M2AN. MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
- MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

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M2AN. Mathematical modelling and numerical analysis - Modélisation mathématique et analyse numérique, tome 24, n° 4 (1990), p. 423-455

<http://www.numdam.org/item?id=M2AN_1990__24_4_423_0>

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(Vol. 24, n° 4, 1990, p. 423 à 455)

DEFECT CORRECTION METHODS FOR CONVECTION DOMINATED CONVECTION-DIFFUSION PROBLEMS (*)

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Communicated by J. DESCLOUX

Abstract. — *Standard Galerkin methods for elliptic problems applied on convection dominated convection-diffusion equations give poor approximations and may even not converge. We prove local and global error estimates for a defect correction method proposed by Hemker and show how the convergence depends on solution regularity, types of layers present and domains/subdomains on which the error is measured.*

In particular, we point out the global regularity associated with problems with characteristic layers alone, and the effect of this on the numerical methods.

Résumé. — *Les méthodes d'éléments finis standard pour des problèmes elliptiques de convection-diffusion dominés par la diffusion donnent de mauvaises approximations et peuvent ne pas converger. Nous établissons des estimations d'erreur locales et globales pour une méthode de « defect correction » proposée par Hemker. Nous analysons la dépendance de la convergence par rapport à la régularité de la solution, aux types de couches limites et aux domaines où l'erreur est considérée.*

En particulier, nous mettons en évidence la régularité globale pour des problèmes de couches limites caractéristiques et ses conséquences pour les méthodes numériques.

1. INTRODUCTION

This paper considers the approximate solution of singularly perturbed, convection diffusion equations

$$\mathscr{L}_\varepsilon u \equiv -\varepsilon \Delta u + \underline{v}(x) \cdot \nabla u + q(x) u = f(x), \quad x \in \Omega, \varepsilon > 0, \quad (1.1)$$

(*) Received in January 1989.

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The work of the second author was partially supported by grant MCS-8202025 from the National Science Foundation of the U S A.

where $\Omega \subset \mathbf{R}^j$, $(j = 1, 2, 3)$ is a regular (*) domain with boundary Γ, $\underline{v}(x)$ is a smooth vector field and $q(x)$ a smooth function on $\bar{\Omega}$. Let $\underline{n}(x)$ be the outward unit normal on Γ. The boundary Γ is partitioned into three sets

$$\Gamma_- = \{x \in \Gamma ; \underline{v}(x) . \underline{n} < 0\} ,$$
$$\Gamma_0 = \{x \in \Gamma ; \underline{v}(x) . \underline{n} = 0\} ,$$
$$\Gamma_+ = \{x \in \Gamma ; \underline{v}(x) . \underline{n} > 0\} ,$$

corresponding to the inflow, characteristic and outflow boundaries, respectively. Dirichlet boundary conditions are imposed on the inflow portion of Γ,

$$u(x) = 0 , \quad x \in \Gamma_- . \tag{1.2}$$

The boundary conditions chosen on Γ_0 and Γ_+ influence the size and nature of the boundary layers occurring in the problem. The method considered herein can be used for various kinds of boundary conditions but to be specific we take Dirichlet boundary conditions on the characteristic boundary and Neuman or Dirichlet boundary conditions on the outflow boundary, i.e.,

$$\begin{aligned} u(x) &= 0 , \quad x \in \Gamma_0 \\ \underline{\nabla} u . \underline{n} &= 0 , \quad x \in \Gamma_+ \end{aligned} \tag{1.3a}$$

or

$$u(x) = 0 , \quad x \in \Gamma_0 \cup \Gamma_+ . \tag{1.3b}$$

In addition, we shall assume that Γ is either a general boundary curve of a regular domain, a convex polygon or very smooth (e.g., C^∞). Throughout the discussion we will assume that all the integrals involved are evaluated exactly and the finite element space is conforming.

The solution of (1.1), (1.2), and (1.3) is characterized by the following weak formulation. $u \in \overset{\circ}{H}{}^1(\Omega)$ satisfies

$$B_\varepsilon(u, v) = (f, w) , \quad \forall u \in \overset{\circ}{H}{}^1(\Omega) . \tag{1.4}$$

Here $\overset{\circ}{H}{}^1(\Omega)$ denotes the subspace of the Sobolev space $H^1(\Omega) \equiv W^{2,1}(\Omega)$ of functions vanishing on $\Gamma_- \cup \Gamma_0$ in case (1.3*a*) and on Γ in case (1.3*b*), and B_ε is the bilinear form

$$B_\varepsilon(u, w) = \int_\Omega [\varepsilon \nabla u . \nabla w + (\underline{v}(x) . \nabla u + q(x) u) w] \, dx . \tag{1.5}$$

(*) A regular domain is one for which the divergence theorem is true.

If S^h denotes a finite dimensional subspace of $\overset{\circ}{H}{}^1(\Omega)$, the usual Galerkin method gives an approximation $U \in S^h$ to u through the equations

$$B_\varepsilon(U, w) = (f, w), \quad \forall w \in S^h. \tag{1.6}$$

In the above formulation, S^h is typically a finite element space consisting of piecewise polynomial functions on a subdivision of $\bar{\Omega}$ with maximum element size h.

Standard finite element methods are not appropriate when $\varepsilon < 0(h)$ for the following two reasons. A generic finite element method for (1.1), (1.2), (1.3) satisfies : for $\varepsilon > 0$ there is an $h_0(\varepsilon)$ such that for $h \leqslant h_0(\varepsilon)$ the method is stable and the error $u - U$ satisfies

$$\|u - U\| \leqslant C(\varepsilon)\, h^{k+1}\, \|u\|_{k+1}, \quad h \leqslant h_0(\varepsilon),$$

where $k + 1$ is the order of accuracy of the method and $\|\cdot\|_{k+1}$ is the $W_2^{k+1} = H^{k+1}$ norm. Here $h_0(\varepsilon) \to 0$ and $C(\varepsilon) \to \infty$ as $\varepsilon \to 0$, essentially because B_ε is not coercive in H^1 uniformly in ε.

The second problem associated with the numerical solution of (1.1), (1.2), (1.3) for $\varepsilon \ll 0(h)$ is approximation theoretic. The solution to (1.1), (1.2), (1.3) is characterized by sharp boundary layers along Γ_0 and Γ_+. It is observed that unless the mesh density around $\Gamma_0 \cup \Gamma_+$ is much greater than in the interior of Ω, the resulting piecewise polynomial approximation is highly oscillatory even in regions in which the true solution is smooth. The global estimates of the form : error $= O(h^{k+1}\varepsilon^{-\gamma})$ for some positive $\gamma = \gamma_k$ are a generic feature of methods for (1.1), (1.2), (1.3) and any appropriate numerical method faces the problem of resolution of layers.

The method studied herein is a combination of defect correction with an artificial viscosity approximation. It computes a sequence $\{U^j\} \subset S^h$ by the following equations : let $\varepsilon_0 = \max\{\varepsilon, h\}$,

$$B_{\varepsilon_0}(U^1, w) = (f, w), \quad \forall w \in S^h$$

$$B_{\varepsilon_0}(U^{j+1} - U^j, w) = (f, w) - B_\varepsilon(U^j, w), \quad \forall w \in S^h. \tag{1.7}$$

If $\varepsilon \ll h$, then, in general, U^1 is only a first order accurate approximation to u. At each correction step the residual is computed and a correction to the current approximation is calculated using the (first order) viscosity approximation with B_{ε_0}.

Defect correction methods of this type have been studied experimentally by e. g., Hemker [6], [7]. He indicated that for problems with layers, the defect corrections produces an acceptable approximation in the interior of Ω after a *few* correction steps, but eventually the numerical approximation approaches the Galerkin solution with its bad oscillatory behavior and deteriorated accuracy. For the analysis he uses « local mode analysis » on a simple model problem.

In this paper we derive local and global error estimates for (1.7) which agree with the observed behavior of the error. We prove detailed global error estimates in section 4, for problems with layers, in $\mathbf{R}^2$, $\mathbf{R}^3$, which are of the general form : $(\nu = 0,1)$

$$\|u - U^j\|_\nu = C(\varepsilon)\left(h^{k + \frac{1-\nu}{2}} + (\varepsilon_0 - \varepsilon)^j\right)$$

without any stability restrictions on h w. r. t. ε. In these, we account for the dependence of $C(\varepsilon)$ upon the various possible types of layers which arise.

In the preliminary section 3 we consider the periodic problem in $1 - D$ with the aim of setting the stage for the later analysis and validating the results of « local-mode-analysis ». The convergence results of section 3 are clearly overly optimistic in a global sense. However, in section 5 we study $2 - D$ problems and we show that, modulo a term of infinite order accuracy which is nonuniform in ε, the convergence in subdomains sufficiently far from layers is analogous to the rates suggested by « local-mode-analysis ». Indeed it is observed in finite element implementations of (1.7) that the error in the method in such subdomains behaves like $0(h^k + (\varepsilon_0 - \varepsilon)^j)$, where $k :=$ formal order of accuracy of the method. Further, it is observed that as j increases we must move to successive subdomains farther from the boundary layers (see [5], [7]) along Γ for the interior estimates above to hold. Under numerous assumptions upon Ω, $\underline{v}$ etc., we show (see Theorem 5.1) that for a subdomain Ω_j', sufficiently far from the characteristic and outflow portions of Γ, the error in H^1 is of the observed form $0(h^k + (\varepsilon_0 - \varepsilon)^j)$ uniformly in ε ; up to a term of infinite order accuracy in h which is nonuniform in ε, for every $s > 0$:

$$\|u - U^j\|_{1,\Omega_j'} \leq C_1(s, f)(h^k + (\varepsilon_0 - \varepsilon)^j) + C_2(\varepsilon)\, h^s .$$

We also give an $L^2(\Omega_j')$ error estimate which is suboptimal by $0(h^{1/2})$, which is typical for these problems.

It is noteworthy that, to achieve this accuracy we must move further inside Ω as we iterate further. This increase in the pollution as we iterate was noticed by Hemker [7]. See also Ervin and Layton [6 ; section 4], for an example in which this effect appears. Computationally attractive modifications of the basic algorithm have been suggested in e.g., Hemker [6], [7], Axelsson [2], Ervin and Layton [4], which seem to slow the spread of the region in which convergence cannot be demonstrated. We consider herein only the basic method, but we believe that several of the modifications can be analyzed by suitably adapting our techniques.

The « streamline diffusion » method represents an entirely different approach to (1.1), (1.2), (1.3) which has been analyzed in, e.g., Navert [12], Johnson and Navert [8], Johnson, Schatz and Wahlbin [10] and Johnson Navert and Pitkaranta [9]. See also Axelsson [1] for a combination of defect correction with this latter method.

2. A PRIORI ESTIMATES

Associated with (1.1), (1.2), (1.3) is the adjoint problem

$$\mathscr{L}_\varepsilon^* u^* = -\varepsilon \Delta u^* - \operatorname{div} (\underline{v}(x) u^*) + q(x) u^* = f, \quad x \in \Omega$$
$$u^* = 0 \text{ on } \Gamma_- \cup \Gamma_0, \quad \text{and either} \tag{2.1}$$

$u^* = 0$ on Γ_+ if (1.3*b*), or $(\varepsilon \nabla u^* + \underline{v} u^*) \cdot \underline{n} = 0$ on Γ_+ if (1.3*a*).

Note that

$$\int_\Omega \mathscr{L}_\varepsilon^* u^* w \, d\Omega = B_\varepsilon(w, u^*) \quad \forall w \in H^1(\Omega).$$

Note also that the convection in the adjoint problem is driven by a velocity field $(-\underline{v}(x))$ in the opposite direction to the original problem and u^* will, in general, have a strong boundary layer along Γ_- of width $O(\varepsilon)$.

We can distinguish between some practical cases of various degrees of smoothness as $\varepsilon \to 0$. As usual, C denotes a generic constant, independent of u, ε and h.

