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TRAVELING WAVES IN A CYLINDER ROLLING ON A FLAT SURFACE (*)

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Abstract. — *We consider the radial deformation of an infinite cylinder rolling on a flat surface, assuming the deformed shape is constant in time. We give an appropriate (unilateral) modelling. For low angular velocities, our problem is elliptic (with possible degeneracy on the boundary). In this case existence and uniqueness of the solution are proved. An iterative method of solution is given, and its convergence is proved. A finite element approximation is formulated, and an error-estimate for the approximated solution is given. The paper closes with the results of several numerical computations.*

Résumé. — *On considère la déformation radiale d'un cylindre infiniment long roulant sur une surface plane, sous l'hypothèse d'une déformation constante dans le temps. On donne le modèle (de problème unilatéral) correspondant. Pour des vitesses angulaires faibles, le problème est elliptique (avec dégénérescence éventuelle à la frontière). On montre l'existence et l'unité de la solution de ce problème. On donne une méthode itérative pour calculer cette solution et on établit la convergence. On introduit une approximation par la méthode des éléments finis et établit une estimation de l'erreur. On conclue par plusieurs exemples numériques.*

1. INTRODUCTION

In this paper we discuss the radial deformation of an infinite cylinder rolling on a flat surface with a constant angular velocity ω. We assume that the deformed shape does not change in time.

The model we give for this problem is a unilateral one, allowing contact between the cylinder and the flat surface in part of the external surface of the cylinder. We study the case of relatively low angular velocity, for which the system obtained is elliptic, with possible degeneracy on the boundary.

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We prove existence and uniqueness of the solution for these velocities. Then we construct a numerical method for approximating this solution, and prove its convergence to the exact solution. We show the results of a few numerical experiments with this method, and estimate the error.

The problem of contact between two solid bodies was first studied by Hetz in 1881 (see reference [9]), who gave a local analytic solution for contact between two bodies at rest. The problem of contact between a rolling cylinder and the surface on which it is rolling has been studied in the last few years, mainly from an engineering point of view — first by Kalker [7], then by Padovan, Tovichakchaikul and Zeid [13], and lately also by Oden and Lin [12], who gave the problem a clearer Mathematical formulation than their precessors. In the last two references there is also a finite element analysis of the problem. But, in all these references, there is no study of the existence and uniqueness of the solution, and the convergence of the numerical schemes was not proved.

Unilateral elliptic problems were studied intensively in the last three decades, as a special case of variational inequalities (see, for example, [5] and [8]). The problems usually considered have a unilateral constraint on all of the boundary, which is not our case. We have also added the possibility of the problem being degenerate-elliptic on the boundary.

2. FORMULATION OF THE PROBLEM

Let $u(r, \theta, t)$ be the deformation of the point (r, θ) of the cylinder at time t in the radial direction r, as measured in a frame of reference attached to the center of a cross-section of the rolling cylinder (but not rolling with it). Neglecting all deformations except the radial one, we get the following model problem :

$$u_{tt} = \Delta u + f(r, \theta, t) = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + f(r, \theta, t) \tag{2.1}$$

where $f(r, \theta, t) \in L^2$ is the force acting on the point (r, θ) of the cylinder at time t.

We look for periodic solutions of (2.1), i.e. solutions which satisfy

$$u(r, \theta^+, t) = u(r, (\theta + 2\pi)^-, t) \tag{2.2}$$

(that is $\lim_{\Delta\theta \to 0^+} u(r, \theta + \Delta\theta, t) = \lim_{\Delta\theta \to 0^-} u(r, \theta + 2\pi + \Delta\theta, t)$).

In a state of constant deformation we have :

$$u(r, \theta, t) = u(r, \theta - \omega t)\,; \quad f(r, \theta, t) = f(r, \theta - \omega t)\,. \tag{2.3}$$

Let $z = \theta - \omega t$. In the (r, z) variables, (2.1) becomes

$$-\left(\frac{1}{r^2} - \omega^2\right) u_{zz} - \frac{1}{r}(r u_r)_r = f(r, z) \tag{2.4}$$

and we assume that u and f have a period of 2π in z.

For $\omega \leqslant \frac{1}{r}$, (2.4) will be elliptic. We shall solve this equation in the ring-shaped domain :

$$\Omega = \{(r, z) \mid a \leqslant r \leqslant b,\ 0 \leqslant z < 2\pi\} \tag{2.5}$$

where a and b are the internal and external radii of the cylinder. Assume the surface on which the cylinder is rolling is at a distance c from the center of the cross-section of the cylinder (see *fig.* 2). Its equation, in the frame of reference we chose, is $y = -c$. The constraint on the radial deformation of the point (b, θ) of the external boundary is

$$(b + u(b, \theta)) \sin\theta \leqslant -c .$$

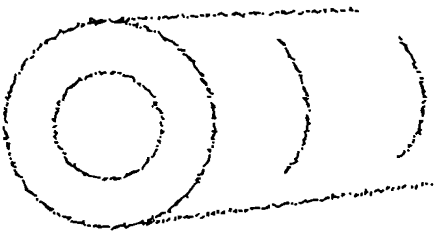


Figure 1.

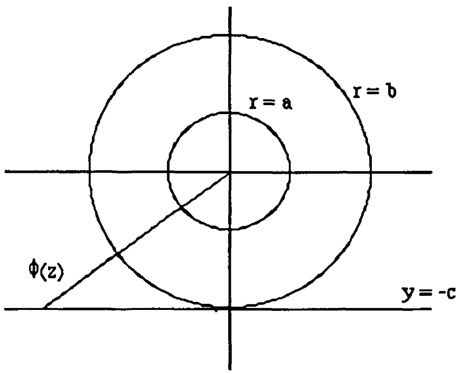


Figure 2.

