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**FINITE ELEMENT APPROXIMATION
 OF A FREE BOUNDARY PROBLEM ARISING IN THE THEORY
 OF LIQUID DROPS AND PLASMA PHYSICS (*)**

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Communiqué par R. TEMAM

Abstract. — *Optimal order error bounds are obtained for a finite element approximation of a variational problem arising in the theory of liquid drops and also in plasma physics. For a bounded domain $\Omega \subset \mathbb{R}^2$ we consider the minimization of $\gamma \langle \nabla \eta, \nabla \eta \rangle + \kappa^2(1 - \gamma) \langle \eta, \eta \rangle - \kappa^4 \langle \mathcal{G} \eta, \eta \rangle$ subject to $\eta \geq 0$ and $\langle 1, \eta \rangle = M/2$; where $\langle \cdot, \cdot \rangle$ denotes the L^2 inner product, $\gamma \geq 0$, $\kappa^2 > 0$ and $M > 0$ are prescribed constants and $\mathcal{G} \in \mathcal{L}(L^2(\Omega), H_0^1(\Omega))$ is the inverse of the Laplacian. The case $\gamma = 1$ corresponds to a model of a liquid drop sitting on a soap film introduced by Benjamin and Cocker. The case $\gamma = 0$ corresponds to the much studied model plasma problem introduced by Temam.*

Résumé. — *Des bornes d'erreur d'ordre optimal sont obtenues pour une approximation par éléments finis d'un problème variationnel qui apparaît dans la théorie des gouttes de liquide ainsi que dans la physique des plasmas. Pour un domaine borné $\Omega \subset \mathbb{R}^2$ nous considérons la minimisation de $\gamma \langle \nabla \eta, \nabla \eta \rangle + \kappa^2(1 - \gamma) \langle \eta, \eta \rangle - \kappa^4 \langle \mathcal{G} \eta, \eta \rangle$ soumis à $\eta \geq 0$ et $\langle 1, \eta \rangle = M/2$; où $\langle \cdot, \cdot \rangle$ désigne le produit scalaire dans L^2 $\gamma \geq 0$, $\kappa^2 > 0$ et $M > 0$ sont des constantes données et $\mathcal{G} \in \mathcal{L}(L^2(\Omega), H_0^1(\Omega))$ est l'inverse du Laplacien. Le cas $\gamma = 1$ correspond à un modèle d'une goutte de liquide posée sur un film savonneux introduit par Benjamin et Cocker. Le cas $\gamma = 0$ correspond au problème tant étudié de modèle de plasma introduit par Temam.*

1. INTRODUCTION

Let Ω be a bounded domain in \mathbb{R}^2 with a Lipschitz boundary $\partial\Omega$. For a prescribed non-negative constant γ and a prescribed positive constant κ we set for $w, \varphi \in H^1(\Omega)$

$$a(w, \varphi) \equiv \gamma \langle \nabla w, \nabla \varphi \rangle + \kappa^2(1 - \gamma) \langle w, \varphi \rangle - \kappa^4 \langle \mathcal{G} w, \varphi \rangle \quad (1.1a)$$

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and

$$J(\varphi) \equiv a(\varphi, \varphi); \quad (1.1b)$$

where $\langle \cdot, \cdot \rangle$ denotes the $L^2(\Omega)$ inner product and $\mathcal{G} \in \mathcal{L}(L^2(\Omega), H_0^1(\Omega) \cap C(\bar{\Omega}))$ is the Green's operator defined by

$$\langle \nabla \mathcal{G}w, \nabla \varphi \rangle = \langle w, \varphi \rangle \quad \forall \varphi \in H_0^1(\Omega). \quad (1.2)$$

Note that $\langle \mathcal{G}w, \varphi \rangle \equiv \langle w, \mathcal{G}\varphi \rangle$ and hence $a(w, \varphi) \equiv a(\varphi, w)$.

It is the purpose of this paper to consider the finite element approximation of the following problems:

(P_γ) ($\gamma > 0$). Find $\eta \in K_M$ such that

$$J(\eta) = \inf_{\varphi \in K_M} J(\varphi),$$

(P_0) ($\gamma = 0$). Find $\eta \in X_M$ such that

$$J(\eta) = \inf_{\varphi \in X_M} J(\varphi),$$

where

$$K \equiv \{\varphi \in H_0^1(\Omega) : \varphi \geq 0 \text{ in } \Omega\} \quad (1.3a)$$

$$K_M \equiv \{\varphi \in K : \langle 1, \varphi \rangle = M/2\} \quad (1.3b)$$

$$X \equiv \{\varphi \in L^2(\Omega) : \varphi \geq 0 \text{ in } \Omega\} \quad (1.4a)$$

$$X_M \equiv \{\varphi \in X : \langle 1, \varphi \rangle = M/2\} \quad (1.4b)$$

and M is a prescribed positive constant.

It follows immediately that solutions of (P_γ) and (P_0) solve the variational inequalities: (Q_γ) ($\gamma > 0$) Find $\eta \in K_M$ such that

$$a(\eta, \varphi - \eta) \geq 0 \quad \forall \varphi \in K_M, \quad (1.5a)$$

(Q_0) ($\gamma = 0$). Find $\eta \in X_M$ such that

$$a(\eta, \varphi - \eta) \geq 0 \quad \forall \varphi \in X_M; \quad (1.5b)$$

since $J(\eta) \leq J(\eta + \varepsilon(\varphi - \eta))$ for all $\varepsilon \in [0, 1]$.

Furthermore for any $\varphi \in K$ ($\varphi \neq 0$), $M\varphi/(2\langle 1, \varphi \rangle) \in K_M$ and so the solution of (1.5a) satisfies

$$a(\eta, \varphi) \geq a(\eta, \eta) 2\langle 1, \varphi \rangle / M \quad \forall \varphi \in K.$$

A similar statement holds for (1.5b) with K replaced by X .

Hence the solutions of (Q_γ) and (Q_0) satisfy

$$(\gamma > 0) \quad a(\eta, \varphi - \eta) \geq \langle -q, \varphi - \eta \rangle \quad \forall \varphi \in K \quad (1.6a)$$

$$(\gamma = 0) \quad a(\eta, \varphi - \eta) \geq \langle -q, \varphi - \eta \rangle \quad \forall \varphi \in X, \quad (1.6b)$$

where

$$q \equiv -2 J(\eta)/M . \tag{1.6c}$$

The motivation for solving (P_γ) and (P_0) comes from two sources. For the present it is convenient to denote by η_γ , $\gamma \geq 0$ the solutions of each problem and q_γ the corresponding constant in (1.6c). Setting

$$\xi_\gamma \equiv [(\gamma - 1) \eta_\gamma + 2 \kappa^2 \mathcal{G}\eta_\gamma]/(\gamma + 1) , \tag{1.7a}$$

$$u_\gamma \equiv \xi_\gamma + \eta_\gamma , \quad v_\gamma \equiv \xi_\gamma - \eta_\gamma ; \tag{1.7b}$$

one can show, see Barrett & Elliott (1989b), that $\{u_\gamma, v_\gamma\}$, for $\gamma > 0$, solve

$$\mathcal{E}(u_\gamma, v_\gamma) = \inf_{\{u, v\} \in K^*} \mathcal{E}(u, v) , \tag{1.8a}$$

where

$$K^* \equiv \{ \{u, v\} \in H_0^1(\Omega) \times H_0^1(\Omega) : u \geq v \text{ in } \Omega, \langle 1, u - v \rangle = M \} \tag{1.8b}$$

and

$$\mathcal{E}(u, v) \equiv \frac{1}{2} \int_\Omega \{ |\nabla u|^2 + \gamma |\nabla v|^2 - \kappa^2(u^2 - v^2) \} . \tag{1.8c}$$

Problem (1.8) was proposed and analysed by Benjamin & Cocker (1984) in the case $\gamma = 1$. It models a liquid drop of soapy water suspended by a soap film which is attached to the fixed frame $\partial\Omega$. In equilibrium the drop is bounded by an upper surface $z = v_\gamma(x_1, x_2)$ and a lower surface $z = u_\gamma(x_1, x_2)$ so that the drop occupies the region $\{(x_1, x_2, z) : v_\gamma(x_1, x_2) < z < u_\gamma(x_1, x_2)\}$, where z denotes the distance below the horizontal plane in which Ω lies. The prescribed constants γ and κ^2 are such that

$$\gamma \equiv \gamma_v/\gamma_u \quad \text{and} \quad \kappa^2 \equiv \rho g/\gamma_u ,$$

where ρ is the density of the liquid and γ_v and γ_u are the coefficients of surface tension for the upper and lower surfaces. The liquid drop having a prescribed mass gives rise to the constraint $\langle 1, u_\gamma - v_\gamma \rangle = M$. Defining the set $\Omega_+ \equiv \{x \in \Omega : u_\gamma(x) > v_\gamma(x)\}$ then the unknown free boundary is $\Gamma \equiv \partial\Omega_+ \cap \Omega$. In the case $\gamma_u = \gamma_v$, i.e. $\gamma = 1$, existence and some properties of the minimisers $\{u_1, v_1\}$ to (1.8) were established by Benjamin & Cocker (1984). In addition Cocker, Friedman & McLeod (1986) proved regularity results for the minimiser and free boundary ; and studied the asymptotic behaviour of the minimisers as $\kappa \rightarrow \infty$.

Barrett and Elliott (1989b) studied problem (1.8) for $\gamma > 0$, proving results concerning existence, uniqueness and regularity of the minimisers $\{u_\gamma, v_\gamma\}$. In addition they showed as $\gamma \rightarrow 0$ $\{u_\gamma, v_\gamma\}$ converged in

$H_0^1(\Omega) \times L^2(\Omega)$ to $\{u_0, v_0\}$, where $\eta_0 = \frac{1}{2}(u_0 - v_0)$ solves (P_0) . It follows immediately from (1.6b) by choosing $\varphi \equiv 0$ and, $\varphi \equiv 2 \eta_0$ that $a(\eta_0, \eta_0) = -\langle q, \eta_0 \rangle$ and hence

$$\begin{aligned} \eta_0 &= \max \left(0, \kappa^2 \mathcal{G}\eta_0 - \frac{q_0}{\kappa^2} \right) \\ &\equiv \left[\kappa^2 \mathcal{G}\eta_0 - \frac{q_0}{\kappa^2} \right]^+ \end{aligned} \tag{1.9a}$$

and from (1.9a) and (1.7) that

$$\begin{aligned} u_0 &= 2 \kappa^2 \mathcal{G}\eta_0 \\ &\equiv \kappa^2 \mathcal{G} \left[u_0 - \frac{2 q_0}{\kappa^2} \right]^+ . \end{aligned} \tag{1.9b}$$

Hence $\left\{ u_0, \frac{2 q_0}{\kappa^2} \right\} \in H_0^1(\Omega) \times \mathbb{R}$ solves the « plasma problem » : given M , $\kappa \in \mathbb{R}^+$, find $\{U, d\} \in H_0^1(\Omega) \times \mathbb{R}$ such that

$$U = \kappa^2 \mathcal{G}[U - d]^+ \tag{1.10a}$$

$$\langle 1, [U - d]^+ \rangle = M . \tag{1.10b}$$

(1.10) is a much studied free boundary problem, see Temam (1975, 1977) and for an account see Friedman (1982). We note that (P_0) is equivalent to the « plasma problem » and this variational approach has been studied by Berestycki & Brezis (1980)

We now state results concerning the existence and uniqueness of solutions to the problems (P_γ) and (P_0) , and (Q_γ) and (Q_0) . For this purpose we introduce the eigenvalues and eigenfunctions $\{\lambda_i^{-1}, \psi_i\}$ of \mathcal{G}

$$\psi_i = \lambda_i \mathcal{G}\psi_i ; \quad |\psi_i|_{0, \Omega} = 1 , \tag{1.11}$$

ordered so that $0 < \lambda_1 < \lambda_2 \leq \dots$. Classical eigenfunction theory yields that λ_1 is simple and ψ_1 can be taken to be positive in Ω and

$$|\varphi|_{1, \Omega}^2 \geq \lambda_1 |\varphi|_{0, \Omega}^2 \quad \forall \varphi \in H_0^1(\Omega) \tag{1.12a}$$

$$|\varphi|_{0, \Omega}^2 \geq \lambda_1 \langle \mathcal{G}\varphi, \varphi \rangle \quad \forall \varphi \in L^2(\Omega) \tag{1.12b}$$

$$|\varphi|_{1, \Omega}^2 \geq \lambda_2 |\varphi|_{0, \Omega}^2 \quad \forall \varphi \in H_0^1(\Omega) \text{ such that } \langle \psi_1, \varphi \rangle = 0 \tag{1.12c}$$

$$|\varphi|_{0, \Omega}^2 \geq \lambda_2 \langle \mathcal{G}\varphi, \varphi \rangle \quad \forall \varphi \in L^2(\Omega) \text{ such that } \langle \psi_1, \varphi \rangle = 0 , \tag{1.12d}$$

where $|\cdot|_{m, \Omega}$ is the standard semi-norm on $H^m(\Omega)$. Equality holds in (1.12a

and b) for $\varphi \equiv \psi_1$ and in (1.12c and d) for $\varphi \equiv \psi_2$. It also follows that there exists λ_* such that

$$|\varphi|_{0,\Omega}^2 \geq \lambda_* \langle \mathcal{G}\varphi, \varphi \rangle \quad \forall \varphi \in L^2(\Omega) \text{ such that } \langle 1, \varphi \rangle = 0 \quad (1.13a)$$

with equality holding in the above for $\varphi \equiv \psi_*$, where ψ_* satisfies

$$\psi_* = \lambda_* \mathcal{G}\psi_* + C; \quad |\psi_*|_{0,\Omega} = 1 \quad (1.13b)$$

for some constant C so that $\langle 1, \psi_* \rangle = 0$. We note that $\lambda_* \in (\lambda_1, \lambda_2]$ since

$$\frac{1}{\lambda_*} \geq \frac{\langle \mathcal{G}\tilde{\psi}, \tilde{\psi} \rangle}{|\tilde{\psi}|_{0,\Omega}^2} = \left(\frac{\sigma^2}{\lambda_1} + \frac{1}{\lambda_2} \right) / (\sigma^2 + 1),$$

where $\tilde{\psi} = \sigma\psi_1 + \psi_2$ with $\sigma \in \mathbb{R}$ chosen so that $\langle 1, \tilde{\psi} \rangle = 0$.

