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NUMERICAL SOLUTION OF SECOND-ORDER ELLIPTIC EQUATIONS ON PLANE DOMAINS (*)

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Abstract. — *The paper presents a general discretization method for convective diffusion equations. The schemes are based on an integral formula and have the following advantages :*

1. *They are effective particularly in the case when convection is dominated ;*
2. *Solutions obtained by them satisfy a discrete conservation law ;*
3. *A discrete maximum principle is valid.*

We show that the finite element solution converges to the exact one with the rate $O(h)$ in $H^1(D)$.

Résumé. — *On propose une méthode générale de discrétisation pour la solution numérique des équations de diffusion avec convection. Les règles pour la discrétisation sont basées sur les formules d'intégration pour les champs et possèdent les avantages suivants :*

1. *Elles restent efficaces aussi dans le cas de la dominance du terme de convection ;*
2. *pour les solutions obtenues on peut vérifier une loi de conservation discrète ;*
3. *on obtient également un principe du maximum discret.*

On démontre la convergence de la solution des éléments finis vers la solution exacte avec la vitesse $O(h)$ concernant la norme en H^1 .

INTRODUCTION

Let us consider the following convective diffusion equation

$$-\Delta u + b \cdot \nabla u + cu = f \quad (1)$$

in some polygonal domain D , where b is a \mathbb{R}^2 -valued function and c, f are real-valued functions. Throughout this paper we assume the validity of the condition

$$(A1) \quad c(x) - \frac{1}{2} \nabla \cdot b(x) \geq a_0 > 0 \quad \text{on } D.$$

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For the sake of simplicity we suppose Dirichlet boundary conditions to be fulfilled. We note, however, that this is not a principal restriction. The discretization method proposed below can be applied e.g. to mixed inhomogeneous boundary conditions, too.

The purpose of this paper is to present a conservative discretization scheme preserving coercivity and monotonicity properties of (1). Especially, the scheme is effective even in the case when the convection is dominated and it gives a nonnegative solution when the source term is nonnegative.

CONSTRUCTION OF THE DISCRETIZATION

Our scheme is based on the so-called control region method which has been theoretically investigated in the case of elliptic equations in the divergence form

$$-\nabla \cdot (k \nabla u) + cu = f,$$

see e.g. [3, 5, 6]

This type of methods is playing an important role in discretizing semiconductor device problems [10, 11, 12]

In the sequel, by n we shall denote the unit outer normal to the boundary of D (or a subset of D) with positive measure

We first triangulate \bar{D} in such a way that for some $h_0 > 0$ the usual regularity condition is satisfied

The triangles T meet only in entire common sides or in vertices. Each triangle contains a circle of radius $c_1 h$ and is contained in a circle of radius $c_2 h$, where $0 < h \leq h_0$ is the mesh parameter and the constants $c_1, c_2 > 0$ do not depend on T and h .

It is wellknown that the condition last mentioned is equivalent to Zlámal's « minimum angle condition », i.e. it is assumed the existence of some number $\alpha_0 > 0$ such that all interior angles of the triangles are not less than α_0 .

Moreover we suppose this triangulation to satisfy the so-called inverse assumption (see [14]) and the following condition

(A2) There exists a number α_1 with $0 < \alpha_1 < \pi/2 - \alpha_0$ such that each angle is either not greater than $\pi/2 - \alpha_1$ or equal to $\pi/2$.

The meaning of this condition will become clear later. Now we define the finite-dimensional subspace X_h^0 of

$$Y^0 = \{v \in H^1(D) \mid v = 0 \text{ on } \partial D\}$$

$$X_h^0 = \{v_h \in Y^0 \mid v_h \text{ is linear on each } T\}$$

By v_{h_i} we shall denote the value of v_h at the nodal point x_i . For the set of indices of the nodes lying in D we shall use the notation V . Finally, let V_i contain the indices of all nodal points adjacent to $x_i \in \bar{D}$.

Before formulating completely the discrete scheme we give a motivation of the proposed construction. We consider for all $i \in V$ the corresponding control regions (*Dirichlet domains*) $D_i = \{x \in D : \text{dist}(x, x_i) < \text{dist}(x, x_j)\}$ for all nodes $x_j \in \bar{D}, j \neq i$. In view of the relation

$$b \cdot \nabla u = \nabla \cdot (bu) - (\nabla \cdot b) u,$$

after integrating (1) over D_i we obtain

$$-\int_{\partial D_i} n \cdot [\nabla u - bu] ds - \int_{D_i} [\nabla \cdot b - c] u dx = \int_{D_i} f dx.$$

With $B_{ij} = \partial D_i \cap \partial D_j$ it follows

$$-\sum_{j \in V_i} \int_{B_{ij}} n \cdot [\nabla u - bu] ds - \int_{D_i} [\nabla \cdot b - c] u dx = \int_{D_i} f dx. \tag{2}$$

At this place, the control region method applied to elliptic equations in the divergent form is continued with replacing the term $k(n \cdot \nabla u)$ to be integrated over B_{ij} by a constant. In practice, this means that instead of $k(n \cdot \nabla u)$ the corresponding difference quotient $k(x_{ij})(u_{h_j} - u_{h_i})/d_{ij}$ is taken, where x_{ij} is the midpoint of the straight-line segment between x_j and x_i and d_{ij} denotes the length of this segment.

In our case we shall use the idea, too.

Therefore we replace the first integrand in (2) by constants, i.e.

$$-n \cdot [\nabla u - bu] \approx S_{ij}.$$

Then we can attempt to seek for a function w^h being the solution of the corresponding differential equation on the straight-line segment between x_i and x_j . With $x = yn, y \in \mathbb{R}^1$, we have

$$-\frac{dw^h}{dy} + (n \cdot b) w^h = S_{ij}.$$

As a rule, this equation cannot be solved analytically for arbitrary b . Since the approximation of the second term in (2) containing $\nabla \cdot b$ is usually performed by means of

$$\begin{aligned} \int_{D_i} (\nabla \cdot b) u dx &\approx u(x_i) \int_{D_i} (\nabla \cdot b) dx = u(x_i) \int_{\partial D_i} (n \cdot b) ds \\ &= u(x_i) \sum_{j \in V_i} \int_{B_{ij}} (n \cdot b) ds \approx u(x_i) \sum_{j \in V_i} N_{ij} m_{ij}, \end{aligned}$$

where N_{ij} is some constant and m_{ij} is the length of B_{ij} (see [13]), it is reasonable to approximate the coefficient $n \cdot b$ in the same way, i.e. we set $n \cdot b \approx N_{ij}$. Then we look for a function w^h satisfying the differential equation

$$-\frac{dw^h}{dy} + N_{ij} w^h = S_{ij}.$$

Obviously, the general solution of this equation is

$$w^h(y) = C_{ij} e^{N_{ij}y} + S_{ij}/N_{ij}.$$

In this representation we have two unknowns: C_{ij} and S_{ij} . Demanding $w^h(y(x_i)) = u_{hi}$ and $w^h(y(x_j)) = u_{hj}$, where u_{hi} and u_{hj} are the values of the final approximate solution u_h at x_i and x_j respectively, we obtain a linear system of algebraic equations to determine C_{ij} and S_{ij} :

$$\begin{cases} u_{hi} = C_{ij} e^{N_{ij}y_i} + S_{ij}/N_{ij} \\ u_{hj} = C_{ij} e^{N_{ij}y_j} + S_{ij}/N_{ij}. \end{cases}$$

It is clear that for different y_i and y_j this system always admits a unique solution.

