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CONCENTRATED FORCES. ASYMPTOTIC STUDY (*)

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Abstract. — *In this work we study the asymptotic behaviour of the solution of the two-dimensional elasticity system. We consider Dirichlet data and the support of the body forces to be of order* ε *(a small parameter approaching zero). For the sake of simplicity, we study first the case of Laplace's equation. We determine the "outer" and the "inner" expansions and we perform their matching. All the terms are well-determined. By introducing a composite expansion, we then show the convergence of the asymptotic process as* ε *tends to zero. Finally, we generalize this results for elasticity's system.*

Résumé. — *Dans ce travail nous étudions le comportement asymptotique de la solution du système de l'élasticité bidimensionnel. Nous considérons des données de Dirichlet et le support des forces volumiques de l'ordre de* ε *(a petit paramètre). Pour plus de simplicité, nous développons l'étude dans le cas de l'équation de Laplace. L'étude asymptotique nous conduit à la détermination des développements extérieur et intérieur et à leur raccordement. Tous les termes sont bien déterminés. Grâce à l'introduction d'un développement composite, nous montrons alors la convergence du processus lorsque* ε *tend vers zéro. Enfin, nous généralisons ces résultats au cas du système de l'élasticité.*

0. INTRODUCTION

We study the asymptotic behaviour of the solutions of

$$\begin{cases} -\Delta u^{\varepsilon} = f^{\varepsilon} \text{ in } \Omega \subset \mathbb{R}^2 \\ u^{\varepsilon} = 0 \text{ on } \partial\Omega \end{cases}$$

where f^{ε} is not identically null only for x in a neighbourhood εD of the origin.

The asymptotic study exhibits the singular character of the solution at the origin : terms in $\log |x|$ and in $|x|^p$ with $p < 0$ show up. This study is in the

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general framework of matched asymptotic expansions (Eckhaus [2], Van Dyke [9]). There are "outer" and "inner" asymptotic expansions in the standard variable x far from the origin and in the "microscopic variable" $y = x/\varepsilon$ near the origin, respectively. All the terms of the expansions are well defined and completely characterized.

After having performed a matching of the "outer" and "inner" solutions we define a new expansion (see Van Dyke [9]) called the composite expansion, valid in the whole domain Ω, and which is of the form

$$u^{\varepsilon} = \varepsilon^2 [u_{\text{reg}}^2(x) + v^2(x/\varepsilon) + c \log \varepsilon - u_{\text{reg}}^2(0)] + \\ + \varepsilon^3 u_{\text{reg}}^3(x) + \varepsilon^4 u_{\text{reg}}^4(x) + \cdots$$

In this expression $u_{\text{reg}}^i(x)$ is the solution of Laplace's equation in Ω, with non homogeneous Dirichlet boundary conditions on $\partial\Omega$, $v^2(y)$ is a solution of an elliptic problem on $\mathbb{R}^2$, regular at the origin and having a constant behaviour at infinity.

All the terms of this expansion are rigorously justified by a convergence theorem. We emphasize this aspect because in general only in very particular cases the first term of the expansion is rigorously justified (Lions [4]).

In section 6 we generalize these results to the case where the force f^{ε} is the form

$$f^{\varepsilon} = \begin{cases} f(x/\varepsilon)\, \varepsilon^{-m}, & \text{for } \ x \in \varepsilon D \\ \Phi(x), & \text{for } \ x \notin \Omega \backslash \varepsilon \bar{D}, \quad m \in \mathbb{Z}. \end{cases}$$

Finally we generalize this problem to the elasticity operator.

A similar study was already considered in [7] and [10] for the case of an elastic two-dimensional body with a small hole.

Notation :

Vectors of the physical space $\mathbb{R}^2$ are written on the form $\underline{u} = (u^1, u^2)$. Upper indices denote terms in an asymptotic expansion, that is

$u^{\varepsilon} = u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \cdots$
n = unit normal to a curve
$|x|$, θ - polar coordinates
δ - Dirac's distribution
$\partial_\alpha = \partial^\alpha / \partial x^\alpha$, $\alpha = (\alpha_1, \alpha_2)$, $|\alpha| = \alpha_1 + \alpha_2$
$\partial_i = \partial_i / \partial x^i$ or $\partial_i / \partial y^i$
δ_{ij} - Kronecker symbol
$[|\cdot|]$ - jump of the enclosed quantities

1. SETTING OF THE PROBLEM

Let Ω be a bounded domain of $\mathbb{R}^2$ containing the origin. Let also D be a bounded domain of the auxiliar space $\mathbb{R}^2$ of variable $y = (y_1, y_2)$ containing also the origin. We denote by $\partial\Omega$ and by Γ the boundaries of Ω and of D, respectively, which we assume to be sufficiently regular.

For ε small enough we have the sheme (*fig.* 1*a*)

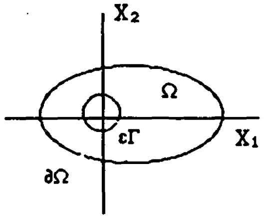

Figure 1a.

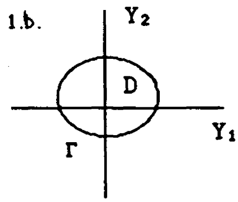

Figure 1b.

and we consider in Ω the following problem

$$-\Delta u^\varepsilon = f^\varepsilon \quad \text{in} \quad \Omega \tag{1.1}$$

$$u^\varepsilon = 0 \quad \text{on} \quad \partial\Omega \tag{1.2}$$

where

$$f^\varepsilon = \begin{cases} f(x/\varepsilon)\,, & \text{for} \quad x \in \varepsilon D \\ 0\,, & \text{for} \quad x \in \Omega \backslash \varepsilon \bar{D} \end{cases}$$

with $f \in L^2(\Omega)$ given.

Problem (1.1)-(1.2) is well posed and possesses a unique solution $u^\varepsilon \in H_0^1(\Omega)$. Our purpose is to study the asymptotic behaviour of u^ε as ε becomes small.