Thus, the boundary conditions are :

$u = 0$ for $r = a$

$\frac{\partial u}{\partial n} = 0$ for $r = b$ outside the contact-zone of the cylinder and the surface

$$u \leqslant \phi(\theta), \quad \frac{\partial u}{\partial n} \leqslant 0, \quad (u - \phi)\cdot\frac{\partial u}{\partial n} = 0 \quad \text{on the contact-zone}$$

where

$$\phi(\theta) = -b - \frac{c}{\sin\theta}. \tag{2.6}$$

In the original variables (r, θ, t), the contact-zone is given by

$$\Gamma_{bc} = \{(b, \theta) \mid \theta_0 \leqslant \theta \leqslant \theta_1\}.$$

In the (r, z) variables, since we assume that the deformed shape is constant, it is sufficient to solve for a contact-zone of the form

$$\Gamma_{bc} = \{(b, z) \mid z_0 \leqslant z \leqslant z_1\}$$

where z_0, z_1 are independent of the time t. The boundary conditions in these variables are :

$$\begin{gathered} u = 0 \quad \text{on} \quad \Gamma_a \\ \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \Gamma_{bf} \\ u \leqslant \phi(z), \quad \frac{\partial u}{\partial n} \leqslant 0, \quad (u - \phi) \cdot \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \Gamma_{bc} \end{gathered} \tag{2.7}$$

where :

$$\partial\Omega = \Gamma_a \cup \Gamma_{bf} \cup \Gamma_{bc} \tag{2.8}$$

Γ_a is the internal boundary $(r = a)$, Γ_{bf} is the free external boundary, and Γ_{bc} is the zone of possible contact between the cylinder and the surface.

We shall call « problem (A) » the following problem :

(A) Find a function u which solves equation (2.4) in Ω, under the boundary conditions (2.7).

3. AN EQUIVALENT VARIATIONAL FORM

We denote by $L^2(\Omega ; r)$ the space of functions v which are measurable on Ω and

$$\int_\Omega r v^2 \, d\omega < \infty$$

$H^1_{2\pi}(\Omega; r)$ is the space of functions v such that v, $\frac{\partial v}{\partial r}$ and $\frac{\partial v}{\partial z}$ are in $L^2(\Omega ; r)$ and are periodic with period 2π.

We define :

$$V = \{v \in H^1_{2\pi}(\Omega ; r) \mid v = 0 \text{ on } \Gamma_a\}. \tag{3.1}$$

For $v \in V$, it is easily verified that

$$\int_\Omega rv^2 \, d\omega \leq C \int_\Omega rv_r^2 \, d\omega$$

and thus the norm of $v \in V$ can be defined as

$$\|v\|^2 = \int_\Omega r\{v_r^2 + v_z^2\} \, d\omega . \tag{3.2}$$

Let K be the following convex subset of V:

$$K = \{v \in V \mid v \leq \phi \text{ on } \Gamma_{bc}\} . \tag{3.3}$$

Problem (B) will be the following:

(B) Find $u \in K$ such that:

$$a(u, v - u) \geq (f, v - u) \quad \forall v \in K \tag{3.4}$$

where

$$(u, v) = \int_\Omega ruv \, d\omega \tag{3.5}$$

$$a(u, v) = \int_\Omega r\left\{\left(\frac{1}{r^2} - \omega^2\right) u_z v_z + u_r v_r\right\} d\omega . \tag{3.6}$$

PROPOSITION 1: *The problems* (A) *and* (B) *are equivalent in the following sense: If u is a solution of* (A) *then u is also a solution of* (B). *If u is a regular solution to* (B) *then it solves also problem* (A).

Proof: Let $\mathscr{D}(\Omega)$ be the space of indefinitely differential functions with compact support in Ω. If u is a regular solution of (B), and $\varphi \in \mathscr{D}(\Omega)$, then $v = u \pm \varphi$ are in K. Taking successively $v = u + \varphi$, $v = u - \varphi$ in (3.4) gives

$$a(u, \varphi) = (f, \varphi) \tag{3.7}$$

and thus the equation (2.4).

Multiplying (2.4) by $r(v - u)$ and integrating over Ω (using Green's formula), leads to

$$a(u, v - u) - \int_{\partial\Omega} r \frac{\partial u}{\partial n} (v - u) \, d\sigma = (f, v - u) .$$

So that (B) is equivalent to (2.4) and

$$\int_{\partial\Omega} r \frac{\partial u}{\partial n} (v - u) \, d\sigma \geq 0 \quad \forall v \in K . \tag{3.8}$$

For $v \in K$, the integral in (3.8) vanishes on Γ_a. Let $v = u \pm \varphi$ where $\varphi = 0$ on Γ_{bc}. This leads to

$$\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \Gamma_{bf} . \tag{3.9}$$

If $v = u + \psi$ where $\psi \leqslant 0$ on Γ_{bc}, then $v \in K$, and from (3.8) we get

$$\int_{\Gamma_{bc}} r \frac{\partial u}{\partial n} \psi \, d\sigma \geqslant 0 . \tag{3.10}$$

Therefore

$$\frac{\partial u}{\partial n} \leqslant 0 \quad \text{on} \quad \Gamma_{bc} . \tag{3.11}$$

Finally, we take $v = u + \psi$ where $\psi = -(u - \phi)$ on Γ_{bc}. Thus $v \in K$, and (3.8) gives

$$\int_{\Gamma_{bc}} r \frac{\partial u}{\partial n} (u - \phi) \, d\sigma \leqslant 0 . \tag{3.12}$$

Comparing this with (3.3) and (3.11), we deduce that

$$(u - \phi) \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \Gamma_{bc}$$

and hence (2.7), and conversely (2.7) implies (3.8).