Thus we get

$$C_{ij} = \frac{u_{hi} - u_{hj}}{e^{N_{ij}y_i} - e^{N_{ij}y_j}}, \quad S_{ij} = \frac{u_{hj} e^{N_{ij}y_i} - u_{hi} e^{N_{ij}y_j}}{e^{N_{ij}y_i} - e^{N_{ij}y_j}}.$$

Now we can replace the integral over B_{ij} by $S_{ij} m_{ij}$. In view of

$$\|x_j - x_i\| = \|n(y_j - y_i)\| = |y_j - y_i| = y_j - y_i$$

we get

$$S_{ij} m_{ij} = \frac{u_{hj} - u_{hi} e^{N_{ij}d_{ij}}}{1 - e^{N_{ij}d_{ij}}} N_{ij} m_{ij}.$$

We note here that the procedure described above for obtaining a suitable approximation of the term

$$-\int_{\partial D_i} n \cdot [\nabla u - bu] ds$$

can be interpreted in various ways. One of these approaches is the application of the idea of exact schemes in connection with appropriate quadrature rules for the occurring integrals, see e.g. [10, 14].

The discretization of the remaining terms

$$\int_{D_i} cu \, dx \quad \text{and} \quad \int_{D_i} f \, dx$$

will be performed via lumping-technique, i.e. we take

$$\int_{D_i} cu \, dx \approx c_i u_i m_i \quad \text{and} \quad \int_{D_i} f \, dx \approx f_i m_i,$$

where m_i denotes the area of D_i . Thus we obtain a system of discrete equations

$$\sum_{j \in V_i} \{1 - [1 - r(N_{ij} d_{ij})] N_{ij} d_{ij}\} (u_{hi} - u_{hj}) m_{ij}/d_{ij} + c_i u_{hi} m_i = f_i m_i, \quad i \in V,$$

with $r(z) = 1 - \frac{1 - B(z)}{z}$ and $B(z) = \frac{z}{e^z - 1}$.

Multiplying the i -th equation by an arbitrary number v_{hi} and adding all these expressions, this system can be rewritten as follows:

The unknown function $u_h \in X_h^0$ satisfies the variational equality

$$a_l(u_h, v_h) = (f, v_h)_l \quad \text{for all } v_h \in X_h^0 \tag{3}$$

where

$$a_l(u_h, v_h) = b_l(u_h, v_h) + c_l(u_h, v_h)$$

with

$$b_l(u_h, v_h) = \sum_{i \in V} \sum_{j \in V_i} \{1 - [1 - r(N_{ij} d_{ij})] N_{ij} d_{ij}\} (u_{hi} - u_{hj}) v_{hi} \frac{m_{ij}}{d_{ij}}$$

and

$$c_l(u_h, v_h) = \sum_{i \in V} c_i u_{hi} v_{hi} m_i, \quad (f, v_h)_l = \sum_{i \in V} f_i v_{hi} m_i.$$

In the following we shall use the notation

$$\|v_h\|_l = \sqrt{(v_h, v_h)_l}.$$

We observe that the function r has the following properties:

- (P1) $r(z)$ is monotone for all real z ,
- (P2) $\lim_{z \rightarrow -\infty} r(z) = 0, \quad \lim_{z \rightarrow \infty} r(z) = 1,$

$$\left. \begin{aligned} \text{(P3)} \quad & 1 + r(z) z \geq 0 \\ \text{(P4)} \quad & [1 - r(z) - r(-z)] z = 0 \\ \text{(P5)} \quad & \left[r(z) - \frac{1}{2} \right] z \geq 0 \end{aligned} \right\} \text{ for all real } z .$$

Moreover, the function r is Lipschitz-continuous on the whole real axis. In view of some practical aspects which will be explained later we shall formulate this property in a weaker variant :

$$\text{(P6)} \quad r(z) z \text{ is Lipschitz-continuous on } \mathbb{R} .$$

As a consequence of (P3) and (P4) we obtain the relation

$$\text{(P7)} \quad 1 - [1 - r(z)] z \geq 0 \text{ for all real } z .$$

Indeed, from (P4) we get

$$1 - [1 - r(z)] z = 1 + r(-z) (-z) \geq 0 .$$

Now we examine the properties of scheme (3).

BASIC PROPERTIES OF THE DISCRETIZATION

LEMMA 1 : *Let the condition (A1) be satisfied.*

If $b \in [W_\infty^2(D)]^2$, $c \in W_\infty^1(D)$ and if the approximations N_{ij} are such that

$$|N_{ij}| \leq \|b\| ; [L_\infty(B_{ij})]^2 \text{ and } N_{ij} m_{ij} - \int_{B_{ij}} (n \cdot b) ds = 0$$

for b linear on B_{ij} , then for sufficiently small $h_0 > 0$ there exists a constant $K > 0$ such that for all $h \in (0, h_0]$ and $u_h \in X_h^0$ the relation

$$a_i(u_h, u_h) \geq |u_h ; H^1(D)|^2 + K \|u_h\|_l^2$$

holds, where the constant K is independent of h . ■

For future reference, we introduce the function

$$R(z) = 1 - \left[\frac{1}{2} - r(z) \right] z$$

and record the following important property of it.

COROLLARY 1 : *The function R satisfies the relation*

$$\max \left\{ 1, \frac{|z|}{2} \right\} \leq R(z) \leq 1 + \frac{|z|}{2} \tag{4}$$

for all real z . ■

Proof: The estimate $R(z) \geq 1$ follows immediately from (P5). The remaining relations will be proved for nonnegative z only. This is sufficient in view of $R(z) = R(-z)$.