LEMMA 2.1 : *Assume that* $q - \frac{1}{p} \operatorname{div} \underline{v} \geq q_0 > 0$, $x \in \Omega$, $p \geq 2$. *Then the solution u of* (1.1), (1.2), (1.3) *satisfies*

$$\|u\|_{L^p(\Omega)} \leq C \|f\|_{L^p(\Omega)}, \quad 2 \leq p \leq \infty, \tag{2.2a}$$

$$\varepsilon^{3/2} \|\Delta u\| + \varepsilon^{1/2} \|u\|_1 + \|u\| \leq C \|f\|, \tag{2.2b}$$

and if p is either a convex polygon or smooth and we have pure Dirichlet boundary conditions

$$\varepsilon^{3/2} \|u\|_2 + \varepsilon^{1/2} \|u\|_1 + \|u\| \leq C \|f\|, \tag{2.2c}$$

Proof: To prove *a*), multiply (1.1) by u^{p-1} and integrate. By use of Green's formula on the first two terms and a Holder inequality on the right hand side (*cf.* Schatz and Wahlbin [13]) we arrive at *a*). The L^∞ estimate follows by letting $p \to \infty$. Similarly, for $p = 2$ we get $\varepsilon^{1/2} \|u\|_1 + \|u\| \leq C \|f\|$. From the differential equation (1.1) we get then

$$\varepsilon \|\Delta u\| \leq C [\|\underline{v} \cdot \nabla u\| + \|u\| + \|f\|] \leq C \varepsilon^{-1/2} \|f\|$$

and b) follows. If $\partial\Omega$ is smooth, or a convex polygon, and we have pure Dirichlet boundary conditions $\|u\|_2 \leqslant C[\|\Delta u\| + \|u\|]$, and c) follows. □

Remark 2.1 : Note that (2.2a) implies that $\|\mathscr{L}_\varepsilon^{-1}\|_{L^p}$ is bounded uniformly in ε for any $p \geqslant 2$, i.e., we have stability with respect to given data in any such norm. This result is already known for $p = 2$ and $p = \infty$, see for instance Miranda [11]. The powers $\varepsilon^{3/2}$, $\varepsilon^{1/2}$ occurring in part b) can be shown to be sharp by considering simple examples in one dimension.

LEMMA 2.2 : *If* $q - \frac{1}{2}\operatorname{div}(\underline{v}) \geqslant q_0 > 0$ *if* $f \in H^{k-2}(\Omega)$ *and* Γ *is smooth the solution to* (1.1), (1.2), (1.3b) *satisfies* $u \in H^k(\Omega) \cap \cup \mathring{H}^1(\Omega)$ *and*

$$\varepsilon^{k-1/2}\|u\|_k \leqslant C\left\{\|f\| + \varepsilon^{3/2}\|f\|_1 + \cdots + \varepsilon^{k-2+1/2}\|f\|_{k-2}\right\}.$$

Proof : Since u satisfies

$$\Delta u = \varepsilon^{-1}\{f - qu - \underline{v}\,.\,\nabla u\} \equiv \tilde{f}\,,$$

the « shift theorem » implies that $\|u\|_{s+2} \leqslant C\,\|\tilde{f}\|_s$, $s \geqslant 0$. Thus,

$$\varepsilon\,\|u\|_{s+2} \leqslant C\left\{\|f\|_s + \|u\|_{s+1}\right\}.$$

The result follows by beginning with Lemma 2.1 and proceeding inductively using the above. □

In the following $u_{\underline{v}}$ denotes the unnormalized directional derivative (streamline derivative) along $\underline{v}$, $u_{\underline{v}} \equiv \underline{v}\,.\,\nabla u$.

Generalized Periodic Boundary Value Problem : The boundary value problem (1.1) is said to be generalized periodic if the data are such that (the trace of) the solutions and its derivatives satisfy $u|_{\Gamma_-} - u|_{\Gamma_+} = \nabla u\,.\,\underline{n}|_{\Gamma_-} - \nabla u\,.\,\underline{n}|_{\Gamma_+} = 0$, and similarly for $\underline{v}$, in the sense that boundary integrals like $\int_{\Gamma_- \cap \Gamma_+} \underline{\nabla} u\,.\,\underline{n} u_{\underline{v}}\, u_x\, d\Gamma$ vanishes. This is valid in particular if Γ_- and Γ_+ are congruent and these functions are equal at corresponding points of Γ_- and Γ_+. An example is illustrated in figure 2.1.

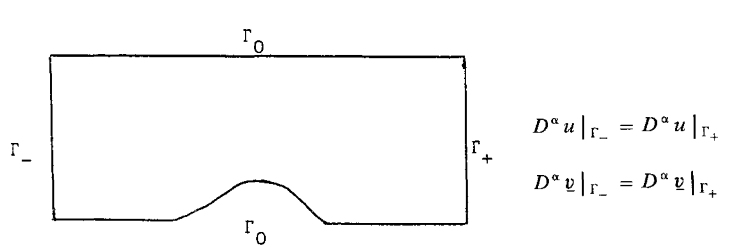

Figure 2.1. — A Generalized Periodic Boundary Value Problem.

LEMMA 2.3 : *Assume that* $u \in H^2(\Omega)$, *that* $(\|f_{\underline{v}}\| + \|f\|)$ *exists and that* $q - \frac{1}{2}\operatorname{div} \underline{v} \geq q_0 > 0$, $x \in \Omega$. *If* (i) $\Gamma_- \cup \Gamma_+$ *is empty or* (ii) *the boundary value problem is generalized periodic, then*

$$\varepsilon \|\Delta u\| + \varepsilon^{1/2}\|u\|_1 + \|u_{\underline{v}}\| + \|u\| \leq C[\|f_{\underline{v}}\| + \|f\|]. \qquad (2.5a)$$

If, in addition, Γ *is smooth or a convex polygon and we have pure Dirichlet boundary conditions then*

$$\varepsilon \|u\|_2 + \varepsilon^{1/2}\|u\|_1 + \|u_{\underline{v}}\| + \|u\| \leq C[\|f_{\underline{v}}\| + \|f\|]. \qquad (2.5b)$$

If, in addition, Γ *is smooth and* $f \in H^{k-2}(\Omega)$

$$\varepsilon^{k-1}\|u\|_k \leq C\left\{\varepsilon^{k-2}\|f\|_{k-2} + \cdots + \varepsilon\|f\|_1 + \|f_{\underline{v}}\| + \|f\|\right\}. \quad (2.5c)$$

Proof : Note at first that $u = 0$ on Γ_0 implies $u_{\underline{v}} = 0$ on Γ_0. For notational simplicity, we consider now only the case $n = 2$ and we denote $x_1 = x$, $x_2 = y$. By direct calculations, $(qu)_{\underline{v}} = q_{\underline{v}} u + q u_{\underline{v}}$ and

$$L_\varepsilon(u_{\underline{v}}) =$$
$$= (L_\varepsilon u)_{\underline{v}} - \varepsilon[2\,\underline{\nabla} v_1 \cdot \underline{\nabla} u_x + 2\,\underline{\nabla} v_2 \cdot \underline{\nabla} u_y + \Delta(v_1)\,u_x + \Delta(v_2)\,u_y] - q_{\underline{v}}\,u\,,$$

where we have $L_\varepsilon(u)_{\underline{v}} = f_{\underline{v}}$. Since by assumption $u \in H^2(\Omega)$ and $f_{\underline{v}} \in L_2(\Omega)$, the right hand side of the above equation is square integrable and Lemma 2.1 is applicable. Multiplying by $u_{\underline{v}}$ and integrating and using partial integration of some of the terms, such as

$$\int_\Omega \nabla v_1 \cdot \nabla u_1\, u_{\underline{v}}\, d\Omega = -\int_\Omega [u_{\underline{v}}\, u_x \Delta v_1 + \nabla u_{\underline{v}} \cdot \nabla v_1\, u_x]\, d\Omega +$$
$$+ \oint_{\Gamma_- \cup \Gamma_+} \nabla v_1 \cdot \underline{n} u_x\, u_{\underline{v}}\, d\Gamma\,,$$

we get, as in Lemma 2.1,

$$\varepsilon\|u_{\underline{v}}\|_1^2 + \|u_{\underline{v}}\|^2 \leq C[\|f_{\underline{v}}\|^2 + \|u\|^2 + \varepsilon\|u_{\underline{v}}\|_1\|u\|_1 + \varepsilon^2\|u\|_1^2]. \quad (2.6)$$

Note namely that the boundary integral on Γ_0 is zero because $u_{\underline{v}} = 0$ and (in case (ii)) the boundary integrals, such as

$$\oint_{\Gamma_- \cup \Gamma_+} \underline{n} \cdot \underline{\nabla} u_{\underline{v}}\, u_{\underline{v}}\, d\Gamma\,,$$

vanish because of the generalized periodic boundary conditions. By the usual inequalities, by (2.4*a*) and by the differential equation, we then get

(2 5*a*) (2 5*c*) follows from the shift theorem and (2 5*b*) as in Lemma 2 2 □

Remark 2 2 Note that in case (i) and case (ii) of Lemma 2 3 only a characteristic layer can occur (in case (i) the outflow boundary is even nonexistent) Case (i) occurs in interior fluid flow problems and case (ii) can occur in a (long) channel flow problem, where the inflow and outflow boundaries have the same shape and area

2.1. Local regularity

Next we give a local regularity result from Navert [12] which we use extensively in deriving local error estimates First we present the notations used

The notation we use is all standard $\|\cdot\|_{k\,D}$, $D \subset \mathbf{R}^2$, denotes the $H^k(D)$ norm (if $D = \Omega$ then we shall omit D) and $|\cdot|_{k\,D}$ the corresponding seminorm Given a weight function $\psi(x)$, $\|w\|_{\psi\,0}$ denotes the usual weighted $L^2(D)$ norm defined as $\left(\int_D w^2 \psi\, dx\right)^{1/2}$ Ω is assumed to be a convex polygon in $\mathbf{R}^2$ We define a finite element space $S^h = S^h(\Omega) \subset \mathring{H}^1(\Omega)$ by first covering Ω with an edge to edge triangulation $\{\tau_l\}$ which is assumed to be quasiuniform We shall assume $S^h|_{\tau_l}$ contains polynomials of degree $\leq k$ so that the usual interpolation result holds for all $u \in H^{k+1}(\Omega) \cap \mathring{H}^1(\Omega)$

$$\|u - I_h(u)\| + h\,\|u - I_h(u)\|_1 \leq Ch^{k+1}|u|_{k+1\,\Omega}\,, \tag{2 7}$$

where $I_h(u) \in S^h$ is the interpolant of u If D is a subdomain whose boundary consists of edges of the triangulation $\{\tau_l\}$ we denote by $S^h(D)$ the restriction of S^h to D We shall use the quasiuniformity assumption upon $\{\tau_l\}$ in the form of the usual inverse estimates for S^h

For a subdomain Ω_j, we let Γ_j^+, Γ_j^-, Γ_j^0 denote, respectively, the outflow, inflow and characteristic portions of $\partial\Omega_j$

$$\Gamma_j^+ = \{x \in \partial\Omega_j \,|\, \underline{v} \cdot \underline{n}_j > 0\}\,, \quad \Gamma_j^- = \{x \in \partial\Omega_j = \Gamma_j \,|\, \underline{v} \cdot \underline{n}_j < 0\}\,,$$
$$\Gamma_j^0 = \{x \in \partial\Omega_j = \Gamma_j \,|\, \underline{v} \cdot \underline{n}_j = 0\}$$

Here $\underline{n}_j$ denotes the outward unit normal to Ω_j

As we will be dealing with interior estimates in section 5, it is useful to briefly collect some of the basic properties of the cutoff functions ψ, introduced in Johnson and Navert [8], Navert [12], which we will use In this work ([8], [12]) the crucial role of subdomains which exclude upstream cutoff was first point out and exploited. For the definition of q_0, see Lemma 2.1.

DEFINITION 2.1 [Nävert [12 : p. 19]] : *A smooth function* ψ *is* $q_0 = \alpha$ *compensating in* $\Omega' \subset \Omega$ *if* $\psi \geq 1$ *in* Ω' *and there is a constant* $\rho \in [0, 1]$, *with* $\rho > 0$ *if* $q_0 = 0$, *such that* $-\frac{\partial}{\partial \underline{v}} \psi \geq \rho\psi$ *in* Ω.

Note that ψ can be viewed as a generalization of the exponential weight function used in Axelsson [2] to get a bilinear form which is coercive uniformly in ε.

DEFINITION 2.2. *A subdomain* $\Omega'' \subset \Omega$ *is said to « exclude upstream cutoff » if all points upstream w.r.t.* $\underline{v}$ *of a point in* Ω'' *belongs to* Ω''. *I.e.*, $(\Gamma'')^- \subset \Gamma$.