2. ASYMPTOTIC EXPANSIONS

It is know ([6] chap. VI. 14) that in the distributional sense f^ε has an expansion, of the form

$$\left(\int_D f(y)\,dy\right)\delta\varepsilon^2 + \cdots + \tag{2.1}$$

$$+ \sum_{|\alpha| = n-2} (-1)^{|\alpha|}(\alpha!)^{-1}\left(\int_D y^\alpha f(y)\,dy\right)\partial_\alpha\,\delta\varepsilon^n + \cdots .$$

This suggests to look for an asymptotic expansion of the solution of the form

$$u^{\varepsilon} = \mu_1(\varepsilon)\, u^1(x) + \varepsilon^2\, u^2(x) + \varepsilon^3\, u^3(x) + \cdots \tag{2.2}$$

where $\mu_1(\varepsilon)$ is unknown at this stage.

Substituting (2.2) into (1.1)-(1.2) we obtain for u^1

$$\begin{cases} -\Delta u^1 = 0 & \text{in } \Omega \\ u^1 = 0 & \text{on } \partial\Omega, \end{cases}$$

from which $u^1 = 0$.

Consequently, expansion (2.2) is really of the form

$$\varepsilon^2\, u^2 + \varepsilon^3\, u^3(x) + \cdots .$$

<u>For $i = 2$,</u> we have

$$-\Delta u^2 = \left(\int_D f(y)\, dy\right)\delta \quad \text{in } \Omega \tag{2.3}$$

$$u^2 = 0 \quad \text{on } \partial\Omega . \tag{2.4}$$

Let $E^2 = (2\pi)^{-1} \log |x|$ be the fundamental solution for Laplace's operator in $\mathbb{R}^2$. Then, the solution of (2.3)-(2.4) is of the form

$$u^2 = c'\, E^2(x) + u^2_{\text{reg}}(x)$$

where u^2_{reg} is the unique solution, regular at the origin, of the following problem,

$$-\Delta u = 0 \quad \text{in } \Omega \tag{2.5}$$

$$u = -c'\, E^2 \quad \text{on } \partial\Omega \tag{2.6}$$

with $c' = -\int_D f(y)\, dy$.

<u>For $i = 3$</u>, we have, taking into account that $-\Delta\, \partial_\alpha E^2 = \partial_\alpha \delta$,

$$-\Delta u^3 = -\sum_{|\alpha| = 1} \left(\int_D y^\alpha f(y)\, dy\right) \partial_\alpha \delta \quad \text{in } \Omega \tag{2.7}$$

$$u^3 = 0 \quad \text{on } \partial\Omega , \tag{2.8}$$

whose solution is of the form

$$u^3 = u^3_{\text{sing}} + u^3_{\text{reg}} ,$$

where

$$u^3_{\text{sing}} = + \sum_{|\alpha| = 1} \left(\int_D y^\alpha f(y)\, dy \right) \partial_\alpha E$$

and where u^3_{sing} is the unique solution, regular at the origin, of

$$\begin{cases} - \Delta u = 0 & \text{in } \Omega \\ u = - u^3_{\text{sing}} & \text{on } \partial\Omega . \end{cases}$$

More generally, u^i_{sing} is

(2.9)

$$u^i_{\text{sing}} = - (2\pi)^{-1} \sum_{|\alpha| = i-2} (-1)^{|\alpha|} (\alpha !)^{-1} \left(\int_D y^\alpha f(y)\, dy \right) \partial_\alpha \log |x| .$$

We see the Dirac distribution, at the origin, together with its derivates, showing up. Expansion (2.2) will be called the outer expansion and will be valid in $\Omega \backslash (0)$.

Let us remark that in $\Omega \backslash (0)$, u^i_{sing} has the form

$$u^i_{\text{sing}} = |x|^{-2(i-2)} P_{(i-2)} (x_1, x_2) , \tag{2.10}$$

where P_λ is an homogeneous polynome of degree λ.

In polar coordinates $(|x|, \theta)$, $(x_1 = |x| \cos \theta\ ,\ x_2 = |x| \sin \theta)$, (2.10) takes the form :

$$u^i_{\text{sing}} = F^i(\theta) |x|^{-(i-2)} , \quad i > 2 .$$

As in [7], in the sequel, we shall write this expression as

$$u^i_{\text{sing}} = u^{i, -(i-2)} (\theta) |x|^{-(i-2)} \tag{2.11}$$

Since (2.2) is singular at the origin, we now look for another expansion valid on its neighbourhood (inner expansion). We shall perform their matching (see Eckhaus [2]), afterwards.

In order to study the behaviour in the neighbourhood of the origin we consider another variable, $y = x/\varepsilon$, called the inner variable, and we seek for an inner expansion of the form

$$u^\varepsilon = \mu_1(\varepsilon)\, v^1(y) + \varepsilon^2 v^2(y) + \varepsilon^3 v^3(y) + \cdots . \tag{2.12}$$

Performing in (1.1)-(1.2) the change of variable $y = x/\varepsilon$ we get

$$- \Delta_y u^\varepsilon = \varepsilon^2 f^2(y) \quad \text{in } D \tag{2.13}$$

$$-\Delta_y u^\varepsilon = 0 \quad \text{in} \quad \varepsilon^{-1}\Omega\backslash\bar{D} \tag{2.14}$$

$$[|u^\varepsilon|] = 0\,;\ [|\partial_n u^\varepsilon|] = 0 \quad \text{on} \quad \Gamma \tag{2.15}$$

$$u^\varepsilon = 0 \quad \text{on} \quad \partial(\varepsilon^{-1}\Omega)\,. \tag{2.16}$$

Substituting (2.12) into (2.13)-(2.16) we obtain, for $i \neq 2$,

$$-\Delta v^i = 0 \quad \text{in} \quad \mathbb{R}^2\,. \tag{2.17}$$

For $i = 2$, we have

$$-\Delta v^2 = f \quad \text{in} \quad D \tag{2.18}$$

$$-\Delta v^2 = 0 \quad \text{in} \quad \mathbb{R}^2\backslash\bar{D} \tag{2.19}$$

$$[|v^2|] = 0\,; \quad [|\partial_n v^2|] = 0 \quad \text{on} \quad \Gamma\,. \tag{2.20}$$

3. MATCHING

In the previous section we saw that all the terms of the outer expansion may be evaluated. For the inner expansion we just established the equations the terms should satisfy.

