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DEPENDENCE OF THE BUCKLING LOAD OF A NONSHALLOW ARCH WITH RESPECT TO THE SHAPE OF ITS MIDCURVE (*)

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Abstract. — *We are interested in a buckling model for a non shallow arch, in which the critical buckling load appears as the smallest generalized eigenvalue of a linear operator in a Hilbert space.*

The mechanical origin of the model is recalled.

Then we give a mathematical proof of existence of a smallest eigenvalue. Then we give a mathematical proof of differentiability of the smallest eigenvalue with respect to the shape of the midsurface. If it is simple, it is Fréchet differentiable, if not, it has a directional derivative which is regular enough to get necessary optimality conditions.

Finally we give an analytical formula for this derivative, and we explain how to compute it numerically despite the heaviness of the computations.

Résumé. — *Nous nous intéressons à un modèle de flambement pour une arche profonde, dans lequel la valeur critique de flambement apparaît comme la plus petite valeur propre d'un opérateur linéaire dans un espace de Hilbert.*

On rappelle tout d'abord l'origine mécanique du modèle.

Ensuite, une démonstration mathématique de l'existence d'une plus petite valeur propre est donnée. Puis on démontre mathématiquement un résultat de différentiabilité de cette plus petite valeur propre par rapport à la forme de la surface moyenne : si elle est simple, elle est Fréchet-différentiable, sinon, elle possède une dérivée directionnelle qui est suffisamment régulière pour donner des conditions nécessaires d'optimalité.

Enfin, on donne une formule explicite pour cette dérivée, et on explique comment en faire un calcul numérique, malgré la lourdeur des calculs.

INTRODUCTION

The strength of an arch depends on its shape, that is the shape of its midcurve and its cross section. As a matter of fact, the dependence on the

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midcurve is much more complicate than the cross section. This is what we are interested in here. The cross section could be added easily.

In reference [1], we have studied the differentiability dependence of the static response of an arch with respect to the shape of its midcurve.

In this paper, we study the dependence of the buckling load of a non-shallow arch with respect to the shape of the midcurve. The model we consider is the simple Eulerian buckling one : the prebuckled state comes from the linear model, the buckling equation is an eigenvalue equation.

It is now well known that in optimization problems involving eigenvalues, these may be repeated at the optimum [2]. A general account of perturbations theory of eigen-elements may be found in [3] ; these general results may be used to obtain semi-differentiability of repeated eigenvalues in shape optimization [4] ; related results may be found in [5] and [6].

In reference [7], we have proved a general result of dependence of eigenvalues and eigenvectors of a linear operator with respect to a functional parameter. We have used this result to study the dependence of the buckling load of a beam with respect to its thickness. In this paper, we use this general result for the arch problem.

The eigenvalue problem is self-adjoint. It has a compactness property, like in the beam case. But the dependence on the midcurve is much more complicated that the dependence on the thickness in the beam problem. Moreover, the buckling equation depends on the prebuckled state, which depends on the midcurve shape.

In paragraph I, we briefly recall the origin of the buckling model we work on.

In paragraph II, we describe the set of eigenvalues of the problem, and prove that there is a smaller one in modulus.

In paragraph III, we study the differentiability. Paragraph III.1 recalls the results of reference [7] we use. Then, in paragraph III.2 we prove the directional differentiability of the eigenvalue if it is multiple, its Fréchet differentiability if it is simple. Then we show how to lead a numerical computation of this derivative.

NOTATIONS

$$V = H_0^1(]0,1[) \times [H_0^1(]0,1[) \cap H^2(]0,1[)]$$

$$W = W^{3,\infty}(]0,1[)$$

$$\varphi \in W^{3,\infty}(]0,1[) \qquad \varphi^\bullet = \frac{d}{dx}\varphi(x)$$

$$S = S(\varphi) = [1 + \varphi^{\bullet 2}]^{1/2}$$

$$\frac{1}{R} = \frac{1}{R(\varphi)} = -\frac{\varphi^{\bullet\bullet}}{S^3}$$

$$\varepsilon(\varphi, y) = \frac{1}{S} y_1^{\bullet} + \frac{1}{R} y_2 = \varepsilon(y)$$