In Nävert [12 ; Lemma 2.1, p. 20] and Johnson and Nävert [8] (see also [9] and [10]), appropriate cutoff functions are constructed for subdomains which exclude upstream cutoff. We recall, for later reference, their properties by quoting the following result :

LEMMA 2.4 [Nävert [12 ; Lemma 2.1, pp. 20, 21]] : *Let* Ω'' *be a subdomain of* Ω *with piecewise smooth boundary* Γ'' *such that* $(\Gamma'')^- \subset \Gamma$. *Let* c, s, d, γ *and* M *be positive numbers with* $d \leq 1/2$, $\gamma \geq 1$ *and* M *an integer. Assume that all points upstream (w.r.t.* $\underline{v}$*) a point on* $(\Gamma'')^0$ *lie on* $(\Gamma'')^0$ *and that* $|\underline{v} \,.\, \underline{n}''| \geq c$ *on* $(\Gamma'')^- \cup (\Gamma'')^+$, *where* $\underline{n}''$ *is the outward unit normal to* $\partial\Omega''$. *Then, there are constants* $C_1 = C_1(s, M, \underline{v}, \Omega'')$, $C_2 = C_2(s, M, \underline{v}, \Omega')$ *and* $C = C(\underline{v}, \Omega'')$ *such that if* Ω' *is any subdomain of* Ω'' *at a distance of at least* $C_1 \gamma d \ln\left(\frac{1}{d}\right)$ *and* $C_2 \gamma \sqrt{d} \ln\left(\frac{1}{d}\right)$ *from* $(\Gamma'')^+$ *and* $(\Gamma'')^0$ *respectively, there exists a cutoff function* ψ *which is* $q_0 = \alpha$ *compensating in* Ω' *with the following properties :*

$$\psi \geq 1 \quad in \ \Omega', \tag{2.8a}$$

$$\psi = 0 \quad in \ \Omega - \Omega'', \tag{2.8b}$$

$$\frac{\partial}{\partial \underline{v}} \psi \leq 0, \quad in \ \Omega, \tag{2.8c}$$

$$\max_{|\underline{y}| \leq \gamma d} \frac{\psi(\underline{x} + \underline{y})}{\psi(\underline{x})} \leq C, \quad in \quad \{\underline{x} \in \Omega \mid \psi(\underline{x}) \geq C d^s\}, \tag{2.8d}$$

$$\max_{|\underline{y}| \leq \gamma d} \frac{-\psi_v(\underline{x} + \underline{y})}{-\psi_{\underline{v}}(x)} \leq C, \quad in \quad \{\underline{x} \in \Omega \mid -\psi_{\underline{v}}(\underline{x}) \geq C d^s\}, \tag{2.8e}$$

$$|D_{\underline{v}}^m \psi| \leq C\{\gamma^{-1} d^{-m} \psi + d^s\} \quad in \ \Omega \ if \ 0 < m \leq M. \tag{2.8f}$$

For any $\underline{v}^{\perp}$ *with* $\underline{v}^{\perp} . \underline{v} = 0$, $|\underline{v}^{\perp}| = 0(1)$,

$$\left|D_{\underline{v}^{\perp}}^{\lambda} \psi\right| \leqslant C\left\{\gamma^{-1} d^{-\frac{1}{2}|\lambda|} \psi - d^{1-|\lambda|} \psi_{\underline{v}} + d^{s}\right\} \quad in \quad \Omega\,, \tag{2.8g}$$

if $0 < |\lambda| < M$.

$$\left|D_{\underline{v}}^{m+1} D_{\underline{v}^{\perp}}^{\lambda} \psi\right| \leqslant C\left(-\gamma^{-1} d^{-m-|\lambda|} \psi_{\underline{v}} + d^{s}\right) \quad \text{in} \quad \Omega\,, \tag{2.8h}$$

if $0 < m + |\lambda| < M$.

Here D_{ξ} *differentiation in the* ξ *direction,* $D_{\xi}\,\omega = \xi . \nabla w$ *etc.,* $m \geqslant 0$ *is an integer and* $\lambda = (\lambda_1, \ldots, \lambda_n)$ *is a multi-index of length* $|\lambda| = \sum \lambda_j$. □

DEFINITION 2.3 [Nävert [12 ; Definition 2.2, p. 25]] : *A cutoff function* ψ *satisfying the conditions of Lemma 2.4 is said to be* $q_0 = \alpha$ *compensating and smooth of order* (d, ψ, s) *on* (Ω', Ω'').

THEOREM 2.1 [Nävert [12 ; Theorem 2.3, p. 26]] : *Assume that either* $\underline{v}$ *has no closed arcs in* $\bar{\Omega}$ *or that* $q_0 > \tilde{C}(\underline{v})$ *where* $\tilde{C}$ *depends upon the first derivatives of* $\underline{v}$. *Let* Ω', Ω'' *satisfy Lemma 2.4, let c, s, m be positive numbers with m an integer and let* $(\Gamma'')^{-} \subset \Gamma^{-}$ *and* Γ^{-} *is part of one of the faces of* Ω *(a polygon). Suppose* Ω *is a convex polygon in* $\mathbf{R}^2$, $\overline{(\Gamma'')^{-}} \cap \overline{\Gamma^{+}}$ *is empty. Then, there are constants* $C_1 = C_1(s, m, \underline{v}, \Omega'')$, $C_2 = C_2(s, m, \underline{v}, \Omega'')$ *and* $C = C(\underline{v}, q, \Omega'', \Omega)$ *such that if the distance from* Ω' *to* $(\Gamma'')^{+}$ *and* $(\Gamma'')^{0}$ *is at least* $C_1\, \varepsilon_0 \ln\left(\frac{1}{\varepsilon_0}\right)$ *and* $C_2 \sqrt{\varepsilon_0} \ln\left(\frac{1}{\varepsilon_0}\right)$ *respectively, the solution of:*

$$L_{\varepsilon_0} u_{\varepsilon_0} = f \quad in\, \Omega\,, \quad w = g \quad on\, \Gamma\,, \tag{2.9}$$

with $f \in H^m(\Omega'')$ *and* $g = 0$, *satisfies*

$$\|u\|_{m, \Omega'} + \|u_{\underline{v}}\|_{m, \Omega'} \leqslant C\, \|f\|_{m, \Omega''} + \varepsilon_0^{s} \|f\|_{\Omega}\,. \quad \square \tag{2.10}$$

3. THE PROBLEM IN ONE SPACE DIMENSION

In this preliminary (introductory) section the problem in one space dimension will be considered. The aims of this section are threefold : to make the « local mode analysis », discussed in the introduction, rigorous, to introduce the basic ideas underlying the method in a simplified context, and to provide, as a point of comparison for later sections, an analysis of the basic method *when the critical effects of boundary conditions and directional bias do not influence the scheme.*

Therefore, the equation

$$L_\varepsilon u \equiv -\varepsilon u''(x) + vu'(x) + qu(x) = f(x), \quad 0 < x < 2\pi \tag{3.1}$$

is considered subject to periodic boundary conditions

$$u(0) - u(2\pi) = u'(0) - u'(2\pi) = 0. \tag{3.2}$$

Assume that v and p are constants such that $v \neq 0$, $q > 0$ is satisfied. $f(x)$ is a 2π-periodic, smooth, known forcing term.

Regularity

In this section, periodic functions will be considered as functions from T (the unit circle) $\to \mathbf{R}$. There is a natural identification between 2π-periodic functions : $\mathbf{R} \to \mathbf{R}$ and functions : $T \to \mathbf{R}$ and we shall pass from one to the other without comment. The Sobolev spaces $H^s(T)$, $-\infty < s < \infty$, are defined in the usual manner. Thus, $w \in H^s(T)$ (considered as a periodic function) if

$$w(x) = \sum_{j \in Z} w_j e^{ijx}, \quad w_j = (w, e^{ijx})$$

$$\|w\|_s^2 \equiv \sum_{j \in Z} (1 + j^2)^s |w_j|^2 < \infty .$$

LEMMA 3.1 : *Assume $f \in H^s(T)$ and $v \neq 0$. Then, the solution u to* (3.1), (3.2) *is in $H^{s+2}(T)$ and*

$$\varepsilon \|u\|_{s+2} + \|u\|_{s+1} \leqslant C \|f\|_s, \quad 0 \leqslant s < \infty$$

where C is independent of ε.

Proof : Expanding f in a Fourier series, the solution to (3.1), (3.2) can easily be calculated

$$u(x) = \sum_{j \in Z} [\varepsilon j^2 + vij + q]^{-1} f_j e^{ijx}, \quad f_j = (f, e^{ijx}).$$

Thus (for the $\varepsilon \|u\|_{s+2}$ term proceed in the same manner),

$$\|u\|_{s+1}^2 = \sum_{j \in Z} (1 + j^2)^{s+1} ((\varepsilon j^2 + q)^2 + v^2 j^2)^{-1} |f_j|^2$$
$$\leqslant \sum_{j \in Z} (1 + j^2)^{s+1} (q^2 + v^2 j^2)^{-1} |f_j|^2 \leqslant C \|f\|_s . \quad \square$$

Note that the above regularity result holds for both the equation for L_ε and for its adjoint L_ε^* (with v replaced by $-v$ in L_ε). *This is true in the periodic case.*

Convergence of the Method

In the periodic case the equation (3.1), (3.2) has the weak formulation

$$B_\varepsilon(u, w) = (f, w), \quad \forall w \in H^1(T)$$

$$B_\varepsilon(u, w) = \int_0^{2\pi} [\varepsilon u' w' + v u' w + q u w] \, dx .$$

Using this weak formulation, the defect correction iterations (1.7) will be considered. Let S^h denote a subspace of $H^1(T)$ and $\varepsilon_0 = \max \{\varepsilon, h\}$. Then, the iterations (1.7) can be rewritten in the following form that is convenient for analysis :

$$\begin{aligned} B_{\varepsilon_0}(U^1, w) &= (f, w), & \forall w \in S^h, \\ (R^j, w) &\equiv (f, w) - B_\varepsilon(U^j, w), & \forall w \in S^h, \\ B_{\varepsilon_0}(E^j, w) &= (R^j, w), & \forall w \in S^h \\ U^{j+1} &= U^j + E^j, & j = 1, 2, ..., \end{aligned} \tag{3.3}$$

where, of course, R^j is the residual and E^j the artificial viscosity approximation to the error. Associated with the discrete sequences (3.3) are continuous defect corrections sequences u^j, e^j, r^j defined in an analogous manner,

$$\begin{aligned} L_{\varepsilon_0} u^1 &= f, \quad r^j = f - L_\varepsilon u^j, \\ L_{\varepsilon_0} e^j &= r^j, \quad u^{j+1} = u^j + e^j, \quad j = 1, 2, ... \end{aligned} \tag{3.4}$$

$$\frac{d^k}{dx^k} (u^j(0) - u^j(2\pi)) = \frac{d^k}{dx^k} (e^j(0) - e^j(2\pi)) = 0, \quad k = 0, 1 .$$

In addition, another auxiliary sequence will be necessary. Define $\xi^j \in S^h$ as the artificial viscosity projection of e_j into S^h :

$$B_{\varepsilon_0}(e^j - \xi^j, w) = (r^j, w) - B_{\varepsilon_0}(\xi^j, w) = 0, \quad \forall w \in S^h. \tag{3.5}$$

The sequences (3.3), (3.4), (3.5) will be used in the subsequent sections to analyze the method in the more general context. (Of course, L_ε, L_{ε_0}, B_ε and B_{ε_0} are to be interpreted in the appropriate manner for the problem under consideration.)

The basic convergence theorem in the periodic case is then

THEOREM 3.1 : *Assume $q > 0$, $v \neq 0$ and that S^h satisfies the approximation assumption (AA) given below. Then, the error in the defect correction* *approximation to* (3.1), (3.2) *is given by*

$$\|u - U^J\|_1 \leq C(\varepsilon_0 - \varepsilon)^J \|f\|_J + Ch^k(\|u^1\|_{k+1} + \cdots + \|u^J\|_{k+1}),$$

where $u^1, ..., u^J$ *are given by* (3.4).

Thus, the method is optimal after k iterations (*after which we stop*). It is optimal (in this case of periodic boundary conditions) both with respect to the power of h occurring in the error estimate and with respect to the regularity required of the continuous iterates u^J. The error $\|u - U^J\|_1$ is broken down into two components : $\|u - u^J\|_1$ and $\|u^J - U^J\|_1$. The first is the error in the continuous defect correction iteration (3.4). Its magnitude depends in a critical way upon the regularity of the continuous problem (3.1), (3.2). As will be seen later, the magnitude of the discretization error $\|u^J - U^J\|_1$ also depends upon the regularity of the adjoint problem. Here they are the same (Lemma 3.1), so optimal rates of convergence are obtained. First, $\|u - u^J\|_s$ will be estimated.

LEMMA 3.2 : *In addition to the assumptions of Lemma 3.1, suppose* $f \in H^{s+J-1}(T)$. *Then*

$$\|u - u^J\|_s \leq C(\varepsilon_0 - \varepsilon)^J \|f\|_{s+J-1}, \quad 0 \leq s < \infty,$$

where C *is independent of* ε, h *and* f.