We remark that some terms of the inner expansion will be completly determined by matching.

We shall use the technique of the intermediate variable which has already been used in [7].

The outer expansion contains both singular and regular terms. The singularities are in log $|x|$ and in $|x|^{-p}, p > 0$. The regular terms behave as $|x|^q$, $q > 0$, at the origin. Consequently, in order to perform the matching, the inner expansion may be "singular" at infinity, with terms in log $|y|$ and in $|y|^m$ $(m > 0)$.

Therefore, we are going to look for v^i of the form

$$v^i(y) = v^i_{\text{reg}}(y) + v^i_{\text{sing}}(y)\,, \tag{3.1}$$

where, for $|y|$ sufficiently large, v^i_{sing} and v^i_{reg} are given by

$$v^i_{\text{sing}} = \sum_{k=1}^{+\infty} v^{i,k}(\theta)|y|^k + c^i \log|y| \tag{3.2}$$

and

$$v^i_{\text{reg}} = \sum_{k=-\infty}^{0} v^{i,k}(\theta)|y|^k\,. \tag{3.3}$$

We shall write now $u^i(x)$ for $|x|$ small.

As u^i_{reg} is a C^∞ function in the neighbourhood of the origin, we may use its Taylor expansion for $|x|$ small.

Taking into account the homogeneous terms we obtain

$$u^i_{\text{reg}} = \sum_{k=0}^{+\infty} u^{i,k}(\theta)|x|^k, \; i \geqslant 2, \; |x| \text{ small}. \tag{3.4}$$

Considering the singular part of u^i, (2.9) and (2.10), we may write

$$u^2 = c \log |x| + \sum_{k=-\infty}^{+\infty} u^{i,k}(\theta)|x|^k \tag{3.5}$$

$$u^i = \sum_{k=-\infty}^{+\infty} u^{i,k}(\theta)|x|^k, \quad i > 2 \tag{3.6}$$

where the sum has at most one negative term not identically zero.

Let us now introduce the new variable z defined by

$$z = x\varepsilon^{-\beta}\,; \quad z = y\varepsilon^{1-\beta}, \quad 0 < \beta < 1 .$$

This intermediate variable rule is based on the principle that the two expansions have to coincide in a region where $|z| = 0(1)$ (i.e. $|x|$ small and $|y|$ large). By considering the outer and inner expansions in the variable z we obtain, respectively

$$\begin{aligned} u^\varepsilon &= \varepsilon^2\left[c \log |z\varepsilon^\beta| + \sum_{k=-\infty}^{+\infty} u^{2,k}(\theta)|z\varepsilon^\beta|^k\right] \\ &\quad + \sum_{i=2}^{+\infty} \varepsilon^i\left[\sum_{k=-\infty}^{+\infty} u^{i,k}(\theta)|z\varepsilon^\beta|^k\right] \\ &= c\beta \log \varepsilon\varepsilon^2 + \varepsilon^2 c \log |z| + \sum_{i=2}^{+\infty}\sum_{k=-\infty}^{+\infty} u^{i,k}(\theta)\,|z|^k\,\varepsilon^{\beta k + i} \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} u^\varepsilon = v(z\varepsilon^{\beta-1})\,\mu_1(\varepsilon) + \sum_{i=2}^{+\infty} (c^i \log |z|\, \varepsilon^i + (\beta - 1)\, c^i \log \varepsilon\varepsilon^i) + \\ + \sum_{i=2}^{+\infty}\sum_{k=-\infty}^{+\infty} v^{i,k}(\theta)|z|^k\,\varepsilon^{\beta k - k + i}\,. \end{aligned} \tag{3.8}$$

Identifying (3.7) and (3.8) we see that

$$c^i = 0 \quad \text{for} \quad i \neq 2 \tag{3.9}$$

$$c^2 = c\,. \tag{3.10}$$

Then choosing

$$\mu_1(\varepsilon) = \varepsilon^2 \log \varepsilon ,$$

we obtain by identifying the coefficients of the term $\varepsilon^2 \log \varepsilon$,

$$v^1(y) = c .$$

It only remains to identify $\sum_{i=2}^{+\infty} \sum_{k=-\infty}^{+\infty} u^{i,k}(\theta)|z|^k \varepsilon^{\beta k + i}$ and $\sum_{i=2}^{+\infty} \sum_{k=-\infty}^{+\infty} v^{i,k}(\theta)|z|^k \varepsilon^{\beta k - k + i}$ for all $|z|$, θ and ε.

Identifying the coefficients of the terms with the same powers of $|z|$ and ε we obtain

$$u^{i,k}(\theta) = v^{i+k,k}(\theta) . \tag{3.11}$$

Remark 3.1 : The regular part of v^i, denoted by v^i_{reg}, is reduced to a constant except for $i = 2$. In this case we have

$$v^2_{\text{reg}} = \sum_{k=-\infty}^{0} u^{2-k,k}(\theta)|y|^k . \quad \blacksquare \tag{3.12}$$

Remark 3.2 : The singular part of v^i $(i > 2)$

$$v^i_{\text{sing}} = u^{i-1,1}(\theta)|y| + u^{i-2,2}(\theta)|y|^2 + \cdots + u^{2,i-2}(\theta)|y|^{i-2} , \tag{3.13}$$

is a solution of $-\Delta u = 0$ in $\mathbb{R}^2$, because

$$u^i = u^{i,-(i-2)}(\theta)|x|^{-(i-2)} + \sum_{k=0}^{\infty} u^{i,k}(\theta)|x|^k$$

is a solution of $-\Delta u = 0$ in $\{x : 0 < |x| < \nu\}$. As each term of u^i is homogeneous of order k, Δu is a sum of homogeneous functions of order $k-2$. We shall then have $-\Delta(u^{i,k}(\theta)|x|^k) = 0$ in $\{x : 0 < |x| < \nu\}$ and of course, $-\Delta(u^{i,k}(\theta)|y|^k) = 0$ in $\mathbb{R}^2$ for $k > 0$. $\blacksquare$

Conclusion : For $i > 2$, the matching gives

$$v^i = u^i_{\text{reg}}(0) + u^{i-1,1}(\theta)|y| + \cdots + u^{2,i-2}(\theta)|y|^{i-2} \tag{3.14}$$

which is a solution of $-\Delta u = 0$ in $\mathbb{R}^2$ from which we conclude that (2.17) holds.