4. EXISTENCE AND UNIQUENESS OF THE SOLUTION

If

$$\omega \leqslant \frac{1}{b} \tag{4.1}$$

the problem (A) is elliptic. For this case, we prove existence and uniqueness of the solution.

PROPOSITION 2 : *Under condition* (4.1), *there exists a unique solution of problem* (B) (*and hence also of problem* (A)).

Proof : The bilinear form $a(v, w)$ defined in (3.6) is the usual scalar product in V with the weight

$$A = \begin{pmatrix} \left(\frac{1}{r^2} - \omega^2 \right) & 0 \\ 0 & 1 \end{pmatrix}$$

i.e.

$$a(v, \omega) = \int_\Omega r\left\{(v_r v_z)\, A \begin{pmatrix} w_r \\ w_z \end{pmatrix}\right\} d\omega .$$

Hence it is possible to define a new scalar product on V by $((v, w)) = a(v, w)$. (For $\omega < 1/b$ this bi-linear form is also coercive, i.e. there exists $m > 0$ s.t.

$$\forall v \in V \quad a(v, v) \geqslant m \| v \|^2$$

and thus the two norms are equivalent).

By the Riesz theorem, there exists $\tilde{f} \in V$ such that for every $v \in V$

$$(f, v) = ((\tilde{f}, v)) .$$

We will designate by $|||\,.\,|||$ the norm induced on V by the scalar product $((.,.))$.

LEMMA 1 : *$\bar{u} = \mathcal{P}_k \tilde{f}$ is the unique solution of the problem* (B), *where $\mathcal{P}_k$ is the projection operator on K, defined by the metric* $|||\,.\,|||$.

Proof : In the scalar product notation, (3.4) becomes

$$((\bar{u}, v - \bar{u})) \geqslant ((\tilde{f}, v - \bar{u})) \quad \forall v \in K . \tag{4.2}$$

Suppose there exists $v_0 \in K$ such that

$$((\tilde{f} - \bar{u}, v_0 - \bar{u})) > 0 .$$

Choose

$$0 < t < \min\left\{1, \frac{2((\tilde{f} - \bar{u}, v_0 - \bar{u}))}{((v_0 - \bar{u}, v_0 - \bar{u}))}\right\} . \tag{4.3}$$

Since K is convex, $\bar{v} \equiv tv_0 + (1 - t)\, \bar{u} \in K$

$$((\tilde{f} - \bar{v}, \tilde{f} - \bar{v})) = ((\tilde{f} - \bar{u} + t(\bar{u} - v_0), \tilde{f} - \bar{u} + t(\bar{u} - v_0))) =$$
$$= ((\tilde{f} + \bar{u}, \tilde{f} - \bar{u})) + 2\, t((\tilde{f} - \bar{u}, \bar{u} - v_0)) + t^2((v_0 - \bar{u}, v_0 - \bar{u})) .$$

By (4.3) $t((v_0 - \bar{u}, v_0 - \bar{u})) \leqslant 2((\tilde{f} - \bar{u}, v_0 - \bar{u}))$, hence

$$2\, t((\tilde{f} - \bar{u}, \bar{u} - v_0)) + t^2((v_0 - \bar{u}, v_0 - \bar{u})) < 0$$

$$|||\tilde{f} - \bar{v}|||^2 = ((\tilde{f} - \bar{v}, \tilde{f} - \bar{v})) < ((\tilde{f} - \bar{u}, \tilde{f} - \bar{u})) = |||\tilde{f} - \bar{u}|||^2 .$$

But this is a contradiction, since $\bar{u} = \mathcal{P}_K \tilde{f}$, and hence (4.2).

To prove uniqueness :

Suppose u_1, u_2 are two solutions of (B). Thus for every $v \in K$.

$$a(u_1, v - u_1) \geq (f, v - u_1) \tag{4.4}$$

$$a(u_2, v - u_2) \geq (f, v - u_2)\,. \tag{4.5}$$

By choosing $v = u_2$ in (4.4) and $v = u_1$ in (4.5) and adding both inequalities, we get

$$|||\, u_2 - u_1 \,|||^2 = a(u_2 - u_1, u_2 - u_1) \leq 0$$

and thus $u_1 \equiv u_2$.

5. ITERATIVE METHODS (CONTINUOUS CASE)

Let $J : V \to \mathbb{R}$ be defined by

$$J(v) = \frac{1}{2} a(v, v) - (f, v)\,. \tag{5.1}$$

PROPOSITION 3 : *The problem*

(C) *Find $u \in K$ such that*

$$J(u) = \min_{v \in K} J(v) \tag{5.2}$$

is equivalent to problem (B) (*and hence also to* (A)).

Proof : $J'(u)$, the Gâteaux derivative of J at u, is given by

$$J'(u)\, v = \frac{d}{d\lambda} J(u + \lambda v)\big|_{\lambda = 0} = a(u, v) - (f, v)\,.$$

LEMMA 2 : *If $J(u) \leq J(v)$ for all $v \in K$, then $J'(u)(v - u) \geq 0$ for all $v \in K$, and conversely.*

Proof :

$$J(u + \theta w) - J(u) = \theta J'(u)\, w + \theta \,\|w\|\, \varepsilon(\theta) \text{ where } \lim_{\theta \to 0} \varepsilon(\theta) = 0\,.$$

Let $w = v - u$ where u is the minimizing function in (5.2), and $\theta > 0$. Then

$$J(u + \theta(v - u)) \geq J(u)\,.$$

Suppose $J'(u)(v-u)<0$. For small enough θ

$$\theta J'(u)(v-u)+\theta\|v-u\|\,\varepsilon(\theta)<0\,.$$

But this leads to

$$J(u+\theta(v-u))<J(u)$$

which contradicts (5.2).