Proof : Let $\delta^J = u - u^J$. Then, δ^J satisfies

$$L_{\varepsilon_0}\delta^1 = (L_{\varepsilon_0} - L_\varepsilon)u, \quad L_{\varepsilon_0}\delta^{J+1} = (L_{\varepsilon_0} - L_\varepsilon)\delta^J.$$

Applying Lemma 3.1 recursively gives

$$\begin{aligned}\|\delta^J\|_s &\leq C(\varepsilon_0 - \varepsilon)\|\Delta\delta^{J-1}\|_{s-1} \leq C(\varepsilon_0 - \varepsilon)\|\delta^{J-1}\|_{s+1} \\ &\leq \ldots \leq C(\varepsilon_0 - \varepsilon)^J \|u\|_{s+J} \leq C(\varepsilon_0 - \varepsilon)^J \|f\|_{s+J-1}. \quad \square\end{aligned}$$

Next, the discretization error will be considered. For this, S^h is assumed to satisfy the standard approximation assumption (AA) : given $u \in H^1(T) \cap H^{r+1}(0, 2\pi)$, then

$$\inf_{\chi \in S^h} \{\|u - \chi\| + h\|u - \chi\|_1\} \leq Ch^{r+1}\|u\|_{r+1}, \quad 0 \leq r \leq k. \quad (AA)$$

This is satisfied, for example, if S^h consists of C^0, periodic, piecewise polynomials of degree k, defined on a quasi-uniform mesh of maximum width h.

Under this approximation property, by a straightforward argument (see Appendix 1 for details) and utilizing that $\varepsilon_0 \geq h$, one proves the optimal order estimate

$$h \|e^J - \xi^J\|_1 + \|e^J - \xi^J\|_0 \leq C h^{r+1} \|e^J\|_{r+1}, \quad 0 \leq r \leq k. \tag{3.6}$$

In the nonperiodic case, the fact that L_ε and L_ε^* have different regularity properties causes the L^2 error estimate in general to be suboptimal by a factor $\varepsilon_0^{-1/2}$, see Nävert [12] for details.

LEMMA 3.3 : *Under the assumptions of Theorem 3.1,*

$$\|u^J - U^J\|_\nu \leq C h^{k+1-\nu} (\|u^1\|_{k+1} + \cdots + \|u^J\|_{k+1}), \quad \nu = 0, 1,$$

where $\{u^J\}$ *is given by* (3.4).

Proof: Since U^1 is the artificial viscosity projection of u^1, (3.6) implies that

$$\|u^1 - U^1\|_\nu \leq C h^{k+1-\nu} \|u^1\|_{k+1}, \quad \nu = 0, 1. \tag{3.7}$$

(Note that the regularity Lemma 3.1 holds for $\{u^J\}$.)

To estimate $\|u^2 - U^2\|_\nu$, note that

$$\|u^2 - U^2\|_\nu \leq \|u^1 - U^1\|_\nu + \|e^1 - E^1\|_\nu,$$

so that all that is needed is an estimate of $\|e^1 - E^1\|_\nu$. By the triangle inequality and (3.6), this reduces to $\|\xi^1 - E^1\|_\nu$. $\xi^J - E^J$ satisfies

$$B_{\varepsilon_0}(\xi^J - E^J, w) = (r^J - R^J, w), \quad \forall w \in S^h,$$

which reduces to here

$$B_{\varepsilon_0}(\xi^1 - E^1, w) = - B_\varepsilon(u^1 - U^1, w), \quad \forall w \in S^h.$$

Setting $w = \xi^1 - E^1 \in S^h$ and using coercivity of B_{ε_0} gives

$$\|\xi^1 - E^1\|_{\varepsilon_0, 1}^2 \leq C |B_\varepsilon(u^1 - U^1, \xi^1 - E^1)|,$$

where

$$\|w\|_{\varepsilon_0, 1}^2 = \varepsilon_0 \|w\|_1^2 + \|w\|^2.$$

Integrating $(u^1 - U^1)'(\xi^1 - E^1)$ in the right hand side of the above by parts and using the Cauchy inequality gives

$$\|\xi^1 - E^1\|_{\varepsilon_0, 1}^2 \leq C \|u^1 - U^1\|_{\varepsilon_0, 1}^2 + C \varepsilon_0^{-1} \|u^1 - U^1\|^2.$$

Since $\varepsilon_0 \geqslant h$, this implies that

$$\|\xi^1 - E^1\|_1 \leqslant C(\|u^1 - U^1\|_1 + h^{-1}\|u^1 - U^1\|).$$

Thus, (3.7) gives

$$\|\xi^1 - E^1\|_1 \leqslant Ch^k \|u^1\|_{k+1}.$$

By a duality argument (see Appendix 1) we also get

$$\|\xi^1 - E^1\|_0 \leqslant Ch^{k+1}\|u^1\|_{k+1}.$$

This implies, by the triangle inequality,

$$\begin{aligned}\|u^2 - U^2\|_\nu &\leqslant Ch^{k+1-\nu}(\|u^1\|_{k+1} + \|e^1\|_{k+1}) \\ &\leqslant Ch^{k+\nu-1}(\|u^1\|_{k+1} + \|u^2\|_{k+1}).\end{aligned}$$

$\xi^2 - E^2$ satisfies the same equation as $\xi^1 - E^1$:

$$B_{\varepsilon_0}(\xi^2 - E^2, w) = -B_\varepsilon(u^2 - U^2, w), \quad \forall w \in S^h,$$

so, to complete the proof, proceed by induction. □

Theorem 3.1 now follows directly from the previous two lemmas.

The regularity assumptions on u^j can be restated in terms of f through the following argument. Notice that $\{u^j\}$ is given by

$$L_{\varepsilon_0} u^1 = f, \quad L_{\varepsilon_0} u^{j+1} = f + (L_{\varepsilon_0} - L_\varepsilon) u^j.$$

Lemma 3.1 gives the requires estimate on u^1. For u^{j+1}, note that if

$$f = \sum_{k \in Z} f_k e^{ikx}, \quad u^j = \sum u_k^j e^{ikx},$$

then u^{j+1} is given by

$$u^{j+1} = \sum_{k \in Z} [\varepsilon_0 k^2 + vik + q]^{-1} [f_k + (\varepsilon_0 - \varepsilon) k^2 u_k^j] e^{ikx}.$$

Thus, using the same argument as in the proof of Lemma 3.1 gives

$$\|u^{j+1}\|_{s+1} \leqslant C(\|f\|_s + \|u^j\|_{s+1}).$$

Using an induction argument with $\|u^1\|_{s+1} \leqslant C\|f\|_s$ as the first step and the above as the general step gives that

$$\|u^j\|_{s+1} \leqslant C\|f\|_s, \quad j = 1, 2, \ldots$$

The above regularity result and Theorem 3.1 gives the following corollary.

COROLLARY 3 1 *In addition to the assumptions of Theorem 3 1, suppose* $f \in H^k(T)$ *Then for* $j = 0, 1, \ldots, k$,

$$\|u - U^j\|_1 \leqslant C((\varepsilon_0 - \varepsilon)^j + h^k)\|f\|_k \quad \square$$

We note again that the assumption $f \in H^k(T)$ implicitly involves the compatibility conditions (clearly quite restrictive if $k > 1$) on f that

$$f(0) - f(2\pi) = f'(0) - f'(2\pi) = \cdots = f^{(k-1)}(0) - f^{(k-1)}(2\pi) = 0$$

4 GLOBAL ERROR ESTIMATES IN HIGHER SPACE DIMENSIONS

In this section the global convergence of the algorithm (1 7) on the problem of interest (1 1), (1 2), (1 3) is considered Following the analysis of the previous section, the error in the method will be broken into two components $\|u - U^J\| \leqslant \|u - u^J\| + \|u^J - U^J\|$, where $\{u^j\}$ is given by (3 4) The *iteration error* $\|u - u^J\|$ is governed by the regularity and possible boundary layers of the continuous problem for $\mathscr{L}_\varepsilon$ The discretization error $\|u^J - U^J\|$ may, in addition, be governed by the regularity properties of the adjoint problem for $\mathscr{L}_\varepsilon^*$ Further, the problem in higher space dimensions has the added feature that the reduced equation ($\varepsilon = 0$ in (1 1)) has different properties along the characteristics than across the characteristics the equation is anisotropic

Throughout this section the coefficients of (1 1) will be assumed to satisfy the hypotheses of Lemma 2 1 so that B_ε will be coercive in the norm $\|\cdot\|_{\varepsilon, 1}$,

$$\|w\|_{\varepsilon, 1} = \left\{\varepsilon \|w\|_1^2 + \|w\|^2 + \oint_{\Gamma_+} \underline{v} \cdot \underline{n} w^2 \, dx\right\}^{1/2} \tag{4 1}$$

In addition q and $\underline{v}$ are assumed to satisfy the smoothness properties required by the various lemmas in Section 2, which are applied in this section

Discretization Error

We assume that $\varepsilon < h$, i e, $\varepsilon_0 = h$ (In case $\varepsilon \geqslant h$, it follows from Appendix 1 that L_2-error estimates of optimal, or almost optimal order, are derived without use of defect-correction)

Define, following Section 3, u^j, $e^j \in \overset{\circ}{H}{}^1(\Omega)$, $\xi^j \in S^h$ by

$$B_{\varepsilon_0}(u^1, v) = (f, v), \quad \forall v \in \overset{\circ}{H}{}^1(\Omega),$$

$$B_{\varepsilon_0}(u^{j+1} - u^j, v) = (f, v) - B_\varepsilon(u^j, v), \quad \forall v \in \overset{\circ}{H}{}^1(\Omega),$$

$$B_{\varepsilon_0}(e^j - \xi^j, v) = 0, \quad \forall v \in S^h, \quad e^j = u^{j+1} - u^j$$

In order to derive a bound on the discretization error $u^J - U^J$, we note that for $j = 1$,

$$B_{\varepsilon_0}(u^1 - U^1, w) = 0\,, \quad \forall w \in S^h \tag{4.2}$$

$$\|u^1 - U^1\|_{\varepsilon_0, 1} \leq C\varepsilon_0^{1/2} h^s \|u^1\|_{s+1}\,, \quad 0 \leq s \leq k \tag{4.3}$$

and in particular

$$\|u^1 - U^1\| \leq C h^{s+\frac{1}{2}} \|u^1\|_{s+1}\,, \quad 0 \leq s \leq k\,. \tag{4.4a}$$

This follows directly from the coercivity of B_{ε_0}. If the boundary conditions are generalized periodic, or if $\Gamma_- \cup \Gamma_+$ is empty this estimate can be improved to (see Appendix 1), since the adjoint problem is then more regular,

$$\|u^1 - U^1\| \leq C h^{s+1} \|u^1\|_{s+1}\,, \quad 0 \leq s \leq k\,. \tag{4.4b}$$

Likewise, in the general case, for $0 \leq s \leq k$,

$$\|\xi^J - e^J\|_{\varepsilon_0, 1} \leq C\varepsilon_0^{1/2} h^s \|e^J\|_{s+1}\,, \tag{4.5}$$

and, if Γ_- is empty or the problem is generalized periodic

$$\|\xi^J - e^J\| \leq C h^{s+1} \|e^J\|_{s+1}\,.$$

THEOREM 4.1 : *Let* $\{u^J\}$, $\{U^J\}$ *be defined as above and* $\varepsilon < h$. *Then,*

$$\|u^J - U^J\|_1 \leq C h^s \sum_{i=1}^{J} \|u^i\|_{s+1}\,, \quad 0 \leq s \leq k\,,$$

$$\|u^J - U^J\| \leq C \varepsilon_0^{-(\mu-1)} h^{s+1} \sum_{i=1}^{J} \|u^i\|_{s+1}\,, \quad 0 \leq s \leq k\,,$$

where $\varepsilon_0 = h$, $\mu = 3/2$, *if* $j \geq 2$, $\mu = 3/2$ *if the assumptions of Lemma 2.1 hold and if* $j = 1$ *and the assumptions of Lemma 2.3 hold then* $\mu = 1$.