On the other hand, for $i = 2$, the matching gives v^2 but only for sufficiently large y,

$$v^2 = c \log |y| + u^2_{\text{reg}}(0) + \sum_{k=-\infty}^{-1} u^{2-k,k}(\theta)|y|^k , \quad (|y| \text{ large}) . \tag{3.15}$$

This expression is not valid at the origin, consequently we have to find v^2, solution of (2.18)-(2.20), with the form (3.15), for sufficiently large y.

4. CALCULATION OF v^2

We are going to define v^2 as a solution of a variational problem.

As the singular part of v^2 (at infinity) behaves like $c \log |y|$, we are going to look for v^2 in the form

$$v^2 = \begin{cases} v & \text{in } D \\ c \log |y| + w(y) & \text{in } \mathbb{R}^2 \backslash \bar{D}, \end{cases} \tag{4.1}$$

where $w(y)$ is regular at infinity, that is, $w(y) \to \text{Cst.}$ for $|y| \to +\infty$.

Substituting (4.1) into (2.18)-(2.20) we obtain

$$-\Delta v = f \quad \text{in} \quad D \tag{4.2}$$

$$-\Delta w = 0 \quad \text{in} \quad \mathbb{R}^2/\bar{D} \tag{4.3}$$

$$v_+ = (c \log |y| + w)_- \; ; \; \partial_n v_+ = \partial_n (c \log |y| + w)_- \quad \text{sur } \Gamma \tag{4.4}$$

$$w(y) \to c^*, \quad \text{for} \quad |y| \to +\infty . \tag{4.5}$$

Denoting by φ a given function in $H^{1/2}(\Gamma)$, let us consider the following problem

$$-\Delta w = 0 \quad \text{in} \quad \mathbb{R}^2 \backslash \bar{D} \tag{4.6}$$

$$w = \varphi \quad \text{on} \quad \Gamma \tag{4.7}$$

$$w(y) \to c^*, \quad \text{for} \quad |y| \to +\infty \tag{4.8}$$

where constant c^* is related to the solution fo Laplace's equation, in an outer domain, using Kelvin's transformations.

Let w^{φ} be the solution of (4.6)-(4.8) and

$$T\varphi = -\partial_n w^{\varphi},$$

then we have

THEOREM 4.1 : *Problem* (4.2)-(4.5) *has the following variational formulation : find* $v \in H^1(D)$ *such that :*

$$\begin{aligned} & \int_D \partial_i v \, \partial_i z \, dy + \langle T(v|_\Gamma), z|_\Gamma \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} - \\ & - \langle T(c \log |y| \, |_\Gamma), z|_\Gamma \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} - \\ & - \int_\Gamma \partial_n (c \log |y|)|_\Gamma \, z|_\Gamma \, d\Gamma = \int_D fz \, dy, \quad \forall z \in H^1(D) . \quad \blacksquare \end{aligned} \tag{4.9}$$

Which may be written in the form

$$a(v,z)=F(z)\,,\quad \forall z\in H^1(D) \tag{4.10}$$

with

$$a(v,z)=\int_D \partial_i v\,\partial_i z\,dy+\left\langle T(v|_\Gamma), z|_\Gamma\right\rangle_{H^{-1/2},H^{1/2}}$$

and

$$F(z)=\int_D fz\,dy+\int_\Gamma \partial_n(c\log|y|)|_\Gamma\, z|_\Gamma\, d\Gamma+ + \left\langle T(c\log|y|\,|_\Gamma), z|_\Gamma\right\rangle_{H^{-1/2},H^{1/2}}\,.$$

As $\lambda=0$ is an eigenvalue of $a(\cdot\,,\,\cdot)$ ([6] Chap. IV.8), problem (4.10) will have a solution iff

$$F(z)=0\,,\quad \forall z\in\langle 0\rangle\,, \tag{4.11}$$

where $\langle 0\rangle$ denotes the eigenspace associated to the eigenvalue $\lambda=0$, and whose elements are constant functions.

From the definition of c we see that (4.11) holds.

Applying Theorem 1 of [3], bearing in mind (2.1) and the expression of u^i_{sing} we see that v^2 has the form (3.15).

5. CONVERGENCE

According to Van Dyke [9], there are situations in which it is possible to define an asymptotic expansion valid in the whole Ω, called the composite expansion.

Let us assume that it is possible to define "corrector function" h^i in such a way that

$$u^\varepsilon=\varepsilon^2[c\log|x|+u^2_{\text{reg}}(x)+h^2(x/\varepsilon)]+ + \varepsilon^3[u^3_{\text{reg}}(x)+u^3_{\text{sing}}(x)+h^3(x/\varepsilon)]+\cdots \tag{5.1}$$

is an asymptotic expansion of the solution of (1.1)-(1.2) in the whole Ω. That is, we assume that it is possible to correct the outer expansion in such a way that one obtains an expansion valid for all x in Ω.

By definition of inner expansion we calculate functions h^i and justify (5.1) by convergence results.