The following theorem is a consequence of the existence and uniqueness results of Temam (1975, 1977) and Berestycki & Brezis (1980) for $\gamma = 0$, Cocker, Friedman & McLeod (1986) for $\gamma = 1$ and Barrett & Elliott (1989b) for $\gamma > 0$. We prove a discrete analogue, Theorem 2.1, in the next section.

THEOREM 1.1 : *If*

$$(\gamma + \kappa^2(1 - \gamma)/\lambda_1) > 0 \quad (1.14)$$

there exists a solution to (P_γ) and (P_0) and hence to (Q_γ) and (Q_0) . Furthermore, under the assumption (1.14).

(i) *If $\kappa^2 < \lambda_2$ the solution to (Q_γ) and (Q_0) is unique and hence $(Q_\gamma) \equiv (P_\gamma)$ and $(Q_0) \equiv (P_0)$. In the case $\gamma = 0$, it holds that*

$$\eta = \left[\kappa^2 \mathcal{G}\eta - \frac{q}{\kappa^2} \right]^+ . \quad (1.15)$$

(ii) *The constant $q \equiv -2J(\eta)/M$ is such that*

$$q(\kappa^2 - \lambda_1) > 0 \text{ unless } \kappa^2 = \lambda_1 \text{ and then } q = 0 . \quad (1.16)$$

(iii) *If $\kappa^2 < \lambda_1$ then $\Gamma = \emptyset$ and the variational inequality problems (Q_γ) and (Q_0) become variational equalities. If $\kappa^2 = \lambda_1$ then $\bar{\psi}_1 \equiv M\psi_1/(2\langle 1, \psi_1 \rangle) \in K_M \subset X_M$ is their unique solution. \square*

In this paper we analyse some finite element approximations using continuous piecewise linears of (P_γ) and (P_0) for $\kappa^2 \in (0, \lambda_2)$. The novelty of this analysis is that each of (1.5a, b) is an example of a non-coercive

variational inequality, and in addition its finite element approximation does not satisfy a discrete maximum principle. Therefore the standard error analysis techniques of Falk (1974), Baiocchi (1977) and Nitsche (1977) and its generalisations, Cortey-Dumont (1985*a* and *b*), do not apply directly. As noted previously (P_0) is a variational formulation of the « plasma problem ». Optimal error bounds for the continuous piecewise linear finite element approximation of the plasma problem with $\kappa^2 \in (0, \lambda_2)$ have been obtained by Barret & Elliott (1989*a*) and Caloz (1984), (1987). The analysis used in these papers is based upon the generalised implicit function theorem introduced by Girault and Raviart (1982) and first applied to the plasma problem by Kikuchi *et al* (1984) and Rappaz (1984). Numerical calculations based on this discretization have been reported by Sermange (1979). The approximation of (P_0) introduced in § 2 and analysed in § 4 is equivalent to this scheme. However, the error analysis presented in § 4 is based on the variational principle (P_0) as opposed to the generalised implicit function theorem and we believe the present approach to be simpler. A minor disadvantage of this error analysis is that it requires the triangulation to consist solely of acute-angled triangles.

The layout of this paper is as follows. In the next section we define our approximations of (P_γ) and (P_0) and prove the discrete analogue of Theorem 1.1 concerning existence and uniqueness of a solution. In § 3 and § 4 we prove optimal error bounds (H^1 , L^2 and L^∞) for these approximations of (P_γ) and (P_0) , respectively. In § 5 we study a more practical approximation of (P_0) involving mass lumping, yielding a scheme for the « plasma problem » as analysed in Kikuchi *et al* (1984), Barrett & Elliott (1989*a*) and Caloz (1988). Once again we prove optimal error bounds (H^1 , L^2 and L^∞) for this fully practical scheme. Finally in § 6 we consider an algorithm for solving a non-convex quadratic programming problem. The method presented is a generalisation of a scheme given in Berestycki & Brezis (1980). This approach yields a globally convergent iterative method for computing the approximations of (P_γ) and (P_0) given in the previous sections.

2 FINITE ELEMENT APPROXIMATION

Throughout this section we assume that either (a) Ω is polygonal or (b) $\partial\Omega \in C^{1,1}$. Let Ω^h be a polygonal approximation to Ω defined by $\bar{\Omega}^h = \bigcup_{\tau \in \mathcal{T}^h} \bar{\tau}$, where \mathcal{T}^h is a quasi-uniform triangulation consisting of acute-

angled triangles τ with maximum diameter not exceeding h . We assume that in case (a) $\Omega^h \equiv \Omega$ and in case (b) that $\text{dist}(\partial\Omega, \partial\Omega^h) \leq Ch^2$ and in addition for ease of exposition that $\Omega^h \subseteq \Omega$. Throughout C denotes a generic constant.

independent of h . Finite elements spaces S^h and S_0^h are defined by

$$S^h = \{ \chi \in C(\bar{\Omega}^h) : \chi|_{\tau} \text{ is linear } \forall \tau \in T^h \} \tag{2.1a}$$

$$S_0^h = \{ \chi \in C(\bar{\Omega}) : \chi|_{\bar{\Omega}^h} \in S^h \text{ and } \chi|_{\bar{\Omega} \setminus \Omega^h} = 0 \} \tag{2.1b}$$

and closed convex sets

$$\begin{aligned} K^h &= \{ \chi \in S_0^h : \chi \geq 0 \text{ in } \Omega \} \\ K_M^h &= \{ \chi \in K^h : \langle 1, \chi \rangle_{\Omega^h} = M/2 \} \end{aligned} \tag{2.2a}$$

and

$$\begin{aligned} X^h &= \{ \varphi \in L^2(\Omega^h) : \varphi \geq 0 \text{ in } \Omega^h \} \\ X_M^h &= \{ \varphi \in X^h : \langle 1, \varphi \rangle_{\Omega^h} = M/2 \} ; \end{aligned} \tag{2.2b}$$

where $\langle \cdot, \cdot \rangle_{\Omega^h}$ denotes the $L^2(\Omega^h)$ inner product. Note that X^h and X_M^h are not finite dimensional.

We note the inverse inequality

$$|\chi|_{1, \Omega} \leq Ch^{-1} |\chi|_{0, \Omega} \quad \forall \chi \in S^h \tag{2.3a}$$

and the discrete Sobolev embedding result

$$\|\chi\|_{0, \infty, \Omega} \leq C \left(\ln \frac{1}{h} \right)^{1/2} |\chi|_{1, \Omega} \quad \forall \chi \in S_0^h, \tag{2.3b}$$

see Thomee (1984) p. 67, for example.

We set $\mathcal{G}^h \in \mathcal{L}(L^2(\Omega^h), S_0^h)$ to be the discrete Green's operator defined by

$$\langle \nabla \mathcal{G}^h w, \nabla \chi \rangle = \langle w, \chi \rangle \quad \forall \chi \in S_0^h. \tag{2.4}$$

It follows from elliptic regularity that $\mathcal{G} \in \mathcal{L}(L^2(\Omega), W^{2,p}(\Omega))$ where in case (a) $p \in \left[\frac{4}{3} - \varepsilon, 2 \right]$, for any $\varepsilon > 0$, and in case (b) $p = 2$; see for example Grisvard (1985). We recall the following well-known inequalities :

$$\|\mathcal{G} - \mathcal{G}^h\|_{\mathcal{L}(L^2(\Omega), H_0^1(\Omega))} \leq Ch^{2-\frac{2}{p}} \tag{2.5a}$$

$$\|\mathcal{G} - \mathcal{G}^h\|_{\mathcal{L}(L^2(\Omega), L^2(\Omega))} \leq C \|\mathcal{G} - \mathcal{G}^h\|_{\mathcal{L}(L^2(\Omega), H_0^1(\Omega))}^2 \tag{2.5b}$$

$$|\mathcal{G}^h w|_{1, \Omega} + \|\mathcal{G}^h w\|_{0, \infty, \Omega} \leq C |w|_{0, \Omega^h} \tag{2.5c}$$

and in case (b) if $\mathcal{G}w \in W^{2,\infty}(\Omega)$

$$\|(\mathcal{G} - \mathcal{G}^h)w\|_{0,\infty,\Omega} \leq Ch^2 \ln \frac{1}{h} \|\mathcal{G}w\|_{2,\infty,\Omega}; \quad (2.5d)$$

see for example Schatz (1985) in case (a) and Crouzeix & Rappaz (1987) in case (b).

Let $\{(\lambda_i^h)^{-1}, \psi_i^h\}$ be the eigenvalues and eigenfunctions of \mathcal{G}^h viz.,

$$\psi_i^h = \lambda_i^h \mathcal{G}^h \psi_i^h; \quad |\psi_i^h|_{0,\Omega^h} = 1, \quad (2.6)$$

ordered so that $0 < \lambda_1^h < \lambda_2^h \leq \dots$. The minimax principle yields that $\lambda_i \leq \lambda_i^h$ for all i . The assumption that T^h in an acute-angled triangulation implies that (2.4) satisfies a discrete maximum principle, see Ciarlet & Raviart (1973), and so the Perron-Frobenius theory applies to \mathcal{G}^h . Therefore it follows that λ_1^h is simple and ψ_1^h can be taken to be positive in Ω^h . The following error estimates hold for h sufficiently small

$$\lambda_1 \leq \lambda_1^h \leq \lambda_1 + \|\mathcal{G} - \mathcal{G}^h\|_{\mathcal{L}(L^2(\Omega), H_0^1(\Omega))}^2 \quad (2.7a)$$

$$|\psi_1 - \psi_1^h|_{1,\Omega} \leq \|\mathcal{G} - \mathcal{G}^h\|_{\mathcal{L}(L^2(\Omega), H_0^1(\Omega))}, \quad (2.7b)$$

see for example Strang & Fix (1973).

In addition the following discrete analogues of (1.12) and (1.13) hold :

$$|\chi|_{1,\Omega}^2 \geq \lambda_1^h |\chi|_{0,\Omega}^2 \quad \forall \chi \in S_0^h \quad (2.8a)$$

$$|\varphi|_{0,\Omega^h}^2 \geq \lambda_1^h \langle \mathcal{G}^h \varphi, \varphi \rangle_{\Omega^h} \quad \forall \varphi \in L^2(\Omega^h) \quad (2.8b)$$

$$|\chi|_{1,\Omega}^2 \geq \lambda_2^h |\chi|_{0,\Omega}^2 \quad \forall \chi \in S_0^h \text{ such that } \langle \psi_1^h, \chi \rangle = 0 \quad (2.8c)$$

$$|\varphi|_{0,\Omega^h}^2 \geq \lambda_2^h \langle \mathcal{G}^h \varphi, \varphi \rangle_{\Omega^h} \quad \forall \varphi \in L^2(\Omega^h) \text{ such that } \langle \psi_1^h, \varphi \rangle_{\Omega^h} = 0 \quad (2.8d)$$

and

$$|\varphi|_{0,\Omega^h}^2 \geq \lambda_*^h \langle \mathcal{G}^h \varphi, \varphi \rangle_{\Omega^h} \quad \forall \varphi \in L^2(\Omega^h) \text{ such that } \langle 1, \varphi \rangle_{\Omega^h} = 0; \quad (2.9a)$$

with $\lambda_1^h < \lambda_*^h \leq \lambda_2^h$. Equality holds in (2.8a and b) for $\chi \equiv \varphi \equiv \psi_1^h$, in (2.8c and d) for $\chi \equiv \varphi \equiv \psi_2^h$ and in (2.9a) for $\varphi \equiv \psi_*^h$, where

$$\psi_*^h = \lambda_*^h \mathcal{G}^h \psi_*^h + C^h; \quad |\psi_*^h|_{0,\Omega^h} = 1 \quad (2.9b)$$

for some constant C^h so that $\langle 1, \psi_*^h \rangle_{\Omega^h} = 0$. It is a simple matter to show that for h sufficiently small

$$|\lambda_* - \lambda_*^h| \leq \|\mathcal{G} - \mathcal{G}^h\|_{\mathcal{L}(L^2(\Omega), H_0^1(\Omega))}^2. \quad (2.9c)$$

We outline the proof. There exist constants δ_i , $i = 1$ and 2 , with $|\delta_i| \leq Ch^2$ such that $\hat{\psi} \equiv \psi_* + \delta_1$ satisfies $\langle 1, \hat{\psi} \rangle_{\Omega^h} = 0$ and hence

$$\frac{1}{\lambda_*^h} \geq \frac{\langle \mathcal{G}^h \hat{\psi}, \hat{\psi} \rangle_{\Omega^h}}{|\hat{\psi}|_{0, \Omega^h}^2} = \frac{\langle \mathcal{G} \hat{\psi}, \hat{\psi} \rangle_{\Omega^h}}{|\hat{\psi}|_{0, \Omega^h}^2} + \frac{\langle (\mathcal{G}^h - \mathcal{G}) \hat{\psi}, \hat{\psi} \rangle_{\Omega^h}}{|\hat{\psi}|_{0, \Omega^h}^2}.$$

Therefore it follows that

$$\frac{1}{\lambda_*^h} \geq \frac{1 + \delta_2}{\lambda_*} - \|\mathcal{G} - \mathcal{G}^h\|_{\mathcal{L}(L^2(\Omega), L^2(\Omega))}.$$

In addition we have

$$\frac{1}{\lambda_*^h} \geq \frac{\langle \mathcal{G} \psi_*^h, \psi_*^h \rangle_{\Omega^h}}{|\psi_*^h|_{0, \Omega^h}^2} = \frac{1}{\lambda_*^h} - \|\mathcal{G} - \mathcal{G}^h\|_{\mathcal{L}(L^2(\Omega), L^2(\Omega))}.$$

Hence we obtain the desired result (2.9c).

For $\gamma \geq 0$, we set for $w, \varphi \in H^1(\Omega^h)$

$$a^h(w, \varphi) \equiv \gamma \langle \nabla w, \nabla \varphi \rangle_{\Omega^h} + \kappa^2(1 - \gamma) \langle w, \varphi \rangle_{\Omega^h} - \kappa^4 \langle \mathcal{G}^h w, \varphi \rangle_{\Omega^h} \tag{2.10a}$$

and

$$J^h(\varphi) \equiv a^h(\varphi, \varphi). \tag{2.10b}$$

Note that $\langle \mathcal{G}^h w, \varphi \rangle_{\Omega^h} = \langle w, \mathcal{G}^h \varphi \rangle_{\Omega^h}$ and hence $a^h(w, \varphi) \equiv a^h(\varphi, w)$.