The left-hand side inequality results from (P7):

$$R(z) = 1 - [1 - r(z)]z + \frac{z}{2} \geq \frac{z}{2}.$$

Further we have

$$R(z) = 1 - \frac{z}{2} + zr(z) \leq 1 - \frac{z}{2} + z = 1 + \frac{z}{2},$$

because (P1) and (P2) imply $r(z) \leq 1$. ■

Proof of Lemma 1: At first we decompose $b_l(u_h, u_h)$ into two parts:

$$b_l(u_h, u_h) = b_l^{(1)}(u_h, u_h) + b_l^{(2)}(u_h, u_h)$$

with

$$b_l^{(1)}(u_h, u_h) = \sum_{i \in V} \sum_{j \in V_i} [R(z_{ij}) u_{hi} - \{1 - [1 - r(z_{ij})] z_{ij}\} u_{hj}] u_{hi} \frac{m_{ij}}{d_{ij}}$$

and

$$b_l^{(2)}(u_h, u_h) = -\frac{1}{2} \sum_{i \in V} \sum_{j \in V_i} u_{hi}^2 N_{ij} m_{ij}, \quad \text{where } z_{ij} = N_{ij} d_{ij}.$$

Now, in order to treat the first component $b_l^{(1)}$ we use a symmetry argument.

Namely, changing the succession of summation and taking into consideration the boundary values of u_h , $b_l^{(1)}$ can be written in the following manner:

$$b_l^{(1)}(u_h, u_h) = \sum_{j \in V} \sum_{i \in V_j} [R(z_{ij}) u_{hi} - \{1 - [1 - r(z_{ij})] z_{ij}\} u_{hj}] u_{hi} \frac{m_{ij}}{d_{ij}}.$$

Furthermore, replacing i by j and, vice versa, j by i , we obtain

$$b_l^{(1)}(u_h, u_h) = \sum_{i \in V} \sum_{j \in V_i} [R(z_{ji}) u_{hj} - \{1 - [1 - r(z_{ji})] z_{ji}\} u_{hi}] u_{hj} \frac{m_{ji}}{d_{ji}}.$$

In view of the relations

$$N_{ij} = -N_{ji}, \quad m_{ij} = m_{ji}, \quad d_{ij} = d_{ji}$$

we have

$$b_l^{(1)}(u_h, u_h) = \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{V}_i} [R(-z_{ij}) u_{hj} - \{1 + [1 - r(-z_{ij})] z_{ij}\} u_{hi}] u_{hj} \frac{m_{ij}}{d_{ij}}.$$

Together with the original expression of $b_l^{(1)}$ this leads to the representation

$$b_l^{(1)}(u_h, u_h) = \frac{1}{2} \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{V}_i} [R(z_{ij}) u_{hi}^2 - \{1 - [1 - r(z_{ij})] z_{ij}\} u_{hi} u_{hj}] + R(-z_{ij}) u_{hj}^2 - \{1 + [1 - r(-z_{ij})] z_{ij}\} u_{hi} u_{hj}] \frac{m_{ij}}{d_{ij}}.$$

Since R is an even function, it follows

$$b_l^{(1)}(u_h, u_h) = \frac{1}{2} \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{V}_i} [R(z_{ij}) (u_{hi}^2 + u_{hj}^2) - \{2 + [r(z_{ij}) - r(-z_{ij})] z_{ij}\} u_{hi} u_{hj}] \frac{m_{ij}}{d_{ij}}.$$

Using (P4), we have

$$b_l^{(1)}(u_h, u_h) = \frac{1}{2} \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{V}_i} [R(z_{ij}) (u_{hi}^2 + u_{hj}^2) - 2 R(z_{ij}) u_{hi} u_{hj}] \frac{m_{ij}}{d_{ij}} = \frac{1}{2} \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{V}_i} R(z_{ij}) (u_{hi} - u_{hj})^2 \frac{m_{ij}}{d_{ij}}.$$

Hence, from Corollary 1 we get

$$b_l^{(1)}(u_h, u_h) \geq \frac{1}{2} \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{V}_i} (u_{hi} - u_{hj})^2 \frac{m_{ij}}{d_{ij}} = |u_h; H^1(D)|^2,$$

where the last relation is a consequence of [8 : Lemma 2.3].

Finally we consider the remaining terms

$$b_l^{(2)}(u_h, u_h) + c_l(u_h, u_h) = \sum_{k=1}^3 \delta_{kh}$$

with

$$\delta_{1h} = -\frac{1}{2} \sum_{i \in \mathcal{V}} u_{hi}^2 \sum_{j \in \mathcal{V}_i} \left[N_{ij} m_{ij} - \int_{B_{ij}} (n \cdot b) ds \right],$$

$$\delta_{2h} = \sum_{i \in \mathcal{V}} \int_{D_i} (c_i - c) u_{hi}^2 dx, \quad \delta_{3h} = \sum_{i \in \mathcal{V}} \int_{D_i} \left[c - \frac{1}{2} \nabla \cdot b \right] u_{hi}^2 dx.$$

The first difference is treated as follows :

$$2 |\delta_{1h}| \leq \sum_{i \in \mathcal{V}} u_{hi}^2 \sum_{j \in \mathcal{V}_i} \left| \int_{B_{ij}} [N_{ij} - n \cdot b] ds \right|.$$

We denote by D_{ij}^+ the interior of the triangle having B_{ij} as one side and x_i as the opposite vertex. Then we can write

$$\int_{B_{ij}} [N_{ij} - n \cdot b] ds = \text{meas} (B_{ij}) \left| \int_{T_0^+} [\tilde{N}_{ij} - n \cdot \tilde{b}] d\tilde{s} \right|,$$

where T_0^+ is the corresponding face of the reference triangle T^+ . Now we can estimate

$$\begin{aligned} \left| \int_{T_0^+} [\tilde{N}_{ij} - n \cdot \tilde{b}] d\tilde{s} \right| &\leq 2 \|\tilde{b}\| ; [L_\infty(T_0^+)]^2 \\ &\leq 2 \|\tilde{b}\| ; [L_\infty(T^+)]^2 \leq 2 \|\tilde{b}\| ; [W_\infty^2(T^+)]^2. \end{aligned}$$

Thus the integral is a linear continuous functional on $[W_\infty^2(T^+)]^2$ vanishing for linear functions \tilde{b} . By Bramble-Hilbert's lemma, we obtain

$$\left| \int_{T_0^+} [\tilde{N}_{ij} - n \cdot \tilde{b}] d\tilde{s} \right| \leq C |\tilde{b}\| ; [W_\infty^2(T^+)]^2.$$

The back-transformation yields

$$\left| \int_{T_0^+} [\tilde{N}_{ij} - n \cdot \tilde{b}] d\tilde{s} \right| \leq C \left\| \left(\frac{\partial F_{ij}^+}{\partial x} \right)^{-1} \right\|^2 |b ; [W_\infty^2(D_{ij}^+)]^2|,$$

where F_{ij}^+ is the affin-linear map realizing the transformation $D_{ij}^+ \rightarrow T^+$. From [4] we conclude that

$$\left| \int_{B_{ij}} [\tilde{N}_{ij} - n \cdot \tilde{b}] ds \right| \leq Ch^3 |b ; [W_\infty^2(D_{ij}^+)]^2|$$

holds. Therefore we have

$$\begin{aligned} |\delta_{1h}| &\leq Ch | b; [W_\infty^2(D)]^2 | \sum_{i \in V} u_{hi}^2 h^2 \\ &\leq Ch | b; [W_\infty^2(D)]^2 | \|u_h\|_l^2. \end{aligned}$$

