds to Γ^{j+1}. Thus, $\chi_j u$ can be considered in a natural way as a function on Γ^{ω_j} and thus also as a pair $((\chi_j u)_-, (\chi_j u)_+)$ of functions on $\mathbb{R}_+$.

With this notation, we may write our solution of (2.9) as

$$\begin{bmatrix} v \\ \psi \end{bmatrix} = \sum_{j=1}^{J} \begin{bmatrix} (\chi_j v)_-, (\chi_j v)_+ \\ (\chi_j \psi)_-, (\chi_j \psi)_+ \end{bmatrix}$$

where the components of v along Γ_1 and ψ along Γ_2 are understood to be zero.

Define for a positive integer k

$$\alpha_{jk} := \begin{cases} \dfrac{k\pi}{\omega_j} & \text{for } j \in D \cup N \text{ and } \omega_j \neq \pi \\ \dfrac{2k-1}{2}\dfrac{\pi}{\omega_j} & \text{for } j \in M \text{ and } \omega_j \notin \left\{\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right\}. \end{cases} \tag{2.13}$$

Let x denote the arc length along Γ^j or Γ^{j+1} with origin at t_j. Then we define the following (local) singular functions:

For $j \in D$ and $\omega_j \neq \pi$:

$$u_{jk} = \alpha_{jk}\begin{bmatrix} 0 & 0 \\ x^{\alpha_{jk}-1} & (-1)^{k+1}\, x^{\alpha_{jk}-1} \end{bmatrix}. \tag{2.14a}$$

For $j \in N$ and $\omega_j \neq \pi$:

$$u_{jk} = \begin{bmatrix} x^{\alpha_{jk}} & (-1)^k\, x^{\alpha_{jk}} \\ 0 & 0 \end{bmatrix}. \tag{2.14b}$$

For $j \in M$ and $\omega_j \notin \left\{\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right\}$:

(i) $j \in M_2$, i.e., $\Gamma^j \subset \Gamma_2$ and $\Gamma^{j+1} \subset \Gamma_1$:

$$u_{jk} = \begin{bmatrix} 0 & (-1)^k\, x^{\alpha_{jk}} \\ \alpha_{jk}\, x^{\alpha_{jk}-1} & 0 \end{bmatrix}. \tag{2.14c}$$

(ii) $j \in M_1$, i.e., $\Gamma^j \subset \Gamma_1$ and $\Gamma^{j+1} \subset \Gamma_2$: as for $j \in M_2$ but the two components of each vector are interchanged.

By A we denote the set of all exceptional exponents defined above, i.e.,

$$A = \{\alpha_{jk} : j \in \{1, 2, \dots, J\}, k \in \mathbb{N}\}.$$

THEOREM 2.4: *Let $g_1 \in H^s(\Gamma_1)$, $g_2 \in H^{s-1}(\Gamma_2)$ for $s \in \left[\frac{1}{2}, \frac{3}{2}\right) \big/ A$. Then the solution of* (2.11) *has the form*

$$\begin{bmatrix} v^* \\ \psi^* \end{bmatrix} = \left[\sum_{j=1}^{J} \sum_{\alpha_{jk} < s - 1/2} c_{jk} \begin{bmatrix} v_{jk} \\ \psi_{jk} \end{bmatrix} \chi_j\right] + \begin{bmatrix} v_s \\ \psi_s \end{bmatrix} \tag{2.15}$$

with $(v_s, \psi_s) \in \tilde{H}^s(\Gamma_2) \times \tilde{H}^{s-1}(\Gamma_1)$, *with* $\begin{bmatrix} v_{jk} \\ \psi_{jk} \end{bmatrix} = u_{jk}$ *as defined above,* $c_{jk} \in \mathbb{R}$ *and* $\{\chi_j : j = 1, \dots, J\}$ *a collection of cut-off functions. Furthermore* *with some constant* C

$$\|v_s\|_{\tilde{H}^s(\Gamma_2)} + \|\psi_s\|_{\tilde{H}^{s-1}(\Gamma_1)} + \sum_{j=1}^{J} \sum_{\alpha_{jk} < s - 1/2} |c_{jk}|^2 \leq$$

$$\leq C(\|g_1\|_{H^s(\Gamma_1)} + \|g_2\|_{H^{s-1}(\Gamma_2)}) . \qquad (2.16)$$

Remark 2.1 : The mixed boundary value problem (2.1) has been studied in [14], using a second kind integral equation for v instead of the first line in (2.9), leading to the system

$$\begin{bmatrix} 1 + K_{22} & -V_{12} \\ -K_{21} & V_{11} \end{bmatrix} \begin{bmatrix} v \\ \psi \end{bmatrix} = \begin{bmatrix} -K_{12} & V_{22} \\ 1 + K_{11} & -V_{21} \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} . \qquad (2.17)$$

This is obtained by taking the limit of $u(z)$ for $z \in \Gamma_1$ instead of $\frac{\partial u}{\partial n}$. This system has to be modified in order to satisfy a Gårding inequality, whereas system (2.9) does this without modification.

In the following we will perform the Galerkin method for system (2.11).

The key of the error analysis of Galerkin's method is the following result by Hildebrandt and Wienholtz [17].

THEOREM 2.5 : *Let H be a Hilbert space with dual H' (not necessarily identified with H) and let $\mathscr{A}$ be injective and continuous from H into H' satisfying a Gårding inequality. Let $u \in H$ denote the solution of*

$$\mathscr{A}u = f \qquad (2.18)$$

where $f \in H'$ and let $u_N \in S_N \subset H$ denote the solution of the Galerkin equations

$$\langle \mathscr{A}u_N, v \rangle = \langle f, v \rangle \text{ for all } v \in S_N \subset H . \qquad (2.19)$$

Furthermore let for any $\phi \in H$ there exist a sequence $\{\phi_N\}$, $\phi_N \in S_N$ with $\phi = \lim_{N \to \infty} \phi_N$ in H. Then for N large enough the Galerkin equations (2.19) *are uniquely solvable and there exists a constant C independent of u, u_N, and N such that*

$$\|u - u_N\| \leq C \inf \{\|u - v_N\| : v_N \in S_N\} \qquad (2.20)$$

where $\|\cdot\|$ denotes the norm in H.

In order to perform the Galerkin method defined above, we need a family of finite-dimensional subspaces $\{\mathscr{V}_N\} \equiv \{\mathscr{V}_{p,h}(\Gamma)\}$ defined on Γ. These are constructed as follows. Let for each $\Gamma^j \subset \Gamma$, there be given a family of grids $\{g_h^j\}$ which partition each Γ^j into N_h^j pieces, $\bar{\Gamma}^j = \bigcup_{i=1}^{N_h^j} \bar{\Gamma}_{h,i}^j$ such that $\Gamma_{h,i}^j$ is an open interval with end points $A_{h,i-1}^j$, $A_{h,i}^j$. We assume that $\{g_h^j\}$ is quasiuniform, in the sense that with $h_i^j = \text{meas}\,(\Gamma_{h,i}^j)$ and $h = \max_{i,j} h_i^j$, there exists a constant τ independent of h such that

$$\frac{h}{h_i^j} \leqslant \tau \quad \text{for all} \quad \Gamma_{h,i}^j\,. \tag{2.21}$$