The converse follows in the same way.

Therefore $J'(u)(v-u)\geqslant 0\quad \forall v\in K$ iff $a(u,v-u)-(f,v-u)>0$ $\forall v\in K$ and hence the equivalence between (C) and (B).

For $v\in V$ and $q\in P$, where

$$P=\{q\in L^2(\Gamma_{bc})\,|\,q\leqslant 0 \text{ } a.e. \text{ on } \Gamma_{bc}\}\,. \tag{5.3}$$

We define the Lagrangian $\mathscr{L}$

$$\mathscr{L}(v,q)=J(v)-[q,v-\phi] \tag{5.4}$$

where

$$[q,v-\phi]=\int_{\Gamma_{bc}} rq(v-\phi)\,d\sigma\,. \tag{5.5}$$

DEFINITION 1 : *(u,p) is called a saddle point of $\mathscr{L}(v,q)$ in $V\times P$ if for every $v\in V$ and $q\in P$*

$$\mathscr{L}(u,q)\leqslant\mathscr{L}(u,p)\leqslant\mathscr{L}(v,p)\,. \tag{5.6}$$

LEMMA 3 : *$\mathscr{L}(v,q)$ has at most one saddle point.*

Proof : Assume (u,p) and $(\bar u,\bar p)$ are two saddle points of $\mathscr{L}$. From (5.6), for every $q\in P$ we have

$$\mathscr{L}(u,p)-\mathscr{L}(u,q)=[q-p,u-\phi]\geqslant 0 \tag{5.7}$$
$$\mathscr{L}(\bar u,\bar p)-\mathscr{L}(\bar u,q)=[q-\bar p,\bar u-\phi]\geqslant 0\,. \tag{5.8}$$

Let $q=\bar p$ in (5.7) and $q=p$ in (5.8). Adding the two inequalities gives

$$[p-\bar p,\bar u-u]\leqslant 0\,. \tag{5.9}$$

Let $\tilde J(v\,;q)=\mathscr{L}(v,q)$. As in lemma 2, since $\tilde J(u)=\min_{v\in V}\tilde J(v)$, we have $\tilde J'(u)\,v=0$ $\forall v\in V$, where $\tilde J'(u)$ is the Gâteaux derivative of $\tilde J(u)$. Thus u is the solution to

(5.10) $$u \in V \quad a(u, v) = (f, v) + \int_{\Gamma_{bc}} rpv\, d\sigma \quad \forall v \in V .$$

Similarly, $\bar{u}$ is the solution to the problem :

(5.11) $$\bar{u} \in V \quad a(\bar{u}, v) = (f, v) + \int_{\Gamma_{bc}} r\bar{p}v\, d\sigma \quad \forall v \in V .$$

Letting $v = \bar{u} - u$ in (5.10), $v = u - \bar{u}$ in (5.11), and adding the two equalities gives

(5.12) $$a(\bar{u} - u, \bar{u} - u) = [\bar{p} - p, \bar{u} - u] .$$

By (5.9), we get $|||\, \bar{u} - u \,|||^{2} \leq 0$. Hence $\bar{u} = u$.

Compare now (5.10) and (5.11). Since $\bar{u} = u$, we get

$$\forall v \in V \quad \int_{\Gamma_{bc}} r(p - \bar{p})\, v\, d\sigma = 0$$

i.e. $p = \bar{p}$.

LEMMA 4 : *$v \in K$ iff $[q, v - \phi] \geq 0$ for every $q \in P$.*

PROPOSITION 4 : *If (u, p) is a saddle point of $\mathscr{L}(v, q)$ in $V \times P$ then $u \in K$, and u the unique solution of problem* (C).

Proof : If (u, p) is a saddle point of $\mathscr{L}(v, q)$ then by (5.7) $[p - q, u - \phi] \leq 0$ for every $q \in P$. Letting successively $q = 0$ and $q = 2p$ gives $[p, u - \phi] = 0$. Again, by (5.7), we have $[q, u - \phi] \geq 0$ for all $q \in P$. Thus, by lemma 4, $u \in K$.

For every $v \in K$, by lemma 4, $[p, v - \phi] \geq 0$. Thus, by (5.4), (5.6) and (5.7) we get

$$J(u) = \mathscr{L}(u, p) \leq (v, p) = J(v) - [p, v - \phi] \leq J(v)$$

and u is a solution to problem (C).

PROPOSITION 5 : *If u, the solution to problem* (C), *is in $H^{2}_{2\pi}(\Omega\,; r)$ and $p = \left.\dfrac{\partial u}{\partial n}\right|_{\Gamma_{bc}}$, then (u, p) is the unique saddle point of $\mathscr{L}(v, q)$ on $V \times P$.*

Proof : If $u \in H^{2}_{2\pi}(\Omega\,; r)$ is a solution to (C) then by Lions-Magenes [11], $\dfrac{\partial u}{\partial n} \in H^{1/2}(\Gamma_{bc}\,; r) \subset L^{2}(\Gamma_{bc}\,; r)$. Thus, by the proof of proposition 1,

$p \in P$ and (u, p) is in the right vector space. Again, from the proof of proposition 1, $[p, u - \phi] = 0$, and hence $\mathscr{L}(u, p) = J(u)$. Furthermore, $u \in K$ and thus by lemma 4 $[q, u - \phi] = 0$ for every $q \in P$. Summarizing the above, we get : for every $q \in P$, $\mathscr{L}(u, q) \leqslant J(u) = \mathscr{L}(u, p)$. Thus we got the left-hand-side of (5.6). For the right-hand-side, we will show

$$\tilde{J}(u\,; p) = \min_{v \in V} \tilde{J}(v\,; p)$$

where $\tilde{J}(v\,; p)$ is defined as in the proof of lemma 3.