We may now define the finite element approximations to (P_γ) and (P_0) :

(P_γ^h) ($\gamma > 0$). Find $\eta^h \in K_M^h$ such that

$$J^h(\eta^h) = \inf_{\chi \in K_M^h} J^h(\chi),$$

(P_0^h) ($\gamma = 0$). Find $\eta^h \in X_M^h$ such that

$$J^h(\eta^h) = \inf_{\varphi \in X_M^h} J^h(\varphi).$$

It follows immediately that solutions of (P_γ^h) and (P_0^h) solve the following variational inequalities, approximations of (Q_γ) and (Q_0) :

(Q_γ^h) ($\gamma > 0$). Find $\eta^h \in K_M^h$ such that

$$a^h(\eta^h, \chi - \eta^h) \geq 0 \quad \forall \chi \in K_M^h, \quad (2.11a)$$

(Q_0^h) ($\gamma = 0$). Find $\eta^h \in X_M^h$ such that

$$a^h(\eta^h, \varphi - \eta^h) \geq 0 \quad \forall \varphi \in K_M^h. \quad (2.11b)$$

Furthermore the solutions of (Q_γ^h) and (Q_0^h) satisfy the analogues of (1.6):

$$(\gamma > 0) \quad a^h(\eta^h, \chi - \eta^h) \geq \langle -q^h, \chi - \eta^h \rangle_{\Omega^h} \quad \forall \chi \in K^h \quad (2.12a)$$

$$(\gamma = 0) \quad a^h(\eta^h, \varphi - \eta^h) \geq \langle -q^h, \varphi - \eta^h \rangle_{\Omega^h} \quad \forall \varphi \in X^h, \quad (2.12b)$$

where

$$q^h \equiv -2 J^h(\eta^h)/M. \quad (2.12c)$$

We now prove a discrete analogue of Theorem 1.1.

THEOREM 2.1: *If*

$$(\gamma + \kappa^2(1 - \gamma)/\lambda_1^h) > 0 \quad (2.13)$$

there exists a solution η^h to (P_γ^h) and (P_0^h) and hence to (Q_γ^h) and (Q_0^h) such that for h sufficiently small

$$(\gamma > 0) \quad \|\eta^h\|_{1, \Omega^h} \leq C, \quad (\gamma = 0) \quad \|\eta^h\|_{0, \Omega^h} \leq C. \quad (2.14)$$

Furthermore, under the assumption (2.13).

(i) *If $\kappa^2 < \lambda_2^h$ the solution to (Q_γ^h) and (Q_0^h) is unique and hence $(Q_\gamma^h) \equiv (P_\gamma^h)$ and $(Q_0^h) \equiv (P_0^h)$. In the case $\gamma = 0$ it holds that*

$$\eta^h = \left[\kappa^2 \mathcal{G}^h \eta^h - \frac{q^h}{\kappa^2} \right]^+. \quad (2.15)$$

(ii) *The constant $q^h \equiv -2 J^h(\eta^h)/M$ is such that*

$$q^h(\kappa^2 - \lambda_1^h) > 0 \quad \text{unless} \quad \kappa^2 = \lambda_1^h \quad \text{and then} \quad q^h = 0. \quad (2.16)$$

(iii) *If $\kappa^2 = \lambda_1^h$ then $\bar{\psi}_1^h \equiv M\psi_1^h/(2\langle 1, \psi_1^h \rangle_{\Omega^h}) \in K_M^h \subset X_M^h$ is the unique solution of (P_γ^h) and (P_0^h) .*

Proof: The proof is a discrete analogue of that given in Barrett & Elliott (1989b) for the continuous problems (P_γ) and (Q_γ) . However, we give an

outline of the proof for completeness and for the statement of some inequalities which will be useful in the error analysis.

For any $\chi \in K_M^h$ or X_M^h it holds that

$$\begin{aligned} \langle \mathcal{G}^h \chi, \chi \rangle_{\Omega^h} &\leq \langle 1, \chi \rangle_{\Omega^h} \| \mathcal{G}^h \chi \|_{0, \infty, \Omega^h} \\ &= \frac{M}{2} \| \mathcal{G}^h \chi \|_{0, \infty, \Omega^h} \leq C | \chi |_{0, \Omega^h}, \end{aligned} \tag{2.17}$$

since (2.5c) holds.

If $\gamma > 1$ then using (2.8a) and (2.17) we obtain

$$\begin{aligned} J^h(\chi) &\geq (\gamma + \kappa^2(1 - \gamma)/\lambda_1^h) | \chi |_{1, \Omega^h}^2 - C | \chi |_{1, \Omega^h} \\ &\geq \frac{1}{2} (\gamma + \kappa^2(1 - \gamma)/\lambda_1^h) | \chi |_{1, \Omega^h}^2 - C_1 \quad \forall \chi \in K_M^h. \end{aligned} \tag{2.18}$$

If $1 \gg \gamma > 0$ then noting (2.17) we obtain

$$\begin{aligned} J^h(\chi) &\geq \gamma | \chi |_{1, \Omega^h}^2 - C | \chi |_{1, \Omega^h} \\ &\geq \frac{1}{2} \gamma | \chi |_{1, \Omega^h}^2 - C_2 \quad \forall \chi \in K_M^h. \end{aligned} \tag{2.19}$$

If $\gamma = 0$ then noting (2.17) we obtain

$$\begin{aligned} J^h(\varphi) &\geq \kappa^2 | \varphi |_{0, \Omega^h}^2 - C | \varphi |_{0, \Omega^h} \\ &\geq \frac{1}{2} \kappa^2 | \varphi |_{0, \Omega^h}^2 - C_3 \quad \forall \varphi \in X_M^h. \end{aligned} \tag{2.20}$$

Therefore under the assumption (2.13) $J^h(\cdot)$ is bounded below on K_M^h for $\gamma > 0$ and on X_M^h for $\gamma = 0$. Standard minimising sequence arguments now yield the existence of a minimiser and hence a solution η^h to (P_γ^h) and (P_0^h) .

It follows directly from (2.6) and (2.8a and b) that for $\gamma \geq 0$

$$J^h(\bar{\Psi}_1^h) = (\gamma \lambda_1^h + \kappa^2) (\lambda_1^h - \kappa^2) | \bar{\Psi}_1^h |_{0, \Omega^h}^2 / \lambda_1^h. \tag{2.21}$$

Using (2.8a and b) we have that

$$(\gamma > 0) \quad J^h(\chi) \geq (\gamma \lambda_1^h + \kappa^2) (\lambda_1^h - \kappa^2) | \chi |_{0, \Omega^h}^2 / \lambda_1^h \quad \forall \chi \in S_0^h \tag{2.22a}$$

$$(\gamma = 0) \quad J^h(\varphi) \geq \kappa^2 (\lambda_1^h - \kappa^2) | \varphi |_{0, \Omega^h}^2 / \lambda_1^h \quad \forall \varphi \in L^2(\Omega^h). \tag{2.22b}$$

Consideration of (2.21) and (2.22) yields (2.16); and the fact if $\kappa^2 = \lambda_1^h$ then $\bar{\Psi}_1^h \in K_M^h$ is a solution of (P_γ^h) and (P_0^h) .

Another consequence of (2.18)-(2.21) is that for h sufficiently small,

noting (2.7), the solutions η^h of (P_γ^h) and (P_0^h) for fixed γ , κ^2 and M are bounded independently of h ; that is, (2.14) holds; since

$$J^h(\eta^h) \leq J^h(\bar{\psi}_1^h) \leq C.$$

Rewriting (2.12b) as $\eta^h \in X^h$ such that

$$\langle \eta^h, \chi - \eta^h \rangle_{\Omega^h} \geq \langle \kappa^2 \mathcal{G}^h \eta^h - q^h/\kappa^2, \chi - \eta^h \rangle_{\Omega^h} \quad \forall \chi \in X^h \quad (2.23)$$

it follows that (2.15) holds.

We now turn to the proof of uniqueness for $\kappa^2 < \lambda_2^h$. Let η_1^h and η_2^h be solutions of (Q_γ^h) , $\gamma \geq 0$, such that

$$a^h(\eta_1^h, \eta_1^h) \leq a^h(\eta_2^h, \eta_2^h). \quad (2.24)$$

It follows from (2.11) that

$$a^h(\eta_i^h, \eta_i^h) \leq a^h(\eta_1^h, \eta_1^h) \quad i = 1, 2. \quad (2.25)$$

For $\beta > 0$ set $\tilde{\eta} = \eta_1^h - \beta \eta_2^h$. It follows from (2.24) and (2.25) that

$$J^h(\tilde{\eta}) \equiv a^h(\tilde{\eta}, \tilde{\eta}) \leq (1 - \beta)^2 a^h(\eta_2^h, \eta_2^h). \quad (2.26)$$

If $\kappa^2 < \lambda_1^h$ then take $\beta = 1$ and recall (2.22). It follows that

$$\begin{aligned} (\gamma > 0) \quad \tilde{\eta} \in S_0^h \quad & |\tilde{\eta}|_{1, \Omega^h}^2 \leq 0 \\ (\gamma = 0) \quad \tilde{\eta} \in L^2(\Omega^h) \quad & |\tilde{\eta}|_{0, \Omega^h}^2 \leq 0, \end{aligned}$$

which implies uniqueness. If $\kappa^2 = \lambda_1^h$ then again take $\beta = 1$ and note (2.26) together with (2.8a and b) imply that $\tilde{\eta} = \alpha \psi_1^h$ for some constant α . Since $\langle 1, \tilde{\eta} \rangle_{\Omega^h} = 0$ it follows that $\alpha = 0$ and we have uniqueness.

If $\kappa^2 \in (\lambda_1^h, \lambda_2^h)$ then choose β so that $\langle \psi_1^h, \tilde{\eta} \rangle_{\Omega^h} = 0$. It follows from (2.8c and d) that

$$\begin{aligned} (\gamma > 0) \quad J^h(\chi) \geq (\gamma \lambda_2^h + \kappa^2) (\lambda_2^h - \kappa^2) |\chi|_{0, \Omega^h}^2 / \lambda_2^h \\ \forall \chi \in S_0^h \quad \text{such that} \quad \langle \psi_1^h, \chi \rangle_{\Omega^h} = 0 \end{aligned} \quad (2.27a)$$

$$\begin{aligned} (\gamma = 0) \quad J^h(\varphi) \geq \kappa^2 (\lambda_2^h - \kappa^2) |\varphi|_{0, \Omega^h}^2 / \lambda_2^h \\ \forall \varphi \in L^2(\Omega^h) \quad \text{such that} \quad \langle \psi_1^h, \varphi \rangle_{\Omega^h} = 0. \end{aligned} \quad (2.27b)$$

Also we have from (2.11) that for $i = 1, 2$ taking $\bar{\psi}_1^h \in K_M^h \subset X_M^h$

$$\begin{aligned} a^h(\eta_i^h, \eta_i^h) &\leq a^h(\eta_i^h, \bar{\psi}_1^h) \\ &= (\gamma \lambda_1^h + \kappa^2) (\lambda_1^h - \kappa^2) \langle \eta_i^h, \bar{\psi}_1^h \rangle / \lambda_1^h \leq 0 \end{aligned} \quad (2.28)$$

and combining this with (2.26) and (2.27) with $\chi = \varphi = \tilde{\eta}$ we obtain that $\eta_1^h = \beta \eta_2^h$. Since $\langle 1, \eta_1^h \rangle_{\Omega^h} = \langle 1, \eta_2^h \rangle_{\Omega^h}$ we have $\beta = 1$ and hence uniqueness. \square

Note since $\lambda_i \leq \lambda_i^h$ if (1.14) holds so does (2.13) and both η and η^h are unique for $\kappa^2 \in (0, \lambda_2)$. In the next two sections we prove the following theorems concerning the solutions η and η^h of (P_γ) and (P_γ^h) , $\gamma \geq 0$, in the case of a smooth boundary with $\Omega^h \subseteq \Omega$ for ease of exposition.

THEOREM 2.2: *Let $\partial\Omega \in C^{1,1}$ then given $\gamma > 0$, $\kappa^2 \in (0, \lambda_2)$ satisfying (1.14) and $M > 0$ there exist positive constants h_0 and C , depending on γ , κ^2 and M , such that the unique solutions η and η^h of $(P_\gamma) \equiv (Q_\gamma)$ and $(P_\gamma^h) \equiv (Q_\gamma^h)$, respectively, satisfy for $h \leq h_0$*

$$|\eta - \eta^h|_{1, \Omega} \leq Ch, \tag{2.29a}$$

$$|\eta - \eta^h|_{0, \Omega} \leq Ch^2 \left(\ln \frac{1}{h} \right)^2, \tag{2.29b}$$

$$\|\eta - \eta^h\|_{0, \infty, \Omega} \leq Ch^2 \left(\ln \frac{1}{h} \right)^{5/2}. \tag{2.29c}$$

\square

It follows immediately that similar bounds hold for $u - u^h$ and $v - v^h$ where, see (1.7),

$$\begin{aligned} \xi^{(h)} &\equiv [(\gamma - 1) \eta^{(h)} + 2 \kappa^2 \mathcal{G}^{(h)} \eta^{(h)}] / (\gamma + 1) \\ u^{(h)} &\equiv \xi^{(h)} + \eta^{(h)}, \quad v^{(h)} \equiv \xi^{(h)} - \eta^{(h)}. \end{aligned} \tag{2.30}$$

THEOREM 2.3: *Let $\partial\Omega \in C^{2,1}$ then given $\kappa^2 \in (0, \lambda_2)$ and $M > 0$ there exist positive constants h_0 and C , depending on κ^2 and M , such that the unique solutions η and η^h of $(P_0) \equiv (Q_0)$ and $(P_0^h) \equiv (Q_0^h)$, respectively, satisfy for $h \leq h_0$*

$$|\eta - \eta^h|_{0, \Omega^h} \leq Ch^2. \tag{2.31}$$

In addition the constants $q \equiv -2 J(\eta)/M$ and $q^h \equiv -2 J^h(\eta^h)/M$ are such that

$$|q - q^h| \leq Ch^2. \tag{2.32}$$

\square

We noted previously in the case $\gamma = 0$, see (1.9), that if we set

$$u \equiv 2 \kappa^2 \mathcal{G} \eta \quad \text{and} \quad d \equiv 2 q / \kappa^2 \tag{2.33}$$

then it follows from (1.15) that $\{u, d\} \in H_0^1(\Omega) \times \mathbb{R}$ solves the « plasma problem » (1.10); that is

$$u = \kappa^2 \mathcal{G}[u - d]^+, \quad \langle 1, [u - d]^+ \rangle = M. \tag{2.34}$$