In a similar way one can derive the estimate

$$|\delta_{2h}| \leq Ch | c; W_\infty^1(D) | \|u_h\|_l^2.$$

Summarizing the results and using assumption (A1) we obtain

$$\begin{aligned} a_l(u_h, u_h) &\geq |u_h; H^1(D)|^2 \\ &\quad + \{a_0 - Ch [| b; [W_\infty^2(D)]^2 | + | c; W_\infty^1(D) |]\} \|u_h\|_l^2. \end{aligned}$$

Now it remains to choose h_0 such that for all $h \in (0, h_0]$ the term in the braces is positive. ■

Remark 1: It is wellknown that the relation

$$C \|u_h; L_2(D)\| \leq \|u_h\|, \quad u_h \in X_h^0,$$

holds [8]. Therefore the functional a_l is X_h^0 -coercive. ■

From assumption (A1) it follows that the «continuous» problem is inverse-isotone. Especially, if there is given a nonnegative right-hand side f , then we obtain a nonnegative solution u , too. The same fact is true for the discrete problem (3).

Before formulating and proving the corresponding result, we need some further notations. Let $I_h: C(\bar{D}) \rightarrow X_h^0$ denote the usual interpolation operator. The symbol u_h^+ we shall understand as the positive part of the function u_h , i.e. $u_h^+ = \max \{0, u_h\}$. The negative part of u_h is defined by $u_h^- = u_h^+ - u_h$.

LEMMA 2: For all $u_h \in X_h^0$ it holds

$$a_l(I_h u_h^+, I_h u_h^-) \leq 0. \quad \blacksquare$$

Proof: It is easy to see that $c_l(I_h u_h^+, I_h u_h^-) = 0$. Now we consider the first term b_l . We have

$$\begin{aligned} b_l(I_h u_h^+, I_h u_h^-) &= \sum_{i \in V} \sum_{j \in V_i} \{1 - [1 - r(N_{ij} d_{ij})] N_{ij} d_{ij}\} (u_{hi}^+ - u_{hj}^+) u_{hi}^- \frac{m_{ij}}{d_{ij}} \\ &= - \sum_{i \in V} \sum_{j \in V_i} \{1 - [1 - r(N_{ij} d_{ij})] N_{ij} d_{ij}\} u_{hj}^+ u_{hi}^- \frac{m_{ij}}{d_{ij}}. \end{aligned}$$

Therefore, in view of the property (P7) of r , we obtain

$$b_l(I_h u_h^+, I_h u_h^-) \leq 0. \quad \blacksquare$$

COROLLARY 2: *Let the assumptions of Lemma 1 be fulfilled. Then the discrete problem (3) is inverse-isotone. \blacksquare*

Proof: Since the probleme is a linear one, it is sufficient to show that a nonnegative right-hand side $f \in C(\bar{D})$ leads to a nonnegative solution $u_h \in X_h^0$.

Let $u_h \in X_h^0$ be the solution of the problem (3) with a nonnegative right-hand side $f \in C(\bar{D})$. Taking the special trial function $v_h = I_h u_h^-$, we obtain :

$$\begin{aligned} 0 &\leq (f, I_h u_h^-)_l = a_l(u_h, I_h u_h^-) \\ &= a_l(I_h u_h^+, I_h u_h^-) - a_l(I_h u_h^-, I_h u_h^-) \leq - a_l(I_h u_h^-, I_h u_h^-). \end{aligned}$$

From Lemma 1 and Remark 1 respectively we conclude that

$$\|I_h u_h^- ; L_2(D)\|^2 \leq 0,$$

i.e. $u_h^- = 0$. \blacksquare

Remark 2: It is wellknown that the inverse isotonicity is an important part of the sufficient conditions which imply L_∞ -stability. \blacksquare

LEMMA 3: *The scheme (3) is conservative. \blacksquare*

Proof: We have to derive an analogous discrete relation to the identity

$$\int_{\partial D} [-n \cdot \nabla u + (n \cdot b) u] ds - \int_D [\nabla \cdot b - c] u dx = \int_D f dx. \quad (5)$$

For this aim we consider (3) with the test function $v_h = i_h$, where i_h is a function from X_h^0 with $i_h(x_j) = 1$ for all $j \in V$. Then we can write

$$b_l(u_h, i_h) = b_l^{(3)}(u_h, i_h) + b_l^{(4)}(u_h, i_h) + b_l^{(5)}(u_h, i_h)$$

with

$$b_l^{(3)}(u_h, i_h) = - \sum_{i \in V} \sum_{j \in V_i} (u_{hj} - u_{hi}) m_{ij} / d_{ij},$$

$$b_l^{(5)}(u_h, i_h) = - \sum_{i \in V} u_{hi} \sum_{j \in V_i} N_{ij} m_{ij}$$

and

$$b_l^{(4)}(u_h, i_h) = b_l(u_h, i_h) - b_l^{(3)}(u_h, i_h) - b_l^{(5)}(u_h, i_h).$$

At first we consider $b_l^{(3)}$. In contrast to the proof of Lemma 1, here we must be more attentive to the situation near the boundary. Denoting by Z the index set of all nodal points lying on ∂D , we have

$$\begin{aligned} b_l^{(3)}(u_h, i_h) &= - \sum_{i \in V} \sum_{j \in V_i \cap V} (u_{hj} - u_{hi}) m_{ij} / d_{ij} \\ &\quad - \sum_{i \in V} \sum_{j \in V_i \cap Z} (u_{hj} - u_{hi}) m_{ij} / d_{ij}. \end{aligned}$$

It is not difficult to see that the first addend in this representation vanishes. The second sum can be rewritten as follows :

$$b_l^{(3)}(u_h, i_h) = - \sum_{j \in Z} \sum_{i \in V_j} (u_{hj} - u_{hi}) m_{ij} / d_{ij}.$$

For $b_l^{(4)}$ we get

$$\begin{aligned} b_l^{(4)}(u_h, i_h) &= \sum_{i \in V} \sum_{j \in V_i \cap V} \{r(N_{ij} d_{ij}) u_{hi} + [1 - r(N_{ij} d_{ij})] u_{hj}\} N_{ij} m_{ij} \\ &\quad + \sum_{i \in V} \sum_{j \in V_i \cap Z} \{r(N_{ij} d_{ij}) u_{hi} + [1 - r(N_{ij} d_{ij})] u_{hj}\} N_{ij} m_{ij} \\ &= b_l^{(41)}(u_h, i_h) + b_l^{(42)}(u_h, i_h). \end{aligned}$$

Applying again a symmetry argument to $b_l^{(41)}$, as in the proof of Lemma 1, we observe that in view of (P4)

$$\begin{aligned} b_l^{(41)}(u_h, i_h) &= \sum_{i \in V} \sum_{j \in V_i \cap V} \{r(N_{ji} d_{ij}) u_{hj} + [1 - r(N_{ji} d_{ij})] u_{hi}\} N_{ji} m_{ij} \\ &= - \sum_{i \in V} \sum_{j \in V_i \cap V} \{[1 - r(N_{ij} d_{ij})] u_{hj} + r(N_{ij} d_{ij}) u_{hi}\} N_{ij} m_{ij} \\ &= - b_l^{(41)}(u_h, i_h) \end{aligned}$$

holds, from which $b_l^{(41)} = 0$ follows.