$$\theta(\varphi, y) = \frac{1}{R} y_1 - \frac{1}{S} y_2^{\bullet} = \theta(y)$$

$$K(\varphi, y) = \frac{1}{S} [\theta(\varphi, y)]^{\bullet} = K(y)$$

$$a(u, v) = a(\varphi ; u, v) = \int_0^1 [C \varepsilon(u) \varepsilon(v) + DK(u) K(v)] S(x)\, dx$$

$$b(v ; y, z) = b(\varphi, v ; y, z) = \int_0^1 C [\varepsilon(v) \theta(y) \theta(z) + \theta(v) \varepsilon(y) \theta(z) + \theta(v) \theta(y) \varepsilon(z)](x) S(x)\, dx$$

$$\bar{b}(\varphi ; y, z) = b(\varphi, u_\varphi ; y, z) = \langle \bar{B}(\varphi) . y, z \rangle = \langle B(\varphi, u_\varphi) . y, z \rangle$$

$$S' = \frac{dS}{d\varphi}(\varphi) . \psi = S'(\varphi) . \psi$$

$$S(\varphi + \psi) = S(\varphi) + S'(\varphi) . \psi + \delta_\psi^2 S(\varphi) .$$

I. RECALL OF THE BUCKLING MODEL

An elastic system is in a stable equilibrium position for a displacement field which minimizes its energy.

Let V be the space of admissible displacements and $\Pi(v)$ the energy of the system for a displacement field $v \in V$. A displacement field u is a stable equilibrium if :

$$\Pi(u + v) > \Pi(u) \quad \forall v \in V, \quad v \neq 0 .$$

We suppose that this energy depends smoothly on v, and we expand it with the Taylor formula :

$$\Pi(u + v) = \Pi(u) + d\Pi(u) . v + \langle d^2\Pi(u) . v, v \rangle + o(\|v\|^2)$$

where :

$\langle . , . \rangle$ is the inner product of V .

A necessary condition for u to be a minimum is :

$$d\Pi(u) . v = 0 \quad \forall v \in V$$

which is the classical Euler equilibrium equation.

If for a solution u of this equation the quadratic form $v \mapsto \langle d^2\Pi(u) . v, v \rangle$ is positive definite, then u is effectively a (local) minimum of Π, and the equilibrium is likely to be stable.

The system is now submitted to a loading λf, where f is a given reference loading, and $\lambda \in \mathbb{R}$. When λ is close enough to 0, we suppose that the equilibrium equation has one and only one solution u_λ. When $|\lambda|$ increases, as long as the quadratic form $\langle d^2\Pi(u_\lambda) . v, v \rangle$ keeps positive definite, u_λ is a stable equilibrium. But, if there exists a λ^* with smallest modulus such that the quadratic form stops being positive definite, unless the second and third order terms in the expansion of $\Pi(u_{\lambda^*} + v)$ are zero, u_{λ^*} is no more a stable equilibrium position. λ^* is the buckling load.

Generally, buckling problems are bifurcation problems.

In this work, we use the classical model in which approximations are done which make λ^* be an eigenvalue of a linear operator. These approximations are of two kinds :

1. The energy functional is chosen as a cubic functional, so that its second derivative is a first degree polynomial in u.
2. The Euler equilibrium equation is linearized. Its solution depends linearly on λ :

$$u_\lambda = \lambda u$$

where u is the displacement field for the loading f in the linear model.

Let us now come to the arch problem.

Let ω be a plane curve, graph of $y = \varphi(x)$, $x \in [0, 1]$. φ is supposed to be of class $W^{3,\infty}$ (3 times differentiable in the sense of distributions, with a bounded 3rd derivative). ω is the midcurve of an arch with thickness h. We suppose that it is attached at $x = 0$ and $x = 1$ but not clamped (for instance). This displacement field $\vec{v}(x)$ of a point of abscissa x of the loaded arch is decomposed on the basis $(\vec{t}(x), \vec{n}(x))$ of the unit tangent and normal vectors :

$$\vec{t}(x) = \frac{1}{S(x)} \begin{bmatrix} 1 \\ \varphi^\bullet(x) \end{bmatrix} \quad \vec{n}(x) = \frac{1}{S(x)} \begin{bmatrix} \varphi^\bullet(x) \\ -1 \end{bmatrix}$$

where

$$\varphi^\bullet(x) = \frac{d\varphi(x)}{dx} \quad S(x) = (1 + \varphi^{\bullet 2}(x))^{1/2} .$$

We have

$$\vec{v}(x) = v_1(x)\, \vec{t}(x) + v_2(x)\, \vec{n}(x)$$

and we denote

$$v = (v_1, v_2) .$$

For the loading f we denote $L(v)$ the virtual work of the exterior forces, in the displacement $\vec{v}$. This virtual work depends linearly on the loading. So, if the loading is λf, the virtual work in the virtual displacement is $\lambda L(v)$.