Proof : The result for $j = 1$ follows from (4.3), (4.4). Consider

$$u^2 - U^2 = u^1 - U^1 + e^1 - \xi^1 + \xi^1 - E^1\,. \tag{4.6}$$

Then, $(\xi^1 - E^1) \in S^h$ satisfies

$$B_{\varepsilon_0}(\xi^1 - E^1, w) = -B_\varepsilon(u^1 - U^1, w)$$
$$= (B_{\varepsilon_0} - B_\varepsilon)(u^1 - U^1, w)\,, \quad \forall w \in S^h,$$

because of (4.2). Hence,

$$B_{\varepsilon_0}(\xi^1 - E^1, w) = (\varepsilon_0 - \varepsilon) \int_\Omega \nabla(u^1 - U^1) \,.\, \nabla w \, dx\,, \quad \forall w \in S^h\,.$$

Setting $w = \xi^1 - E^1$ we get

$$\|\xi^1 - E^1\|_{\varepsilon_0, 1} \leqslant C \left(\frac{\varepsilon_0 - \varepsilon}{\varepsilon_0^{1/2}} \right) \|u^1 - U^1\|_1 \leqslant C \left(\frac{\varepsilon_0 - \varepsilon}{\varepsilon_0^{1/2}} \right) h^s \|u^1\|_{s+1}\,,$$

and in particular,

$$\|\xi^1 - E^1\|_1 \leqslant C h^s |u^1|_{s+1}$$
$$\|\xi^1 - E^1\| \leqslant C h^{s+1/2} |u^1|_{s+1}\,.$$

Hence by (4.5), (4.3), (4.6),

$$\|u^2 - U^2\|_l \leqslant C h^{s+(1-l)/2} (\|u^1\|_{s+1} + \|e^1\|_{s+1})\,, \quad l = 0, 1\,,$$
$$\leqslant C h^{s+(1-l)/2} (\|u^1\|_{s+1} + \|u^2\|_{s+1})\,, \quad l = 0, 1\,.$$

The estimate for $\|u^j - U^j\|_1$, $j \geqslant 3$ follows by induction. □

Remark 4.1 : The regularity required in Theorem 4.1 is stated in terms of u^j instead of u (or f). Heuristically, one would expect $L_{\varepsilon_0}^{-1}[(\varepsilon_0 - \varepsilon)(-\Delta)]$ to be a bounded operator (in fact, slightly smoothing since $\varepsilon_0 > (\varepsilon_0 - \varepsilon)$) so that u^j and u would have qualitatively similar regularity properties.

Iteration Error

Consider now the *iteration error* $\delta^j = u - u^j$. By definition of $\{u^j\}$ we have

$$L_{\varepsilon_0} u^1 = f\,, \quad L_{\varepsilon_0} \delta^1 = (\varepsilon_0 - \varepsilon)(-\Delta u)\,, \quad L_{\varepsilon_0} \delta^j = (\varepsilon_0 - \varepsilon)(-\Delta \delta^{j-1}) \tag{4.7}$$

where $j = 2, 3, \ldots$, and δ^j satisfies the same (homogeneous) boundary conditions as u.

THEOREM 4.2 : *If Γ is a regular curve, then*

$$\|u - u^j\| \leqslant C \left(\frac{\varepsilon_0}{\varepsilon} \right)^r \left(\frac{\varepsilon_0 - \varepsilon}{\varepsilon_0^r} \right)^j \{\|f\| + (3/2 - r)\|f_{\mathrm{L}}\|\}$$

where $r = 3/2$ if Lemma 2.1 (b) holds and $r = 1$ if Lemma 2.3 holds.

Proof : By Lemma 2.1 or 2.3

$$\|u\| \leqslant C \|f\| \quad \text{and}$$

$$\|\delta^1\| \leqslant C(\varepsilon_0 - \varepsilon)\|\Delta u\| \leqslant C(\varepsilon_0 - \varepsilon)\|\Delta u\| \leqslant C \left(\frac{\varepsilon_0}{\varepsilon} \right)^r (\varepsilon_0 - \varepsilon)\, \varepsilon_0^{-r} \|f\|\,.$$

In the same way,

$$\|\delta^2\| \leq C(\varepsilon_0 - \varepsilon)\|\Delta\delta^1\| \leq C(\varepsilon_0 - \varepsilon)\,\varepsilon_0^{-r}\|(\varepsilon_0 - \varepsilon)\,\Delta u\|$$
$$\leq C\left(\frac{\varepsilon_0}{\varepsilon}\right)^r\left(\frac{\varepsilon_0 - \varepsilon}{\varepsilon_0^r}\right)^2\|f\|$$

and by induction

$$\|\delta^j\| \leq C\left(\frac{\varepsilon_0}{\varepsilon}\right)^r((\varepsilon_0 - \varepsilon)/\varepsilon_0^r)^j\|f\|. \quad \square$$

Remark 4.3 : Since, in general, $r \geq 1$, this theorem would indicate that the defect correction method does converge very slowly. Local estimates will be improved in section 5 by utilizing the fact that Lemmas 2.1 and 2.3 describe the lack of regularity due to the boundary layers.

By a similar technique, we derive the following regularity result for u^j which will be useful in the next section.

PROPOSITION 4.1 : *Suppose* $q_0 > 0$, $\Omega \subset \mathbf{R}^2$ *is a convex polygon and we have pure Dirichlet boundary conditions. Then, for* $1 \leq j \leq J$,

$$\|\Delta u^j\| \leq C\varepsilon_0^{-1-j/2}\|f\|$$

Proof : From Lemma 2.1 equation (2.2*b*)

$$\|\Delta u^1\| \leq C\varepsilon_0^{3/2}\|f\|.$$

Similarly

$$\|\Delta u^j\| \leq C\varepsilon_0^{-3/2}[\|f\| + (\varepsilon_0 - \varepsilon)\|\Delta u^{j-1}\|]$$
$$\leq C\varepsilon_0^{-3/2}\sum_{l=0}^{j-1}(\varepsilon_0^{-1/2})^l\|f\|,$$

and the result follows. $\square$

5. LOCAL ERROR ESTIMATES IN $\mathbf{R}^2$

In this section, we restrict our discussion to homogeneous Dirichlet boundary conditions and convex polygonal domains in $\mathbf{R}^2$.

We begin by giving a precise statement of the main local error estimate. Let,

$$\Omega = \Omega_0'' = \Omega_0' \supset \Omega_1'' \supset \Omega_1' \supset \dots \supset \Omega_J'' \supset \Omega_J' \tag{5.1a}$$

be a sequence of subdomains of Ω such that each Ω_j'', Ω_j' satisfy the hypotheses of Lemma 2.4 with $d = \varepsilon_0 \geq h$ and does not admit upstream cutoff,

$$j = 1, 2, \dots, J. \tag{5.1b}$$

In particular, we assume that the boundaries of Γ_j', Γ_j'' are piecewise smooth with

$$(\Gamma_j'')^- \subset (\Gamma_{j-1}')^- \subset \Gamma^- \tag{5.2a}$$

all points upstream of a point on $(\Gamma_j'')^0$ belongs to $(\Gamma_j'')^0$ (5.2*b*)

$$|\underline{v} \cdot \underline{n}_j''| \geq c \text{ on } (\Gamma_j'')^- \cup (\Gamma_j'')^+, \tag{5.2c}$$

where, $\underline{n}_j$ is the outward unit normal to Ω_j.

Suppose also :

$\underline{v}$ has no close arcs in $\bar{\Omega}$ *or* $q_0 > \tilde{C}(\underline{v})$ where $\tilde{C}$ depends upont the first derivatives of $\underline{v}$ (5.3*a*)

$$\overline{(\Gamma_j'')^-} \cap \overline{\Gamma^+} \text{ is empty}, \tag{5.3b}$$

Ω_j'' is convex and $\Gamma_j'' = \partial\Omega_j''$ consists of gridlines of the mesh covering Ω

$$\text{dist}\,(\Omega_j' ; (\Gamma_j'')^+) \geq C_1\, \varepsilon_0 \ln\left(\frac{1}{\varepsilon_0}\right), \text{ and}$$

$$\text{dist}\,(\Omega_j' ; (\Gamma_j'')^0) \geq C_2 \sqrt{\varepsilon_0} \ln\left(\frac{1}{\varepsilon_0}\right). \tag{5.3c}$$

$$\Gamma_j'' \text{ consists of grid lines}. \tag{5.3d}$$

Remark : The constraint that Γ_j'' consist of grid lines in (5.3*d*) is not necessary for the H^1 estimate in Theorem 5.1 below but only for the L^2 estimate.

THEOREM 5.1 : *Suppose that* (5.1*a*) *through* (5.3*c*) *above hold. Then, for all* $s > 0$, $j = 1, \dots, J$, *it follows that for* $r = 0$:

$$\begin{aligned}
\varepsilon_0^{1/2} \| u - U^j \|_{1,\Omega_j'} + r \| u - U^j \|_{\Omega_j'} &\leq C (\varepsilon_0 - \varepsilon)^j \, [\varepsilon_0^{1/2} \| f \|_{2j+1,\Omega_0'} + r \| f \|_{2j,\Omega_0'}] + \\
&+ C h^{k+1/2} \left[\sum_{p=1}^{j-1} \sum_{l=0}^{p-1} (\varepsilon_0 - \varepsilon)^l \, \| f \|_{k+2+2l,\Omega_{p-l-1}'} \right] \\
&+ C \varepsilon_0^s \Big[((\varepsilon_0 - \varepsilon)^{j+1} (\varepsilon^{-3/2})^{j+1} + 1) \| f \|_{0,\Omega} \\
&+ \sum_{p=1}^{j-1} \sum_{l=0}^{p} (\varepsilon_0 - \varepsilon)^l \, \| f \|_{2+2l,\Omega_{p-l-1}''} \Big].
\end{aligned}$$

If (5.3*d*) holds additionally then the above holds for $r = 1$.

We shall prove Theorem 5.1 in two steps in, respectively, Theorem 5.3 and Theorem 5.4.

To prove the theorem it is first necessary to establish some local regularity properties of the continuous defect correction sequence, defined by: $u^0 = 0$ and

$$L_{\varepsilon_0} u^j = f - (\varepsilon_0 - \varepsilon)\, \Delta u^{j-1}, \quad \text{in } \Omega, \quad u^j = 0 \text{ on } \partial\Omega. \tag{5.4}$$

THEOREM 5.2: *Suppose* (5.1*a*) *to* (5.3*c*) *hold. Then, for any positive integer m and positive number s, there are constants* $C_1 = C_1(s, m, \underline{v}, \Omega_j'')$ *the solutions* u^j *of* (5.4) *satisfy for* $j = 1, \dots, J < \infty$,

$$\|u^j\|_{m, \Omega_j'} + \left\|u^j_{\underline{v}}\right\|_{m, \Omega_j'} \le C \left(\sum_{l=0}^{j-1} (\varepsilon_0 - \varepsilon)^l \|f\|_{m+2l, \Omega_{j-l}''} + \right.$$
$$\left. + \varepsilon_0^s \|f\|_\Omega + \varepsilon_0^{s+1} \sum_{l=1}^{j-1} \|\Delta u^l\|_\Omega \right)$$

Proof: Note that by Theorem 2.1

$$\|u^1\|_{m, \Omega_1'} + \left\|u^1_{\underline{v}}\right\|_{m, \Omega_1'} \le C\,(\|f\|_{m, \Omega_1''} + \varepsilon_0^s \|f\|_\Omega).$$

Similarly,

$$\|u^j\|_{m, \Omega_j'} + \left\|u^j_{\underline{v}}\right\|_{m, \Omega_j'} \le C\,(\|f\|_{m, \Omega_j''} + \varepsilon_0^s \|f\|_\Omega +$$
$$+ (\varepsilon_0 - \varepsilon)(\|u^{j-1}\|_{m+2, \Omega_j''} + \varepsilon_0^s \|\Delta u^{j-1}\|_\Omega)).$$

Thus, by backwards recursion:

$$\|u^j\|_{m, \Omega_j'} + \left\|u^j_{\underline{v}}\right\|_{m, \Omega_j'} \le C \left(\sum_{l=0}^{j-1} (\varepsilon_0 - \varepsilon)^l \|f\|_{m+2l, \Omega_{j-1}''} + \varepsilon_0^s \|f\|_\Omega + \right.$$
$$\left. + (\varepsilon_0 - \varepsilon)\, \varepsilon_0^s \sum_{l=1}^{j-1} \|\Delta u^l\|_\Omega \right). \quad \square$$

We remark that the estimates in the above theorem can be improved or shown to hold under somewhat relaxed assumptions in several special cases (e.g., Courant elements). These extensions would utilize key modifications introduced for streamline diffusion methods in e.g., Johnson Schatz and Wahlbin [10], and Johnson and Nävert [8].

The following result of Nävert [12] will also be useful in the sequel.