On setting

$$u^h \equiv 2 \kappa^2 \mathcal{G}^h \eta^h \quad \text{and} \quad d^h \equiv 2 q^h / \kappa^2 \tag{2.35}$$

it follows from (2.15) that $\{u^h, d^h\} \in S_0^h \times \mathbb{R}$ satisfies

$$u^h = \kappa^2 \mathcal{G}^h [u^h - d^h]^+, \quad \langle 1, [u^h - d^h]^+ \rangle_{\Omega^h} = M \tag{2.36}$$

and is therefore the standard piecewise linear finite element approximation to the « plasma problem » as studied by Sermange (1979) and analysed by Barrett & Elliott (1989a) and Caloz (1987) using the generalised implicit function theorem. An immediate consequence of (2.31), (2.32), (2.33), (2.35) and (2.5) is that under the assumptions of Theorem 2.3 $u \in C^{2, \alpha}(\bar{\Omega})$, $0 < \alpha < 1$, and hence for $h \leq h_0$:

$$|d - d^h| + |u - u^h|_{0, \Omega} + h |u - u^h|_{1, \Omega} \leq Ch^2 \tag{2.37a}$$

$$\|u - u^h\|_{0, \infty, \Omega} \leq Ch^2 \ln \frac{1}{h}. \tag{2.37b}$$

In addition we note that since

$$\eta = \frac{1}{2} [u - d]^+ \quad \text{and} \quad \eta^h = \frac{1}{2} [u^h - d^h]^+, \tag{2.38}$$

it follows from (2.37) that

$$\|\eta - \eta^h\|_{0, \infty, \Omega^h} \leq Ch^2 \ln \frac{1}{h}. \tag{2.39}$$

Finally we note the free boundary regularity result of Kinderlehrer & Spruck (1978): for $\kappa^2 \in (\lambda_1, \lambda_2)$ Γ is an analytic curve in the interior of Ω , $|\nabla u| \neq 0$ on Γ and the sets Ω_+ , $\Omega_0 \equiv \Omega \setminus \bar{\Omega}_+$ are connected. In addition we note that $\int_{\Gamma} \frac{\partial u}{\partial \nu} ds = - \int_{\partial \Omega} \frac{\partial u}{\partial \nu} ds = \kappa^2 M$, where $\underline{\nu}$ is the outward unit normal to Ω_0 . Hence it follows that Γ has finite length.

On setting

$$\Gamma^h \equiv \{x \in \Omega^h : u^h(x) = d^h\}, \tag{2.40}$$

one can show using the above results concerning Γ and (2.37) that

$$\begin{aligned} \text{meas}(\Gamma^h) &= 0, \quad \Gamma^h \text{ has finite length} \\ \text{dist}(\Gamma, \Gamma^h) &\leq Ch^2 \ln \frac{1}{h}; \end{aligned} \tag{2.41}$$

see Barrett & Elliott (1989a) for details.

3. ERROR BOUNDS FOR (P_γ^h)

In this section we prove Theorem 2.2 under its stated assumptions. It is convenient to introduce

$$f \equiv -\kappa^2(1 - \gamma) \eta + \kappa^4 \mathcal{G}\eta; \tag{3.1}$$

so that η , the solution of $(P_\gamma) \equiv (Q_\gamma)$ (cf. (1.6a)) is the unique solution of the obstacle problem : find $\eta \in K$ such that

$$\gamma \langle \nabla \eta, \nabla \varphi - \nabla \eta \rangle \geq \langle f - q, \varphi - \eta \rangle \quad \forall \varphi \in K. \tag{3.2}$$

Standard regularity theory for variational inequalities, Rodrigues (1987), implies that

$$\eta \in W^{2,p}(\Omega), \quad 1 < p < \infty, \quad \text{and hence } \eta \in C^{1,\alpha}(\bar{\Omega}), \tag{3.3}$$

$$0 < \alpha < 1,$$

under the assumption $\partial\Omega \in C^{1,1}$.

The main idea of the proof is to estimate $E^h = \bar{\eta}^h - \eta^h$ using a modification of the uniqueness proof for (P_γ^h) where $\bar{\eta}^h \in K^h$ is the unique solution of

$$\gamma \langle \nabla \bar{\eta}^h, \nabla \chi - \nabla \bar{\eta}^h \rangle_{\Omega^h} \geq \langle f - q, \chi - \bar{\eta}^h \rangle_{\Omega^h} \quad \forall \chi \in K^h, \tag{3.4}$$

and to note that standard arguments (Falk (1974)) yield

$$|\eta - \bar{\eta}^h|_{1,\Omega} \leq Ch |\eta|_{2,\Omega} \tag{3.5a}$$

and an L^∞ error analysis using the discrete maximum principle yields

$$\|\eta - \bar{\eta}^h\|_{0,\infty,\Omega} \leq C \left(h \ln \frac{1}{h} \right)^2 \tag{3.5b}$$

see Cortey-Dumont (1985b), Nitsche (1977) and Baiocchi (1977).

Setting

$$M^h \equiv 2 \langle 1, \bar{\eta}^h \rangle_{\Omega^h} \tag{3.6a}$$

$$f^h \equiv -\kappa^2(1 - \gamma) \bar{\eta}^h + \kappa^4 \mathcal{G}^h \bar{\eta}^h \tag{3.6b}$$

it is clear that

$$a^h(\bar{\eta}^h, \chi - \bar{\eta}^h) \geq \langle f - f^h - q, \chi - \bar{\eta}^h \rangle_{\Omega^h} \quad \forall \chi \in K^h. \tag{3.7}$$

It is convenient to prove now the following lemma, the results of which will be needed later.

LEMMA 3.1: *There exist positive constants h_0 , C and δ such that for $h \leq h_0$*

$$|M - M^h| \leq C \left(h \ln \frac{1}{h} \right)^2 \quad \text{and} \quad M^h \geq \delta > 0 \quad (3.8)$$

$$|f - f^h|_{0, \Omega} \leq C \left(h \ln \frac{1}{h} \right)^2 \quad (3.9)$$

and

$$|q - q^h| \leq C \left(h \ln \frac{1}{h} \right)^2. \quad (3.10)$$

Proof: By definition

$$M - M^h = 2 \langle 1, \eta - \bar{\eta}^h \rangle_{\Omega^h}$$

and (3.8) is an immediate consequence of (3.5b).

Similarly

$$f - f^h = -\kappa^2(1 - \gamma) (\eta - \bar{\eta}^h) + \kappa^4 (\mathcal{G} - \mathcal{G}^h) \eta + \kappa^4 \mathcal{G}^h (\eta - \bar{\eta}^h)$$

and (3.9) follows from (2.5) and (3.5b).

In order to prove (3.10) we estimate $q - q^h$ from above and below. By definition

$$q^h - q = \frac{2}{M} [J(\eta) - J^h(\eta^h)].$$

Since $K_M^h \subseteq K_M$ it follows that $J(\eta) \leq J(\eta^h)$ and hence

$$\begin{aligned} q^h - q &\leq \frac{2}{M} \kappa^4 \langle (\mathcal{G}^h - \mathcal{G}) \eta^h, \eta^h \rangle_{\Omega^h} \\ &\leq Ch^2 \end{aligned}$$

by (2.5) and (2.14). Similarly, setting

$$\eta_*^h = M \bar{\eta}^h / M^h$$

we have that $\eta_*^h \in K_M^h$ and hence $J^h(\eta^h) \leq J^h(\eta_*^h)$. Noting that taking $\chi \equiv 0$ and $\chi \equiv 2 \bar{\eta}^h$ in (3.7) yields

$$a^h(\bar{\eta}^h, \bar{\eta}^h) = \langle f - f^h - q, \bar{\eta}^h \rangle_{\Omega^h}$$

and so we obtain

$$\begin{aligned} q - q^h &= q + \frac{2}{M} J^h(\eta^h) \leq q + \frac{2}{M} J^h(\eta_*^h) \\ &= q \left(1 - \frac{M}{M^h} \right) + \frac{2}{M} \left(\frac{M}{M^h} \right)^2 \langle f - f^h, \bar{\eta}^h \rangle_{\Omega^h}. \end{aligned}$$

Applying (3.8) and (3.9) yields the desired result (3.10). \square

In order to prove Theorem 2.2 it is sufficient to show that

$$|E^h|_{1, \Omega^h} \leq C \left(h \ln \frac{1}{h} \right)^2, \tag{3.11}$$

where $E^h \equiv \bar{\eta}^h - \eta^h$. The $\|\cdot\|_{0, \infty, \Omega}$ bound then follows from the discrete Sobolev embedding result (2.3b). As previously mentioned the method we use for proving (3.11) is a modification of the uniqueness proof for (P_γ^h) . Setting

$$\tilde{E}^h \equiv \bar{\eta}^h - \beta \eta^h \tag{3.12}$$

for some positive constant β to be determined, we have

$$a^h(\tilde{E}^h, \tilde{E}^h) = -a^h(\bar{\eta}^h, \beta \eta^h - \bar{\eta}^h) - \beta^2 a^h(\eta^h, \bar{\eta}^h/\beta - \eta^h) \tag{3.13}$$

and applying the variational inequalities (2.12a) and (3.7) it follows that

$$\begin{aligned} a^h(\tilde{E}^h, \tilde{E}^h) &\leq \beta \langle q^h, \tilde{E}^h \rangle_{\Omega^h} + \langle f - f^h - q, \tilde{E}^h \rangle_{\Omega^h} \\ &= [\beta(q^h - q) + (\beta - 1)q] \langle 1, \tilde{E}^h \rangle_{\Omega^h} \\ &\quad + \langle f - f^h, \tilde{E}^h \rangle_{\Omega^h}. \end{aligned} \tag{3.14}$$

We consider first the case $\kappa \in (0, \lambda_1]$. We set

$$\beta = M^h/M \quad \text{and so} \quad \langle 1, \tilde{E}^h \rangle_{\Omega^h} = 0. \tag{3.15}$$

It follows from (2.7a), (2.8a), (2.9a and c) and $\lambda_1 < \lambda_*$ that for h sufficiently small

$$\begin{aligned} a^h(\tilde{E}^h, \tilde{E}^h) &\geq [\gamma \lambda_1^h + \kappa^2(1 - \gamma) - \kappa^4/\lambda_*^h] |\tilde{E}^h|_{0, \Omega^h}^2 \\ &= \frac{1}{\lambda_1^h} \{ [\gamma \lambda_1^h + \kappa^2] [\lambda_1^h - \kappa^2] + \kappa^4(1 - \lambda_1^h/\lambda_*^h) \} |\tilde{E}^h|_{0, \Omega^h}^2 \\ &\geq \frac{\kappa^4}{\lambda_1^h} (1 - \lambda_1^h/\lambda_*^h) |\tilde{E}^h|_{0, \Omega^h}^2 \\ &\geq C |\tilde{E}^h|_{0, \Omega^h}^2, \end{aligned} \tag{3.16}$$

for some constant C independent of h . Therefore combining (3.9), (3.14), (3.15), (3.16) yields that

$$\begin{aligned} |\tilde{E}^h|_{0, \Omega^h}^2 &\leq C a^h(\tilde{E}^h, \tilde{E}^h) \\ &\leq C \left(h \ln \frac{1}{h} \right)^2 |\tilde{E}^h|_{0, \Omega^h} \end{aligned} \tag{3.17a}$$

and hence

$$|\tilde{E}^h|_{0, \Omega^h} \leq C \left(h \ln \frac{1}{h} \right)^2. \tag{3.17b}$$

Furthermore, since

$$\begin{aligned} \gamma |\tilde{E}^h|_{1, \Omega^h}^2 &\leq -\kappa^2(1 - \gamma) |\tilde{E}^h|_{0, \Omega^h}^2 + \kappa^4 |\mathcal{G}^h \tilde{E}^h|_{0, \Omega^h} |\tilde{E}^h|_{0, \Omega^h} \\ &\quad + a^h(\tilde{E}^h, \tilde{E}^h) \end{aligned} \tag{3.18}$$

it follows from (3.17) and (2.5c) that

$$|\tilde{E}^h|_{1, \Omega^h} \leq C \left(h \ln \frac{1}{h} \right)^2. \tag{3.19}$$

We finally obtain (3.11) for $\kappa^2 \in (0, \lambda_1]$ by noting that

$$E^h = \tilde{E}^h + \left(\frac{M^h}{M} - 1 \right) \eta^h$$

and applying the bounds (2.14), (3.8) and (3.19).

We consider now the case $\kappa^2 \in (\lambda_1, \lambda_2)$. We set

$$\beta = \langle \psi_1^h, \bar{\eta}^h \rangle_{\Omega^h} / \langle \psi_1^h, \eta^h \rangle_{\Omega^h} \tag{3.20a}$$

and so

$$\langle \psi_1^h, \tilde{E}^h \rangle_{\Omega^h} = 0. \tag{3.20b}$$

Clearly $\beta > 0$ is well-defined by the positivity of ψ_1^h and the non-negativity of η^h over Ω^h . Furthermore (2.7) and (2.14) imply that β is uniformly bounded independently of h . Also we observe that

$$\langle 1, \tilde{E}^h \rangle_{\Omega^h} = \frac{1}{2} (M^h - \beta M) = \frac{1}{2} (1 - \beta) M + \frac{1}{2} (M^h - M) \tag{3.21a}$$

and noting (3.8) and that $q > 0$, see (1.16), we obtain that

$$\begin{aligned} (\beta - 1) q \langle 1, \tilde{E}^h \rangle_{\Omega^h} &\leq \frac{q}{2} (\beta - 1) (M^h - M) \\ &= q \frac{(M^h - M)}{M} \left\{ \frac{1}{2} (M^h - M) - \langle 1, \tilde{E}^h \rangle_{\Omega^h} \right\} \\ &\leq C \left(h \ln \frac{1}{h} \right)^4 + C \left(h \ln \frac{1}{h} \right)^2 |\tilde{E}^h|_{0, \Omega^h}. \end{aligned} \tag{3.21b}$$

Hence from (3.14), (3.9), (3.10) and (3.21) it follows that

$$a^h(\tilde{E}^h, \tilde{E}^h) \leq C \left(h \ln \frac{1}{h} \right)^2 |\tilde{E}^h|_{0, \Omega^h} + C \left(h \ln \frac{1}{h} \right)^4. \tag{3.22}$$

It follows from (2.27a) and (3.20b) that

$$\begin{aligned} a^h(\tilde{E}^h, \tilde{E}^h) &\geq (\gamma\lambda_2^h + \kappa^2) (\lambda_2^h - \kappa^2) |\tilde{E}^h|_{0, \Omega^h/\lambda_2^h}^2 \\ &\geq C |\tilde{E}^h|_{0, \Omega^h}^2 \end{aligned} \tag{3.23}$$

for some constant C , independent of h , depending on $\kappa^2 < \lambda_2 \leq \lambda_2^h$. Combining (3.22) and (3.23) we obtain

$$|\tilde{E}^h|_{0, \Omega^h} \leq C \left(h \ln \frac{1}{h} \right)^2 \tag{3.24a}$$

and from (3.18) with (3.22) it follows that

$$|\tilde{E}^h|_{1, \Omega^h} \leq C \left(h \ln \frac{1}{h} \right)^2 \tag{3.24b}$$

Therefore we finally obtain (3.11) for $\kappa^2 \in (\lambda_1, \lambda_2)$ by noting that (3.21a), (3.8) and (3.24a) imply that

$$|1 - \beta| \leq C \left(h \ln \frac{1}{h} \right)^2 \tag{3.25}$$

and hence

$$\begin{aligned} |E^h|_{1, \Omega^h} &\leq |\tilde{E}^h|_{1, \Omega^h} + |1 - \beta| |\eta^h|_{1, \Omega^h} \\ &\leq C \left(h \ln \frac{1}{h} \right)^2, \end{aligned} \tag{3.26}$$

where we have noted (2.14).