For any interval I, $\mathscr{P}_p(I)$ will denote the set of all polynomials of degree $\leqslant p$ on I. Then for $p \geqslant 0$, $S_{p,h}(\Gamma^j)$ will denote the set of all functions v defined on Γ^j such that the restriction $v|_{\Gamma_{h,i}^j}$ to $\Gamma_{h,i}^j \subset \Gamma^j$ belongs to $\mathscr{P}_p(\Gamma_{h,i}^j)$. Also, $S_{p,h}^0(\Gamma^j)$ denotes those functions in $S_{p,h}(\Gamma^j)$ which are continuous over Γ^j. Next, we define for $p \geqslant 0$, $h > 0$, $V_{p,h}(\Gamma_1)$ to be the set of functions on Γ_1 whose restrictions to $\Gamma^j \subset \Gamma^1$ belong to $S_{p,h}(\Gamma^j)$. Also, for $p \geqslant 1$, $V_{p,h}^0(\Gamma_2)$ will denote those continuous functions on Γ_2 whose restrictions to $\Gamma^j \subset \Gamma_2$ belong to $S_{p,h}^0(\Gamma^j)$ and which vanish at the two end points of Γ_2 if $\Gamma \neq \Gamma_2$. We then define for $p \geqslant 1$

$$\mathscr{V}_{p,h}(\Gamma) = \{(w_{p,h}, \phi_{p,h}) : w_{p,h} \in V_{p,h}^0(\Gamma_2), \phi_{p,h} \in V_{p-1,h}(\Gamma_1)\}\,.$$

Note that for $p \geqslant 1$, $V_{p,h}^0(\Gamma_2) \subset \tilde{H}^{1/2}(\Gamma_2)$ and $V_{p-1}(\Gamma_1) \subset \tilde{H}^{-1/2}(\Gamma_1)$ so that

$$\mathscr{V}_{p,h}(\Gamma) \subset \tilde{H}^{1/2}(\Gamma_2) \times \tilde{H}^{-1/2}(\Gamma_1)\,.$$

With the above definition of our approximate subspaces, the h-p version *Galerkin method* for (2.9) reads : *Find*

$$U_{p,h} = \begin{bmatrix} v_{p,h} \\ \psi_{p,h} \end{bmatrix} \in \mathscr{V}_{p,h}(\Gamma)$$

such that with $F = (f_1, f_2) \in H^{-1/2}(\Gamma_2) \times H^{1/2}(\Gamma_1)$ *given by the right hand side in* (2.11), *for all* $\Phi = \begin{bmatrix} w \\ \phi \end{bmatrix} \in \mathscr{V}_{p,h}(\Gamma)$ *there holds*

$$\left\langle \begin{bmatrix} D_{22} & K'_{12} \\ -K_{21} & V_{11} \end{bmatrix} \begin{bmatrix} v_{p,h} \\ \psi_{p,h} \end{bmatrix}, \begin{bmatrix} w \\ \phi \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} w \\ \phi \end{bmatrix} \right\rangle \tag{2.22}$$

or in short,

$$\langle AU_{p,h}, \Phi \rangle = \langle F, \Phi \rangle\,. \tag{2.23}$$

3. THE RATE OF CONVERGENCE OF THE *h-p* METHOD

By Theorem 2.5, it is seen that the rate of convergence of the *h-p* method is determined entirely by the question of how well the solution can be approximated by functions in $\mathcal{V}_{p,h}(\Gamma)$. Using Theorem 2.4, it is seen that we must consider the approximation of the singular functions ψ_{jk} and v_{jk} and of the smooth functions ψ_s and v_s in appropriate spaces.

In the sequel, we will denote $I = [0, 1]$ and $I_h = [0, h]$. First we need to introduce appropriate norms on the spaces used.

By Theorems 9.1 and 9.2 in [18] and Theorem 2 on p. 318 in [25] the spaces $H^s(\Gamma^j)$, $\tilde{H}^s(\Gamma^j)$ can be obtained as real interpolation spaces with $s \geq 0$, $k \geq s$, $k \in \mathbb{N}$

$$H^s(\Gamma^j) = [L^2(\Gamma^j), H^k(\Gamma^j)]_{(s/k,2)}$$
$$\tilde{H}^s(\Gamma^j) = [L^2(\Gamma^j), \mathring{H}^k(\Gamma^j)]_{(s/k,2)}$$

where $\mathring{H}^k(\Gamma^j)$ is the closure of $C_0^\infty(\Gamma^j)$ in the norm of $H^k(\Gamma^j)$. Therefore we can also use on $H^s(\Gamma^j)$, $\tilde{H}^s(\Gamma^j)$ the norms which are defined by interpolation. In the following we use these norms for $H^s(\Gamma^j)$, $\tilde{H}^s(\Gamma^j)$ and also for $H^s(\Gamma^j_{h,i})$ $\tilde{H}^s(\Gamma^j_{h,i})$ and $H^s(I_h)$, $\tilde{H}^s(I_h)$. The latter spaces are for $s \geq 0$, $k \geq s$, $k \in \mathbb{N}$ defined as (for example)

$$H^s(\Gamma^j_{h,i}) := [L^2(\Gamma^j_{h,i}), H^k(\Gamma^j_{h,i})]_{(s/k,2)}$$
$$\tilde{H}^s(\Gamma^j_{h,i}) := [L^2(\Gamma^j_{h,i}), \mathring{H}^k(\Gamma^j_{h,i})]_{(s/k,2)}$$

with the induced interpolation norms. For $s < 0$ we define

$$H^s(\Gamma^j_{h,i}) := (\tilde{H}^{-s}(\Gamma^j_{h,i}))', \; \tilde{H}^s(\Gamma^j_{h,i}) := (H^{-s}(\Gamma^j_{h,i}))' .$$

Remark 3.1 : It is crucial for Lemma 3.3 and 3.4 to define the norms by interpolation. The norms induced on $\Gamma^j_{h,i}$ by (2.4), (2.5) would give rise to equivalent norms, but with equivalence constants depending on h. Therefore in Lemma 3.3 and 3.4 one would obtain constants C depending on h.

The following scaling results will be needed.

LEMMA 3.1 : *Let $\hat{v}$ be a function defined on I, $\hat{v} \in \tilde{H}^{1/2}(I)$. Then the function v defined on I_h by $v(x) = \hat{v}(x/h)$ belongs to $\tilde{H}^{1/2}(I_h)$ and*

$$\|v\|_{\tilde{H}^{1/2}(I_h)} \leq C \|\hat{v}\|_{\tilde{H}^{1/2}(I)} . \tag{3.1}$$

where C is a constant independent of v and h.

Proof: By Theorem 3.1.2 of [11], we have

$$\| v \|_{H^0(I_h)} \leq C h^{1/2} \| \hat{v} \|_{H^0(I)} \quad \forall \hat{v} \in H^0(I) \tag{3.2}$$

and

$$\| v \|_{H^1(I_h)} \leq C h^{-1/2} \| \hat{v} \|_{H^1(I)} \quad \forall \hat{v} \in H^1(I) . \tag{3.3}$$

Interpolating between $H^0(I)$ and $\mathring{H}^1(I)$ gives the result. □

LEMMA 3.2 : *Let $\hat{v}$ and v be as in Lemma 3.1. Then for $k \geq 0$,*

$$\inf_{\hat{p} \in \mathcal{P}_p(I)} \| \hat{v} - \hat{p} \|_{H^k(I)} \leq C h^{\mu - 1/2} \| v \|_{H^k(I_h)} \tag{3.4}$$

where $\mu = \min(p+1, k)$ and C depends on k but is independent of p, h and μ.