By definition of inner expansion function $v^2(y)$ is given by

$$v^2(y)=\text{inner lim}\,\frac{u^\varepsilon(x)-c\varepsilon^2\log\varepsilon}{\varepsilon^2}=\lim_{\substack{\varepsilon\to0\\ y\text{ fixed}}}\frac{u^\varepsilon(\varepsilon y)-c\varepsilon^2\log\varepsilon}{\varepsilon^2}\,.$$

Using for u^ε expression (5.1) and bearing in mind that

$$u^i_{\text{sing}}(\varepsilon y) = \varepsilon^{-(i-2)} |y|^{-2(i-2)} p_{(i-2)}(y_1, y_2) = \varepsilon^{-(i-2)} u^i_{\text{sing}}(y)$$

we obtain

$$h^2(y) = v^2(y) - c \log |y| - u^2_{\text{reg}}(0) - \sum_{i \geq 3} u^i_{\text{sing}}(y) .$$

Substituting $h^2(x/\varepsilon)$ into (5.1) we find

$$u^\varepsilon = \varepsilon^2 \left[u^2_{\text{reg}}(x) + v^2(x/\varepsilon) + c \log \varepsilon - u^2_{\text{reg}}(0) - \sum_{i \geq 3} u^i_{\text{sing}}(x/\varepsilon) \right] + \\ + \varepsilon^3 [u^3_{\text{reg}}(x) + u^3_{\text{sing}}(x) + h^3(x/\varepsilon)] + \cdots .$$

As

$$\varepsilon^2 \sum_{i \geq 3} u^i_{\text{sing}}(x/\varepsilon) = \sum_{i \geq 3} \varepsilon^i u^i_{\text{sing}}(x)$$

we get

$$(5.2) \quad u^\varepsilon = \varepsilon^2 [u^2_{\text{reg}}(x) + v^2(x/\varepsilon) + c \log \varepsilon - u^2_{\text{reg}}(0)] + \\ + \varepsilon^3 [u^3_{\text{reg}}(x) + h^3(y)] + \cdots .$$

Following the same process, once again, we have

$$v^3(y) = \lim_{\substack{\varepsilon \to 0 \\ y \text{ fixed}}} \frac{u^2_{\text{reg}}(\varepsilon y) - c\varepsilon^2 \log \varepsilon - \varepsilon^2 v^2(y)}{\varepsilon^3}$$

which becomes, taking into account (5.2),

$$v^3(y) = \lim_{\substack{\varepsilon \to 0 \\ y \text{ fixed}}} \left\{ \varepsilon^{-1} [u^2_{\text{reg}}(\varepsilon y) - u^2_{\text{reg}}(0)] + u^3_{\text{reg}}(\varepsilon y) + h^3(y) \right\} .$$

Considering now the Taylor expansion of u^2_{reg}, at the origin, we have

$$v^3(y) = \sum_{|\alpha| = 1} \partial_\alpha u^2_{\text{reg}}(0)\, y^\alpha + u^3_{\text{reg}}(0) + h^3(y) .$$

From (3.14) we get $h^3(y) = 0$. Going on with this process we obtain $h^i(y) = 0$ for $i \geq 3$. We then have

$$(5.3) \quad u^\varepsilon = \varepsilon^2 [u^2_{\text{reg}}(x) + v^2(x/\varepsilon) + c \log \varepsilon - u^2_{\text{reg}}(0)] + \varepsilon^3 u^3_{\text{reg}}(x) + \cdots .$$

Remark 5.1 : Relation (4.1) shows that the behaviour of $v^2(y)$, at the origin, is not logarithmic. Consequently function (5.3) is defined for all x in Ω. As v^2 is a solution of $-\Delta u = f$ in D, with $f \in L^2(D)$, we know ([1] page 336) that v^2 is continuous. Thus we may define its value at the origin. ■

Let now

$$z_2^\varepsilon(x) = u_{\text{reg}}^2(x) + v^2(x/\varepsilon) + c \log \varepsilon - u_{\text{reg}}^2(0) \tag{5.4}$$

$$z_i^\varepsilon(x) = u_{\text{reg}}^i(x) , \quad i \geqslant 3 \tag{5.5}$$

and u^ε be the unique solution of (1.1)-(1.2), we then have

LEMMA 5.1 :

(i) $u^\varepsilon \underset{\varepsilon \to 0}{\to} 0 \quad H^1(\Omega)$ - *strong*

(ii) $\varepsilon^{-1} U^\varepsilon \underset{\varepsilon \to 0}{\to} 0 \quad H^1(\Omega)$ - *weak* .

Proof: (i) From (1.1)-(1.2) we get

$$- \int_\Omega \Delta u^\varepsilon \, u^\varepsilon \, dx = \int_\Omega f^\varepsilon \, u^\varepsilon \, dx$$

from which we obtain

$$\begin{aligned} \int_\Omega |\partial_i u^\varepsilon|^2 \, dx &\leqslant \left(\int_\Omega |f^\varepsilon|^2 \, dx \right)^{1/2} \left(\int_\Omega |u^\varepsilon|^2 \, dx \right)^{1/2} \\ &= \varepsilon \left(\int_\Omega |f(y)|^2 \, dy \right)^{1/2} \|u^\varepsilon\|_{L^2(\Omega)} . \end{aligned}$$

that is

$$\|u^\varepsilon\|^2_{L^2(\Omega)} \leqslant \varepsilon c \, \|u^\varepsilon\|_{L^2(\Omega)} .$$

Using Poincaré's inequality it follows that

$$\|u^\varepsilon\|_{H^1(\Omega)} \leqslant c_2 \, \varepsilon . \tag{5.6}$$

(ii) We may then extract a subsequence of $\varepsilon^{-1} u^\varepsilon$, denoted by v^ε such that

$$v^\varepsilon \underset{\varepsilon \to 0}{\to} v^* \quad H^1(\Omega) \text{ - weak} . \tag{5.7}$$

Let w be an arbitrary function in $\mathscr{D}(\Omega)$, then from (1.1)-(1.2) we have

$$\int_\Omega \partial_i v^* \, \partial_i w \, dx = \varepsilon^{-1} \int_\Omega f^\varepsilon \, w \, dx .$$

Passing to the limit, as $\varepsilon \to 0$, and bearing in mind (5.7), we obtain for the first member

$$\int_\Omega \partial_i v^* \, \partial_i w \, dx .$$

Performing the change of variable $y = x/\varepsilon$ (using the Taylor expansion of $w \in \mathscr{D}(\Omega)$, at the origin), we see that the second member converges to zero.
By virtue of the density of $\mathscr{D}(\Omega)$ in $H_0^1(\Omega)$ we obtain

$$\int_\Omega \partial_i v^* \, \partial_i w \, dx = 0 , \quad \forall w \in H_0^1(\Omega)$$

from which we conclude that $v^* = 0$.