The energy functional we consider at a first step is the following :

$$\bar{\Pi}(v) = 1/2\, \bar{a}(v, v) - \lambda L(v)$$

where :

$$\bar{a}(u, v) = \int_0^1 [C \bar{\varepsilon}(u) \bar{\varepsilon}(v) + DK(u) K(v)](x) S(x)\, dx$$

with :

$C = Eh$ (E is the Young modulus of the material)

$D = E \dfrac{h^3}{12}$

$\dfrac{1}{R} = -\dfrac{\varphi^{\bullet\bullet}}{S^3}$ is the curvature of ω

$\bar{\varepsilon}(u) = \varepsilon(u) + \dfrac{\theta^2(u)}{2}$

$\varepsilon(u) = \dfrac{1}{S} u_1^\bullet + \dfrac{1}{R} u_2$ (membrane strain)

$\theta(u) = \dfrac{1}{R} u_1 - \dfrac{1}{S} u_2^\bullet$ (rotation of the normal)

$K(u) = \dfrac{1}{S} \left(\dfrac{1}{R} u_1 - \dfrac{1}{S} u_2^\bullet \right)^\bullet$ (bending strain).

This functional is of fourth degree in v. We approximate $\bar{a}(v, v)$ by its 3rd degree approximation $\tilde{a}(v)$, and get the energy functional :

$$\Pi(v) = \frac{1}{2} \tilde{a}(v) - \lambda L(v).$$

The linearized equilibrium equation is then :

$$\text{(I.1)} \qquad a(u_\lambda, v) = \lambda L(v) \quad \forall v \in V$$

where

$$\text{(I.2)} \qquad a(u, v) = \int_0^1 [C \varepsilon(u) \varepsilon(v) + DK(u) K(v)](x) S(x)\, dx.$$

We notice that $u_\lambda = \lambda u$, where u is the solution of :

$$\text{(I.1')} \qquad a(u, v) = L(v) \quad \forall v \in V.$$

Then, we have to study the second order terms in $\Pi(v)$. The calculation gives :

$$\begin{aligned}\langle d^2\Pi(u_\lambda)\, v, v\rangle &= \frac{1}{2}\,[a(v,v) + b(u_\lambda\,; v, v)]\\ &= \frac{1}{2}\,[a(v,v) + \lambda b(u\,; v, v)]\end{aligned}$$

where a is defined before, u is the solution of (I.1′), and b is defined by :

$$\forall v, y, z \in V: \quad b(v\,; y, z) = \int_0^1 C\,[\varepsilon(v)\,\theta(y)\,\theta(z) + \theta(v)\,\varepsilon(y)\,\theta(z) + \\ + \theta(v)\,\theta(y)\,\varepsilon(z)](x)\,S(x)\,dx$$

(which is linear in v, y, z).

The buckling load is the number λ^* of smallest modulus (if it exists) such that there exists $y \in V$, $y \neq 0$ such that :

$$\langle d^2\Pi(\lambda^* u)\,.\,y, y\rangle \leqslant 0\,.$$

We have the following property :

LEMMA I.1 : *If* (λ^*, y) *is such a pair,* λ^* *is the real number of smallest modulus such that there exists* $y \in V$, $y \neq 0$ *such that :*

$$d^2\Pi(\lambda^* u)\,.\,y = 0\,.$$

Proof: The linear operator $d^2\Pi(\lambda u)$ depends on λ in a continuous manner. By definition of λ^*, we know that :

$$\left.\begin{array}{l}\forall z \in V, z \neq 0\\ \forall\,|\lambda| < \lambda^*\end{array}\right\} \quad \langle d^2\Pi(\lambda u)\,.\,z, z\rangle > 0\,.$$

So :

$$\forall z \in V \qquad \langle d^2\Pi(\lambda^* u)\,.\,z, z\rangle \geqslant 0\,.$$