Hence we get the relation

$$b_l^{(41)}(u_h, i_h) = - \sum_{j \in Z} \sum_{i \in V_j} \{[1 - r(N_{ij} d_{ij})] u_{hj} + r(N_{ij} d_{ij}) u_{hi}\} N_{ij} m_{ij}.$$

Taking into consideration the remaining terms in (3), we have

$$\begin{aligned} \sum_{j \in V} \sum_{i \in V_j} \left\{ - \frac{u_{hj} - u_{hi}}{d_{ij}} + N_{ij} u_{hij} \right\} m_{ij} - \\ - \sum_{i \in V} \left\{ \sum_{j \in V_i} N_{ij} m_{ij} - c_i m_i \right\} u_{hi} = \sum_{i \in V} f_i m_i, \end{aligned}$$

where $u_{hij} = r(N_{ij} d_{ij}) u_{hi} + [1 - r(N_{ij} d_{ij})] u_{hj}$ can be regarded as an approximation of u_h on B_{ij} , and this is the desired identity. ■

CONVERGENCE IN THE H^1 -NORM

THEOREM : *Suppose that the assumptions of Lemma 1 are fulfilled and that $f \in W_q^1(D)$ with some $q > 2$. Furthermore, let the assumption (A2) be satisfied and let the « continuous » solution u of (1) belong to $Y^0 \cap H^2(D)$.*

Then, for sufficiently small $h_0 > 0$ the estimate

$$\|u - u_h; H^1(D)\| \leq Ch$$

holds, where $h \in (0, h_0]$ and C is a constant independent of h . ■

Proof : From assumption (A1) it follows that the bilinear form

$$a(u, v) = \int_D \nabla u \cdot \nabla v \, dx + \int_D [b \cdot \nabla u + cu] v \, dx$$

is H^1 -coercive. In particular, it holds

$$C \|u_h - w_h; H^1(D)\|^2 \leq a(u_h - w_h, u_h - w_h), \tag{6}$$

where w_h is an arbitrary function from X_h^0 . The right-hand side of this inequality can be treated as follows :

$$\begin{aligned} a(u_h - w_h, u_h - w_h) &= a(u - w_h, u_h - w_h) - a(u - u_h, u_h - w_h) \\ &= a(u - w_h, u_h - w_h) + a(u_h, u_h - w_h) - a_l(u_h, u_h - w_h) \\ &\quad + (f, u_h - w_h)_l - (f, u_h - w_h). \end{aligned}$$

Now we have to consider the expressions

$$\delta a = a(u_h, v_h) - a_l(u_h, v_h)$$

and

$$\delta f = (f, v_h) - (f, v_h)_l,$$

where we set $v_h = u_h - w_h$. Making use of the relation

$$\int_D \nabla u_h \cdot \nabla v_h \, dx = \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{V}_i} (u_{hi} - u_{hj}) v_{hi} m_{ij} / d_{ij}$$

(see [8 : Lemma 2.3]), the term δa can be represented as follows .

$$\begin{aligned} \delta a = & \int_D \nabla \cdot (bu_h) v_h dx - \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{V}_i} [r_{ij} u_{hi} + (1 - r_{ij}) u_{hj}] N_{ij} v_{hi} m_{ij} \\ & - \int_D [\nabla \cdot b - c] u_h v_h dx + \sum_{i \in \mathcal{V}} \left\{ \sum_{j \in \mathcal{V}_i} N_{ij} m_{ij} - c_i m_i \right\} u_{hi} v_{hi} , \end{aligned}$$

where $r_{ij} = r(N_{ij} d_{ij})$.

At first we consider the term

$$\delta b_1 = \int_D \nabla \cdot (bu_h) v_h dx - \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{V}_i} [r_{ij} u_{hi} + (1 - r_{ij}) u_{hj}] N_{ij} v_{hi} m_{ij}$$

and decompose it additively into the following parts :

$$\begin{aligned} \delta b_{11} &= \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{V}_i} \int_{B_{ij}} \{u_h - r_{ij} u_{hi} - [1 - r_{ij}] u_{hj}\} N_{ij} v_{hi} ds , \\ \delta b_{12} &= \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{V}_i} \int_{B_{ij}} (n \cdot b - N_{ij}) u_h v_{hi} ds , \\ \delta b_{13} &= \sum_{i \in \mathcal{V}} \int_{D_i} \nabla \cdot (bu_h) (v_h - v_{hi}) dx . \end{aligned}$$

Now we start the detailed estimations. For δb_{11} we have, by a symmetry argument,

$$\begin{aligned} \delta b_{11} &= \frac{1}{2} \sum_{i \in \mathcal{V} \cup \mathcal{Z}} \sum_{j \in \mathcal{V}_i} \int_{B_{ij}} \{[u_h - r_{ij} u_{hi} - (1 - r_{ij}) u_{hj}] N_{ij} v_{hi} \\ & \quad + [u_h - r_{ji} u_{hj} - (1 - r_{ji}) u_{hi}] N_{ji} v_{hj}\} ds \\ &= \frac{1}{2} \sum_{i \in \mathcal{V} \cup \mathcal{Z}} \sum_{j \in \mathcal{V}_i} \int_{B_{ij}} \{[u_h - r_{ij} u_{hi} - (1 - r_{ij}) u_{hj}] N_{ij} (v_{hi} - v_{hj}) \\ & \quad + (1 - r_{ij} - r_{ji}) N_{ij} (u_{hi} - u_{hj}) v_{hj}\} ds . \end{aligned}$$

Since $(1 - r_{ij} - r_{ji}) N_{ij} d_{ij} = 0$, in view of (P4), we obtain

$$\begin{aligned} \delta b_{11} = \frac{1}{2} \sum_{i \in \mathcal{V} \cup \mathcal{Z}} \sum_{j \in \mathcal{V}_i} \int_{B_{ij}} [u_h - r_{ij} u_{hi} - (1 - r_{ij}) u_{hj}] \times \\ \times N_{ij} (v_{hi} - v_{hj}) ds . \quad (7) \end{aligned}$$

For the sake of simplicity in the notation we shall investigate the integral

$$\int_{B_{ij}} u_{hj} v_{hj} ds ,$$

where $u_{hj} = u_h - r_{ij} u_{hi} - (1 - r_{ij}) u_{hj}$
 and $v_{hj} = N_{ij}(v_{hi} - v_{hj})$.