THEOREM 5.1 : *For* $p = 1, 2, \ldots$

$$w_p^\varepsilon \underset{\text{def}}{=} \frac{u^\varepsilon(x) - \sum_{i=2}^{p+1} z_i^\varepsilon(x)\, \varepsilon^i}{\varepsilon^{p+1}} \underset{\varepsilon \to 0}{\to} 0 \quad H^1(\Omega)\text{-}strong .$$

Proof: This is an immediate consequence of [5] remark 7.2 and of the following Lemma :

LEMMA 5.2 : *For* $p = 1, 2, \ldots$ w_p^ε *verifies*

$$\begin{cases} -\Delta w_p^\varepsilon = 0 & in \quad \Omega \\ w_p^\varepsilon = g_p^\varepsilon & on \quad \partial\Omega \end{cases}$$

with $g_p^\varepsilon \to 0$, $H^{1/2}(\partial\Omega)$-*strong, for* $\varepsilon \to 0$.

Proof: We have

$$\begin{aligned} -\Delta w_p^\varepsilon &= \varepsilon^{-(p+1)} f^\varepsilon(x) + \varepsilon^{2-(p+1)} \Delta z_2^\varepsilon(x) \\ &= \varepsilon^{-(p+1)} f^\varepsilon(x) - \varepsilon^{-(p+1)} f(x/\varepsilon) . \end{aligned}$$

From the definition of $f^\varepsilon(x)$ we conclude that $-\Delta w_p^\varepsilon = 0$.
On the boundary we obtain

$$w_p^\varepsilon|_{\partial\Omega} = -\left[u_{\text{reg}}^2|_{\partial\Omega} + v^2(x/\varepsilon)|_{x \in \partial\Omega} + c \log \varepsilon - u_{\text{reg}}^2(0)\right] \varepsilon^{2-(p+1)} - \\ - \sum_{i=3}^{p+1} z_i^\varepsilon|_{\partial\Omega}\, \varepsilon^{i-(p+1)} .$$

As for $x \in \partial\Omega$ and ε sufficiently small, x/ε is large, we get

$$v^2(x/\varepsilon) = u^2_{\text{reg}}(0) + c \log |x/\varepsilon| + \sum_{|\alpha| \geq 1} m_\alpha \, \partial_\alpha(\log |y| (x/\varepsilon)) .$$

On the other hand

$$z_i^\varepsilon \big|_{\partial\Omega} = - u^i_{\text{sing}} \big|_{\partial\Omega} = - \sum_{|\alpha| = i-2} m_\alpha \, \partial_\alpha(\log |x|)\big|_{x \in \partial\Omega} .$$

Therefore,

$$w_p^\varepsilon \big|_{\partial\Omega} = - \sum_{|\alpha| \geq p} m_\alpha \, \partial_\alpha(\log |x|)\big|_{x \in \partial\Omega} \, \varepsilon^{|\alpha| + 2 - (p-1)} \underset{\text{def.}}{=} g_p^\varepsilon .$$

<u>*Let us now proof that*</u> $g_p^\varepsilon \to 0$, $H^{1/2}(\partial\Omega)$ - strong, for $\varepsilon \to 0$.
We are going to construct a function $\Phi_p^\varepsilon \in H^1(\Omega)$ verifying :

1. $\Phi_p^\varepsilon \big|_{\partial\Omega} = g_p^\varepsilon$
2. $\|\Phi_p^\varepsilon\|_{H^1(\Omega)} \underset{\varepsilon \to 0}{\to} 0$.

For $a > 0$, let

$$v_a(\partial\Omega) = \{x \in \Omega : \text{dist } (x, \partial\Omega) < a\} .$$

We consider $\rho(x) \in C^\infty(\Omega)$ such that $0 \leq \rho(x) \leq 1$ and

$$\rho(x) = \begin{cases} 1 , & \text{for } \; x \in V_{\delta/2}(\partial\Omega) \\ 0 , & \text{for } \; x \in \Omega \backslash V_\delta(\partial\Omega) . \end{cases}$$

If δ and ε are sufficiently small and $x \in V_\delta(\partial\Omega)$ then x/ε is large. From [3] page 80 we see that we may define the u^{p-1} at point x/ε. Denoting it by φ_p^ε we have

$$\varphi_p^\varepsilon = \sum_{|\alpha| \geq p} m_\alpha \, \partial_\alpha(\log |y|)(x/\varepsilon)$$

with

$$|\varphi_p^\varepsilon| = O(\varepsilon^p) ; \quad |\partial_i \varphi_p^\varepsilon| = O(\varepsilon^p) .$$

Let us consider

$$\Phi_p^\varepsilon(x) = \varphi_p^\varepsilon(x) \, \rho(x) \, \varepsilon^{1-p} ,$$

which is well defined, and show that it verifies 1, and 2.