Letting u_{ε_0} denote the solution to: $L_{\varepsilon_0} u_{\varepsilon_0} = f$ in Ω, $u_{\varepsilon_0} = g$ on $\partial\Omega$, with $\varepsilon_0 \ge h$ define $U_{\varepsilon_0} \in S^h$ by

$$\begin{cases} B_{\varepsilon_0}(u_{\varepsilon_0} - U_{\varepsilon_0}, w) = 0, & \forall w \in S^h \cap \mathring{H}^1(\Omega), \\ U_{\varepsilon_0} = I_h(g) \;\; \text{(interpolant) on } \Gamma. \end{cases} \tag{5.5}$$

Thus, U_{ε_0} is the usual Galerkin projection of u_{ε_0} into S^h. (5.5) can also be thought of as the Galerkin artificial viscosity of u_{ε_0} into S^h, as $\varepsilon_0 \geqslant h$.

PROPOSTION 5.1 [Nävert [12; Corollary 4.3, p. 52]] : *Let* $\varepsilon_0 \geqslant h$, $\Omega' \subset \Omega'' \subset \Omega$ *be as in Lemma 2.1, and* $u_{\varepsilon_0} \in H^{m+1}(\Omega)$, $g \in H^{m+1}(\Gamma)$, $0 \leqslant m \leqslant k$, *integers. Let* $s > 0$ *be given. Then, there are constants* $C_1 = C_1(s, \underline{v}, \Omega'')$, $C_2 = C_2(s, \underline{v}, \Omega'')$ *and* $C = C(\underline{v}, q, \Omega'', \Omega, T)$ *such that if the distance from* Ω' *to* $(\Gamma'')^0$ *and* $(\Gamma'')^+$ *is at least* $C_1 \sqrt{\varepsilon_0} \ln\left(\frac{1}{\varepsilon_0}\right)$ *and* $C_2\, \varepsilon_0 \ln\left(\frac{1}{\varepsilon_0}\right)$, *resp., then*

$$\|u_{\varepsilon_0} - U_{\varepsilon_0}\|_{1,\Omega'} \leqslant C\left(h^k |u_{\varepsilon_0}|_{k+1,\Omega''} + \varepsilon_0^s h^m |u_{\varepsilon_0}|_{m+1,\Omega}\right). \tag{5.6a}$$

Moreover, if Ω' is convex, Γ'' consists of gridlines and $u|_{\Gamma''} \in H^{k+1}(\Gamma'')$ then,

$$\|u_{\varepsilon_0} - U_{\varepsilon_0}\|_{\Omega'} \leqslant C\left\{\frac{h^{k+1}}{\varepsilon_0}\left(|u|_{k+1,\Omega''} + |u|_{k+1,\Gamma''}\right) + \right. \\ \left. + \varepsilon_0^s h^{m+1}\left(|u|_{m+1,\Omega} + |g|_{m+1,\Gamma}\right)\right\}. \quad \square \tag{5.6b}$$

Following the strategy of section 3, we decompose the error $u - U^j$ into $(u - u^j)$ and $u^j - U^j$. Thus, Theorem 5.1 is proven in two steps, the first of which is given by :

THEOREM 5.3 [Iteration Error] : *Suppose* (5.1*a*, *b*) (5.2*a*, *b*, *c*) *hold. Suppose also that* $\underline{v}$ *has no closed arcs in* $\bar{\Omega}$ *or that* $q_0 > \bar{C}(\underline{v})$ *where* $\bar{C}(\underline{v})$ *depends upon the first derivatives of* $\underline{v}$. *Assume* $\overline{(\Gamma_j'')^-} \cap \overline{\Gamma^+}$ *is empty and* $s > 0$. *Then, there are constants* $C_1 = C_1(s, \underline{v}, \Omega_j'')$ $C_2 = C_2(s, \underline{v}, \Omega_j'')$ *and* $C = C(\underline{v}, q, \Omega_j'', \Omega, J)$ *such that if*

$$\operatorname{dist}(\Omega_j', (\Gamma_j'')^+) \geqslant C_1\, \varepsilon_0 \ln\left(\frac{1}{\varepsilon_0}\right),$$

$$\operatorname{dist}(\Omega_j', (\Gamma_j'')^0) \geqslant C_2 \sqrt{\varepsilon_0} \ln\left(\frac{1}{\varepsilon_0}\right),$$

it follows that for $m = 0, 1, 2$

$$\|\delta_j\|_{m,\Omega_j'} + \left\|\frac{\partial}{\partial \underline{v}} \delta_j\right\|_{m,\Omega_j'} \leqslant C\left\{(\varepsilon_0 - \varepsilon)^j \|f\|_{2j+m,\Omega_0'} + \varepsilon^s \|f\|_{0,\Omega}\right\} + \\ + C\varepsilon_0^s(\varepsilon_0 - \varepsilon)^{j+1}\left\{(\varepsilon^{-3/2})^{j+1} \|f\|_{0,\Omega}\right\}, \\ j = 1, \dots, J.$$

Proof: Letting u^j denote the continuous defect correction iteration, $\delta^j := u - u^j$ is easily seen to satisfy the following

$$\begin{cases} L_{\varepsilon_0} \delta^j = (\varepsilon_0 - \varepsilon)(-\Delta \delta^{j-1}) \text{ in } \Omega, \\ \delta^j = 0 \text{ on } \partial\Omega, \ \delta^0 = u, \ j = 1, \dots, J. \end{cases}$$

Applying Nävert [12, Theorem 2.3, p. 26], which is given in Section 2, Theorem 2.1, to the iteration (5.7) upon the subsets Ω_j', Ω_j'' of Ω gives (for $m = 0, 1, 2$):

$$\begin{aligned} \|\delta_j\|_{m, \Omega_j'} + \left\| \frac{\partial}{\partial \underline{v}} \delta_j \right\|_{m, \Omega_j'} &\leq \\ &\leq C[(\varepsilon_0 - \varepsilon)\|\delta_{j-1}\|_{2+m, \Omega_j''} + \varepsilon_0^s(\varepsilon_0 - \varepsilon)\|\Delta\delta_{j-1}\|_\Omega] \\ &\leq (\text{iterating backward to } \delta_0 = u)\dots \\ &\leq C(\varepsilon_0 - \varepsilon)^j \left\{ \|f\|_{2j+m, \Omega_0'} + \varepsilon^s \|f\|_{0, \Omega} \right\} \\ &\quad + C\varepsilon_0^s(\varepsilon_0 - \varepsilon)\left(\sum_{l=0}^{J} \|\Delta\delta_l\|_\Omega \right). \end{aligned}$$

the last term is bounded using the global estimates for the iteration error given in Corollary 4.2. Thus, as $0 \leq l \leq j \leq J$, $\sum \|\Delta\delta_l\| \leq C\varepsilon^{-3/2}\left(\frac{\varepsilon_0 - \varepsilon}{\varepsilon^{3/2}} \right)^j \|f\|$ and the result follows.

Combining the previous result with Proposition 4.1 gives:

COROLLARY 5.1: *Under the assumptions* (5.1*a*) *to* (5.3*c*) *of Theorem 5.1, for* $j = 1, \dots, J < \infty$,

$$\begin{aligned} \|u^j\|_{m, \Omega_j'} + \|u^j_{\underline{v}}\|_{m, \Omega_j'} \leq C &\left\{ \sum_{l=0}^{j-1} (\varepsilon_0 - \varepsilon)^l \|f\|_{m+2l, \Omega_{j-l}''} + \right. \\ &\left. + \varepsilon_0^s \left[1 + \left(\varepsilon_0^{-1/2} \varepsilon_0^{-\left(\frac{j-1}{s}\right)} \right) \right] \|f\|_\Omega \right\} \end{aligned}$$

and for $j = 1, \dots, J < \infty$

$$\begin{aligned} \|u^j\|_{m, \Omega_j''} + \|u^j_{\underline{v}}\|_{m, \Omega_j''} \leq C &\left\{ \sum_{l=0}^{j-1} (\varepsilon_0 - \varepsilon)^l \|f\|_{m+2l, \Omega_{j-l-1}'} + \right. \\ &\left. + \varepsilon_0^s \left[1 + \left(\varepsilon_0^{-1/2} \varepsilon_0^{-\left(\frac{j-1}{s}\right)} \right) \right] \|f\|_\Omega \right\}. \quad \square \end{aligned}$$

The discretization error $u^J - U^J$ is bounded via an induction argument. For this, we require the following special notation:

$$|||w|||^2 := \varepsilon_0 \|\nabla w\|^2 + \|w\|^2,$$

$$|||w|||^2_\psi := \varepsilon_0 \|\nabla w\|^2_\psi + \|w\|^2_\psi, \quad \|w\|^2_\psi = \int_\Omega w^2 \psi\, dx,$$

$$|||w|||^2_{\tilde\Omega} := \varepsilon_0 \|\nabla w\|^2_{L^2(\tilde\Omega)} + \|w\|^2_{L^2(\tilde\Omega)}$$

$$|||w|||^2_{\psi,\tilde\Omega} := \varepsilon_0 \|\nabla w\|^2_{\psi,\tilde\Omega} + \|w\|_{\psi,\tilde\Omega}.$$

Further, define the operator

$$\mu_j : H^1(\Omega) \to S^h(\Omega) \cap \{\phi \mid \phi = 0 \text{ on } \Omega - \Omega_j''\}$$

by

$$\mu_j(u) = \tilde w$$

where $\tilde w$ satisfies: for all $w \in S^h(\Omega_j'') \cap \mathring{H}^1(\Omega_j'')$

$$\begin{cases} 0 = B_{\varepsilon_0, \Omega_j''}(u - \tilde w, w) \\ \quad := \displaystyle\int_{\Omega_j''} \varepsilon_0 \nabla(u - \tilde w) \cdot \nabla w + \underline{v} \cdot \nabla(u - \tilde w)\, w + q\,(u - \tilde w)\, w\, dx \\ \tilde w = I_h(u) \text{ (interpolant) on } \Omega - \Omega_j'' \end{cases} \tag{5.8}$$

THEOREM 5.4: *Suppose the hypotheses of Lemma 2.4 are satisfied by* Ω_j *and* (5.1*a*) *to* (5.3*c*) *hold. Suppose* $f(\underline{x})$ *is sufficiently smooth* ($f \in H^m(\Omega)$ *for* m *sufficiently large*). $\varepsilon_0 \geq h$. *Suppose that each* Ω_j', Ω_j'' *satisfies the hypotheses of Theorem 2.1 and Proposition 5.1. In particular:* $(\Gamma_j'')^- \subset \Gamma^-$, $\overline{(\Gamma_j'')^-} \cap \overline{\Gamma^+}$ *is empty,* Γ^- *is part of the faces of* Ω *and* Γ_j'' *consists of grid lines.*

Then, for any $s > 0$ *there is a* $C(s)$ *such that*

$$|||u^J - U^J|||_{\Omega_j'} \leq C h^{k+1/2} \left[\sum_{p=1}^{j-1} \sum_{l=0}^{p-1} (\varepsilon_0 - \varepsilon)^l \|f\|_{k+2+2l, \Omega''_{p-l-1}} \right] + $$
$$+ C \varepsilon_0^s \left[\sum_{p=1}^{j-1} \sum_{l=0}^{p-1} (\varepsilon_0 - \varepsilon)^l \|f\|_{2+2l, \Omega''_{p-l-1}} + \|f\|_\Omega \right]. \tag{5.9}$$

In particular, for any $s > 0$ *there are constants* $\tilde C_1 = \tilde C_1(s, f, C_1, C_2)$ $\tilde C_2 = \tilde C_2(f, C_1, C_2)$ *such that*

$$\|u^J - U^J\|_{1, \Omega_j'} \leq \tilde C_1(s, f, C_1, C_2)\, h^k + \tilde C_2(f, C_1, C_2)\, \varepsilon_0^s, \tag{5.10a}$$

and

$$\|u^J - U^J\|_{\Omega_j'} \leq \tilde C_1(s, f, C_1, C_2)\, h^{k+1/2} + \tilde C_2(f, C_1, C_2)\, \varepsilon_0^s. \tag{5.10b}$$

Proof. When $j = 1$ we have:

$$B_{\varepsilon_0}(u^1 - U^1, w) = 0\,, \quad \forall w \in S^h(\Omega)\,, \quad \varepsilon_0 = 0(h)\,.$$

Thus from, e.g., Nävert [12 ; Corollary 4.3, p. 52 ; Theorem 4.8, p. 49] we have, for integer m, $0 \leqslant m \leqslant k$,

$$\|u^1 - U^1\|_{1,\Omega_1'} \leqslant C\left\{h^k |u^1|_{k+1,\Omega_1''} + \varepsilon_0^s h^m |u^s|_{m+1,\Omega}\right\}, \tag{5.11a}$$

$$\|u^1 - U^1\|_{1,\Omega_1'} \leqslant C\left\{h^k |u^1|_{k+1,\Omega_1''} + \varepsilon_0^s \|f\|_{0,\Omega}\right\}, \tag{5.11b}$$

$$\|u^1 - U^1\|_{0,\Omega_1'} \leqslant C\left\{h^{k+1/2}(|u^1|_{k+1,\Omega_1''} + |u^1|_{k+1,\Gamma_1''}) + \varepsilon_0^s \|f\|_{0,\Omega}\right\} \tag{5.12a}$$

$$\|u^1 - U^1\|_{0,\Omega_1'} \leqslant C\left\{h^{k+1/2}(|u^1|_{k+1,\Omega_1''} + |u^1|_{k+1,\Gamma_1''}) \right. \\ \left. \varepsilon_0^s h^{m+1}(|u^1|_{m+1,\Omega} + |u^1|_{m+1,\Gamma})\right\} \tag{5.12b}$$

We now proceed by induction, by showing that, loosely speaking, $|||U^j - u^j|||_{\Omega_j'}$ is of the same order as $|||U^{j-1} - u^{j-1}|||_{\Omega_{j-1}'}$.