If $\bar{u}$ satisfies $\tilde{J}(\bar{u}) = \min\limits_{v \in V} \tilde{J}(v)$, then $\bar{u}$ is a generalized solution to the problem :

$$\begin{cases} -\left(\dfrac{1}{r^2} - \omega^2\right) v_{zz} - \dfrac{1}{r}(r v_r)_r = f & \text{in } \Omega \\ v(a, z) = 0 & \text{on } \Gamma_a \\ \dfrac{\partial v}{\partial n} = 0 & \text{on } \Gamma_{bf} \\ \dfrac{\partial v}{\partial n} = p & \text{on } \Gamma_{bc}\,. \end{cases}$$

Obviously u is also a solution to this problem, and the solution is unique. Hence $u = \bar{u}$, and $\mathscr{L}(u, p) = \tilde{J}(u) \leqslant \tilde{J}(v) = \mathscr{L}(v, p)$ for every $v \in V$. Thus (u, p) is a saddle point. Since by lemma 3 the saddle point is unique, we have concluded the proof.

By the Min Max theorem

$$\mathscr{L}(u, p) = \sup_{q \in P} \inf_{v \in V} \mathscr{L}(v, q) = \inf_{v \in V} \sup_{q \in P} \mathscr{L}(v, q)\,. \tag{5.13}$$

Thus, we formulate problem (D) :

(D) Find $(u, p) \in V \times P$ s.t.

$$\mathscr{L}(u, p) = \sup_{q \in P} \inf_{v \in V} \mathscr{L}(v, q) \tag{5.14}$$

which we will solve in an iterative way. The algorithm will be the following :

$$p^0 = 0\,; \quad \mathscr{L}(u^0, p^0) = \inf_{v \in V} \mathscr{L}(v, p^0)\,. \tag{5.15}$$

Then we proceed by induction. Assume (u^n, p^n) are known. We obtain u^{n+1}, p^{n+1} by :

(5.16)

$$p^{n+1} = \mathscr{P}_P((p^n - \rho(u^n - \phi))|_{\Gamma_{bc}}) ; \quad \mathscr{L}(u^{n+1}, p^{n+1}) = \inf_{v \in V} \mathscr{L}(v, p^{n+1}) .$$

Where $\mathscr{P}_P$ is the projection operator on P, i.e.

$$\mathscr{P}_P(v) = v^- , \quad v(b,z)^- = \min \{v(b,z), 0\} .$$

Let $\gamma : V \to L^2(\Gamma_{bc} ; r)$ be the trace operator, $\gamma u = u|_{\Gamma_{bc}}$. γ is a continuous linear operator, that is

(5.17) $$\forall v \in V \quad |\gamma v| \leq \|\gamma\| \|v\| .$$

Where $|.|$ is the norm in $L^2(\Gamma_{bc} ; r)$

$$|q|^2 = \int_{\Gamma_{bc}} rq^2 \, d\sigma .$$

PROPOSITION 6 : *If $a(.,.)$ is coercive, that is $\omega < 1/b$ and*

(5.18) $$\exists m > 0 \quad a(v,v) \geq m \|u\|^2 \quad \forall v \in V$$

then for

(5.19) $$0 < \rho < \frac{2m}{\|\gamma\|^2}$$

the sequence $\{u^n\}$ defined by the iteration process given above, converges (strongly) to u.

Proof : u^n satisfy

(5.20) $$\forall v \in V \quad a(u^n, v) = (f, v) + \int_{\Gamma_{bc}} rp^n v \, d\sigma$$

u satisfies

(5.21) $$\forall v \in V \quad a(u, v) = (f, v) + \int_{\Gamma_{bc}} rpv \, d\sigma .$$

We have seen that for every $q \in P$, $\mathscr{L}(u, q) \leq \mathscr{L}(u, p)$ implies

$$\int_{\Gamma_{bc}} r(q - p)(u - \phi) \, d\sigma \geq 0 .$$

Writing this in another form, we have

$$\int_{\Gamma_{bc}} r(q-p)(p-\rho(u-\phi)-p)\,d\sigma \leq 0 \quad \forall q \in P \quad \rho > 0 . \tag{5.22}$$

But this implies

$$p = \mathscr{P}_P(p - \rho(u-\phi)) . \tag{5.23}$$

Let $\bar{u}^n = u^n - u$, $\bar{p}^n = p^n - p$. The projection operator $\mathscr{P}$ is a contraction. Therefore, by (5.16) and (5.23), we have

$$\begin{aligned} |\bar{p}^{n+1}|^2 = |p^{n+1} - p|^2 &\leq |\bar{p}^n - \rho\bar{u}^n|^2 \\ &= |\bar{p}^n|^2 + \rho^2|\gamma\bar{u}^n|^2 - 2\rho \int_{\Gamma_{bc}} r\bar{p}^n\,\bar{u}^n\,d\sigma \end{aligned}$$

or

$$|\bar{p}^n|^2 - |\bar{p}^{n+1}|^2 \geq 2\rho \int_{\Gamma_{bc}} r\bar{p}^n\,\bar{u}^n\,d\sigma - \rho^2|\gamma\bar{u}^n|^2 . \tag{5.24}$$