1. $$\Phi_p^\varepsilon|_{\partial\Omega} = \varepsilon^{1-p} \sum_{|\alpha| \geqslant p} m_\alpha \, \partial_\alpha(\log|x|)|_{x \in \partial\Omega} \, \varepsilon^{|\alpha|} = \sum_{|\alpha| \geqslant p} m_\alpha \, \partial_\alpha(\log|x|)|_{x \in \partial\Omega} \, \varepsilon^{|\alpha| - p + 1} = g_p^\varepsilon(x) .$$

2. $$\int_\Omega |\partial_i \Phi_p^\varepsilon|^2 \, dx = \int_{V_\delta(\partial\Omega)} |\partial_i \varphi_p^\varepsilon \, \rho(x) + \varphi_p^\varepsilon(x) \, \partial_i \rho(x)|^2 \, \varepsilon^{2-2p} \, dx \leqslant \left[\int_{V_\delta(\partial\Omega)} |\partial_i \varphi_p^\varepsilon|^2 \, dx + c \int_{V_\delta(\partial\Omega)} |\varphi_p^\varepsilon(x)|^2 \, dx \right] \varepsilon^{2-2p} .$$

The last inequality is a consequence of $0 \leqslant \rho(x) \leqslant 1$ and of $|\partial_i \rho|^2 \leqslant c$, $\forall x \in V_\delta(\partial\Omega)$. As $|\varphi_p^\varepsilon|$ and $|\partial_i \varphi_p^\varepsilon|$ are $O(\varepsilon^p)$ we have

$$\int_\Omega |\partial_i \Phi_p^\varepsilon|^2 \, dx \leqslant O(\varepsilon^{2p}) \, \varepsilon^{2-2p} .$$

From which we conclude that

$$\int_\Omega |\partial_i \Phi_p^\varepsilon|^2 \, dx \to 0 , \quad \text{for} \quad \varepsilon \to 0 .$$

We also obtain

$$\int_\Omega |\Phi_p^\varepsilon|^2 \, dx \leqslant \int_{V_\delta(\partial\Omega)} |\varphi_p^\varepsilon(x) \, \varepsilon^{1-p}|^2 \, dx \to 0 , \quad \text{for} \quad \varepsilon \to 0 .$$

Thus, we see that

$$\|\Phi_p^\varepsilon\|_{H^1(\Omega)} \underset{\varepsilon \to 0}{\to} 0 .$$

Finally, the result follows from the definition of $\|\cdot\|_{H^{1/2}(\partial\Omega)}$.

6. GENERALIZATIONS

The previous techniques may be applied to other problems. In this section we give some examples and a general idea of the calculations.

Example 1 : We consider the same problem as in (1.1)-(1.2) but with the force f^ε defined by

$$f^\varepsilon(x) = \begin{cases} f(x/\varepsilon) \, \varepsilon^{-m} , & \text{for} \quad x \in \varepsilon D \\ \Phi(x) , & \text{for} \quad x \notin \varepsilon D , \end{cases} \qquad m \in \mathbb{Z} , \tag{6.1}$$

where f and Φ are given functions belonging to $L^2(D)$ and $L^2(\Omega)$ respectively. We suppose that Φ is defined at the origin and that it is of class C^∞ in its neighbourhood.

We consider the following decomposition of f^ε

$$f^\varepsilon(x) = f^\varepsilon_\cdot(x) + \Phi(x) - \Phi^\varepsilon(x),$$

with

$$f^\varepsilon_\cdot = \begin{cases} f(x/\varepsilon)\,\varepsilon^{-m}, & \text{for} \quad x \in \varepsilon D \\ 0, & \text{for} \quad x \notin \varepsilon D, \end{cases}$$

and

$$\Phi^\varepsilon(x) = \begin{cases} \Phi(x), & \text{for} \quad x \in \varepsilon D \\ 0, & \text{for} \quad x \notin \varepsilon D. \end{cases}$$

From the previous example we see that it suffices to study the behaviour of the solution of

$$\begin{cases} -\Delta u^\varepsilon = \Phi^\varepsilon & \text{in} \quad \Omega \\ u^\varepsilon = 0 & \text{on} \quad \partial\Omega \end{cases}$$

as $\varepsilon \to 0$.

For this purpose we consider the "moment" expansion of Φ^ε

$$\int_D \Phi(0)\,dy\,\partial\varepsilon^2 + \cdots + \left[\int_D \int_{|\alpha|=0} (\alpha!)^{-1}\,\partial_\alpha\Phi(0)\,y^\alpha \times \right. \\ \left. \times \sum_{|\beta|=n-|\alpha|} (-1)^{|\beta|}(\beta!)^{-1}\,y^\beta\,dy\,\partial_\beta\delta\right]\varepsilon^{n+2} + \cdots \tag{6.1}$$

which is established in the same way as (2.1).

We can see from (6.1) that in this case all the terms of the outer expansion have a logarithmic term. This is only difference between this case and the previous one.

Example 2 : (Linear elasticity-isotropic homogeneous case)

Let a_{ijmn} be the elasticity coefficients in the isotropic case denoting the Lamé constants by λ and μ, they are of the form,

$$a_{ijmn} = \lambda\delta_{ij}\,\delta_{mn} + \mu(\delta_{im}\,\delta_{jn} + \delta_{in}\,\delta_{jm}).$$

Denoting by e_{ij} and σ_{ij} the strain and the stress tensor components, respectively, and by $\underline{u}$ this displacement vector, we have

$$e_{ij}(\underline{u}) = 1/2(\partial_j u_i + \partial_i u_j)\,; \quad \sigma_{ij}(\underline{u}) = a_{ijmn}\,e_{mn}(\underline{u}).$$

We consider the elasticity system

$$\begin{cases} -\partial_j \sigma_{ij}(\underline{u}^\varepsilon) = \underline{f}^\varepsilon & \text{in } \Omega \\ \underline{u}^\varepsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{6.2}$$

where

$$\underline{f}^\varepsilon(x) = \begin{cases} (f_1(x/\varepsilon), f_2(x/\varepsilon)), & \text{for } x \in \varepsilon D \\ (0,0) & \text{for } x \notin \varepsilon D, \end{cases}$$

and f_1, f_2 are given functions of $L^2(D)$.