4. ERROR BOUNDS FOR (P_0^h)

In this section we prove Theorem 2.3 under its stated assumptions. From (1.15) we have η the unique solution of $(P_0) \equiv (Q_0)$, is such that

$$\eta = \left[\kappa^2 \mathcal{G}\eta - \frac{q}{\kappa^2} \right]^+ \tag{4.1}$$

Since $w \in W^{1,p}(\Omega)$, $1 \leq p \leq \infty$, $\Rightarrow [w]^+ \in W^{1,p}(\Omega)$, see for example Kinderlehrer & Stampacchia (1980), p. 50 ; we have that

$$\eta \in W^{1,\infty}(\Omega) \text{ and hence } \eta \in C^{0,\alpha}(\bar{\Omega}), \quad 0 < \alpha < 1, \tag{4.2}$$

under the assumption $\partial\Omega \in C^{2,1}$. It follows that

$$u \equiv 2 \kappa^2 \mathcal{G}\eta \in C^{2,\alpha}(\bar{\Omega}), \quad 0 < \alpha < 1. \tag{4.3}$$

In proving the error bound it is convenient first to establish the following estimate.

LEMMA 4.1: *There exist positive constants h_0 , C and δ such that for $h \leq h_0$*

$$0 \leq M - M^h \leq Ch^2 \quad \text{and} \quad M^h \geq \delta > 0 \quad (4.4a)$$

and

$$|q - q^h| \leq Ch^2, \quad (4.4b)$$

where

$$M^h \equiv 2 \langle 1, \eta \rangle_{\Omega^h}. \quad (4.5)$$

Proof. The result (4.4a) follows directly from $\Omega^h \subseteq \Omega$, $\text{dist}(\partial\Omega, \partial\Omega^h) \leq Ch^2$ and $\eta \in C(\bar{\Omega})$. Let us now prove (4.4b). We define

$$\bar{\eta}^h \equiv \begin{cases} \eta^h & \text{in } \bar{\Omega}^h \\ 0 & \text{in } \Omega \setminus \bar{\Omega}^h. \end{cases} \quad (4.6)$$

By definition

$$q^h - q = \frac{2}{M} [J(\eta) - J^h(\eta^h)]$$

and since $\bar{\eta}^h \in X_M$

$$\begin{aligned} q^h - q &\leq \frac{2}{M} [J(\bar{\eta}^h) - J^h(\eta^h)] \\ &= \frac{2}{M} \kappa^4 \langle (\mathcal{G}^h - \mathcal{G}) \bar{\eta}^h, \bar{\eta}^h \rangle \\ &\leq Ch^2, \end{aligned}$$

where we have applied (2.5) and the uniform boundedness of $|\eta^h|_{0, \Omega^h}$, (2.14). Similarly we have, noting that $M\eta/M^h \in X_M^h$

$$\begin{aligned} q - q^h &= \frac{2}{M} [J^h(\eta^h) - J(\eta)] \\ &\leq \frac{2}{M} \left[J^h \left(\frac{M\eta}{M^h} \right) - J(\eta) \right] \\ &= \frac{2}{M} \left\{ \left[\left(\frac{M}{M^h} \right)^2 - 1 \right] J^h(\eta) + [J^h(\eta) - J(\eta)] \right\} \\ &= \frac{2}{M} \left\{ \frac{(M + M^h)(M - M^h)}{(M^h)^2} J^h(\eta) + \kappa^4 \langle (\mathcal{G} - \mathcal{G}^h) \eta, \eta \rangle \right\}. \end{aligned}$$

It follows from (4.4a) and (2.5) that

$$q - q^h \leq Ch^2,$$

which completes the proof of (4.4b) \square

Once again the method we use for proving the error bound is a modification of the uniqueness proof for (P_0^h) . Setting

$$\tilde{E}^h = \begin{cases} \eta - \beta\eta^h & \text{in } \Omega^h \\ 0 & \text{in } \Omega \setminus \bar{\Omega}^h \end{cases} \tag{4.7}$$

for some positive constant β to be determined, we obtain

$$\begin{aligned} a^h(\tilde{E}^h, \tilde{E}^h) &= -a(\eta, \beta\eta^h - \eta) - \beta^2 a^h(\eta^h, \eta/\beta - \eta^h) \\ &\quad + \kappa^4 \langle (\mathcal{G} - \mathcal{G}^h), \eta, \tilde{E}^h \rangle \end{aligned}$$

and applying the variational inequalities (1.6b), (2.12b) and (2.5) it follows that

$$a^h(\tilde{E}^h, \tilde{E}^h) \leq [\beta(q^h - q) + (\beta - 1)q] \langle 1, \tilde{E}^h \rangle_{\Omega^h} + Ch^2 |\tilde{E}^h|_{0, \Omega^h} \tag{4.8}$$

We consider first the case $\kappa^2 \in (0, \lambda_1]$. We set

$$\beta = M^h/M \quad \text{and so} \quad \langle 1, \tilde{E}^h \rangle_{\Omega^h} = 0 \tag{4.9}$$

It follows from (2.7a), (2.9a) and c) and $\lambda_1 < \lambda_*$ that for h sufficiently small

$$\begin{aligned} a^h(\tilde{E}^h, \tilde{E}^h) &\geq \frac{\kappa^4}{\lambda_1^h} (1 - \lambda_1^h/\lambda_*^h) |\tilde{E}^h|_{0, \Omega^h}^2 \\ &\geq C |\tilde{E}^h|_{0, \Omega^h}^2 \end{aligned} \tag{4.10}$$

for some constant C independent of h . Therefore combining (4.8), (4.9) and (4.10) yields that

$$|\tilde{E}^h|_{0, \Omega^h} \leq Ch^2 \tag{4.11}$$

and hence for $\kappa^2 \in (0, \lambda_1]$ we have shown that

$$\begin{aligned} |\eta - \eta^h|_{0, \Omega'} &\leq |\tilde{E}^h|_{0, \Omega^h} + \left| 1 - \frac{M^h}{M} \right| |\eta^h|_{0, \Omega^h} \\ &\leq Ch^2, \end{aligned} \tag{4.12}$$

where we have noted (2.14) and (4.4a)

We consider now the case $\kappa^2 \in (\lambda_1, \lambda_2)$. We set

$$\beta = \langle \psi_1^h, \eta \rangle_{\Omega^h} / \langle \psi_1^h, \eta^h \rangle_{\Omega^h} \tag{4 13a}$$

so that

$$\langle \psi_1^h, \tilde{E}^h \rangle_{\Omega^h} = 0 \tag{4 13b}$$

Clearly β is well-defined by the positivity of ψ_1^h and the non-negativity of η^h over Ω^h and is uniformly bounded independently of h . Applying the same argument as in (3 21) we obtain

$$\langle 1, \tilde{E}^h \rangle_{\Omega^h} = \frac{1}{2} (1 - \beta) M + \frac{1}{2} (M^h - M) \tag{4 14a}$$

and noting (4 4a) and that $q > 0$, see (1 16), yields

$$(\beta - 1) q \langle 1, \tilde{E}^h \rangle_{\Omega^h} \leq Ch^4 + Ch^2 |\tilde{E}^h|_0_{\Omega^h} \tag{4 14b}$$

Hence from (4 8), (4 4b), (4 14b) and (3 23) we obtain

$$C |\tilde{E}^h|_0_{\Omega^h}^2 \leq a^h(\tilde{E}^h, \tilde{E}^h) \leq Ch^2 |\tilde{E}^h|_0_{\Omega^h} + Ch^4$$

and thus

$$|\tilde{E}^h|_0_{\Omega^h} \leq Ch^2 \tag{4 15}$$

Combining (4 4a), (4 14a) and (4 15) yields that

$$|1 - \beta| \leq Ch^2 \tag{4 16}$$

and hence we obtain the desired result for $\kappa^2 \in (\lambda_1, \lambda_2)$

$$\begin{aligned} |\eta - \eta^h|_0_{\Omega^h} &\leq |\tilde{E}^h|_0_{\Omega^h} + |1 - \beta| |\eta^h|_0_{\Omega^h} \\ &\leq Ch^2, \end{aligned} \tag{4 17}$$

where we have noted (2 14)

5 A MORE PRACTICAL APPROXIMATION OF (P_0)

Whereas (P_γ^h) leads to a fully practical method for obtaining approximations to (P_γ) , see § 6, the approximation (P_0^h) to (P_0) introduced in § 2 and analysed in § 4 requires the term $\langle 1, \eta^h \rangle_{\Omega^h} \equiv \left\langle 1, \left[\kappa^2 \mathcal{G}^h \eta^h - \frac{q}{\kappa^2} \right]^+ \right\rangle_{\Omega^h}$ to be integrated exactly. To obtain the approximation η^h the globally convergent iterative method presented in § 6 requires one to solve a

sequence of problems of the following type : given $\chi \in S_0^h$ find $\mu \in \mathbb{R}$ such that

$$\langle 1, [\chi - \mu]^+ \rangle_{\Omega^h} = M/2 . \tag{5.1}$$

Although this is possible it is computationally more convenient to consider a scheme where numerical integration is applied to this term. Below we introduce and analyse such a scheme. We assume that Ω and Ω^h satisfy the assumptions as stated in the opening paragraph of § 2.

With $\bar{\Omega}^h = \bigcup_{\tau \in T^h} \bar{\tau}$ and $\{a_i\}_{i=1}^3$ being the vertices of τ we define the quadrature rule for $w \in C(\bar{\tau})$

$$I_\tau(w) \equiv \frac{1}{3} \text{meas}(\tau) \sum_{i=1}^3 w(a_i) \tag{5.2}$$

approximating $\int_\tau w$, and then set for $w, \varphi \in C(\bar{\Omega}^h)$

$$\langle w, \varphi \rangle^h \equiv \sum_{\tau \in T^h} I_\tau(w\varphi) \tag{5.3}$$

as an approximation to $\langle w, \varphi \rangle_{\Omega^h}$. On setting $|w|_h \equiv [\langle w, w \rangle^h]^{1/2}$, it holds that for $\chi, \varphi \in S^h$

$$(i) \quad |\chi|_{0, \Omega^h} \leq |\chi|_h \leq 2 |\chi|_{0, \Omega^h} \tag{5.4a}$$

$$(ii) \quad \left| \int_\tau \varphi \chi - I_\tau(\varphi \chi) \right| \leq Ch^2 |\varphi|_{1, \tau} |\chi|_{1, \tau} , \tag{5.4b}$$

see for example Kikuchi *et al.* (1984).

Given any $\varphi \in C(\bar{\Omega}^h)$ we denote by $\pi_h \varphi$ that element of S^h such that

$$\varphi(a_i) = \pi_h \varphi(a_i) \quad i = 1 \rightarrow 3, \quad \forall \tau \in T^h .$$

We introduce the discrete Green's operator, in the presence of numerical integration, $\hat{\mathcal{G}}^h \in \mathcal{L}(C(\bar{\Omega}^h), S_0^h)$ defined by

$$\langle \nabla \hat{\mathcal{G}}^h w, \nabla \chi \rangle = \langle w, \chi \rangle^h \quad \forall \chi \in S_0^h . \tag{5.5}$$

The following well-known inequalities for $\chi \in S^h$ follow immediately from (2.4), (2.5), (2.3), (5.4) and (5.5)

$$|(\mathcal{G}^h - \hat{\mathcal{G}}^h) \chi|_{1, \Omega^h} \leq Ch^2 |\chi|_{1, \Omega^h} \tag{5.6a}$$

$$|\hat{\mathcal{G}}^h \chi|_{1, \Omega^h} + \|\hat{\mathcal{G}}^h \chi\|_{0, \infty, \Omega^h} \leq C |\chi|_{0, \Omega^h} . \tag{5.6b}$$

Let $\{(\hat{\lambda}_i^h)^{-1}, \hat{\psi}_i^h\}$ be the eigenvalues and eigenfunctions of $\hat{\mathcal{G}}^h$

$$\hat{\psi}_i^h = \hat{\lambda}_i^h \hat{\mathcal{G}}^h \hat{\psi}_i^h; \quad |\psi_i^h|_h = 1, \tag{5.7}$$

ordered so that $0 < \hat{\lambda}_1^h < \hat{\lambda}_2^h \leq \dots$. As T^h is an acute angled triangulation (5.5) satisfies a discrete maximum principle and so Perron-Frobenius theory applies to $\hat{\mathcal{G}}^h$, see for example Barrett & Elliott (1989a). Therefore it follows that $\hat{\lambda}_1^h$ is simple and $\hat{\psi}_1^h$ can be taken to be positive in Ω^h . The following error estimates hold for h sufficiently small.