Let $v = \bar{u}^n$ in (5.20) and (5.21). Subtracting (5.21) from (5.20) gives

$$a(\bar{u}^n, \bar{u}^n) = \int_{\Gamma_{bc}} r\bar{p}^n\,\bar{u}^n\,d\sigma .$$

Therefore

$$m\,\|\bar{u}^n\|^2 \leq \int_{\Gamma_{bc}} r\bar{p}^n\,\bar{u}^n\,d\sigma . \tag{5.25}$$

Using (5.17), (5.25) in (5.24) we get :

$$|\bar{p}^n|^2 - |\bar{p}^{n+1}|^2 \geq \rho(2m - \rho\,\|\gamma\|^2)\,\|\bar{u}^n\|^2 .$$

Assuming (5.19) the sequence $\{|\bar{p}^n|^2\}$ is decreasing and hence converges. Therefore we have

$$\lim_{n\to\infty} (|\bar{p}^n|^2 - |\bar{p}^{n+1}|^2) = 0$$

so that

$$\lim_{n\to\infty} \|\bar{u}^n\|^2 = \lim_{n\to\infty} \|u^n - u\| = 0 .$$

For $\omega = 1/b$ a similar estimation is derived using the norm $|||\,.\,|||$ instead of $\|\cdot\|$. Define

$$(5.26) \qquad |||\,\gamma\,||| = \sup_{v \in V} \frac{|\gamma v|}{|||\,v\,|||}\,.$$

PROPOSITION 7 : *If*

$$(5.27) \qquad 0 < \rho < \frac{2\,m}{|||\,v\,|||^2}$$

the sequence u^n defined by the iteration process above converges to u in the $|||\,.\,|||$ norm, i.e. $\lim\limits_{n \to \infty} |||\,u^n - u\,||| = 0$.

Proof: The proof is similar to the proof of proposition 5.

6. FINITE ELEMENTS APPROXIMATION

As we have seen in § 5, for every step of the iteration process, we have a minimization problem of the form

$$\mathcal{L}(u^n, p^n) = \min_{v \in V} \mathcal{L}(v, p^n)\,.$$

Or, using the notation we had in proposition 4,

$$(6.1) \qquad \tilde{J}(u^n ; p^n) = \min_{v \in V} \tilde{J}(v ; p^n)$$

where

$$(6.2) \qquad \tilde{J}(v ; p) = \frac{1}{2} a(v, v) - (f, v) - \int_{\Gamma_{bc}} rp(v - \phi)\, d\sigma\,.$$

We shall approximate this minimization problem by a discrete problem, which can be solved by the finite elements method. Let $\mathcal{T}_h$ be a division of Ω into four-node elements, such that $\Omega_h = \bigcup_{T \in \mathcal{T}_h} T$ satisfies :

(6.3) (1) $\forall x \in \Omega_h$ dist $(x, \Omega) < h$; $\forall x \in \Omega$ dist $(x, \Omega_h) < h$.
(2) All nodes of Ω_h are in Ω, all nodes of $\partial\Omega_h$ are in $\partial\Omega$.
(3) $h = \max\limits_{T \in \mathcal{T}_h} \{\text{diam } T\}$.
(4) $\forall T_1, T_2 \in \mathcal{T}_h$, $T_1 \neq T_2 \Rightarrow \mathring{T}_1 \cap \mathring{T}_2 = \varnothing$, and exactly one of the following conditions holds :

(*a*) $T_1 \cap T_2 = \emptyset$
(*b*) T_1 and T_2 have exactly one common node
(*c*) T_1 and T_2 have one common side.

We will approximate our previous spaces as follows :

$$H^1_{2\pi}(\Omega\,;r) \mapsto H_h = \left\{v_h \in C(\Omega_h\,;r) \colon v_h|_T \in \mathbf{P}_k \ \ \forall T \in \mathcal{T}_h\right\}$$

$$V \mapsto V_h = \{v_h \in H_h : v_h = 0 \text{ on } \Gamma_{a,h}\}$$

$$P \mapsto P_h = \left\{q_h \in C(\Gamma_h\,;r) \colon q_h|_{\Gamma_h} \leqslant 0 \ \ q_h|_{\gamma^h} \in \mathbf{P}_k \ \ \forall \gamma^h \subset \Gamma_h\right\}.$$

Where $\mathbf{P}_k$ = space of polynomials of degree $\leqslant k$ in r and z (usually we will use $k = 1$)

$$\Gamma_{a,h} = \bigcup \{\gamma^h : \exists T \in \mathcal{T}_h \text{ s.t. } \gamma^h \text{ is a side of } T\,;$$
$$\gamma^h \subset \partial\Omega_h, \text{ both nodes of } \gamma^h \text{ are in } \Gamma_a\}$$

The scalar product and norm on V_h are :

$$\langle v_h, w_h\rangle = \int_{\Omega_h} r\{v_{h,r}\, w_{h,r} + v_{h,z}\, w_{h,z}\}\, d\omega$$

$$\|v_h\|^2 = \langle v_h, v_h\rangle\,.$$

For $k = 1$ we define

$$K_h = \{v_h \in V_h | v_h(b,z) \leqslant \phi(z) \quad \text{at nodes on } \Gamma_h\}$$

and for $k = 2$ the convex set will be

$$K_h = \{v_h \in V_h | v_h(Q) \leqslant \phi(Q) \ \ \forall Q \in \Sigma_h \cap \Gamma_{bc}\}$$

where Σ_h is the union of the set of nodes of Γ_h with the set of midpoints of sides in Γ_h.