For fixed ε problem (6.2) is well posed problem possesses a unique solution $\underline{u}^\varepsilon$. We are interested in the asymptotic behaviour of $\underline{u}^\varepsilon$ as $\varepsilon \to 0$.

From the fundamental solution of the elasticity operator in $\mathbb{R}^2$

$$\Gamma^{ij} = -[8\pi\mu(\lambda+2\mu)]^{-1} \begin{Bmatrix} (2\lambda+6\mu)\log|x| + (2\lambda+2\mu)|x|^{-2}x_2^2 & -2(\lambda+\mu)|x|^{-2}x_1x_2 \\ -2(\lambda+\mu)|x|^{-2}x_1x_2 & (2\lambda+6\mu)\log|x| + (2\lambda+2\mu)|x|^{-2}x_1^2 \end{Bmatrix}$$

we obtain

$$\underline{u}^\varepsilon = \varepsilon^2(u_1^2, u_2^2) + \varepsilon^3(u_1^3, u_2^3) + \cdots$$

with

$$u_i^n = u_{i\,\text{reg}}^n + u_{i\,\text{sing}}^n, \quad i = 1, 2$$

where

$$u_{i\,\text{sing}}^n = \sum_{|\alpha| = n-2} \sum_{j=1,2} \partial_\alpha \Gamma^{ij} C_j^\alpha, \quad (C_j^\alpha \text{ Cst.}) \quad i = 1, 2$$

and u_{reg}^n is the unique solution, regular at the origin, of

$$\begin{cases} -\partial_j \sigma_{ij}(\underline{u}) = 0 & \text{in } \Omega \\ \underline{u} = -u_{\text{sing}}^n & \text{on } \partial\Omega. \end{cases}$$

For the inner expansion we obtain

$$\underline{u}^\varepsilon = \varepsilon^2 \log \varepsilon\,(v_1^1(y), v_2^1(y)) + \varepsilon^2(v_1^2(y), v_2^2(y)) + \varepsilon^3(v_1^3(y), v_2^3(y)) + \cdots$$

where, for $k \neq 2$, $\underline{v}^k$ satisfies

$$-\partial_j(\sigma_{ij}^y(\underline{v}^k)) = 0 \quad \text{in } \mathbb{R}^2$$

and where for $k = 2$,

$$(6.3) \qquad - \partial_j(\sigma^y_{ij}(\underline{v}^2)) = 0 \quad \text{in} \quad D$$

$$(6.4) \qquad - \partial_j(\sigma^y_{ij}(\underline{v}^2)) = 0 \quad \text{in} \quad \mathbb{R}^2 \backslash \bar{D}$$

$$(6.5) \qquad [|v^2|] = 0 \, ; \quad [|\sigma_{ij}(\underline{v}^2) \, n_j|] = 0 \quad \text{in} \quad \Gamma \, .$$

For $k \neq 2$, the $\underline{v}^k(y)$ are completely defined by matching. On the other hand, in order to prove the existence of $\underline{v}^2$ satisfying the matching conditions, together with (6.3)-(6.5), we look for $\underline{v}^2$ in the form

$$(6.6) \qquad v_1^2 = \begin{cases} v_1(y) & \text{in} \quad D \\ z_1(y) + w_1(y) & \text{in} \quad \mathbb{R}^2 \backslash \bar{D} \, , \end{cases}$$

$$(6.7) \qquad v_2^2 = \begin{cases} v_2(y) & \text{in} \quad D \\ z_2(y) + w_2(y) & \text{in} \quad \mathbb{R}^2 / \bar{D} \, , \end{cases}$$

with

$$z_i(y) = d_i \log |y| + d_i^i \, y_i^2 |y|^{-2} + d_i^j \, y_i \, y_j |y|^{-2} + d_i^{ij} \, y_j^2 |y|^{-2}$$

and

$$(6.8) \quad d_i = - (2\lambda + 6\mu)[8 \pi\mu(\lambda + 2\mu)]^{-1} \int_D f_i(y) \, dy$$

$$d_i^{ij} = - (4\pi\mu)^{-1} \int_D f_i(y) \, dy$$

$$(6.9) \quad d_j^i = - (\lambda + \mu)[4 \pi\mu(\lambda + 2\mu)]^{-1} \int_D f_i(y) \, dy$$

$$d_i^i = - [4 \pi(\lambda + 2\mu)]^{-1} \int_D f_i(y) \, dy$$

$i, j \in [1, 2]$, and $i \neq j$.

Functions $w_i(y)$, which will be defined later, are assumed to be regular at infinity, in the sense that $\underline{w} \in \underline{W}(\mathbb{R}^2 \backslash \bar{D})$; the completion of $\mathcal{D}(\mathbb{R}^2 \backslash \bar{D})$ for the norm

$$\|u\|_{\underline{W}}^2 = \sum_{i,j} \int_{\mathbb{R}^2 \backslash \bar{D}} |\partial_j u^i|^2 \, dy$$

(see Tchatat, H. [8]).

By substituting (6.6)-(6.7) into (6.3)-(6.5) we obtain

$$(6.10) \qquad - \partial_j(\sigma^y_{ij}(v_1 \, v_2)) = f_i \quad \text{in} \quad D$$

(6.11) $\quad -\partial_j(\sigma^y_{ij}(w_1, w_2)) = 0 \quad \text{in} \quad \mathbb{R}^2 \backslash \bar{D}$

(6.12) $\quad v_{i+} = (z_i + w_i)_- \ ; \ \sigma_{ij}(v_1, v_2)\, n_{j+} = \sigma_{ij}(z_1 + w_1, z_2 + w_2)\, n_j \ \text{in} \ \Gamma$

(6.13) $\quad w_i$ regular at infinity .

We solve a problem in $\mathbb{R}^2 \backslash \bar{D}$ (see [8]) and we transform (6.10)-(6.13) into a variational problem posed in D. Using (6.8)-(6.9) we show that (6.10)-(6.13) has a solution of the form (6.8)-(6.9), unique up to a constant, determined by matching.

We then proced as in the previous cases. ■

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