$$|\lambda_i^h - \hat{\lambda}_i^h| \leq Ch^2 \quad i = 1 \quad \text{and} \quad 2 \tag{5.8a}$$

$$|\psi_1^h - \hat{\psi}_1^h|_{0,\Omega} \leq Ch^2. \tag{5.8b}$$

In addition the following analogues of (2.8) and (2.9) hold

$$|\chi|_{1,\Omega}^2 \geq \hat{\lambda}_1^h |\chi|_h^2 \quad \forall \chi \in S_0^h \tag{5.9a}$$

$$|\varphi|_h^2 \geq \hat{\lambda}_1^h \langle \hat{\mathcal{G}}^h \varphi, \varphi \rangle^h \quad \forall \varphi \in C(\bar{\Omega}^h) \tag{5.9b}$$

$$|\chi|_{1,\Omega}^2 \geq \hat{\lambda}_2^h |\chi|_h^2 \quad \forall \chi \in S_0^h \text{ such that } \langle \hat{\psi}_1^h, \chi \rangle^h = 0 \tag{5.9c}$$

$$|\varphi|_h^2 \geq \hat{\lambda}_2^h \langle \hat{\mathcal{G}}^h \varphi, \varphi \rangle^h \quad \forall \varphi \in C(\bar{\Omega}^h) \text{ such that } \langle \hat{\psi}_1^h, \varphi \rangle^h = 0, \tag{5.9d}$$

and

$$|\varphi|_h^2 \geq \hat{\lambda}_*^h \langle \hat{\mathcal{G}}^h \varphi, \varphi \rangle^h \quad \forall \varphi \in C(\bar{\Omega}^h) \text{ such that } \langle 1, \varphi \rangle^h = 0, \tag{5.10a}$$

with

$$\hat{\lambda}_1^h < \hat{\lambda}_*^h \leq \hat{\lambda}_2^h.$$

Equality holding in (5.9a and b) for $\chi \equiv \varphi \equiv \hat{\psi}_1^h$, in (5.9c and d) for $\chi \equiv \varphi \equiv \hat{\psi}_2^h$ and in (5.10a) for $\varphi = \hat{\psi}_*^h$, where

$$\hat{\psi}_*^h = \hat{\lambda}_*^h \hat{\mathcal{G}}^h \hat{\psi}_*^h + \hat{C}^h; \quad |\hat{\psi}_*^h|_h = 1 \tag{5.10b}$$

for some constant \hat{C}^h so that $\langle 1, \hat{\psi}_*^h \rangle^h = 0$. Applying a similar argument to that for $|\lambda_* - \hat{\lambda}_*^h|$ in § 2 it is a simple matter to show that for h sufficiently small

$$|\lambda_*^h - \hat{\lambda}_*^h| \leq Ch^2. \tag{5.10c}$$

We set for $w, \varphi \in C(\bar{\Omega}^h)$

$$\hat{d}^h(w, \varphi) \equiv \kappa^2 \langle w - \kappa^2 \mathcal{G}^h w, \varphi \rangle^h \tag{5.11a}$$

and

$$\hat{J}^h(\varphi) \equiv \hat{d}^h(\varphi, \varphi). \tag{5.11b}$$

Note that $\langle \mathcal{G}^h w, \varphi \rangle^h \equiv \langle w, \mathcal{G}^h \varphi \rangle^h$ and hence $\hat{d}^h(w, \varphi) \equiv \hat{d}^h(\varphi, w)$. We now define a more practical finite element approximation to (P_0) than (\hat{P}_0^h) :

(\hat{P}_0^h) Find $\hat{\eta}^h \in \hat{X}_M^h$ such that

$$\hat{J}^h(\hat{\eta}^h) = \inf_{\chi \in \hat{X}_M^h} \hat{J}^h(\chi);$$

where

$$\hat{X}^h = \{ \chi \in S^h : \chi \geq 0 \text{ in } \Omega^h \} \tag{5.12a}$$

$$\hat{X}_M^h = \{ \chi \in \hat{X}^h : \langle 1, \chi \rangle_{\Omega^h} = M/2 \}. \tag{5.12b}$$

Solutions of (\hat{P}_0^h) solve

(\hat{Q}_0^h) Find $\hat{\eta}^h \in \hat{X}_M^h$ such that

$$\hat{d}^h(\hat{\eta}^h, \chi - \hat{\eta}^h) \geq 0 \quad \forall \chi \in \hat{X}_M^h. \tag{5.13}$$

Furthermore, solutions of (\hat{Q}_0^h) satisfy

$$\hat{d}^h(\hat{\eta}^h, \chi - \hat{\eta}^h) \geq \langle -\hat{q}^h, \chi - \hat{\eta}^h \rangle^h \quad \forall \chi \in \hat{X}^h, \tag{5.14a}$$

where

$$\hat{q}^h \equiv -2 \hat{J}^h(\hat{\eta}^h)/M. \tag{5.14b}$$

We have the following analogue of Theorem 2.1 for $\gamma = 0$.

THEOREM 5.1 : *There exists a solution $\hat{\eta}^h$ to (\hat{P}_0^h) and (\hat{Q}_0^h) , such that for h sufficiently small*

$$|\hat{\eta}^h|_{0, \Omega^h} \leq C. \tag{5.15}$$

Furthermore,

(i) *If $\kappa^2 < \hat{\lambda}_2^h$ the solution to (\hat{Q}_0^h) is unique and satisfies*

$$\hat{\eta}^h = \pi_h \left[\kappa^2 \mathcal{G}^h \hat{\eta}^h - \frac{\hat{q}^h}{\kappa^2} \right]^+. \tag{5.16}$$

Hence for $\kappa^2 < \hat{\lambda}_2^h$ it holds that $(\hat{Q}_0^h) \equiv (\hat{P}_0^h)$

(ii) The constant $\hat{q}^h \equiv -2 \hat{J}^h(\hat{\eta}^h)/M$ is such that

$$\hat{q}^h(\kappa^2 - \hat{\lambda}_1^h) > 0 \text{ unless } \kappa^2 = \hat{\lambda}_1^h \text{ and then } \hat{q}^h = 0 \tag{5 17}$$

(iii) If $\kappa^2 = \hat{\lambda}_1^h$ then $M\hat{\psi}_1^h / \left(2 \langle 1, \hat{\psi}_1^h \rangle_{\Omega^h} \right) \in \hat{X}_M^h$ is the unique solution of (\hat{P}_0^h)

Proof The proof follows in exactly the same way as the proof of Theorem 2 1 and hence is omitted \square

It follows from (5 11), (5 4a) and (5 6b) that

$$|\hat{J}^h(\chi)| \leq C |\chi|_0^2_{\Omega^h} \quad \forall \chi \in S^h \tag{5 18}$$

Hence it follows from (5 14b), (5 18) and (5 15) that \hat{q}^h is bounded independently of h for h sufficiently small, since

$$|\hat{q}^h| \leq \frac{2}{M} |\hat{J}^h(\hat{\eta}^h)| \leq C |\hat{\eta}^h|_0^2_{\Omega^h} \leq C \tag{5 19}$$

A simple calculation yields that

$$|\pi_h[\chi]^+|_1_{\Omega^h} \leq |\chi|_1_{\Omega^h} \quad \forall \chi \in S^h \tag{5 20}$$

Hence it follows from (5 16), (5 6b) and (5 15) that $|\hat{\eta}^h|_1_{\Omega^h}$ is also bounded independently of h for h sufficiently small, since

$$|\hat{\eta}^h|_1_{\Omega^h} \leq \left| \kappa^2 \mathcal{G}^h \hat{\eta}^h - \frac{\hat{q}^h}{\kappa^2} \right|_1_{\Omega^h} \leq C \tag{5 21}$$

We now prove error bounds for $\hat{\eta}^h$ assuming $\partial\Omega \in C^{2-1}$ and $\Omega \subseteq \Omega^h$ for ease of exposition

LEMMA 5 1 *Let $\partial\Omega \in C^{2-1}$ then, given $\kappa^2 \in (0, \lambda_2)$ and $M > 0$, there exist positive constants h_0 and C such that*

$$|\eta - \pi_h \eta|_0_{\Omega'} \leq Ch^2 \tag{5 22a}$$

and for $h \leq h_0$

$$|M - \hat{M}^h| \leq Ch^2, \tag{5 22b}$$

where η is the unique solution of $(P_0) \equiv (Q_0)$ and

$$\hat{M}^h = 2 \langle 1, \pi_h \eta \rangle_{\Omega^h} \tag{5 22c}$$

Proof: Defining u and d using (2.33), we set

$$\bar{\Omega}_+^h \equiv \bigcup_{\tau \in T_+^h} \bar{\tau}, \quad \bar{\Omega}_0^h = \bigcup_{\tau \in T_0^h} \bar{\tau}, \quad \bar{\Omega}_-^h \equiv \bigcup_{\tau \in T_-^h} \bar{\tau}, \tag{5.23a}$$

where

$$\begin{aligned} T_+^h &\equiv \{ \tau \in T^h : u(x) > d \quad \forall x \in \bar{\tau} \} \\ T_0^h &\equiv \{ \tau \in T^h : \Gamma \cap \bar{\tau} \neq \emptyset \} \\ T_-^h &\equiv \{ \tau \in T^h : u(x) < d \quad \forall x \in \bar{\tau} \}. \end{aligned} \tag{5.23b}$$

It follows that

$$\bar{\Omega}^h \equiv \bar{\Omega}_+^h \cup \bar{\Omega}_-^h \cup \bar{\Omega}_0^h \tag{5.24a}$$

and recalling that Γ has finite length (see the end of § 2) yields that

$$\text{meas}(\Omega_0^h) \leq Ch. \tag{5.24b}$$

It holds on Ω_+^h that $\eta \equiv \frac{1}{2}(u - d) \in C^{2,\alpha}(\bar{\Omega}_+^h)$, $0 < \alpha < 1$, and hence

$$|\eta - \pi_h \eta|_{0, \Omega_+^h} \leq Ch^2. \tag{5.25}$$

On Ω_-^h $\eta \equiv 0$ and on Ω_0^h we have $\eta \equiv \frac{1}{2}[u - d]^+ \in W^{1,\infty}(\Omega_0^h)$ and hence

$$\begin{aligned} |\eta - \pi_h \eta|_{0, \Omega_0^h} &\leq Ch |\eta - \pi_h \eta|_{0, \infty, \Omega_0^h} \\ &\leq Ch^2. \end{aligned} \tag{5.26}$$

Therefore combining these results we obtain the desired result (5.22a). Noting that $M - \hat{M}^h = (M - M^h) + (M^h - \hat{M}^h)$ where M^h is defined by (4.5), the desired result (5.22b) follows from (4.4a) and (5.22a). \square

Let $\bar{\lambda}_2^h \equiv \min \{ \lambda_2, \hat{\lambda}_2^h \}$. Then for $\kappa^2 \in (0, \bar{\lambda}_2^h)$ both η and $\hat{\eta}^h$, the solutions of (P_0) and (\hat{P}_0^h) , are unique.

LEMMA 5.2 : *Let $\partial\Omega \in C^{2,1}$ then given $\kappa^2 \in (0, \bar{\lambda}_2^h)$ and $M > 0$ there exist positive constants h_0 and C such that for $h \leq h_0$*

$$|q^h - \hat{q}^h| \leq Ch^2. \tag{5.27}$$

Proof: By definition

$$\begin{aligned} \hat{q}^h - q^h &= \frac{2}{M} [J^h(\eta^h) - \hat{J}^h(\hat{\eta}^h)] \\ &\leq \frac{2}{M} [J^h(\hat{\eta}^h) - \hat{J}^h(\hat{\eta}^h)], \end{aligned} \tag{5.28}$$

since $\hat{\eta}^h \in \hat{X}_M^h \subset X_M^h$. From (2.10), (5.11), (5.4) and (5.6) it follows that

$$|J^h(\chi) - \hat{J}^h(\chi)| \leq Ch^2 |\chi|_{1, \Omega^h}^2 \quad \forall \chi \in S^h. \quad (5.29)$$

Therefore noting (5.28), (5.29) and (5.21) we obtain

$$\hat{q}^h - q^h \leq Ch^2. \quad (5.30)$$

Similarly we have, noting that $M(\pi_h \eta)/\hat{M}^h \in \hat{X}_M^h$,

$$\begin{aligned} q^h - \hat{q}^h &= \frac{2}{M} [J^h(\hat{\eta}^h) - J^h(\eta^h)] \\ &\leq \frac{2}{M} \left[\hat{J}^h \left(\frac{M}{\hat{M}^h} (\pi_h \eta) \right) - J^h(\eta^h) \right] \\ &= \frac{2}{M} \left[\left\{ \left(\frac{M}{\hat{M}^h} \right)^2 - 1 \right\} J^h(\pi_h \eta) + \{ \hat{J}^h(\pi_h \eta) - J^h(\pi_h \eta) \} + \right. \\ &\quad \left. + \{ J^h(\pi_h \eta) - J^h(\eta^h) \} \right] \\ &= s_1 + s_2 + s_3. \end{aligned} \quad (5.31)$$

Noting (5.22), (5.18), (5.29) and (4.2) we have that $|s_1| + |s_2| \leq Ch^2$.

From (2.10), (2.5c), (2.31) and (5.22a) we obtain that $|s_3| \leq Ch^2$. Thus we have

$$q^h - \hat{q}^h \leq Ch^2$$

and hence the desired result (5.27) holds. \square

Given $f \in C(\bar{\Omega}^h)$ then the solution of the variational inequality: find $F \in \hat{X}^h$ such that

$$\langle F - f, \chi - F \rangle^h \geq 0 \quad \forall \chi \in \hat{X}^h \quad (5.32)$$

is $F \equiv \pi_h[f]^+$, since choosing $\chi \equiv 0$ and $\chi \equiv 2F$ in (5.32) yields $\langle F - f, F \rangle^h = 0$. Therefore it follows from (1.15) that $\pi_h \eta \in \hat{X}^h$ satisfies

$$\left\langle \pi_h \eta - \left[\kappa^2 \mathcal{G} \eta - \frac{q}{\kappa^2} \right], \chi - \pi_h \eta \right\rangle^h \geq 0 \quad \forall \chi \in \hat{X}^h \quad (5.33)$$

and hence

$$\hat{a}^h(\pi_h \eta, \chi - \pi_h \eta) \geq \langle \kappa^4 (\mathcal{G} - \hat{\mathcal{G}}^h) \eta - q, \chi - \pi_h \eta \rangle^h \quad \forall \chi \in \hat{X}^h, \quad (5.34)$$

noting that $\hat{\mathcal{G}}^h \pi_h \eta \equiv \hat{\mathcal{G}}^h \eta$.