The above lemma has been stated and proved for two-dimensional grids in [10] (Lemma 4.4). The proof is essentially the same for one dimension.

The following lemmas from [19] allow us to derive the error estimates for the h-p version by piecing together the estimates on each subinterval $\Gamma^j_{h,i}$ of the segments Γ^j.

LEMMA 3.3 : *Let $f \in \tilde{H}^s(\Gamma^j)$, $s \in \mathbb{R}$, be such that $f\big|_{\Gamma^j_{h,i}} \in \tilde{H}^s(\Gamma^j_{h,i})$ for $i = 1, \dots, N^j_h$. Then*

$$\| f \|^2_{\tilde{H}^s(\Gamma^j)} \leq C \sum_{i=1}^{N^j_h} \| f \|^2_{\tilde{H}^s(\Gamma^j_{h,i})} \tag{3.5}$$

where C is independent of h and f.

LEMMA 3.4 [19] : *Let $f \in H^s(\Gamma^j)$, $s \in \mathbb{R}$. Then there holds*

$$\sum_{i=1}^{N^j_h} \| f \|^2_{H^s(\Gamma^j_{h,i})} \leq C \| f \|^2_{H^s(\Gamma^j)} \tag{3.6}$$

with a constant C independent of h and f.

The following is a modified version of Lemma 3.2 from [21].

LEMMA 3.5 : *Let $f \in \tilde{H}^{1/2}(I_h)$. Then $f' \in \tilde{H}^{-1/2}(I_h)$ and*

$$\| f' \|_{\tilde{H}^{-1/2}(I_h)} \leq C \| f \|_{\tilde{H}^{1/2}(I_h)}$$

where C is a constant independent of f and h.

Proof: Let $D_{(1)} : \mathring{H}^1(I_h) \to L^2(I_h)$ denote the differentiation operator : $D_{(1)}\, g := g'$ for $g \in \mathring{H}^1(I_h)$. Then we have

$$\| D_{(1)}\, g \|_{L^2(I_h)} \leq \| g \|_{\mathring{H}^1(I_h)},$$

i.e., $\|D_{(1)}\| \leq 1$. Denote by $\tilde{D} : H^1(I_h) \to L^2(I_h)$ the differentiation operator : $\tilde{D} g := g'$ for $g \in H^1(I_h)$. Then we also have $\|\tilde{D}\| \leq 1$. Therefore the adjoint operator $\tilde{D}' : L^2(I_h) \to \tilde{H}^{-1}(I_h)$ (with the duality induced by the scalar product on $L^2(I_h)$) satisfies $\|\tilde{D}'\| \leq 1$.

Now we show that for $g \in \mathring{H}^1(I_h)$ there holds $-\tilde{D}'\, g = D_{(1)}\, g$: Using the definition of the adjoint operator and integration by parts we obtain for all $f \in H^1(I_h)$

$$\langle \tilde{D}'\, g, f \rangle := \langle g, \tilde{D} f \rangle =$$
$$= \int_{I_h} g(x)\, f'(x)\, dx = - \int_{I_h} g'(x)\, f(x)\, dx = - \langle D_{(1)}\, g, f \rangle .$$

Therefore $D_{(0)} := -\tilde{D}' :\ L^2(I_h) \to \tilde{H}^{-1}(I_h)$ satisfies $D_{(0)}\, g = D_{(1)}\, g$ for $g \in \mathring{H}^1(I_h)$. Furthermore there holds $\|D_{(0)}\| \leq 1$. Now interpolating between $D_{(0)}$ and $D_{(1)}$ gives for $0 \leq s \leq 1$ an operator

$$D_{(s)} : \tilde{H}^s(I_h) \to \tilde{H}^{s-1}(I_h),\ \|D_{(s)}\| \leq 1 .$$

For $g \in \mathring{H}^1(I_h)$ there holds $D_{(s)}\, g = g'$, hence $D_{(s)}$ is the extension of the classical derivative to $\tilde{H}^s(I_h)$. With $s = 1/2$ we obtain the assertion of the lemma. □

The approximation of the smooth functions v_s by polynomials in $V^0_{p,h}(\Gamma_2)$ can be treated using the following result.

THEOREM 3.1 : *Let* $r > \frac{1}{2}$, $p \geq 1$. *Then for* $v \in H^r(\Gamma^j)$ *there exists* $v^j_{p,h} \in S^0_{p,h}(\Gamma^j)$ *such that*

$$v^j_{p,h}(t_\ell) = v(t_\ell) \quad for \quad \ell = j-1, j \tag{3.7}$$

$$\| v - v^j_{p,h} \|_{\tilde{H}^{1/2}(\Gamma^j)} \leq C h^{\mu - 1/2} p^{-(r-1/2)} \log^{1/2} p\, \| v \|_{H^r(\Gamma^j)} \tag{3.8}$$

where $\mu = \min\{r, p+1\}$.

Proof: Let us consider any interval of Γ^j, say the first one, $\Gamma^j_{h,1}$ assumed to be I_h. Let $\hat{v}$ denote the function $v\left(\dfrac{x}{h}\right)$, so that $\hat{v} \in H^r(I)$. By Theorem

3.1 of [9], there exists a projection $P_p^{1/2} : H^r(I) \to \mathscr{P}_p(I)$ satisfying for all $\hat{w} \in H^r(I)$,

$$P_p^{1/2} \hat{w} = \hat{w} \quad \text{at} \quad x = 0, x = 1 \tag{3.9}$$

$$P_p^{1/2} \hat{w} = \hat{w} \quad \text{for} \quad \hat{w} \in \mathscr{P}_p(I) \tag{3.10}$$

$$\| \hat{w} - P_p^{1/2} \hat{w} \|_{\tilde{H}^{1/2}(I)} \leq C p^{-(r-1/2)} \log^{1/2} p \| \hat{w} \|_{H^r(I)} . \tag{3.11}$$

Hence for any $\hat{S} \in \mathscr{P}_p(I)$,

$$\begin{aligned} \| \hat{v} - P_p^{1/2} \hat{v} \|_{\tilde{H}^{1/2}(I)} &= \| (\hat{v} - \hat{S}) - P_p^{1/2} (\hat{v} - \hat{S}) \|_{\tilde{H}^{1/2}(I)} \\ &\leq C p^{-(r-1/2)} \log^{1/2} p \inf_{\hat{S} \in \mathscr{P}_p(I)} \| \hat{v} - \hat{S} \|_{H^r(I)} \\ &\leq C p^{-(r-1/2)} h^{\mu - 1/2} \log^{1/2} p \| v \|_{H^r(I_h)} \end{aligned}$$

by (3.4). Hence, by Lemma 3.1 we obtain

$$\| v - P_p^{1/2} v \|_{\tilde{H}^{1/2}(I_h)} \leq C p^{-(r-1/2)} h^{\mu - 1/2} \log^{1/2} p \| v \|_{H^r(I_h)} . \tag{3.12}$$

We may repeat this over each subinterval. Using Lemma 3.3, this gives the required $v_{p,h}^j \in S_{p,h}^0(\Gamma^j)$ satisfying (3.7), (3.8). □

Lemma 3.2 may be used separately on each $\Gamma^j \subset \Gamma_2$. Then defining $v_{p,h} \in V_{p,h}^0(\Gamma_2)$ by

$$v_{p,h} |_{\Gamma^j} = v_{p,h}^j$$

we obtain for $v_s \in H^s(\Gamma_2)$

$$\| v_s - v_{p,h} \|_{\tilde{H}^{1/2}(\Gamma_2)} \leq C h^{\mu - 1/2} p^{-(s-1/2)} \log^{1/2} p \| v_s \|_{H^s(\Gamma_2)} \tag{3.13}$$

where $\mu = \min \{ s, p+1 \}$.