On the other hand, by definition of y, we have :

$$\langle d^2\Pi(\lambda^* u)\,.\,y, y\rangle \leqslant 0$$

so :

$$\langle d^2\Pi(\lambda^* u)\,.\,y, y\rangle = 0\,.$$

Then, y is the minimum of the mapping $z \mapsto \langle d^2\Pi(\lambda^* u)\,.\,z, z\rangle$ which is differentiable. So :

$$\frac{d}{dz}\,\langle d^2\Pi(\lambda^* u)\,.\,z, z\rangle\,\big|_{z=y} = 0$$

which is

$$\langle d^2\Pi(\lambda^* u) \,.\, y, z\rangle = 0 \quad \forall z \in V$$

or :
$$d^2\Pi(\lambda^* u) \,.\, y = 0 \,.$$

This proves lemma I.1. ■

COROLLARY I.2 : *Let a be the bilinear form defined in* (I.2) *and b defined in* (I.3). λ^* *is the buckling load of the arch if and only if it is the real number of smallest modulus such that there exists* $y \in V$, $y \neq 0$, *such that :*

$$a(y, z) = \lambda^* \, b(u \,; y, z) \quad \forall z \in V$$

where u is the solution of the equilibrium equation (I.1').

This is the direct application of lemma I.1 to the arch energy functional. (Notice that it should be $a(y, z) = -\lambda^* \, b(u \,; y, z)$ but these 2 problems have the same set of eigenvalues). ■

Now, to end this modelization paragraph, let us precise the space V of admissible displacements.

The linear equilibrium equation is classically posed in the space :

$$V = H_0^1(]0, 1[) \times [H_0^1(]0, 1[) \cap H^2(]0, 1[)] \,.$$

It is known that it has one and only one solution (ref. [8]). The bilinear form a is symmetric, continuous coercive on $V \times V$.

On the other side, for any $v \in V$, the trilinear form $b(v \,; y, z)$ is well defined on $V \times V \times V$. This is because :

$$\begin{aligned} v \in V &\Rightarrow \varepsilon(v) \in L^2 \\ y \in V &\Rightarrow \theta(y) \in H^1 \,. \end{aligned}$$

But we know that $H^1(]0, 1[) \subset L^\infty(]0, 1[)$. So :

$$\left.\begin{aligned} \theta(y) \in L^\infty \\ \varepsilon(v) \in L^2 \end{aligned}\right\} \Rightarrow \theta(y)\, \varepsilon(v) \in L^2$$

$$\left.\begin{aligned} \theta(z) \in L^2 \\ S \in L^\infty \end{aligned}\right\} \Rightarrow S\theta(z) \in L^2 \,.$$

This proves that

$$\int_0^1 [\varepsilon(v)\, \theta(y)\, \theta(z)\, S](x)\, dx = \langle \theta(y)\, \varepsilon(v), S\theta(z)\rangle_{L^2, L^2}$$

is well defined. The other terms of b behave the same.

So, the buckling problem is :

Find $\lambda \in \mathbb{R}$ of smallest modulus such that there exists $y \in V$, $y \neq 0$ s.t. :

$$a(y, z) = \lambda b(u ; y, z) \quad \forall z \in V$$

where

- $a(y, z) = \int_0^1 [C\varepsilon(y)\,\varepsilon(z) + DK(y)\,K(z)](x)\,S(x)\,dx$
- $u \in V$ is the solution of $a(u, v) = L(v) \quad \forall v \in V$
- $b(u ; y, z) = C \int_0^1 [\varepsilon(u)\,\theta(y)\,\theta(z) + \theta(u)\,\varepsilon(y)\,\theta(z) + \theta(u)\,\theta(y)\,\varepsilon(z)](x)\,S(x)\,dx$
- $\varepsilon(v) = \frac{1}{S} v_1^{\bullet} + \frac{1}{R} v_2$

$$\theta(v) = \frac{1}{R} v_1 - \frac{1}{S} v_2^{\bullet}$$

$$K(v) = \frac{1}{S}\left(\frac{1}{R} v_1 - \frac{1}{S} v_2^{\bullet}\right)^{\bullet} = \frac{1}{S}\,[\theta(v)]^{\bullet}$$

$$S = (1 + \varphi^{\bullet 2})^{1/2} \qquad \frac{1}{R} = -\frac{\varphi^{\bullet\bullet}}{S^3}$$