Note that v_{hj} can also be written as

$$v_{hj} = N_{ij} d_{ij}(n \cdot \nabla v_h) |_{D_{ij}} .$$

Obviously, the point x_i and the segment B_{ij} define some triangle \bar{D}_{ij}^+ in \bar{D}_i in such a way that B_{ij} is one side and x_i is the corresponding vertex (see Lemma 1). A second triangle lying in \bar{D}_j is defined by x_j and B_{ij} . We denote the interior of the union of these two triangles by D_{ij} .

It is clear that \bar{D}_{ij} can be represented in another manner, namely as the union of two triangles $\bar{D}_{ij}^{(1)}$ and $\bar{D}_{ij}^{(2)}$ having the common vertices x_i and x_j .

Further we set $B_{ij}^{(k)} = B_{ij} \cap \bar{D}_{ij}^{(k)}$, $k = 1 ; 2$. Let us mention that from assumption (A2) the regularity of the resulting triangulation follows. For each $\bar{D}_{ij}^{(k)}$ we are able to find an affin-linear mapping $\bar{F}_{ij}^{(k)}$ which transforms the triangle $\bar{D}_{ij}^{(k)}$ onto a reference element T :

$$\bar{F}_{ij}^{(k)} : \quad \bar{D}_{ij}^{(k)} \rightarrow T .$$

By $\bar{G}_{ij}^{(k)}$ we denote the restriction of $\bar{F}_{ij}^{(k)}$ to $B_{ij}^{(k)}$, i.e. $\bar{G}_{ij}^{(k)} : B_{ij}^{(k)} \rightarrow T_0$. Now we return to the estimation of the integral over B_{ij} . We have

$$\begin{aligned} \int_{B_{ij}} u_{hj} v_{hj} ds &= \sum_{k=1,2} \int_{B_{ij}^{(k)}} u_{hj} v_{hj} ds \\ &= \sum_{k=1,2} \text{meas} (B_{ij}^{(k)}) \int_{T_0} \tilde{u}_{hj}^{(k)} \tilde{v}_{hj}^{(k)} d\tilde{s} . \end{aligned}$$

The desired estimate follows from the study of the functional

$$J(\tilde{u}_h^{(k)}) = \int_{T_0} \tilde{u}_{hj}^{(k)} \tilde{v}_{hj}^{(k)} d\tilde{s} ,$$

where we assume temporarily that v_h is fixed. Obviously, $\tilde{u}_{hj}^{(k)}$ can be

supposed to belong to $H^2(T)$ since it is affin-linear on T . Moreover, $\tilde{v}_{hj}^{(k)}$ is constant on T . Therefore it holds

$$\begin{aligned} |J(\tilde{u})| &\leq 2 \|\tilde{u}; L_\infty(T)\| \int_{T_0} |\tilde{v}_{hj}^{(k)}| \, d\tilde{s} \\ &= 2 \frac{\text{meas}(T_0)}{\text{meas}(T)} \|\tilde{u}; L_\infty(T)\| \|\tilde{v}_{hj}^{(k)}; L_1(T)\|. \end{aligned}$$

From Sobolev's imbedding theorem it follows

$$|J(\tilde{u})| \leq C \|\tilde{u}; H^2(T)\| \|\tilde{v}_{hj}^{(k)}; L_1(T)\|.$$

Hence J is a linear continuous functional on $H^2(T)$ with the norm

$$\|J; (H^2(T))^*\| \leq C \|\tilde{v}_{hj}^{(k)}; L_1(T)\|.$$

In addition, we observe that $J(\tilde{u})$ vanishes for all functions \tilde{u} being constant on T . By a modification of the wellknown lemma by Bramble & Hilbert [6] we obtain

$$|J(\tilde{u})| \leq C \|\tilde{v}_{hj}^{(k)}; L_1(T)\| \{|\tilde{u}; H^1(T)|^2 + |\tilde{u}; H^2(T)|^2\}^{1/2}.$$

In view of $|\tilde{u}; H^2(T)| = 0$ it follows

$$|J(\tilde{u}_{1/h}^{(k)})| \leq C \|\tilde{v}_{hj}^{(k)}; L_1(T)\| |\tilde{u}_{1/h}^{(k)}; H^1(T)|.$$

Therefore we have

$$\begin{aligned} \left| \int_{B_{ij}^{(k)}} u_{hj} v_{hj} \, ds \right| &\leq \\ &\leq C \|(\mathcal{D}F_{ij}^{(k)})^{-1}\| |\det \mathcal{D}F_{ij}^{(k)}|^{3/2} |u_h; H^1(D_{ij}^{(k)})| \|v_{hj}; L_1(D_{ij}^{(k)})\| \end{aligned}$$

with $\mathcal{D}F_{ij}^{(k)} = \left(\frac{\partial F_{ij}^{(k)}}{\partial x} \right)$, where we used the transformation theorem from [4]. The same book gives the estimate

$$\|(\mathcal{D}F_{ij}^{(k)})^{-1}\| \leq C h_{ij}^{(k)},$$

where $h_{ij}^{(k)}$ denotes the diameter of $D_{ij}^{(k)}$ and $C > 0$ is a constant independent of $D_{ij}^{(k)}$. Furthermore it is known that under assumption (A2) the relation

$$|\det \mathcal{D}F_{ij}^{(k)}| \leq C (h_{ij}^{(k)})^{-2}$$

with a independent of $D_{ij}^{(k)}$ constant $C > 0$ holds. Hence we obtain

$$\left| \int_{B_{ij}^{(k)}} u_{hj} v_{hj} ds \right| \leq C (h_{ij}^{(k)})^{-2} |u_h; H^1(D_{ij}^{(k)})| \|v_{hj}; L_1(D_{ij}^{(k)})\| .$$

Now we derive an estimate for the L_1 -norm of \bar{v}_{hj} on $D_{ij}^{(k)}$:

$$\begin{aligned} \|v_{hj}; L_1(D_{ij}^{(k)})\| &\leq d_{ij} |N_{ij}| \|n \cdot \nabla v_h; L_2(D_{ij}^{(k)})\| \|1; L_2(D_{ij}^{(k)})\| \\ &\leq C (h_{ij}^{(k)})^2 |N_{ij}| |v_h; H^1(D_{ij}^{(k)})| . \end{aligned}$$