Next, let us consider the singular functions v_{jk}. Let us define $\overline{S_{p,h}^0}(\Gamma^j) \subset S_{p,h}^0(\Gamma^j)$ to be the subset of functions vanishing at the end points of Γ^j. Then it suffices to look at the approximation by polynomials in $\overline{S_{p,h}^0}(\Gamma^j)$ of a function v defined on Γ^j (assumed to be $(-1, 1)$) by

$$v(x) = (1 + x)^\alpha \chi(x) \tag{3.14}$$

where $\alpha > 0$ and χ is a C^∞ cut-off function satisfying $\chi = 1$ for $x \leq -1/2$, $\chi = 0$ for $x \geq 0$.

In order to prove our desired results for the approximation of functions like (3.14), we will need to consider their regularity in a class of weighted spaces, $W^s(\mu, \nu)$ with $\mu, \nu \in \mathbb{R}$, defined as the completion of C^∞ under the norm $\|\cdot\|_{W^s(\mu,\nu)}$ defined for integer $s > 0$ by

$$\|u\|^2_{W^s(\mu,\nu)} = \int_{-1}^{+1} \left[(1-x^2)^{-\mu} \left[\frac{d^s u}{dx^s} \right]^2 + (1-x^2)^{-\nu} u^2 \right] dx .$$

For $s = 0$, we use $\mu = \nu$ and

$$\|u\|^2_{W^0(\mu,\nu)} = \int_{-1}^{+1} (1-x^2)^{-\mu} u^2 \, dx .$$

For $0 < s = [s] + \{s\}$ where $[s]$ is an integer and $0 < \{s\} < 1$, we define

$$\|u\|^2_{W^s(\mu,\nu)} = \|u\|^2_{W^{[s]}(\mu,\nu)} + \\ + \int_{-1}^{+1} \int_{-1}^{+1} \frac{[(1-x^2)^{-\mu/2} u^{(s)}(x) - (1-y^2)^{-\mu/2} u^{(s)}(y)]^2 \, dx \, dy}{|x-y|^{1+2\{s\}}}$$

where $u^{(s)}(x) = \dfrac{d^s u}{dx^s}$. The relevant interpolation properties of the subspaces are summarized in [7].

We will also use $\overset{\circ}{W^s}(\mu, \nu)$ to denote the completion of the set $\{u \in C_0^\infty \mid \|u\|_{H^s(\mu,\nu)} < \infty\}$.

The use of these weighted spaces is essential to prove that the p-version results in double the rate of convergence for singular functions shown by the h-version. Carrying out the analysis in the usual H^s Sobolev spaces leads to exactly the same rate of convergence for the two versions.

We now prove the following result, which is crucial to the analysis of errors for singular functions. This generalizes Theorem 3.3 of [21] which dealt only with the special case $\alpha = 1/2$ in (3.14) and was the main approximation result in that reference.

LEMMA 3.6 : *Let $\hat{w}$ be defined on $I = (-I, 1)$ by* (3.14) *with $\alpha > 0$. Then for $p \geqslant 1$ and* $\min(3/2, 2\alpha) > \varepsilon > 0$, *there exists a sequence of functions $\hat{w}_p \in \mathcal{P}_p(I)$ satisfying*

$$\hat{w}_p(\pm 1) = \hat{w}(\pm 1) = 0 \tag{3.15}$$

$$\|\hat{w} - \hat{w}_p\|_{\tilde{H}^{1/2}(I)} \leqslant C p^{-2\alpha+\varepsilon} \tag{3.16}$$

where C depends on α and ε but is independent of p.

Proof : We first calculate the regularity of $\hat{w}$ in terms of $\overset{\circ}{W^s}(\mu, \nu)$ spaces. We have for $s \geqslant 0$ integer

$$\|\hat{w}\|_{\mathring{W}^s(\mu - s, \mu)} \leqslant$$
$$\leqslant C \int_{-1}^{+1} [(1+x)^{-(\mu - s)}(1+x)^{2(\alpha - s)} + (1+x)^{-\mu}(1+x)^{2\alpha}]\, dx$$

which is finite provided that

$$2\alpha - s - \mu + 1 > 0 \quad \text{and} \quad 2\alpha - \mu + 1 > 0$$

that is

$$s < 2\alpha - \mu + 1 \quad \text{and} \quad \alpha > \frac{\mu - 1}{2}.$$

We will choose $\mu = 1/2 + \varepsilon/3$. Then we see that

$$\hat{w} \in \mathring{W}^s(\mu - s, \mu)$$

for any integer $s < 2\alpha + \frac{1}{2} - \frac{\varepsilon}{3}$ provided $\alpha > -\frac{1}{4} + \frac{\varepsilon}{6}$. This will also hold for noninteger s, as can be seen by interpolation.

Using Theorem 2.9 of [7], we see that for any $u \in \mathring{W}^s(\mu - s, \mu)$, $s > \mu$, and any $\frac{1}{2} \geqslant \tilde{\varepsilon} > 0$, there exists a polynomial u_p such that $u_p = u = 0$ at ± 1 and

$$\|u - u_p\|_{\mathring{H}^{1/2+\tilde{\varepsilon}}(I)} \leqslant C p^{-(s - 1/2) + \tilde{\varepsilon}} \|u\|_{\mathring{W}^s(\mu - s, \mu)} \tag{3.17}$$

(where $u_p = \mathring{\mathscr{P}}_p^{\mu} u$ in the notation of [7]).

We now apply (3.17) to $\hat{w}$, taking $\tilde{\varepsilon} = \varepsilon/3$, $\mu = 1/2 + \tilde{\varepsilon}$ and $s = 2\alpha + 1/2 - 2\varepsilon/3$. Since $2\alpha > \varepsilon$, we see that $s - \mu = 2\alpha - \varepsilon > 0$ so that $s > \mu$ as required. This gives a $u_p = \hat{w}_p \in \mathscr{P}_p(I)$ such that

$$\|\hat{w} - \hat{w}_p\|_{\mathring{H}^{1/2+\tilde{\varepsilon}}(I)} \leqslant C p^{-2\alpha + \varepsilon}.$$

The assertion (3.16) now follows by noting that for any $\tilde{\varepsilon} > 0$,

$$\|\hat{w} - \hat{w}_p\|_{\tilde{H}^{1/2}(I)} \leqslant \|\hat{w} - \hat{w}_p\|_{\mathring{H}^{1/2+\tilde{\varepsilon}}(I)}. \qquad \square$$

In the sequel, we will find it more convenient to define $I = (0, 1)$ so that our function v in (3.14) takes the form

$$v(x) = x^{\alpha} \tilde{\chi}(x). \tag{3.18}$$

with a suitable C^{∞} cut-off function $\tilde{\chi}$ satisfying $\tilde{\chi} = 1$ for $x \leqslant 1/4$ and $\tilde{\chi} = 0$ for $x \geqslant 1/2$. Obviously, Lemma 3.6 is applicable once more.