On setting

$$\hat{E}^h = \pi_h \eta - \beta \hat{\eta}^h \tag{5.35}$$

for some positive constant β to be determined we obtain from (5.14a) and (5.34) that

$$\hat{a}^h(\hat{E}^h, \hat{E}^h) \leq \kappa^4 \langle (\mathcal{G} - \hat{\mathcal{G}}^h) \eta, \hat{E}^h \rangle^h + [\beta \hat{q}^h - q] \langle 1, \hat{E}^h \rangle_{\Omega^h}. \tag{5.36}$$

It follows from (5.4a) that

$$\left| \langle (\mathcal{G} - \hat{\mathcal{G}}^h) \eta, \hat{E}^h \rangle^h \right| \leq 4 \left| \pi_h \mathcal{G} \eta - \hat{\mathcal{G}}^h \eta \right|_{0, \Omega^h} \left| \hat{E}^h \right|_{0, \Omega^h}. \tag{5.37}$$

Now

$$\left| \pi_h \mathcal{G} \eta - \hat{\mathcal{G}}^h \eta \right|_{0, \Omega^h} \leq \left| (I - \pi_h) \mathcal{G} \eta \right|_{0, \Omega^h} + \left| (\mathcal{G} - \hat{\mathcal{G}}^h) \eta \right|_{0, \Omega^h}. \tag{5.38}$$

The first term on the right-hand side of (5.38) can be bounded by using interpolation error bounds and elliptic regularity :

$$\left| (I - \pi_h) \mathcal{G} \eta \right|_{0, \Omega^h} \leq Ch^2 \left| \mathcal{G} \eta \right|_{2, \Omega} \leq Ch^2 \left| \eta \right|_{0, \Omega}. \tag{5.39}$$

The second term can be bounded simply using the splitting approach of Crouzeix and Rappaz (1987) p. 43 and the bounds (5.22a), (2.5) and (5.6)

$$\begin{aligned} \left| (\mathcal{G} - \hat{\mathcal{G}}^h) \eta \right|_{0, \Omega^h} &\leq \left| \mathcal{G} [(I - \pi_h) \eta] \right|_{0, \Omega^h} + \left| (\mathcal{G} - \hat{\mathcal{G}}^h) \pi_h \eta \right|_{0, \Omega^h} \\ &\quad + \left| (\mathcal{G}^h - \hat{\mathcal{G}}^h) \pi_h \eta \right|_{0, \Omega^h} \\ &\leq Ch^2. \end{aligned} \tag{5.40}$$

Hence combining (5.36)-(5.40) we obtain

$$\hat{a}^h(\hat{E}^h, \hat{E}^h) \leq [\beta(\hat{q}^h - q) + (\beta - 1)q] \langle 1, \hat{E}^h \rangle_{\Omega^h} + Ch^2 \left| \hat{E}^h \right|_{0, \Omega^h}. \tag{5.41}$$

A direct analogue of the argument given in § 4, (4.9)-(4.17), for the bound $\left| \eta - \hat{\eta}^h \right|_{0, \Omega^h}$ yields that

$$\left| \pi_h \eta - \hat{\eta}^h \right|_{0, \Omega^h} \leq Ch^2.$$

Combining this bound with (5.22a) we have the following result.

THEOREM 5.2 : *Let $\partial\Omega \in C^{2,1}$ then given $\kappa^2 \in (0, \bar{\lambda}_2^h)$ and $M > 0$ there exist positive constants h_0 and C , depending on κ^2 and M , such that the unique*

solutions η and $\hat{\eta}^h$ of $(P_0) \equiv (Q_0)$ and $(\hat{P}_0^h) \equiv (\hat{Q}_0^h)$, respectively, satisfy for $h \leq h_0$

$$|\eta - \hat{\eta}^h|_{0,\Omega} \leq Ch^2. \quad (5.42)$$

On setting □

$$\hat{u}^h \equiv 2\kappa^2 \hat{\mathcal{G}}^h \hat{\eta}^h \quad \text{and} \quad \hat{d}^h \equiv 2\hat{q}^h/\kappa^2 \quad (5.43)$$

it follows from (5.16) that $\{\hat{u}^h, \hat{d}^h\} \in S_0^h \times \mathbb{R}$ satisfies

$$\hat{u}^h = \kappa^2 \hat{\mathcal{G}}^h[\hat{u}^h - \hat{d}^h]^+, \quad \langle 1, [\hat{u}^h - \hat{d}^h]^+ \rangle^h = M \quad (5.44)$$

and is the piecewise linear finite element approximation in the presence of numerical integration to the « plasma problem » as studied by Kikuchi *et al.* (1984), Barrett & Elliott (1989a) and Caloz (1988) using the generalised implicit function theorem. From (2.36) and (5.43) we have that

$$u^h - \hat{u}^h = 2\kappa^2 \mathcal{G}^h(\eta^h - \hat{\eta}^h) + 2\kappa^2(\mathcal{G}^h - \hat{\mathcal{G}}^h) \hat{\eta}^h. \quad (5.45)$$

Hence it follows from (2.5c), (5.6a), (2.31), (5.42) and (5.21) that

$$|u^h - \hat{u}^h|_{1,\Omega} \leq Ch^2. \quad (5.46)$$

Therefore combining Theorems 2.3 and 5.2, Lemma 5.2 and the Sobolev discrete embedding inequality (2.3b) we obtain the following result : under the assumptions of Theorem 5.2 and for $h \leq h_0$

$$|d - \hat{d}^h| + |u - \hat{u}^h|_{0,\Omega} + h|u - \hat{u}^h|_{1,\Omega} \leq Ch^2 \quad (5.47a)$$

$$\|u - \hat{u}^h\|_{0,\infty,\Omega} \leq Ch^2 \ln \frac{1}{h}. \quad (5.47b)$$

Finally we note that on setting

$$\hat{\Gamma}^h \equiv \{x \in \Omega^h : \hat{u}^h(x) = \hat{d}^h\} \quad (5.48)$$

one can show using the regularity results concerning Γ , see the end of § 2, and (5.47) that

$$\begin{aligned} \text{meas}(\hat{\Gamma}^h) &= 0, & \hat{\Gamma}^h &\text{ has finite length} \\ \text{dist}(\Gamma, \hat{\Gamma}^h) &\leq Ch^2 \ln \frac{1}{h}; \end{aligned} \quad (5.49)$$

once again see Barrett & Elliott (1989a) for details.

6. AN ALGORITHM FOR A NON-CONVEX QUADRATIC PROGRAMMING PROBLEM

In this section we consider an algorithm for solving a non-convex quadratic programming problem. The method presented is a generalisation of a scheme given in Berestycki & Brezis (1980). This approach yields a globally convergent iterative method for computing the approximation of (P_γ) and (P_0) given in the previous sections.

Let V and H be real Hilbert spaces such that $V \subseteq H$ with the injection being compact. Let $b(\cdot, \cdot)$ and $c(\cdot, \cdot)$ be symmetric continuous bilinear forms on V and H respectively with the properties

$$\exists \alpha > 0 \text{ such that } b(\varphi, \varphi) \geq \alpha \|\varphi\|_V^2 \quad \forall \varphi \in V \tag{6.1a}$$

$$c(\varphi, \varphi) \geq 0 \quad \forall \varphi \in H \text{ and we set } |\varphi|_c \equiv [c(\varphi, \varphi)]^{1/2}. \tag{6.1b}$$

We set

$$a(w, \varphi) \equiv b(w, \varphi) - c(w, \varphi). \tag{6.2}$$

The optimization problem we wish to consider is :

(P) Find $w \in W$ such that

$$I(w) = \inf_{\varphi \in W} I(\varphi); \tag{6.3}$$

where W is a closed convex non-empty subset of V ,

$$I(\varphi) \equiv a(\varphi, \varphi) - 2 \ell(\varphi) \tag{6.4}$$

and $\ell(\cdot) : V \rightarrow \mathbb{R}$ is a bounded linear functional. It follows that a solution of (P) also solves the variational inequality :

(Q) Find $w \in W$ such that

$$b(w, \varphi - w) \geq c(w, \varphi - w) + \ell(\varphi - w) \quad \forall \varphi \in W. \tag{6.5}$$

Any solution of (6.5) is said to be a **critical point** of (P). We consider the following iterative procedure to solve (Q) :

(A) Given $w_0 \in W$, construct the sequence $\{w_n\}_{n=1}^\infty \in W$ by solving for each $n \geq 1$

$$b(w_n, \varphi - w_n) \geq c(w_{n-1}, \varphi - w_n) + \ell(\varphi - w_n), \quad \forall \varphi \in W. \tag{6.6}$$

THEOREM 6.1 : *Assume that there exist positive constants α_0 and C_0 such that*

$$I(\varphi) \geq \alpha_0 \|\varphi\|_V^2 - C_0 \quad \forall \varphi \in W. \tag{6.7}$$

Then every sequence $\{w_n\}$ generated by Algorithm (A) possesses a subsequence convergent in V to a critical point of (P) . Also the limit point of any subsequence of $\{w_n\}$ weakly convergent in V , and hence strongly convergent in H , is a critical point of (P) . Furthermore if the critical points of (P) are isolated then the whole sequence converges in V to a critical point of (P) .

Proof : Since (6.1a) holds there exists a unique solution to the variational inequality (6.6) which satisfies

$$b(w_n, w_n) - \ell(w_n) - c(w_{n-1}, w_n) \leq b(w_{n-1}, w_n) - \ell(w_{n-1}) - c(w_{n-1}, w_{n-1})$$

and, upon rearranging the above inequality, for $n \geq 1$

$$I(w_n) + |w_n - w_{n-1}|_c^2 + \alpha \|w_n - w_{n-1}\|_V^2 \leq I(w_{n-1}). \tag{6.8}$$

After summation we obtain for all $n \geq 1$

$$I(w_n) + \sum_{k=1}^n \{ |w_k - w_{k-1}|_c^2 + \alpha \|w_k - w_{k-1}\|_V^2 \} \leq I(w_0). \tag{6.9}$$

It follows from (6.9) and (6.7) that

$$\|w_n\|_V \leq C(w_0), \quad \forall n \geq 1 \tag{6.10}$$

where $C(w_0)$ is a positive constant depending on w_0 , and

$$\lim_{n \rightarrow \infty} \|w_n - w_{n-1}\|_V = 0 = \lim_{n \rightarrow \infty} |w_n - w_{n-1}|_c. \tag{6.11}$$

Since V is compactly imbedded in H it follows from (6.10) that there exists a subsequence $\{w_{n_p}\}_{p=1}^\infty$ of $\{w_n\}$ such that as $n_p \rightarrow \infty$

$$w_{n_p} \rightarrow w_* \text{ weakly in } V \text{ and strongly in } H, \tag{6.12}$$

and $w_* \in W$ since W is a closed convex subset of V . The strong convergence of $\{w_{n_p}\}$ in H and (6.11) yields that for any $\varphi \in W$

$$\lim_{n_p \rightarrow \infty} |c(w_{n_p} - w_{n_p-1}, \varphi - w_{n_p})| \leq \lim_{n_p \rightarrow \infty} |w_{n_p} - w_{n_p-1}|_c |\varphi - w_{n_p}|_c = 0.$$

Hence we may pass to the limit in

$$\begin{aligned} b(w_{n_p}, \varphi - w_{n_p}) - \ell(\varphi - w_{n_p}) &\geq c(w_{n_p-1}, \varphi - w_{n_p}) \\ &= c(w_{n_p}, \varphi - w_{n_p}) + c(w_{n_p-1} - w_{n_p}, \varphi - w_{n_p}) \end{aligned}$$

for each $\varphi \in W$ using the continuity of $b(\cdot, \varphi)$, the lower semi-continuity of $b(\cdot, \cdot)$ on V , the continuity of $c(\cdot, \cdot)$ on H and the continuity of $\ell(\cdot)$ on V in order to obtain (6.5) for $w = w_*$. Therefore w_* is a critical point of (P) . The same argument applies to any subsequence of $\{w_n\}$ satisfying (6.12). Furthermore it follows from (6.5) and (6.6) that

$$b(w - w_n, w - w_n) \leq c(w - w_{n-1}, w - w_n) \tag{6.13}$$

and hence from (6.1a) that

$$\alpha \|w_* - w_{n_p}\|_V^2 \leq [|w_* - w_{n_p}|_c^2 + |w_{n_p} - w_{n_p-1}|_c |w_* - w_{n_p}|_c] .$$

Therefore from (6.11) and the strong convergence in H it follows that $\{w_{n_p}\}$ converges strongly in V to w_* .

We now consider the situation where the critical points of (P) are isolated, in which case there exists $\delta > 0$ such that each critical point is the centre of a ball in V of radius 3δ containing no other critical point. Let $\sigma(w_0)$ be the set of limit points of $\{w_n\}$. Suppose $\hat{w} \in \sigma(w_0)$. It follows from (6.13) and (6.1a) that there exists a constant $\mu > 1$ such that for $n \geq 0$

$$\|\hat{w} - w_{n+1}\|_V \leq \mu \|\hat{w} - w_n\|_V . \tag{6.14}$$

Set

$$B(\hat{w}, \varepsilon) \equiv \{ \varphi \in V ; \|\hat{w} - \varphi\|_V < \varepsilon \}$$

and let $\{w_{n_q}\}$ be that subsequence of $\{w_n\}$ such that $\{w_{n_q}\} \in B(\hat{w}, \delta/\mu)$ for all q . It follows from (6.14) that

$$\{w_{n_q+1}\} \in B(\hat{w}, \delta) . \tag{6.15}$$

We wish to show that $\sigma(w_0)$ consists of the singleton \hat{w} and therefore the whole sequence converges to \hat{w} . Now from (6.15) either there exists an infinite subsequence $\{w_{n_k}\} \in B(\hat{w}, \delta) \setminus B(\hat{w}, \delta/\mu)$ of $\{w_{n_q+1}\}$ or not. If not this implies that the whole sequence $\{w_n\}$ converges to \hat{w} as required. If there does exist $\{w_{n_k}\}$ as above then it possesses a subsequence with limit point $w_* \in B(\hat{w}, \delta) \setminus B(\hat{w}, \delta/\mu) \subset B(\hat{w}, 3\delta)$ but not equal to \hat{w} , which is a contradiction to \hat{w} being isolated. Therefore $\sigma(w_0)$ consists of a singleton. \square

Algorithm (A) applies directly to the problems (P_γ) , (P_0) , (P_γ^h) , (P_0^h) and (\hat{P}_0^h) met in the previous sections. Throughout these examples