$$C = Eh \qquad D = E\,\frac{h^3}{12}.$$

So λ is an eigenvalue. We will call it an eigenvalue of the problem :

$$a(y, z) = \lambda b(u ; y, z) \quad \forall z \in V .$$

II. DESCRIPTION OF THE EIGENVALUES

The first thing we have to do is to prove the existence of a smallest eigenvalue (in modulus) of the problem :

$$a(y, z) = \lambda b(u ; y, z)$$

where $u \in V$ is the unique solution of the equation :

$$a(u, v) = L(v) \quad \forall v \in V .$$

As u is fixed in V, we can denote :

$$\bar{b}(y, z) = b(u ; y, z)$$

and we have to study the eigenvalues of the problem :

$$a(y, z) = \lambda \bar{b}(y, z) \quad \forall z \in V .$$

This has been done in reference [7], in a general abstract frame. Let us recall the result of reference [7] we are going to use :

V : is a Hilbert space with inner product $\langle \cdot\,, \cdot \rangle$
a : $V \times V \to \mathbb{R}$ is bilinear, symmetric, continuous, coercive,
$\bar{b}$: $V \times V \to \mathbb{R}$ is bilinear, symmetric, continuous, not necessarily positive.

By the Riesz representation formula, there exist 2 operators A and $\bar{B}$ belonging to $\mathcal{L}(V)$, space of linear continuous operators from V into V, such that :

$$a(y, z) = \langle Ay, z \rangle$$
$$\bar{b}(y, z) = \langle \bar{B}y, z \rangle .$$

Both are selfadjoint. The first one is positive invertible.

Reference [7] shows that if $\bar{B} : V \to V$ is compact, then the set of eigenvalues of

$$a(y, z) = \lambda \bar{b}(y, z) \quad \forall z \in V \quad (\text{or } Ay = \lambda \bar{B} y)$$

is made of a sequence of non zero real numbers, which goes to infinity. So there exists an eigenvalue with smallest modulus. Also an eigenvalue is necessarily of finite multiplicity.

The main idea leading to this result is to consider the square root S of A, which is invertible like A, and the operator $K = S^{-1} BS^{-1}$ which is compact. It is clear that :

$$Ay = SSy = \lambda \bar{B} y \Leftrightarrow \begin{cases} S^{-1} \bar{B} S^{-1} z = Kz = \dfrac{1}{\lambda} z \\ z = Sy . \end{cases}$$

This relates the eigenvalues of our problem to the eigenvalues of the operator K. Then, as K is compact and selfadjoint, the properties of its spectrum are well known (see ref. [9]).

Let us mention that if the bilinear form $\bar{b}$ is neither positive nor negative, there can be eigenvalues of both signs. We are just sure that O is not an eigenvalue because A is injective.

So, for the arch buckling problem, we have to check that for the given u, the bilinear form :

$$\bar{b} : y, z \mapsto b(u ; y, z) = \bar{b}(y, z)$$

is symmetric, continuous, and that the associated operator $\bar{B}$ is compact.

The hypothesis on a is known to be satisfied, $\bar{b}$ is obviously bilinear symmetric.

II.1. Continuity of $\bar{b}$

The form :

$$v, y, z \mapsto b(v\,; y, z) : V \times V \times V \to \mathbb{R}$$

is trilinear. We prove that it is trilinear continuous, which will prove that $\bar{b}$ is continuous. This is a consequence of the following lemma.

LEMMA II.1 : *For any $y \in V$, we have :*

i) $\varepsilon(y) \in L^2$ *and* $\|\varepsilon(y)\|_{L^2} \leq C\,\|y\|_V$

ii) $\theta(y) \in H^1$ *and* $\|\theta(y)\|_{H^1} \leq C\,\|y\|_V$

iii) $\theta(y) \in L^\infty$ *and* $\|\theta(y)\|_{L^\infty} \leq C\,\|y\|_V$.