We define :

$$a_h(v_h, \omega_h) = \int_{\Omega_h} r\left\{\left(\frac{1}{r^2} - \omega^2\right) v_{h,z}\, w_{h,z} + v_{h,r}\, w_{h,r}\right\} d\omega$$

$$(f, v_h)_h = \int_{\Omega_h} r f v_h\, d\omega$$

$$J_h(v_h) = \frac{1}{2} a_h(v_h, v_h) - (f, v_h)_h$$

$$\mathcal{L}_h(v_h, q_h) = \tilde{J}_h(v_h\,;q_h) = J_h(v_h) - [q_h, v_h - \phi]_h$$

where $[p, q]_h$ is the numerical integration of rpq on Γ_h — by the complex trapezoidal method on the nodes of Γ_h for $k = 1$ and by the complex Simpson method for $k = 2$. As in § 5, if (u_h, p_h) is a saddle point of $\mathscr{L}_h(v_h ; q_h)$ in $V_h \times P_h$ then

$$J_h(u_h) = \min_{v_h \in K_h} J_h(v_h)$$

and

$$a_h(u_h, v_h - u_h) \geqslant (f, v_h - u_h)_h \tag{6.4}$$

for every $v_h \in K_h$. By Céa, [2], such a saddle point exists. The iterative process will be :

$$p_h^0 = 0$$

for $n \geqslant 0$

$$\tilde{J}_h(u_h^n ; p_h^n) = \min_{v_h \in V_h} \tilde{J}_h(v_h ; p_h^n) ; \; p_h^{n-1} = (p_h^n - \rho(u_h^n - \phi)|_{\Gamma_h})^-$$

where ρ is chosen to be « small enough » so that the iterations will converge (such a ρ exists as in Propositions 6 and 7).

7. ERROR-ESTIMATE

In order to estimate the error of the approximated solution from the exact one, we first have to know the regularity of the exact solution. For a convex-set of the form

$$K = \{v \in H^1(\Omega) \,|\, v \leqslant 0 \quad \text{on} \quad \partial\Omega\}$$

Lions [9] has proved $H^2(\Omega) \cap W^{1,\infty}(\Omega)$ regularity of the solution. A similar proof can be applied to the set

$$K = \{v \in V \,|\, v \leqslant 0 \quad \text{on} \quad \Gamma_b\} .$$

In our case, since the boundary conditions are more complicated, we cannot give a similar regularity result. However, for a sufficiently smooth f we give an error-estimate of the approximated solution, after Brezzi, Hager and Raviart [1].

In order to simplify the proof, we assume that $\Omega_h = \Omega$ (this is the case when the finite element mesh is generated using a cylindrical coordinate system — see *fig.* 3).

PROPOSITION 6 : *If f is such that $u \in H^{1+\alpha}(\Omega ; r) \cap W^{1,\infty}(\Omega)$ for some $0 < \alpha \leqslant 1$, (5.17) is satisfied, and the number of points in Γ_{bc} where the constraint changes from binding to non-binding is finite, then for a piecewise linear finite element approximation u_h, $\|u - u_h\| = O(h^\alpha)$.*

Proof : For every $v_h \in V_h$, by (5.10) and (6.4) we have (since $\Omega = \Omega_h$) :

$$\begin{aligned}
(7.1)\quad & a(u-u_h, u-u_h) = \\
&= a(u-u_h, u-v_h) + a(u, v_h - u_h) - a(u_h, v_h - u_h) \\
&= a(u-u_h, u-v_h) + \int_{\Gamma_{bc}} rp(v_h - u_h)\, d\sigma + (f, v_h - u_h) - a(u_h, v_h - u_h) \\
&\leqslant a(u-u_h, u-v_h) + \int_{\Gamma_{bc}} rp(v_h - u_h)\, d\sigma .
\end{aligned}$$

Let $v_h = u_I$, where u_I is the piecewise linear interpolate to u on the nodes of Ω_h. Let ϕ_I be the interpolate of ϕ. In Γ_{bc}, $u_h \leqslant \phi_I$ and $p \leqslant 0$. Hence

$$\begin{aligned}
(7.2)\quad & \int_{\Gamma_{bc}} rp(u_I - u_h)\, d\sigma = \\
&= \int_{\Gamma_{bc}} rp\{(u_I - \phi_I) - (u - \phi)\} + \int_{\Gamma_{bc}} rp(u-\phi)\, d\sigma + \int_{\Gamma_{bc}} rp(\phi_I - u_h)\, d\sigma \\
&\leqslant \int_{\Gamma_{bc}} rp\{(u_I - \phi_I) - (u-\phi)\}\, d\sigma .
\end{aligned}$$

But

$$\begin{aligned}
(7.3)\quad & \int_{\Gamma_{bc}} rp\{(u_I - \phi_I) - (u-\phi)\}\, d\sigma = \\
&\qquad = \sum_{\gamma \in \Gamma_{bc,h}} \int_\gamma rp\{(u_I - \phi_I) - (u-\phi)\}\, d\sigma .
\end{aligned}$$