$\ell(\cdot) \equiv 0$. For problem (P_γ) we set $V \equiv H_0^1(\Omega)$, $H \equiv L^2(\Omega)$, $a(\cdot, \cdot) \equiv a(\cdot, \cdot)$, $I(\cdot) \equiv J(\cdot)$,

$$b(w, \varphi) \equiv \gamma \langle \nabla w, \nabla \varphi \rangle + \kappa^2(1 - \gamma) \langle w, \varphi \rangle \quad \forall w, \varphi \in H_0^1(\Omega) \quad (6.16a)$$

and

$$c(w, \varphi) \equiv \kappa^4 \langle \mathcal{G}w, \varphi \rangle \quad \forall w, \varphi \in L^2(\Omega). \quad (6.16b)$$

Under the assumption (1.14) conditions (6.1a) and (6.7) hold; and clearly (6.1b) holds since $\langle \mathcal{G}w, \varphi \rangle \equiv \langle \nabla \mathcal{G}w, \nabla \mathcal{G}\varphi \rangle$. For problem (P_γ^h) we set $V \equiv H \equiv S_0^h$, $a(\cdot, \cdot) \equiv a^h(\cdot, \cdot)$, $I(\cdot) \equiv J^h(\cdot)$, $b(\cdot, \cdot)$ and $c(\cdot, \cdot)$ are as in (6.16) with \mathcal{G} replaced by \mathcal{G}^h . Under the assumption (2.13) conditions (6.1a) and (6.7) hold, see (2.18) and (2.19), and similarly to the above (6.1b) holds. We have used algorithm (A) to compute solutions to (P_γ^h) even for $\kappa^2 > \lambda_2^h$ see the numerical examples later in this section. At each step of the algorithm, see (6.6), given $\eta_n^h \in K_M^h$ one finds $\eta_{n+1}^h \in K_M^h$ such that

$$\begin{aligned} & \gamma \langle \nabla \eta_{n+1}^h, \nabla (\chi - \eta_{n+1}^h) \rangle_{\Omega^h} + \kappa^2(1 - \gamma) \langle \eta_{n+1}^h, \chi - \eta_{n+1}^h \rangle_{\Omega^h} \\ & \geq \kappa^4 \langle \mathcal{G}^h \eta_n^h, \chi - \eta_{n+1}^h \rangle_{\Omega^h} \quad \forall \chi \in K_M^h. \end{aligned} \quad (6.17)$$

Having obtained $\mathcal{G}^h \eta_n^h \in S_0^h$ (6.17) is equivalent to the minimization of a quadratic functional subject to a linear constraint and a non-negativity constraint. Efficient algorithms for solving this type of problem can be obtained by combining Uzawa's method, see Ciarlet (1988) Chapter 9 for example and the iterative schemes of Dyn & Ferguson (1983) for the problem in the absence of the inequality constraint, see Chakrabarti (1988) for details. Thus we see that (P_γ^h) and algorithm (A) is a fully practical method of obtaining approximations to (P_γ) .

For the problem (P_0) we set $V \equiv L^2(\Omega)$, $H \equiv H^{-1}(\Omega)$ $a(\cdot, \cdot) \equiv a(\cdot, \cdot)$, $I(\cdot) \equiv J(\cdot)$ and $b(\cdot, \cdot)$ as in (6.16a) with $\gamma = 0$ and

$$c(w, \varphi) \equiv \kappa^4 \langle \nabla \mathcal{G}w, \nabla \mathcal{G}\varphi \rangle \quad \forall w, \varphi \in H^{-1}(\Omega) \quad (6.18a)$$

$$\equiv \kappa^4 \langle \mathcal{G}w, \varphi \rangle \quad \text{if } \varphi \in L^2(\Omega); \quad (6.18b)$$

where we are viewing $\mathcal{G} \in \mathcal{L}(H^{-1}(\Omega), H_0^1(\Omega))$. Clearly conditions (6.1) and (6.7) hold. For problem (P_0^h) we set $V \equiv L^2(\Omega^h)$, $H \equiv H^{-1}(\Omega^h)$, $a(\cdot, \cdot) \equiv a^h(\cdot, \cdot)$, $I(\cdot) \equiv J^h(\cdot)$,

$$b(w, \varphi) \equiv \kappa^2 \langle w, \varphi \rangle_{\Omega^h} \quad \forall w, \varphi \in L^2(\Omega^h) \quad (6.19a)$$

and

$$c(w, \varphi) \equiv \kappa^4 \langle \nabla \mathcal{G}^h w, \nabla \mathcal{G}^h \varphi \rangle_{\Omega^h} \quad \forall w, \varphi \in H^{-1}(\Omega^h) \quad (6.19b)$$

$$\equiv \kappa^4 \langle \mathcal{G}^h w, \varphi \rangle_{\Omega^h} \quad \text{if } \varphi \in L^2(\Omega^h); \quad (6.19c)$$

where we are viewing $\mathcal{G}^h \in \mathcal{L}(H^{-1}(\Omega^h), S_0^h)$. Conditions (6.1a) and (6.7) hold, see (2.20). Clearly from (6.19b) we see that (6.1b) holds. For problem (\hat{P}_0^h) we set $V \equiv H \equiv S^h$, $a(\cdot, \cdot) \equiv \hat{a}^h(\cdot, \cdot)$, $I(\cdot) \equiv \hat{J}^h(\cdot)$,

$$b(\chi, \varphi) \equiv \kappa^2 \langle \chi, \varphi \rangle^h \tag{6.20a}$$

$$\forall \chi, \varphi \in S^h$$

$$c(\chi, \varphi) \equiv \kappa^4 \langle \hat{\mathcal{G}}^h \chi, \varphi \rangle^h. \tag{6.20b}$$

It follows from (5.4a) and (5.5) that (6.1) and (6.7) hold.

We have used algorithm (A) to compute solutions to (P_0^h) and (\hat{P}_0^h) . For (P_0^h) : at each step of the algorithm, see (6.6), given $\eta_n^h \in X_M^h$ one finds $\eta_{n+1}^h \in X_M^h$ such that

$$\langle \eta_{n+1}^h - \kappa^2 \mathcal{G}^h \eta_n^h, \varphi - \eta_{n+1}^h \rangle_{\Omega^h} \geq 0 \quad \forall \varphi \in X_M^h; \tag{6.21}$$

whereas for (\hat{P}_0^h) : given $\hat{\eta}_n^h \in \hat{X}_M^h$ one finds $\hat{\eta}_{n+1}^h \in \hat{X}_M^h$ such that

$$\langle \hat{\eta}_{n+1}^h - \kappa^2 \hat{\mathcal{G}}^h \hat{\eta}_n^h, \chi - \hat{\eta}_{n+1}^h \rangle^h \geq 0 \quad \forall \chi \in \hat{X}_M^h. \tag{6.22}$$

Having obtained $\mathcal{G}^h \eta_n^h \in S_0^h$ (6.21) is equivalent to finding $q_{n+1}^h \in \mathbb{R}$ such that

$$\left\langle 1, \left[\kappa^2 \mathcal{G}^h \eta_n^h - \frac{q_{n+1}^h}{\kappa^2} \right]^+ \right\rangle_{\Omega^h} = M/2 \tag{6.23a}$$

and then setting

$$\eta_{n+1}^h = \left[\kappa^2 \mathcal{G}^h \eta_n^h - \frac{q_{n+1}^h}{\kappa^2} \right]^+. \tag{6.23b}$$

Whereas, having obtained $\hat{\mathcal{G}}^h \hat{\eta}_n^h \in S_0^h$ (6.22) is equivalent to finding $\hat{q}_{n+1}^h \in \mathbb{R}$ such that

$$\left\langle 1, \left[\kappa^2 \hat{\mathcal{G}}^h \hat{\eta}_n^h - \frac{\hat{q}_{n+1}^h}{\kappa^2} \right]^+ \right\rangle^h = M/2 \tag{6.24a}$$

and then setting

$$\hat{\eta}_{n+1}^h = \pi_h \left[\kappa^2 \hat{\mathcal{G}}^h \hat{\eta}_n^h - \frac{\hat{q}_{n+1}^h}{\kappa^2} \right]^+. \tag{6.24b}$$

Although it is possible to solve the problem (6.23a) it is far simpler to solve (6.24a). Therefore the approximation (\hat{P}_0^h) is computationally simpler than

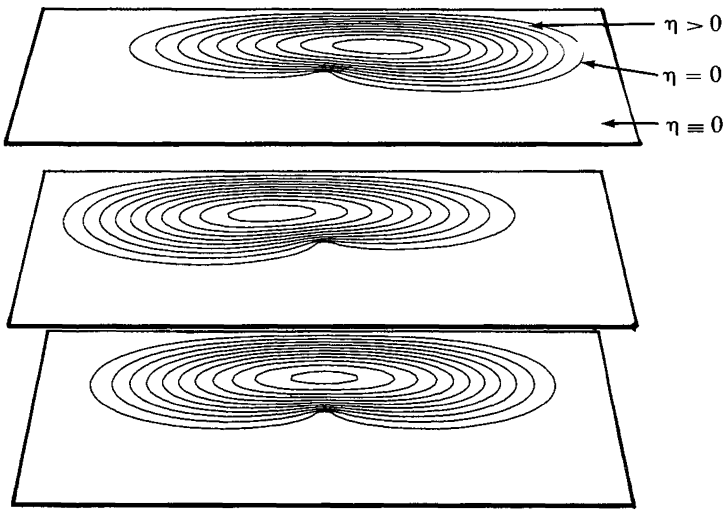


Figure 6.1. — Contour plots for the symmetric and two anti-symmetric solutions with $\kappa = 8$.

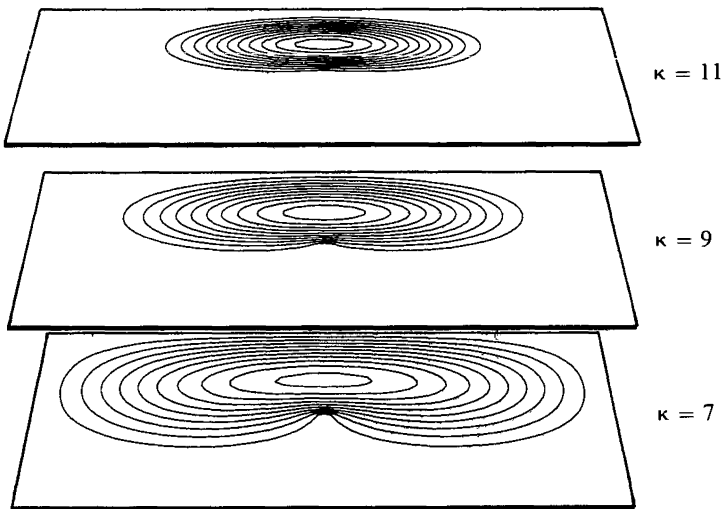


Figure 6.2. — Contour plots for the symmetric solution with $\kappa = 7, 9, 11$.

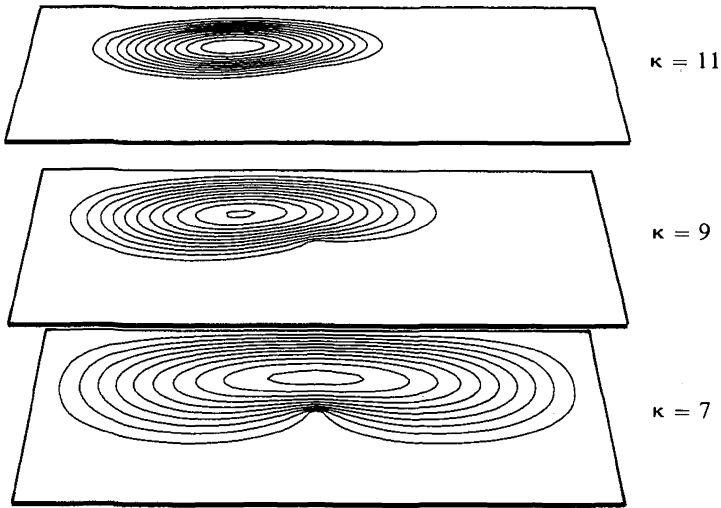


Figure 6.3. — Contour plots for an anti-symmetric solution with $\kappa = 7, 9, 11$.

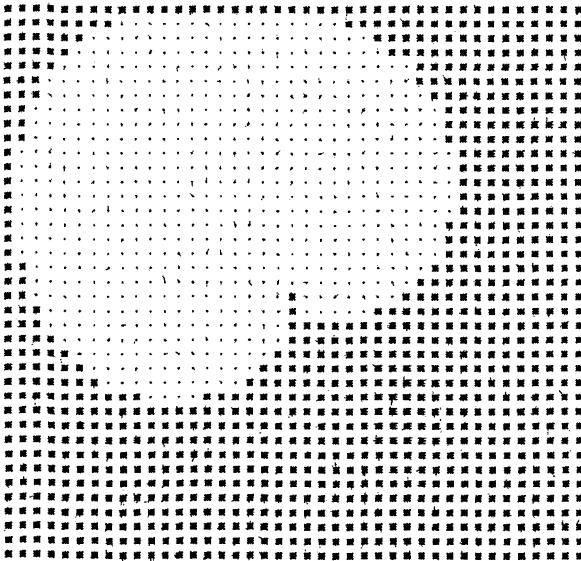


Figure 6.4. — The positive and zero mesh points for an anti-symmetric solution with $\kappa = 9$.

● positive mesh points, ■ zero mesh points.

(P_0^h) and as we have seen in § 5 there is no loss accuracy for this gain in simplicity

We now report on a numerical computation with (P_γ^h) with $\gamma = 1$ and $M = 2$. The boundary of the domain Ω was a square of unit area with a cut from the centre to the midpoint of an edge. We note that the existence and uniqueness results in § 2 are applicable to this domain. A uniform right angled triangulation with $h = 0.025$ was used. Computations were performed for various values of κ . It was found that there is a critical value of κ at which a symmetry breaking bifurcation takes place, for $\kappa \leq \kappa_c$ there is a unique symmetric solution and for $\kappa > \kappa_c$ there are three solutions. In figures 6.1, 6.2, 6.3 and 6.4 we display contour plots for η^h and also the regions of positive and zero mesh points. It was observed that (i) the free boundary first occurs for $6.10 < \kappa < 6.20$ (ii) $7.60 < \kappa_c < 7.75$ (iii) total detachment of the bubble (free boundary) from the frame ($\partial\Omega$) occurs for $12.75 < \kappa < 12.90$. The iterative method of § 6 performed well away from the bifurcation point but needed more iterations in the neighbourhood of κ_c . The symmetric solution for $\kappa > \kappa_c$ was obtained by enforcing symmetry since it is unstable.

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