We now consider the approximation by the *h-p* version of the singular function (3.18) over the side Γ^j (assumed to be the interval $I = (0, 1)$) when a quasiuniform family of meshes with meshsizes h and polynomials of degree p are used.

THEOREM 3.2 : *Let v be given by* (3.18) *on* $\Gamma^j = (0, 1)$. *Then for* $\alpha > 0$ *there exists* $v_{p,h} \in \overline{S^0_{p,h}}(\Gamma^j)$ *satisfying*

$$\|v - v_{p,h}\|_{\tilde{H}^{1/2}(\Gamma^j)} \leq C \max \{h^\alpha p^{-2\alpha+\varepsilon}, \min \{h^\alpha, h^{p+1/2} p^{-2\alpha} \log^{1/2} p\}\} \tag{3.19}$$

where the constant C depends on the exponent α *in* (3.18) *and on the constant* τ *in* (2.21) *but is independent of p and h.*

Proof. : We split the function (3.18) into two portions w_1 and w_2 where w_1 has support only in I_h,

$$w_1(x) = v(x)\,\chi\left(\frac{x}{h}\right) \tag{3.20}$$

$$w_2(x) = v(x)\left(1 - \chi\left(\frac{x}{h}\right)\right). \tag{3.21}$$

For h small enough, w_1 considered as a function on I_h is related to the function $\hat{w}$ in Lemma 3.6 by

$$w_1(x) = x^\alpha \chi\left(\frac{x}{h}\right) = h^\alpha \hat{w}\left(\frac{x}{h}\right).$$

Define $w(x)$ on I_h by $w(x) = \hat{w}\left(\frac{x}{h}\right)$. Then using Lemmas 3.6 and 3.1, there exists $w_p(x) = \hat{w}_p\left(\frac{x}{h}\right)$ in $\mathcal{P}_p(I_h)$ satisfying

$$w_p = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = h$$

and

$$\|w - w_p\|_{\tilde{H}^{1/2}(I_h)} \leq C p^{-2\alpha+\varepsilon}.$$

Then, taking $w^1_{p,h}(x) = h^\alpha w_p(x)$ we see that $w^1_{p,h}(x) \in \mathcal{P}_p(I_h)$ satisfies

$$\|w_1 - w^1_{p,h}\|_{\tilde{H}^{1/2}(I_h)} \leq C h^\alpha p^{-2\alpha+\varepsilon}. \tag{3.22}$$

Extending $w^1_{p,h}$ by 0, we get a function in $\overline{S^0_{p,h}}(\Gamma^j)$ such that (3.22) holds in $\tilde{H}^{1/2}(\Gamma^j)$.

We now approximate w_2, which has support in $[h/4, 1]$ and is obviously a C^∞ function on Γ^j. By Theorem 3.1, for any $r > 1/2$, there exists $w_{p,h}^2 \in \overline{S_{p,h}^0}(\Gamma^j)$ satisfying

$$\| w_2 - w_{p,h}^2 \|_{\tilde{H}^{1/2}(\Gamma^j)} \leq C(r)\, h^{\mu - 1/2} p^{-(r - 1/2)} \log^{1/2} p \, \| w_2 \|_{H^r(\Gamma^j)} . \quad (3.23)$$

Also, using (3.18) and (3.14), we see that for $s \geq 0$ integer,

$$\left| \frac{d^s w_2}{dx^s} \right| \leq C x^{\alpha - s} \quad \text{on} \quad \Gamma^j$$

so that for $r \geq \alpha + 1/2$

$$\| w_2 \|_{H^r(\Gamma^j)} \leq C(r)\, h^{\alpha + 1/2 - r} . \quad (3.24)$$

Using (3.23), (3.24) we get

$$\| w_2 - w_{p,h}^2 \|_{\tilde{H}^{1/2}(\Gamma^j)} \leq C(r)\, h^{\mu - r + \alpha} p^{-(r - 1/2)} \log^{1/2} p . \quad (3.25)$$

Taking $r = 2\alpha + 1/2$ in (3.25) gives

$$\| w_2 - w_{p,h}^2 \|_{\tilde{H}^{1/2}(\Gamma^j)} \leq C(r)\, h^{\min(\alpha, p - \alpha + 1/2)} p^{-2\alpha} \log^{1/2} p .$$

When p is small with respect to α, we can select r so that $h^{\mu - r + \alpha} p^{-(r - 1/2)}$ is minimal. For example, taking $r = 2$ we get

$$\| w_2 - w_{p,h}^2 \|_{\tilde{H}^{1/2}(\Gamma^j)} \leq C h^{\alpha} .$$

Combining the estimates for w_1 and w_2, and setting $v_{p,h} = w_{p,h}^1 + w_{p,h}^2$ completes the proof. □

Once we have estimates for the approximation of smooth and singular functions v_s and v_{jk} (as given in (2.16)) in the $\tilde{H}^{1/2}(\Gamma^j)$ norm, corresponding estimates for ψ_s and ψ_{jk} in the $\tilde{H}^{-1/2}(\Gamma^j)$ norm follow by the arguments of Theorem 3.4 in [21]. The underlying idea is that the antiderivative v of ψ_s (for instance) can be approximated in the $\tilde{H}^{1/2}(\Gamma^j)$ norm by a polynomial $v_{p,h}$ using the estimate (3.13). Then $\psi_{p,h}$, defined to be the derivative of $v_{p,h}$ (with respect to arc length), will approximate ψ_s in the $\tilde{H}^{-1/2}(\Gamma^j)$ norm with the same accuracy. We illustrate this technique below in Theorem 3.3 for the function ψ_s. (Theorem 3.4 follows similarly.)

THEOREM 3.3 : *Let $r > -1/2$. Then for $\psi \in H^r(\Gamma^j)$, there exists $\psi_{p,h}^j \in S_{p,h}(\Gamma^j)$ such that*

$$\| \psi - \psi_{p,h}^j \|_{\tilde{H}^{-1/2}(\Gamma^j)} \leq C h^{\mu + 1/2} p^{-(r + 1/2)} \log^{1/2} p \, \| \psi \|_{H^r(\Gamma^j)} \quad (3.26)$$

where $\mu = \min(r, p + 1)$.

Proof : We consider any interval of Γ^j, say the first one, $\Gamma^j_{h,1}$ assumed to be I_h. Let $\psi \in H^r(I_h)$ with $r > 0$ and let $\bar{\psi} = \int_0^h \psi(t)\,dt$. Define

$$v(x) = \int_0^x (\psi - \bar{\psi})(t)\,dt\,.$$

Then obviously $v \in H^{r+1}(I_h) \cap \tilde{H}^{1/2}(I_h)$. By (3.12), there exists a polynomial $P_p^{1/2} v \in \mathcal{P}_{p+1}(I_h)$ such that

$$\| v - P_p^{1/2} v \|_{\tilde{H}^{1/2}(I_h)} \leqslant C p^{-(r+1/2)} h^{\mu + 1/2} \log^{1/2} p \, \| v \|_{H^{r+1}(I_h)}$$

with $\mu = \min \{ r, p+1 \}$. By Lemma 3.5 taking $\psi_p = (P_p^{1/2} v)' + \bar{\psi}$ we have

$$\| \psi - \psi_p \|_{\tilde{H}^{-1/2}(I_h)} \leqslant C \| v - P_p^{1/2} v \|_{\tilde{H}^{1/2}(I_h)} \leqslant$$
$$\leqslant C p^{-(r+1/2)} h^{\mu + 1/2} \log^{1/2} p \, \| \psi \|_{H^r(I_h)}\,.$$