Define $\theta_j = \mu_j(u^j) - U^j$ so that

$$|||u^j - U^j|||_{\Omega_j'} \leqslant |||u^j - \mu_j(u^j)|||_{\Omega_j'} + |||\theta_j|||_{\Omega_j'}\,, \tag{5.13}$$

and θ_j satisfies for all $w \in S^h(\Omega_j'')$

$$B_{\varepsilon_0,\Omega_j''}(\theta_j, w) = (\varepsilon_0 - \varepsilon)(\nabla(u^{j-1} - U^{j-1}), \nabla w)_{\Omega_j''}\,. \tag{5.14}$$

Introduce the cutoff function ψ_j which is $q_0 = \alpha$ compensating of order $(\varepsilon_0, \gamma, s)$ in (Ω_j', Ω_j''). Note that $\psi\theta_j = 0$ on Γ^- so we obtain, following, e.g., Nävert [12], Johnson, Nävert, Pitkaranta [9],

$$B_{\varepsilon_0,\Omega_j''}(\theta_j, \psi\theta_j) \geqslant \varepsilon_0 \|\nabla\theta_j\|_\psi^2 + \varepsilon_0(\nabla\theta_j, \theta_j \nabla\psi) + \|\theta_j\|^2_{q_0\psi - \frac{1}{2}\frac{\partial}{\partial \underline{v}}\psi}\,.$$

Hence, for γ sufficiently large, we find, as in [9], [12],

$$B_{\varepsilon_0,\Omega_j''}(\theta_j, \psi\theta_j) \geqslant C_1(|||\theta_j|||_\psi^2 + \|\theta_j\|^2_{-\psi_{\underline{v}}}) - C_2\,\varepsilon_0^s\,|||\theta_j|||^2_{\Omega_j''}\,.$$

Therefore,

$$C(\varepsilon_0^s\,|||\theta_j|||^2_{\Omega_j''} + B_{\varepsilon_0,\Omega_j''}(\theta_j, \psi\theta_j)) \geqslant |||\theta_j|||_\psi^2 + \|\theta_j\|^2_{-\psi_{\underline{v}}}\,. \tag{5.15}$$

From (4.8),

$$B_{\varepsilon_0,\Omega_j''}(\theta_j, I_h(\psi\theta_j)) = -(\varepsilon_0 - \varepsilon)(\nabla(u^{j-1} - U^{j-1}), \nabla(I_h(\psi\theta_j)))_{\Omega_j''} = 0\,,$$

so that

$$B_{\varepsilon_0, \Omega_0''}(\theta_j, \psi\theta_j) \leq B_{\varepsilon_0, \Omega_j''}(\theta_j, (\psi\theta_j) - I_h(\psi\theta_j)) -$$
$$- (\varepsilon_0 - \varepsilon)(\nabla(u^{J-1} - U^{J-1}), \nabla(I_h(\psi\theta_j)))_{\Omega_j''}$$
$$\leq - (\varepsilon_0 - \varepsilon)(\nabla(u^{J-1} - U^{J-1}), \nabla(I_h(\psi\theta_j)))_{\Omega_j''} +$$
$$+ C\gamma^{-1/2}\left\{\varepsilon_0^{1/2}\|\nabla\theta_j\|_\psi + \frac{h}{\varepsilon_0}\left(\left\|\frac{\partial}{\partial \underline{v}}\theta_j\right\|_\psi + \|\theta\|_\psi\right)\right\} \times$$
$$\times (\|\theta\|_\psi + \|\theta\|_{-\psi_{\underline{L}}}) . \tag{5.16}$$

Inserting (5.16) into (5.15) gives

$$|||\theta_j|||_\psi^2 + \|\theta_j\|_{-\psi_{\underline{v}}}^2 \leq - (\varepsilon_0 - \varepsilon)(\nabla(u^{J-1} - U^{J-1}), \nabla(I_h(\psi\theta_j)))_{\Omega_j''} +$$
$$+ C\varepsilon_0^s |||\theta_j|||_{\Omega_j''}^2 + c\gamma^{-1}\left\{\varepsilon_0\|\nabla\theta_j\|_\psi^2\right.$$
$$\left. + \frac{h^2}{\varepsilon_0}\left(\left\|\frac{\partial}{\partial \underline{v}}\theta_j\right\|_\psi^2 + \|\theta_j\|_\psi^2\right)\right\}. \tag{5.17}$$

As $\varepsilon_0 \geq 0(h)$, taking γ sufficiently large to hide the appropriate terms in (5.17) gives

$$|||\theta_j|||_{\Omega_j'}^2 \leq C(|||\theta_j|||_\psi^2 + \|\theta\|_{\psi_{\underline{v}}}^2)$$
$$\leq C\varepsilon_0^s |||\theta_j|||_{\Omega_j''}^2 +$$
$$+ C(\varepsilon_0 - \varepsilon)\left|(\nabla(u^{J-1} - U^{J-1}), \nabla(I_h(\psi\theta_j)))_{\Omega_j''}\right| . \tag{5.18}$$

We now focus our attention on the last term in the R.H.S. of (5.18).

Expanding :

$$(\varepsilon_0 - \varepsilon)(\nabla(u^{J-1} - U^{J-1}), \nabla(I_h(\psi\theta_j)))_{\Omega_j''} = (\varepsilon_0 - \varepsilon)(T_1 + T_2) \tag{5.19}$$

where :

$$T_1 := (\nabla(u^{J-1} - U^{J-1}), \nabla(\psi\theta_j))_{\Omega_j''}$$
$$T_2 := (\nabla(u^{J-1} - U^{J-1}), \nabla\{I_h(\psi\theta_j) - \psi\theta_j\})_{\Omega_j''} .$$

Consider T_1 :

$$T_1 = (\nabla(u^{J-1} - U^{J-1}), \psi\nabla\theta_j)_{\Omega_j''} + (\nabla(u^{J-1} - U^{J-1}), \theta_j \nabla\psi)_{\Omega_j'}$$
$$\leq \|\nabla(u^{J-1} - U^{J-1})\|_\psi \cdot \|\nabla\theta_j\|_\psi + (\nabla(u^{J-1} - U^{J-1}), \theta_j \nabla\psi)_{\Omega_j'}$$
$$\leq \|\nabla(u^{J-1} - U^{J-1})\|_\psi(\|\nabla\theta_j\|_\psi + \|\theta_j \nabla\psi\|_{\psi^{-1}, \Omega_j''}) . \tag{5.20}$$

With $\underline{v}^{\perp}$ the velocity orthogonal to $\underline{v}$:

$$\|\theta_J \nabla\psi\|^2_{\psi^{-1},\Omega_J''} \leq C\left\{\left\|\theta_J \frac{\partial}{\partial \underline{v}}\psi\right\|^2_{\psi^{-1},\Omega_J''} + \left\|\theta_J \frac{\partial}{\partial \underline{v}^{\perp}}\psi\right\|^2_{\psi^{-1},\Omega_J''}\right\}$$

$$\leq C\left\{(\gamma h)^{-1}\|\theta_J\|^2_{-\psi_{\underline{v}}} + \gamma^{-1}h^{-1}\|\theta_J\|^2_{-\psi_{\underline{v}}}\right\} + Ch^s\|\theta_J\|^2_{\psi}, \tag{5.21a}$$

where we have used the properties of ψ given in Nävert [12], see also Section 2, Lemma 2.1.

Inserting (5.21) into (5.20) gives the following bound for T_1 of (5.19);

$$T_1 \leq \|\nabla(u^{J-1} - U^{J-1})\|_{\psi}(\|\nabla\theta_J\|_{\psi} + C(\gamma h)^{-1/2}\|\theta_J\|_{-\psi_{\underline{v}}} + Ch^s\|\theta_J\|_{\psi}). \tag{5.21b}$$

We bound T_2 using similar techniques as in (5.16), (5.17). Indeed,

$$T_2 \leq \|\nabla(u^{J-1} - U^{J-1})\|_{\psi}\|\nabla(\psi\theta - I_h(\psi\theta))\|_{\psi^{-1},\Omega_J''}.$$

By interpolation theory :

$$\|\nabla((\psi\theta_J) - I_h(\psi\theta_J))\|^2_{\psi^{-1},\Omega_J''} \leq Ch^2 \sum_{\tau_l \text{ in triangulation of } \Omega_J'} \times$$

$$\times \sum_{|\alpha| = 2}\|D^{\alpha}(\psi\theta_J)\|_{\psi^{-1},\tau_l} \leq$$

$$\leq Ch^2\left\{\sum_{\tau_l}\sum_{|s+t| = 2}(\|\psi D^s\theta_J\|^2_{\psi^{-1},\tau_l} + \|\theta_J D^t\psi\|^2_{\psi^{-1},\tau_l})\right\}. \tag{5.22}$$

Analyzing the components in the sum on the R.H.S. of (5.22) individually : from Nävert [12] quoted in Section 2 as Lemma 2.1,

$$\sum_{\tau_l}\sum_{|s| = 2}\|\theta_J D^s\psi\|^2_{\psi^{-1},\tau_l} \leq Ch^{-2}\|\theta_J \psi_{\underline{v}}\|^2_{\psi^{-1},\Omega_J''}$$

$$\leq C\gamma^{-1}\text{h}^{-3}\|\theta_J\|^2_{-\psi_{\underline{v}}\,\Omega_J} + Ch^s\|\theta_J\|^2_{\psi}. \tag{5.23a}$$

From Lemma 2.1 also :

$$\|\theta_{J,\underline{v}}\psi_{\underline{v}}\|^2_{\psi^{-1},\Omega_J''} \leq C(\gamma h)^{-2}\|\theta_{J,\underline{v}}\|^2_{\psi} + Ch^s\|\theta_J\|^2_{\psi}$$

and from the inverse estimate for S^h and Lemma 2.1

$$\|\theta_{J,\underline{v}^{\perp}}\psi_{\underline{v}}\|^2_{\psi^{-1},\Omega_J''} \leq C\gamma^{-1}h^{-3}\|\theta_J\|^2_{-\psi_{\underline{v}}} + Ch^s\|\theta_J\|_{\psi}.$$

Further, Lemma 2.1 gives,

$$\left\{\|\theta_{J,\underline{v}^{\perp}}\psi_{\underline{v}}\|^2_{\psi^{-1},\Omega_J''} + \|\theta_{J,\underline{v}}\psi_{\underline{v}^{\perp}}\|^2_{-\psi^{-1},\Omega_J''}\right\} \leq C\gamma^{-1}h^{-3}\|\theta_J\|^2_{-\psi_{\underline{v}}} + Ch^s\|\theta_J\|^2_{\psi}.$$

The previous three inequalities then give :

$$\sum_{\tau_j} \sum_{\substack{|s| = 1 \\ |t| = 1}} \| D^s \psi D^t \theta_j \|^2_{\psi^{-1}} \leq C \left\{ (\gamma h)^{-2} \| \theta_{j,\, \underline{v}} \|^2_{\psi} + \right.$$

$$\left. + \gamma^{-1} h^{-3} \| \theta_j \|^2_{-\psi_{\underline{v}}} + \gamma^{-1} h^{-3} \| \theta \|^2_{-\psi_{\underline{v}}} \right\} + C h^s \| \theta_j \|^2_{\psi} . \quad (5.23b)$$