Thus we have

$$\begin{aligned} \left| \int_{B_{ij}} u_{hj} v_{hj} ds \right| &\leq \\ &\leq C \sum_{k=1,2} \text{meas}(B_{ij}^{(k)}) |N_{ij}| |u_h; H^1(D_{ij}^{(k)})| |v_h; H^1(D_{ij}^{(k)})| \\ &\leq Ch \|b; [L_\infty(D)]^2\| \sum_{k=1,2} |u_h; H^1(D_{ij}^{(k)})| |v_h; H^1(D_{ij}^{(k)})| . \end{aligned}$$

Applying Cauchy's inequality, we get finally

$$\left| \int_{B_{ij}} u_{hj} v_{hj} ds \right| \leq Ch \|b; [L_\infty(D)]^2\| |u_h; H^1(D_{ij})| |v_h; H^1(D_{ij})| .$$

The same inequality applied to $|\delta b_{11}|$ with δb_{11} from (7) leads to

$$|\delta b_{11}| \leq Ch \|b; [L_\infty(D)]^2\| |u_h; H^1(D)| |v_h; H^1(D)| .$$

The next step consists in estimating δb_{12} . We have

$$\begin{aligned} |\delta b_{12}| &\leq \frac{1}{2} \sum_{i \in \mathcal{V} \cup \mathcal{Z}} \sum_{j \in \mathcal{V}_i} \int_{B_{ij}} |(n \cdot b - N_{ij}) u_h (v_{hi} - v_{hj})| ds \\ &\leq \frac{1}{2} \sum_{i \in \mathcal{V} \cup \mathcal{Z}} \sum_{j \in \mathcal{V}_i} \|u_h; L_\infty(B_{ij})\| |v_{hi} - v_{hj}| \int_{B_{ij}} |n \cdot b - N_{ij}| ds . \end{aligned}$$

In the proof of Lemma 1 (see the estimate of δ_{1h}) we obtained the relation

$$\int_{B_{ij}} |n \cdot b - N_{ij}| ds \leq Ch^3 |b; [W_\infty^2(D_{ij})]^2| ,$$

and from this we conclude that

$$|\delta b_{12}| \leq Ch^2 |b; [W_\infty^2(D)]^2| \|u_h; L_\infty(D)\| \sum_{i \in \mathcal{V} \cup \mathcal{Z}} \sum_{j \in \mathcal{V}_i} |v_{hi} - v_{hj}| h$$

holds. In view of

$$|v_{hi} - v_{hj}| = d_{ij} |n \cdot \nabla v_h| \Big|_{D_{ij}}$$

and

$$d_{ij} h \leq C \operatorname{meas} (D_{ij})$$

it follows

$$|\delta b_{12}| \leq Ch^2 |b; [W_\infty^2(D)]^2| \|u_h; L_\infty(D)\| |v_h; H^1(D)|.$$

The norm-equivalence theorem (see [4]) implies

$$\|u_h; L_\infty(D)\| \leq Ch^{-1} \|u_h; L_2(D)\|,$$

i.e. we get

$$|\delta b_{12}| \leq Ch |b; [W_\infty^2(D)]^2| \|u_h; L_2(D)\| |v_h; H^1(D)|.$$

The last term δb_{13} can be estimated as follows :

$$\begin{aligned} |\delta b_{13}| &\leq \sum_{i \in \mathcal{V}} \|\nabla \cdot (bu_h); L_2(D_i)\| \|v_h - v_{hi}; L_2(D_i)\| \\ &\leq \|\nabla \cdot (bu_h); L_2(D)\| \left\{ \sum_{i \in \mathcal{V}} \|v_h - v_{hi}; L_2(D_i)\|^2 \right\}^{1/2}, \end{aligned}$$

where we applied Cauchy's inequality. The term in the braces satisfies the relation

$$\left\{ \sum_{i \in \mathcal{V}} \|v_h - v_{hi}; L_2(D_i)\|^2 \right\}^{1/2} \leq Ch |v_h; H^1(D)|,$$

see [8]. Thus we obtain

$$|\delta b_{13}| \leq Ch \|\nabla \cdot (bu_h); L_2(D)\| |v_h; H^1(D)|,$$

and therefore it holds

$$|\delta b_1| \leq Ch \|v_h; H^1(D)\|. \quad (8)$$

The next step consists in estimating the expression

$$\begin{aligned} \delta b_2 &= \int_D [\nabla \cdot b - c] u_h v_h dx - \sum_{i \in \mathcal{V}} \left\{ \sum_{j \in \mathcal{V}_i} N_{ij} m_{ij} - c_i m_i \right\} u_{hi} v_{hi} \\ &= \int_D (\nabla \cdot b) u_h v_h dx - \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{V}_i} N_{ij} m_{ij} u_{hi} v_{hi} \\ &\quad - \left\{ \int_D cu_h v_h dx - \sum_{i \in \mathcal{V}} c_i u_{hi} v_{hi} m_i \right\} = \delta b_{21} - \delta b_{22}. \end{aligned}$$

The term δb_{21} can be handled as follows :

$$\begin{aligned} \delta b_{21} &= \sum_{i \in \mathcal{V}} \int_{D_i} (\nabla \cdot b) u_h (v_h - v_{hi}) dx \\ &\quad + \sum_{i \in \mathcal{V}} \int_{D_i} (\nabla \cdot b) (u_h - u_{hi}) v_{hi} dx \\ &\quad + \sum_{i \in \mathcal{V}} u_{hi} v_{hi} \sum_{j \in \mathcal{V}_i} \left[\int_{B_j} (n \cdot b) ds - N_{ij} m_{ij} \right] \\ &= \delta b_{211} + \delta b_{212} + \delta b_{213} . \end{aligned}$$

Now we have (see, e.g. [8])

$$|\delta b_{211}| \leq Ch |b; [W_\infty^1(D)]^2| \|u_h; L_2(D)\| \|v_h; H^1(D)\|$$

and

$$\begin{aligned} |\delta b_{212}| &\leq Ch |b; [W_\infty^1(D)]^2| \|u_h; H^1(D)\| \|v_h\|_l \\ &\leq Ch |b; [W_\infty^1(D)]^2| \|u_h; H^1(D)\| \|v_h; L_2(D)\| . \end{aligned}$$

Using the same argument as in the proof of Lemma 1, where we estimated δ_{1h} , we obtain

$$|\delta b_{213}| \leq Ch |b; [W_\infty^2(D)]^2| \|u_h; L_2(D)\| \|v_h; L_2(D)\| .$$