We may repeat this over each interval. Using Lemmas 3.3, 3.4 this gives the required $\psi^j_{p,h} \in S_{p,h}(\Gamma^j)$ satisfying (3.26). □

THEOREM 3.4 : *Let $\psi(x) = x^\alpha \chi(x)$ on $\Gamma^j = (0, 1)$. Then for $\alpha > -1$, there exists $\psi_{p,h} \in S_{p,h}(\Gamma^j)$ satisfying*

$$\| \psi - \psi_{p,h} \|_{\tilde{H}^{-1/2}(\Gamma^j)} \leqslant C \max \times$$
$$\times \{ h^{\alpha+1} p^{-2(\alpha+1)+\varepsilon}, \min \{ h^{\alpha+1}, h^{p-\alpha-1/2} p^{-2(\alpha+1)} \log^{1/2} p \} \}\,. \quad (3.27)$$

The above theorem is obtained by splitting ψ into a singular and a smooth portion, as in Theorem 3.2 and then using an argument analogous to that in Theorem 3.3 for each portion.

We may now use the results from Theorem 3.1 through Theorem 3.4 to obtain the following theorem, which gives the rate of convergence when the *h-p* version (with quasiuniform mesh) is used to approximate the solution of system (2.11).

THEOREM 3.5 : *Let $U = \begin{bmatrix} v^* \\ \psi^* \end{bmatrix}$, the solution of* (2.11), *satisfy* (2.14), (2.15). *Then for $p \geqslant 1$ large enough or $h > 0$ small enough, the solution $U_{p,h} = \begin{bmatrix} v_{p,h} \\ \psi_{p,h} \end{bmatrix} \in \mathcal{V}_{p,h}(\Gamma)$ of the h-p version* (2.22) *exists and for any $\varepsilon > 0$ there exists a constant $C = C(\varepsilon)$, independent of h and p, such that*

$$\| v^* - v_{p,h} \|_{\tilde{H}^{1/2}(\Gamma_2)} + \| \psi^* - \psi_{p,h} \|_{\tilde{H}^{-1/2}(\Gamma_1)} \leqslant$$
$$\leqslant C \max \{ e_1(\alpha_0), e_2(\alpha_0), e_3(s) \} \; K \quad (3.28)$$

where $\alpha_0 = \min\{\alpha_{j1}\}$ *and*

$$e_1(\alpha_0) = h^{\alpha_0} p^{-2\alpha_0+\varepsilon} \tag{3.29}$$

$$e_2(\alpha_0) = \min(h^{\alpha_0}, h^{p-\alpha_0+1/2} p^{-2\alpha_0} \log^{1/2} p) \tag{3.30}$$

$$e_3(s) = h^{\min(s-1/2,\, p+1/2)} p^{-(s-1/2)} \log^{1/2} p \tag{3.31}$$

and

$$K = \|g_1\|_{H^s(\Gamma_1)} + \|g_2\|_{H^{s-1}(\Gamma_2)} . \tag{3.32}$$

Proof: The existence and convergence of solutions of the Galerkin scheme (2.22) follow from Theorem 2.5 using the Gårding inequality (2.12) and choosing $\mathcal{A} = A$ given by (2.10) and $H = \tilde{H}^{1/2}(\Gamma_2) \times \tilde{H}^{-1/2}(\Gamma_1)$ with dual $H' = H^{-1/2}(\Gamma_2) \times H^{1/2}(\Gamma_1)$. As $p \to \infty$ or $h \to 0$, $\{\mathcal{V}_{p,h}(\Gamma)\}$ is a sequence of approximating subspaces for $\tilde{H}^{1/2}(\Gamma_2) \times \tilde{H}^{-1/2}(\Gamma_1)$. The estimate (3.28) follows by the quasioptimality (2.20), the regularity result (2.15), (2.16) and the approximation results given by Theorems 3.1-3.4. $\square$

Remark 3.2: In most cases the rate of convergence in (3.28) is limited by the approximability of the singular function. As a consequence we obtain for s and p large enough a convergence rate of $O(h^{\alpha_0} p^{-2\alpha_0+\varepsilon})$.

Using Theorem 3.5, we may also predict the rates of convergence when the p-version and h-version are used, by respectively taking h or p to be constant in (3.28)-(3.31).

Higher convergence rates for the Galerkin error measured in lower order Sobolev norms can be obtained by Aubin-Nitsche type duality arguments. The following theorem gives such rates of convergence, which depend on both the solution of the interior problem on Ω and of the same problem on the exterior domain Ω^c. For the latter, we obtain singularity functions given by (2.13), (2.14) with ω_j replaced by $2\pi - \omega_j$.

THEOREM 3.6: *Under the assumptions of Theorem* 3.5 *there holds for the Galerkin error of the h-p method on a quasiuniform mesh for any* $\varepsilon > 0$ *and* $\sigma > 0$

$$\|v^* - v_{p,h}\|_{\tilde{H}^{1/2-\sigma}(\Gamma_2)} + \|\psi^* - \psi_{p,h}\|_{\tilde{H}^{-1/2-\sigma}(\Gamma_1)}$$

$$\leq C \max\{e_1(\alpha_0), e_2(\alpha_0), e_3(s)\} \max\left\{e_1(\tilde{\alpha}_0), e_2(\tilde{\alpha}_0), e_3\left(\sigma + \frac{1}{2}\right)\right\} \tag{3.33}$$

where e_1, e_2, e_3 *are as defined in* (3.29)-(3.31) *and* α_0 *is the smallest singular exponent of the interior problem,* $\tilde{\alpha}_0$ *is the smallest of* all *singular exponents of both the interior and exterior problem and* $C = C(\varepsilon)$ *is independent of h and p.*

Proof. : Firstly, we observe that the operator A in (2.10) satisfies for any $v_1, v_2 \in \tilde{H}^{1/2}(\Gamma_2)$ and $\psi_1, \psi_2 \in \tilde{H}^{-1/2}(\Gamma_1)$

$$\begin{aligned}\left\langle A\begin{bmatrix} v_1 \\ \psi_1 \end{bmatrix}, \begin{bmatrix} v_2 \\ -\psi_2 \end{bmatrix}\right\rangle &= \langle D_{22} v_1 + K'_{12} \psi_1, v_2\rangle + \langle -K_{21} v_1 + V_{11} \psi_1, (-\psi_2)\rangle \\ &= \langle v_1, D_{22} v_2 + K'_{12} \psi_2\rangle + \langle (-\psi_1), -K_{21} v_2 + V_{11} \psi_2\rangle \\ &= \left\langle \begin{bmatrix} v_1 \\ -\psi_1 \end{bmatrix}, A\begin{bmatrix} v_2 \\ \psi_2 \end{bmatrix}\right\rangle \end{aligned} \tag{3.34}$$