Let Γ_c and Γ_f denote the parts of Γ_{bc} where $u = \phi$ and $u < \phi$ respectively. For a side γ of $\Gamma_{bc,h}$, if $\gamma \subset \Gamma_c$ then $u = \phi$ and $u_I = \phi_I$ on γ. If $\gamma \subset \Gamma_f$ then $p = 0$ on γ and again the integral in (7.3) is equal to zero. If $\gamma \not\subset \Gamma_c \cup \Gamma_f$ then there exists a point $Q \in \gamma$ such that $u(Q) = \phi(Q)$. Since $u \in W^{1,\infty}$ near Γ_{bc}, both $u - \phi = O(h)$ and $u_I - \phi_I = O(h)$ on γ. Since $p \in L^\infty$, we obtain

$$\int_\gamma rp\{(u_I - \phi_I) - (u-\phi)\}\, d\sigma = O(h^2) .$$

By our assumption, the number of points of change from Γ_c to Γ_f is finite, and thus (7.2) and (7.3) imply

$$\int_{\Gamma_{bc}} rp(u_I - u_h)\, d\sigma = O(h^2)\,. \tag{7.4}$$

By interpolation theory (see for example Strang and Fix [12]), $\|u - u_I\| = O(h^\alpha)$. $a(.,.)$ is a bounded form, that is

$$a(v, w) \leq M \|v\| \; \|w\|\,. \tag{7.5}$$

Thus, combining (7.1), (7.4) and (7.5) we get

$$\|u - u_h\| = O(h^\alpha)\,.$$

8. NUMERICAL RESULTS

The numerical results given here were computed with a computer-program written by the method described above. The deformations were computed for a cylinder with an internal radius of $a = 0.5$ cm and an external radius of $b = 1$ cm. The finite element mesh used is shown in figure 3. The contact-zone Γ_{bc} is the external boundary between the points z_0 and z_1 in this figure.

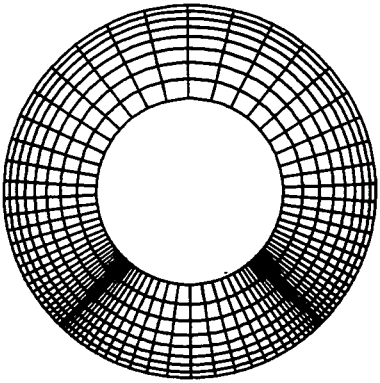
Figure 4 shows the deformed cylinder when the distance of the surface from the center is $c = 0.8$ cm, and no external force is applied to the cylinder, i.e. $f = 0$. The angular velocity assumed in this example is $\omega = 0.5$ rad/s.

In figure 5 we have the radial deformation in the same conditions, except that here the angular velocity is higher : $\omega = 1$ rad/s. In this velocity the system has a parabolic degeneracy on the boundary.

Figure 6 shows again the deformation for $\omega = 0.5$, but this time a force of $f = 1$, is applied to the cylinder.

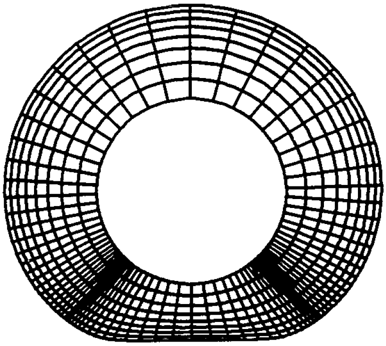
Figures 7 and 8 show the deformation for $\omega = 0.5$ rad/s, $f = 0$, and $c = 0.85$ cm and 0.75 cm respectively.

The number of iterations required is dependent on the projection coefficient, ρ. Numerical calculations showed that the optimal value lies between two and three. In figure 9 there is a table of the number of iterations required for the different values of ρ corresponding to the data in figures 4 and 5.



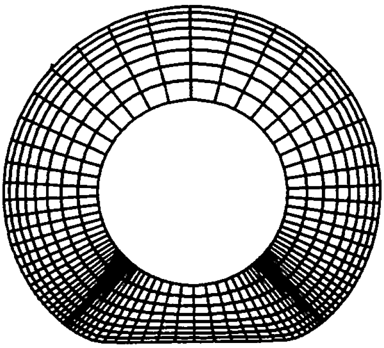
finite element mesh

Figure 3.



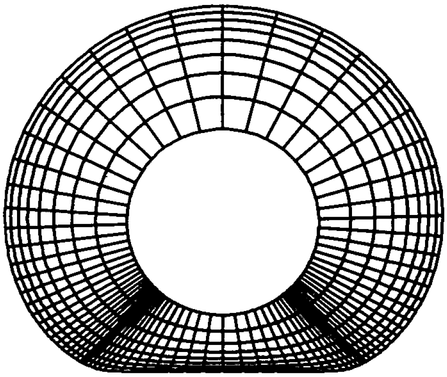
f = 0. ; c = 0.8 : w = 0.5

Figure 4.



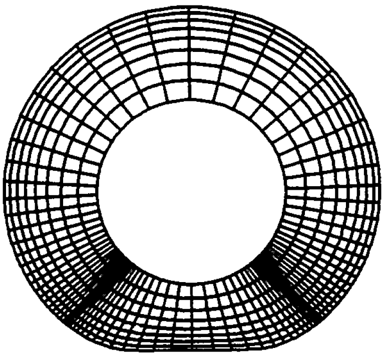
f = 0. ; c = 0.8 : w = 1.

Figure 5.



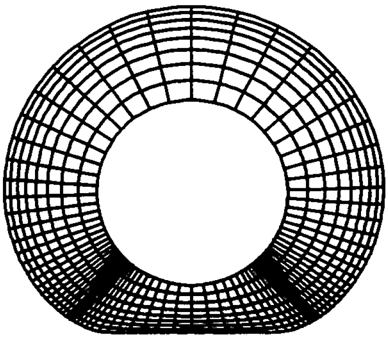
f = 1. ; c = 0.8 : w = 0.5

Figure 6.



f = 0. ; c = 0.85 ; w = 0.5

Figure 7.



f = 0. ; c = 0.75 ; w = 0.5

Figure 8.

For the data of Fig. 4 :

ρ	No. of iterations
0.5	37
1.	28
2.	19
2.5	17
2.75	16
3.	14
3.25	18
3.5	28

For the data of Fig. 5 :

ρ	No. of iterations
0.5	29
1.	19
1.5	15
2.	12
2.5	14
2.75	21
3.	37

Figure 9.

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