Finally we need an estimate for δb_{22} . This is not difficult to achieve :

$$|\delta b_{22}| = \delta b_{221} + \delta b_{222} + \delta b_{223},$$

where

$$\begin{aligned} |\delta b_{221}| &= \left| \sum_{i \in \mathcal{V}} \int_{D_i} c u_h (v_h - v_{hi}) dx \right| \\ &\leq Ch \|c; L_\infty(D)\| \|u_h; L_2(D)\| \|v_h; H^1(D)\| \end{aligned}$$

and

$$\begin{aligned} |\delta b_{222}| &= \left| \sum_{i \in \mathcal{V}} \int_{D_i} c (u_h - u_{hi}) v_{hi} dx \right| \\ &\leq Ch \|c; L_\infty(D)\| \|u_h; H^1(D)\| \|v_h; L_2(D)\| . \end{aligned}$$

For

$$\delta b_{223} = \sum_{i \in \mathcal{V}} \int_{D_i} (c - c_i) u_{hi} v_{hi} dx$$

we take the corresponding result for δ_{2h} from the proof of Lemma 1 :

$$|\delta b_{223}| \leq Ch |c; W_\infty^1(D)| \|u_h; L_2(D)\| \|v_h; L_2(D)\| .$$

Thus we get

$$|\delta b_2| \leq Ch \|v_h; H^1(D)\| . \quad (9)$$

Now it remains to consider the right-hand side difference δf :

$$\begin{aligned} \delta f &= \int_D f v_h dx - \sum_{i \in V} f_i v_{hi} m_i \\ &= \sum_{i \in V} \left\{ \int_{D_i} (f - f_i) v_h dx + \int_{D_i} f_i (v_h - v_{hi}) dx \right\} . \end{aligned}$$

The first sum admits the estimation (see [8] again)

$$\left| \sum_{i \in V} \int_{D_i} (f - f_i) v_h dx \right| \leq Ch |f; W_q^1(D)| \|v_h; L_2(D)\| ;$$

for the second one we have

$$\left| \sum_{i \in V} \int_{D_i} f_i (v_h - v_{hi}) dx \right| \leq Ch |f; W_q^1(D)| \|v_h; H^1(D)\| .$$

Therefore it holds

$$|\delta f| \leq Ch |f; W_q^1(D)| \|v_h; H^1(D)\| . \quad (10)$$

Summarizing the estimates (8)-(10) and using the boundedness of a , we obtain from (6)

$$\|u - u_h; H^1(D)\| \leq \|u - w_h; H^1(D)\| + \|u_h - w_h; H^1(D)\|$$

Since we may assume $u_h - w_h \neq 0$, it follows

$$\|u_h - w_h; H^1(D)\| \leq C [\|u - w_h; H^1(D)\| + h] .$$

The triangle inequality

$$\|u - u_h; H^1(D)\| \leq \|u - w_h; H^1(D)\| + \|u_h - w_h; H^1(D)\|$$

yields

$$\|u_h - u_h; H^1(D)\| \leq C [\|u - w_h; H^1(D)\| + h] .$$

If we take $w_h = I_h u$, then it is wellknown that the relation

$$\|u - w_h; H^1(D)\| \leq Ch \|u; H^2(D)\|$$

holds. Thus we get

$$\|u - u_h; H^1(D)\| \leq Ch. \blacksquare$$

CONCLUSIVE REMARKS

In the formulation of our theorem we did not attend to the qualitative dependence of the constant C on the norms $|b; [W_\infty^2(D)]^2|$, $|c; W_\infty^1(D)|$ and $|f; W_q^1(D)|$ respectively.

However, the interested reader can understand it without serious difficulties from the detailed proof.

In the theory of singular perturbations of elliptic differential equations often there are considered problems of the type

$$-\varepsilon \Delta u + b \cdot \nabla u + cu = f$$

with $0 < \varepsilon \ll 1$ and $\|b\|^2 + c^2 + f^2 = O(1)$ for $\varepsilon \rightarrow 0$. The corresponding discrete bilinear form a_i^ε can be constructed in the described way, i.e. we set

$$a_i^\varepsilon(u_h, v_h) = b_i^\varepsilon(u_h, v_h) + c_i(u_h, v_h),$$

where

$$b_i^\varepsilon(u_h, v_h) = \sum_{i \in V} \sum_{j \in V_i} \{ \varepsilon - [1 - r(N_{ij} d_{ij}/\varepsilon)] N_{ij} d_{ij} \} (u_{hi} - u_{hj}) v_{hi} \frac{m_{ij}}{d_{ij}}.$$

Then it is convenient to investigate the coercivness of a_i^ε with respect to the so-called « ε -weighted » norm defined by

$$\|u\|_\varepsilon^2 = \varepsilon |u; H^1(D)|^2 + \|u; L_2(D)\|^2, \quad u \in H^1(D).$$

Requiring the assumption (A1) to be fulfilled and analyzing the proof of Lemma 1 we conclude that a_i^ε is coercive and, what is significant, that the bound h_0 does not depend on the parameter ε .

All assertions concerning the scheme (3) are proved under the assumption that the control function $r(z)$ satisfies the properties (P1)-(P5). The property (P6) was added for the sake of completeness only.

Now one can ask, of course, whether or not there exist further and, may be, more simple functions r satisfying the properties mentioned above. Here we give some examples of such functions :

$$r(z) = \begin{cases} 1, & z \geq 0 \\ 0, & z < 0, \end{cases} \quad [2, 13, 14]$$

$$r(z) = \frac{1}{2} (1 + \tanh z), \quad [9]$$

$$r(z) = \begin{cases} 0, & z < -m \\ \frac{z+m}{2m}, & |z| \leq m, \\ 1, & z > m \end{cases} \quad (0 < m \leq 8) \quad [1]$$

$$r(z) = \begin{cases} 0, & z < -m \\ 0.5, & |z| \leq m, \\ 1, & z > m \end{cases} \quad (0 \leq m \leq 2). \quad [1]$$

Heinrich [7] takes the function

$$r(z) = \begin{cases} (1 + \sigma)/2, & z \geq 0 \\ (1 - \sigma)/2, & z < 0, \end{cases}$$

where $\sigma = \sigma(z)$ is chosen such that the relation

$$\sigma(z) \geq \sigma_\delta(z) = \max \left\{ 0, 1 - \frac{2}{(1 + \delta) |z|} \right\}$$

holds for some small $\delta > 0$.

This function satisfies the properties (P1)-(P7), too.

Let us add that in our situation it is legitimate to permit $\delta = 0$. Then the resulting scheme corresponds to the scheme E (« partial upwind ») proposed in [8].

At this place we notice that the first and the last of these examples, properly speaking, are the reason for the necessity to formulate the property (P6) of r exactly in the variant given above.

Finally we want to accentuate that the discretization method and the results can be extended easily to the three-dimensional case. An application of the method for two- and three-dimensional domains D to the spatial discretization of the fundamental equations for the carrier transport in semiconductors was given in [1].

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