Let $\begin{bmatrix} v_{p,h} \\ \psi_{p,h} \end{bmatrix} \in \mathcal{V}_{p,h}(\Gamma)$ denote the Galerkin solution of (2.22). We now want to estimate the Galerkin error in the space $\tilde{H}^{1/2-\sigma}(\Gamma_2) \times \tilde{H}^{-1/2-\sigma}(\Gamma_1)$, $\sigma > 0$. We have

$$\begin{aligned}\left\| \begin{bmatrix} v^* \\ \psi^* \end{bmatrix} - \begin{bmatrix} v_{p,h} \\ \psi_{p,h} \end{bmatrix} \right\|_{\tilde{H}^{1/2-\sigma}(\Gamma_2) \times \tilde{H}^{-1/2-\sigma}(\Gamma_1)} \\ := \| v^* - v_{p,h} \|_{\tilde{H}^{1/2-\sigma}(\Gamma_2)} + \| \psi^* - \psi_{p,h} \|_{\tilde{H}^{-1/2-\sigma}(\Gamma_1)} \\ = \sup \left\langle \begin{bmatrix} \phi \\ w \end{bmatrix}, \begin{bmatrix} v^* - v_{p,h} \\ \psi^* - \psi_{p,h} \end{bmatrix}\right\rangle \\ = \sup \left\langle \begin{bmatrix} \phi \\ -w \end{bmatrix}, \begin{bmatrix} v^* - v_{p,h} \\ -(\psi^* - \psi_{p,h}) \end{bmatrix}\right\rangle \end{aligned} \tag{3.35}$$

where the sup is taken over all $\phi \in H^{-1/2+\sigma}(\Gamma_2)$, $w \in H^{1/2+\sigma}(\Gamma_1)$ with

$$\| \phi \|_{H^{-1/2+\sigma}(\Gamma_2)} = 1\,, \quad \| w \|_{H^{1/2+\sigma}(\Gamma_1)} = 1\,.$$

Next, we introduce the auxiliary problem : *For given $\phi \in H^{-1/2+\sigma}(\Gamma_2)$ and $w \in H^{1/2+\sigma}(\Gamma_1)$ find $\tilde{v}$, $\tilde{\psi}$ satisfying*

$$A\begin{bmatrix} \tilde{v} \\ \tilde{\psi} \end{bmatrix} = \begin{bmatrix} \phi \\ -w \end{bmatrix}. \tag{3.36}$$

With the methods in [14] one can conclude that the solution is of the form

$$\begin{bmatrix} \tilde{v} \\ \tilde{\psi} \end{bmatrix} = \begin{bmatrix} \tilde{v}_s \\ \tilde{\psi}_s \end{bmatrix} + \sum_{j=1}^{J} \left\{ \sum_{\gamma_{jk} < \sigma} \tilde{c}_{jk}\, \tilde{u}_{jk}\, \chi_j \right\} \tag{3.37}$$

where $\tilde{u}_{jk}$ are either the singularity functions of the original boundary value problem in Ω given by (2.13), (2.14) or the singularity functions of the corresponding problem in the exterior domain Ω^c.

We have therefore with (3.36) and (3.34) and arbitrary $\begin{bmatrix} \tilde{v}_{p,h} \\ -\tilde{\psi}_{p,h} \end{bmatrix} \in \mathcal{V}_{p,h}(\Gamma)$

$$\begin{aligned}
\left\langle \begin{bmatrix} \phi \\ -w \end{bmatrix}, \begin{bmatrix} (v^* - v_{p,h}) \\ -(\psi^* - \psi_{p,h}) \end{bmatrix} \right\rangle &= \left\langle A \begin{bmatrix} \tilde{v} \\ \hat{\psi} \end{bmatrix}, \begin{bmatrix} v^* - v_{p,h} \\ -(\psi^* - \psi_{p,h}) \end{bmatrix} \right\rangle \\
&= \left\langle \begin{bmatrix} \tilde{v} \\ -\tilde{\psi} \end{bmatrix}, A \begin{bmatrix} v^* - v_{p,h} \\ \psi^* - \psi_{p,h} \end{bmatrix} \right\rangle \\
&= \left\langle \begin{bmatrix} \tilde{v} - \tilde{v}_{p,h} \\ -\tilde{\psi} + \tilde{\psi}_{p,h} \end{bmatrix}, A \begin{bmatrix} v^* - v_{p,h} \\ \psi^* - \psi_{p,h} \end{bmatrix} \right\rangle \\
&\leqslant \left\| A \begin{bmatrix} v^* - v_{p,h} \\ \psi^* - \psi_{p,h} \end{bmatrix} \right\|_{H^{-1/2}(\Gamma_2) \times H^{1/2}(\Gamma_1)} \left\| \begin{bmatrix} \tilde{v} - \tilde{v}_{p,h} \\ -\tilde{\psi} + \tilde{\psi}_{p,h} \end{bmatrix} \right\|_{\tilde{H}^{1/2}(\Gamma_2) \times \tilde{H}^{-1/2}(\Gamma_1)} \\
&\leqslant C \left\| \begin{bmatrix} v^* - v_{p,h} \\ \psi^* - \psi_{p,h} \end{bmatrix} \right\|_{\tilde{H}^{1/2}(\Gamma_2) \times \tilde{H}^{-1/2}(\Gamma_1)} \left\| \begin{bmatrix} \tilde{v} - \tilde{v}_{p,h} \\ -\tilde{\psi} + \tilde{\psi}_{p,h} \end{bmatrix} \right\|_{\tilde{H}^{1/2}(\Gamma_2) \times \tilde{H}^{-1/2}(\Gamma_1)}
\end{aligned} \tag{3.38}$$

Now we use the special form of $\begin{bmatrix} v^* \\ \psi^* \end{bmatrix}$ in (2.15) and $\begin{bmatrix} \tilde{v} \\ \tilde{\psi} \end{bmatrix}$ in (3.37) to obtain approximation rates for the two terms on the right hand side of (3.38) which *together* give the convergence rate for the Galerkin error in the norm of $\tilde{H}^{1/2-\sigma}(\Gamma_2) \times \tilde{H}^{-1/2-\sigma}(\Gamma_1)$. The convergence rate of $\begin{bmatrix} v^* - v_{p,h} \\ \psi^* - \psi_{p,h} \end{bmatrix}$ in the energy space $\tilde{H}^{1/2}(\Gamma_2) \times \tilde{H}^{-1/2}(\Gamma_1)$ is determined by the singular exponents α_{jk} in (2.13), (2.14) and the regularity of the given data $g_1 \in H^s(\Gamma_1)$, $g_2 \in H^{s-1}(\Gamma_2)$. The convergence rate of $\begin{bmatrix} \tilde{v} - \tilde{v}_{p,h} \\ \tilde{\psi} - \tilde{\psi}_{p,h} \end{bmatrix}$ is determined by the singular exponents of both the interior and the exterior mixed boundary value problem, α_{jk} and $\tilde{\alpha}_{jk}$, and the « shift parameter » σ, since the right hand side in (3.36) belongs to $H^{-1/2+\sigma}(\Gamma_2) \times H^{1/2+\sigma}(\Gamma_1)$. Therefore application of the estimates (3.28)-(3.31) in (3.38) yields the desired estimate (3.33) □

Remark 3.3 : For simplicity we make the following assumptions :

(i) The worst singularity occurs in the interior, not the exterior problem, i.e., $\alpha_0 = \tilde{\alpha}_0$.

(ii) p is high enough : $p + 1/2 \geq 2\alpha_0$. This implies $p + 1/2 \geq \alpha_0$ and $p + 1/2 - \alpha_0 \geq \alpha_0$.