From the higher order inverse estimate :

$$| \theta_j |_{2, \tau_j} \leq C h^{-1} \| \nabla \theta_j \|_{\tau_j} , \quad \theta_j \in S^h ,$$

we bound the last component of (5.22) as :

$$\sum_{\tau_j} \sum_{|\alpha| = 2} \| \psi D^\alpha \theta_j \|^2_{\psi^{-1}, \Omega_j''} \leq C h^{-2} \| \nabla \theta_j \|^2_{\psi} + C h^2 \| \theta_j \|^2_{\psi} . \quad (5.23c)$$

Combining (5.23*a*, *b*, *c*) into (5.22) gives

$$\| \nabla (\psi \theta_j - I_h (\psi \theta_j)) \|^2_{\psi^{-1}} \leq C h^2 \left\{ \gamma^{-1} h^{-3} \| \theta_j \|^2_{-\psi_{\underline{v}}} + (\gamma h)^{-2} \| \theta_{j,\, \underline{v}} \|^2_{\psi} + \right.$$

$$\left. + h^{-2} \| \nabla \theta_j \|^2_{\psi} \right\} + C h^s \| \theta_j \|^2_{\psi} . \quad (5.24)$$

Thus, as $\varepsilon_0 = 0(h)$,

$$(\varepsilon_0 - \varepsilon)\, T_2 \leq (\varepsilon_0 - \varepsilon)\, C(\gamma) \| \nabla (u^{j-1} - U^{j-1}) \|^2_{\psi} +$$

$$+ C \left\{ \gamma^{-1} \| \theta_j \|^2_{-\psi_{\underline{v}}} + (\varepsilon_0 - \varepsilon)\, \gamma^{-2} \| \theta_{j,\, \underline{v}} \|^2_{\psi} \right.$$

$$\left. + \gamma^{-1} (\varepsilon_0 - \varepsilon) \| \nabla \theta_j \|^2_{\psi} \right\} + C h^s \| \theta_j \|^2_{\psi} \quad (5.25)$$

using (5.25) and

$$(\varepsilon_0 - \varepsilon)\, T_1 \leq (\varepsilon_0 - \varepsilon)\, C(\gamma) \| \nabla (u^{j-1} - U^{j-1}) \|^2_{\psi} +$$

$$+ (\varepsilon_0 - \varepsilon)\, \gamma^{-1} \| \nabla \theta_j \|^2_{\psi} + \gamma^{-1} \| \theta_j \|^2_{-\psi_{\underline{v}}} + C h^s \| \theta_j \|^2_{\psi}$$

(which follow from (5.21*b*)) by γ sufficiently large in (5.18) gives :

$$|\!|\!| \theta_j |\!|\!|^2_{\Omega_j'} \leq C \left(|\!|\!| \theta_j |\!|\!|^2_{\psi} + \| \theta \|^2_{-\psi_{\underline{v}}} \right)$$

$$\leq C \varepsilon_0^s |\!|\!| \theta_j |\!|\!|^2_{\Omega_j''} + C (\varepsilon_0 - \varepsilon) \| \nabla (u^{j-1} - U^{j-1}) \|^2_{\psi} . \quad (5.26)$$

As $\| w \|_{\psi} \leq C \| w \|_{\Omega_j''}$ we have

$$|\!|\!| \theta_j |\!|\!|^2_{\Omega_j'} \leq C \varepsilon_0^s |\!|\!| \theta_j |\!|\!|^2_{\Omega_j''} + C (\varepsilon_0 - \varepsilon) \| \nabla (u^{j-1} - U^{j-1}) \|^2_{\Omega_j''} . \quad (5.27)$$

To bound the first term in the R.H.S. of the above, note that

$$||| \theta_j |||_{\Omega_j''} \leqslant ||| u^j - \mu_j(u^j) |||_{\Omega_j''} + ||| u^j - U^j |||_{\Omega_j''} . \tag{5.28}$$

From the global estimates for $u^j - U^j$ in Theorem 4.1 applied to Ω_j'' we have

$$||| u^j - U^j |||_{\Omega_j''} \leqslant ||| u^j - U^j |||_{\Omega} \leqslant C h^{k+1/2} \left(\sum_{l=1}^{j} \| u^l \|_{k+1} \right) . \tag{5.29}$$

Letting $\tilde{g}_j$ be some $S^h(\Omega_j)$ function which interpolates u^j on $\partial\Omega_j'$ we then have, again from the global estimates in Appendix 1 applied to Ω_j'' that

$$||| u^j - \mu^j(u^j) |||_{\Omega_j''} \leqslant C h^{k+1/2} | u^j - \tilde{g}_j |_{k+1, \Omega_j''} . \tag{5.30}$$

Using (5.30), (5.29) and (5.28) in (5.27) gives :

$$\begin{aligned} ||| \theta_j |||^2_{\Omega_j} \leqslant\ & C(\varepsilon_0 - \varepsilon) \| \nabla(u^{j-1} - U^{j-1}) \|^2_{\Omega_j''} + \\ & + C \varepsilon_0^s \left\{ h^{k+1/2} \sum_{l=1}^{j} \| u^l \|_{k+1} + C h^{k+1/2} | u^j - \tilde{g}_j |_{k+1, \Omega_j} \right\}^2 \\ \leqslant\ & C(\varepsilon_0 - \varepsilon) \| \nabla(u^{j-1} - U^{j-1}) \|^2_{\Omega_j''} \times \\ & \times C \varepsilon_0^s \left\{ h^{2k+1} \sum_{l=1}^{j} \| u^l \|^2_{k+1} + h^{2k+1} | u^j - \tilde{g}_j |^2_{k+1, \Omega_j''} \right\} . \end{aligned} \tag{5.31}$$

Local error estimates for the artificial viscosity projection operator μ_j follow from e.g. Nävert [12 ; Corollary 4.3, p. 52]

$$\begin{aligned} ||| u^j - \mu_j(u^j) |||_{\Omega_j'} \leqslant C \left\{ h^{k+1/2} (| u^j |_{k+1, \Omega_j''} + | u^j |_{k+1, \Gamma_j''}) \right\} + \\ + \varepsilon_0^{s+1/2} (| u^j |_{1, \Omega_{j-1}'} + | u^j |_{1, \Gamma_{j-1}'}) . \end{aligned} \tag{5.32}$$

Using the triangle inequality :

$$||| u^j - U^j |||_{\Omega_j'} \leqslant ||| \theta_j |||_{\Omega_j'} + ||| u^j - \mu_j(u^j) |||_{\Omega_j''} ,$$

(5.31) and (5.32) we obtain a bound for $||| u^j - U^j |||_{\Omega_j'}$:

$$\begin{aligned} ||| u^j - U^j |||_{\Omega_j'} \leqslant\ & C ||| u^{j-1} - U^{j-1} |||_{\Omega_j''} + \\ & + C h^{k+1/2} (| u^j |_{k+1, \Omega_j''} + | u^j |_{k+1, \Gamma_j''}) \\ & + C \varepsilon_0^s \left\{ \sum_{l=1}^{j} ||| u^l ||| + h^{1/2} | \tilde{g}_j |_{1, \Omega_j''} \right. \\ & \left. + | u^j |_{1, \Omega_{j-1}'} + | u^j |_{1, \Gamma_{j-1}'} \right\} . \end{aligned} \tag{5.33}$$

Letting $\tilde{g}_j = I_h(u^j)$, which satisfies : $|\tilde{g}_j|_{1, \Omega_j''} \leq C \|u^j\|_{1, \Omega_j''}$ applying the trace theorem, using Corollary 5.1 and picking sufficiently high reduces (5.33) to :

$$\||u^j - U^j|\|_{\Omega_j'} \leq C \||u^{j-1} - u^{j-1}|\|_{\Omega_{j-1}'} + \\ + C h^{k+1/2} \left[\sum_{l=0}^{j-1} (\varepsilon_0 - \varepsilon)^l \|f\|_{k+1+2l, \Omega_{j-l-1}'} \right] \\ + C \varepsilon_0^s \left[\sum_{l=0}^{j-1} (\varepsilon_0 - \varepsilon)^l \|f\|_{2+2l, \Omega_{j-l-1}''} \right].$$

The theorem now follows by induction. □

CONCLUSION

We have performed an analysis of the global and local convergence properties of the defect correction method with artificial viscosity. On the basis of this, the method studied herein seems to have the following characteristics. When the true solution is smooth uniformly in ε, the method convergences with optimal rates after a few iterations. When there are characteristic (parabolic) boundary layers the method converges slowly globally with no mesh refinement. When there are downstream layers the method will not converge globally, in general, unless the mesh is refined in the layer or an exponential weighting function is incorporated in the layer element. If the mesh is refined suitably in the boundary layers, the convergence is as in the case when u is smooth uniformly in ε. A convergence is to be understood as $h \to 0$ for j fixed, and $j \leq k$.

In addition, the dominant error term in the method when sufficiently far from the layers is independent of ε. However, as j increases, we must move farther into the interior of Ω to obtain this nice rate of convergence. Computational experiments, reported in [6, 7, 4, 5] also show this effect clearly.

We have considered the most straightforward artificial viscosity + defect correction formulation. This is because the defect correction method (1.7) is easy to implement when a finite element package is available to solve convection diffusion equations for large ε. If better results are desired the defect correction idea could be used with methods that are « tailored » for the problem, such as the streamline diffusion method studied by Nävert [18] (see also Axelsson [1, 2]).

APPENDIX 1

Optimal Order L_2-Error Estimate

Let $\theta = U - U_I$, $\eta = u - U_I$, where U_I is the interpolant of u on S^h. Then by (1.4), (1.5), (1.6),

$$B_\varepsilon(\theta, \theta) = B_\varepsilon(\eta, \theta) \leqslant \varepsilon |\theta|_1 |\theta|_1 + \varepsilon^{-1/2} \|\eta\| \varepsilon^{1/2} |\theta|_1 + + C \|\eta\| \|\theta\| + \oint_{\Gamma^+} |\underline{v} . \underline{n} \eta\theta| \, ds .$$

By the usual inequalities

$$\|\theta\|_{1,\varepsilon} \equiv \varepsilon^{1/2} |\theta|_1 + |\theta| + \left\{ \oint_{\Gamma^+} \underline{v} . \underline{n} \theta^2 \right\}^{1/2} \leqslant C \left[\varepsilon^{1/2} |\eta|_1 + \varepsilon^{-1/2} \|\eta\| + \left\{ \int_{\Gamma^+} \underline{v} . \underline{n} \eta^2 \right\}^{1/2} \right] .$$

Here $|.|_1$ is the first order Sobolev seminorm. Then if $\varepsilon \geqslant h$, it follows that

$$\|\theta\|_{1,\varepsilon} \leqslant C \varepsilon^{1/2} h^r |u|_{r+1} ,$$

and by a triangle inequality

$$\|u - U\|_{1,\varepsilon} \leqslant C \varepsilon^{1/2} h^r |u|_{r+1} , \quad r \leqslant k . \tag{1}$$

The L_2-error estimate part of this can be improved a little (by $h^{1/2}$ if $\varepsilon = h$) if we use the usual duality argument.

Let then

$$\mathscr{L}_\varepsilon^* \phi = u - U , \quad \phi = 0 \text{ on } \Gamma_- , \quad \varepsilon \, \underline{\nabla} \phi . n + v . n \phi = 0 \text{ on } \Gamma / \Gamma_- .$$

We assume that $\mathscr{L}_\varepsilon^*$ satisfies the regularity estimate

$$\|\phi\|_2 \leqslant C \varepsilon^{-1} \|u - U\| . \tag{2}$$

(Note that by the results of Section 2, this is satisfied when we have only parabolic layers.) Then

$$(u - U, u - U) = \int_\Omega \mathscr{L}_\varepsilon^* \phi (u - U) \, d\Omega = B_\varepsilon(u - U, \phi) = B_\varepsilon(u - U, \phi - \phi_h) \quad \forall \phi_h \in S^h ,$$

and we obtain

$$\|u - U\|^2 \leq (\varepsilon^{1/2}|\phi - \phi_h|_1) + (\varepsilon^{-1/2}\|\phi - \phi_h\|)\ \|u - U\|_{1,\varepsilon}, \quad \forall \phi_h \in S^h .$$

Since $\inf_{\phi_h \in S^h} \|\phi - \phi_h\|_\nu \leq Ch^{2-\nu}|\phi|_2$, $\nu = 0, 1$, we finally get by use of (1) and (2),

$$\begin{aligned}\|u - U\| &\leq (\varepsilon^{1/2} h + C\varepsilon^{-1/2} h^2)\,\varepsilon^{-1/2} h^r |u|_{r+1} \\ &\leq Ch^{r+1}|u|_{r+1}, \quad r \leq k .\end{aligned}$$

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