(iii) The given data g_1, g_2 in (2.9) are smooth enough : $s - 1/2 > 2\alpha_0$.

Then we have with $\varepsilon > 0$

$$\left\| \begin{bmatrix} v^* \\ \psi^* \end{bmatrix} - \begin{bmatrix} v_{p,h} \\ \psi_{p,h} \end{bmatrix} \right\|_{\tilde{H}^{1/2-\sigma}(\Gamma_2) \times \tilde{H}^{-1/2-\sigma}(\Gamma_1)} \leq$$

$$\leq C h^{\alpha_0 + \min\{\sigma, \alpha_0\}} p^{-2\alpha_0 - \min\{\sigma, 2\alpha_0\} + \varepsilon} \quad (3.39)$$

Hence we obtain for $\sigma = 2\alpha_0$

$$\left\| \begin{bmatrix} v^* \\ \psi^* \end{bmatrix} - \begin{bmatrix} v_{p,h} \\ \psi_{p,h} \end{bmatrix} \right\|_{\tilde{H}^{1/2-2\alpha_0}(\Gamma_2) \times \tilde{H}^{-1/2-2\alpha_0}(\Gamma_1)} \leq C h^{2\alpha_0} p^{-4\alpha_0 + \varepsilon}.$$

Let Ω_0 be a compact subdomain of Ω. Theorem 3.6 may be applied to obtain an L^∞ estimate on Ω_0 for the error between the exact solution u of the original mixed boundary value problem (2.1) and its approximation $u_{p,h}$ defined below.

Here u is given by the representation formula (2.8) for $z \in \Omega$ as

$$2u(z) \equiv \int_{\Gamma_1} \left[\psi(\zeta)\, G(z,\zeta) - g_1(\zeta) \frac{\partial}{\partial n_\zeta} G(z,\zeta) \right] ds_\zeta +$$

$$+ \int_{\Gamma_2} \left[g_2(\zeta)\, G(z,\zeta) - v(\zeta) \frac{\partial}{\partial n_\zeta} G(z,\zeta) \right] ds_\zeta \quad (3.40)$$

where (v, ψ) solve the system (2.9).

An approximation $u_{p,h}$ for u in Ω is obtained by inserting the Galerkin solution $(v_{p,h}\ \psi_{p,h})$ of (2.22) into (3.40) instead of (v, ψ). Then making the assumptions (i)-(iii) of Remark 3.3 we obtain from the estimate (3.33) by application of standard arguments (as in [26]) for any $\Omega_0 \subset\subset \Omega$ and $\varepsilon > 0$

$$\| u - u_{p,h} \|_{L^\infty(\Omega_0)} \leq C h^{2\alpha_0} p^{-4\alpha_0 + \varepsilon}.$$

Here the constant $C = C(d, \varepsilon)$ is independent of h and p but $C(d, \varepsilon) \to \infty$ as $d \to 0$ where d is the distance between Ω_0 and the boundary Γ of Ω.

Often, one is interested in estimating the Galerkin error in higher order norms than the energy norm. To this end, we prove the following.

THEOREM 3.7 : *Let* ψ^*, *the component of the solution of* (2.11) *on* Γ_1, *behave like* $O(x^{\alpha_j - 1})$ *near the vertices* t_j *with* $\alpha := \min \alpha_j > 1/2$. *Let* $\psi_{p,h} \in V_{p-1,h}(\Gamma_1)$ *be its Galerkin approximation from* (2.22). *Then for any*

$\varepsilon > 0$ *there exists a constant* $C = C(\varepsilon)$ *independent of* p *and* h *such that (for* p *large enough)*

$$\| \psi^* - \psi_{p,h} \|_{L^2(\Gamma_1)} \leq C h^{\alpha - 1/2} p^{-2(\alpha - 1/2) - \varepsilon} .$$

Proof. : Making use of the Galerkin operator $G: \; \psi^* \to \psi_{p,h}$ we obtain with an arbitrary $\phi \in V_{p-1,h}(\Gamma_1)$

$$\begin{aligned}
\| \psi^* - \psi_{p,h} \|_{L^2(\Gamma_1)} &\leq \| \psi^* - \phi \|_{L^2(\Gamma_1)} + \| \phi - \psi_{p,h} \|_{L^2(\Gamma_1)} \\
&\leq \| \psi^* - \phi \|_{L^2(\Gamma_1)} + \| G(\phi - \psi^*) \|_{L^2(\Gamma_1)} \\
&\leq \| \psi^* - \phi \|_{L^2(\Gamma_1)} + h^{-1/2} p \| G(\phi - \psi^*) \|_{H^{-1/2}(\Gamma_1)} \\
&\leq \| \psi^* - \phi \|_{L^2(\Gamma_1)} + C h^{-1/2} p \| \phi - \psi^* \|_{\tilde{H}^{-1/2}(\Gamma_1)} \\
&\leq C h^{\alpha - 1/2} p^{-2\alpha + 1 + \varepsilon} .
\end{aligned}$$

□

Remark 3.4 : In the above proof we applied the inverse assumption which is well-known for the h-version and gives for the p-version for any $\phi \in V_{p-1,h}(\Gamma_1)$

$$\| \phi \|_{H^k(\Gamma_1)} \leq C p^{2(k-s)} \| \phi \|_{H^s(\Gamma_1)} \quad \text{for all} \quad k \geq s . \tag{3.41}$$

For $k \geq s \geq 0$, (3.41) follows from [15] and for the remaining indices it is a direct consequence of Theorem 4.1.3 in [5].

We also made use of the estimate

$$\| \psi - \phi \|_{L^2(\Gamma_1)} \leq C h^{\alpha - 1/2} p^{-2\alpha + 1 + \varepsilon}$$

for $\psi = O(x^{\alpha_j - 1})$ near the vertices t_j with $\alpha := \min \alpha_j > 1/2$, and $\phi \in V_{p-1,h}(\Gamma_1)$, which follows from [16].

Finally, we comment on a numerical example performed by E. Rank [20]. He uses the boundary element Galerkin method to solve the mixed boundary value problem (2.1) for an L-shaped domain Ω. The boundary conditions are suitably chosen to yield the solution

$$u = r^{2/3} \sin (2 \phi / 3)$$

where (r, ϕ) are polar coordinates centered at the reentrant corner of Ω. Rank obtains experimental convergence rates of order 1/6 for the h-version and of order 1/3 for the p-version for the sum of the L^2 errors of u and $\frac{\partial u}{\partial n}$ and their respective Galerkin approximations $v_{p,h}$ and $\psi_{p,h}$. We note that Rank has implemented the system (2.17) and not the system (2.11) which we have analyzed above. Nevertheless, the results for (2.11) may be used as guidelines for (2.17). Since $r^{2/3} \in H^{7/6}(\Gamma)$ and $r^{-1/3} \in H^{1/6}(\Gamma)$, application of Theorem 3.5 and Theorem 3.7 yields with $\varepsilon > 0$

$$\| v^* - v_{p,h} \|_{L^2(\Gamma_2)} \leq \| v^* - v_{p,h} \|_{\tilde{H}^{1/2}(\Gamma_2)} \leq C h^{2/3} p^{-4/3+\varepsilon}$$

$$\| \psi^* - \psi_{p,h} \|_{L^2(\Gamma_1)} \leq C h^{1/6} p^{-1/3+\varepsilon}$$

which is in agreement with the experimental convergence rates computed in [20].

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