Proof:
i) $\varepsilon(y) = \frac{1}{S} y_1^\bullet + \frac{1}{R} y_2$ with

$$\begin{aligned} &y_1^\bullet \in L_2, \quad y_2 \in H^2 \subset L^2 \\ &S = 1 + \varphi^{\bullet 2} \in W^{2,\infty} \subset L^\infty \\ &\frac{1}{R} = -\frac{\varphi^{\bullet\bullet}}{S^3} \in W^{1,\infty} \subset L^\infty . \end{aligned}$$

So $\varepsilon(y) \in L^2$ and :

$$\begin{aligned} \|\varepsilon(y)\|_{L^2} &\leq \left\|\frac{1}{S}\right\|_{L^\infty} \|y_1^\bullet\|_{L^2} + \left\|\frac{1}{R}\right\|_{L^\infty} \|y_2\|_{L^2} \\ &\leq C\,\|y\|_V , \end{aligned}$$

ii) $\theta(y) = \frac{1}{R} y_1 + \frac{1}{S} y_2^\bullet$ with

$$\begin{aligned} &y_1 \in H^1 , \quad y_2 \in H^1 \subset L^2 \\ &\frac{1}{R} \in W^{1,\infty} , \quad \frac{1}{S} \in W^{2,\infty} \subset W^{1,\infty} . \end{aligned}$$

It is well known that (see ref. [10], theorem 1.4.4.2) :

$$f \in W^{1,\infty} , g \in H^1 \Rightarrow \begin{cases} fg \in H^1 \\ \|fg\|_{H^1} \leq C\,\|f\|_{W^{1,\infty}}\,\|g\|_{H^1} . \end{cases}$$

This gives the result.

iii) We have :

$$H^1(]0,1[) \subset L^\infty(]0,1[)$$

and the corresponding injection is continuous (ref. [9], p. 129).

So :

$$\|\theta(y)\|_{L^\infty} \leqslant C \|\theta(y)\|_{H^1} < C \|y\|_V . \qquad \blacksquare$$

From this, we get :

PROPOSITION II.2 : *The trilinear form :*

$$v, y, z \mapsto b(v\,; y, z) : V \times V \times V \to \mathbb{R}$$

is continuous.

Proof: We have :

$$b(v\,; y, z) = b_1(v\,; y, z) + b_2(v\,; y, z) + b_3(v\,; y, z)$$

with

$$b_1(v\,; y, z) = \int_0^1 C[\varepsilon(v)\,\theta(y)\,\theta(z)](x)\,S(x)\,dx$$

$$b_2(v\,; y, z) = \int_0^1 C[\theta(v)\,\varepsilon(y)\,\theta(z)](x)\,S(x)\,dx$$

$$b_3(v\,; y, z) = \int_0^1 C[\theta(v)\,\theta(y)\,\varepsilon(z)](x)\,S(x)\,dx .$$

We can write :

$$b_1(v\,; y, z) = \langle C\theta(y)\,\varepsilon(v), S\theta(z)\rangle_{L^2, L^2} \quad \text{(inner product of } L^2)$$

with

$$\theta(y) \in L^\infty, \varepsilon(v) \in L^2, S \in L^\infty, \theta(z) \in H^1 \subset L^2 ,$$

and using lemma II.1

$$|b_1(v\,; y, z)| \leqslant C \|\theta(y)\|_{L^\infty} \|\varepsilon(v)\|_{L^2} \|S\|_{L^\infty} \|\theta(z)\|_{L^2}$$
$$\leqslant C \|v\|_V \|y\|_V \|z\|_V .$$

So b_1 is continuous. By symmetry the 2 others are also continuous. ■

II.2. Compactness of $\bar{B}$

Let us recall that $\bar{B} \in \mathcal{L}(V)$ is the linear continuous operator on V such that

$$\forall y, z \in V \quad b(u ; y, z) = \langle \bar{B} . y, z \rangle$$

(of course $\bar{B}$ depends on u).
We show first :

LEMMA II.3 : *Let H be a Banach space such that :*

$$V \hookrightarrow H \qquad (\textit{with compact injection}) .$$

We suppose that the functional $y, z \mapsto \bar{b}(y ; z) : V \times V \to \mathbb{R}$ can be extended in :

$$y, \tilde{z} \mapsto \tilde{b}(y, \tilde{z}) : V \times H \to \mathbb{R}$$

which is bilinear continuous. Then the operator $\bar{B}$ is compact.

Proof: Let H' be the topological dual space of H. From the Riesz theorem, there exists a linear continuous operator :

$$\tilde{B} : V \to H'$$

such that :

$$\forall y \in V, \forall \tilde{z} \in H : \quad \tilde{b}(y, \tilde{z}) = \langle \tilde{B}y, \tilde{z} \rangle_{H', H} .$$

Let us consider the canonical injection :

$$i : V \to H : z \mapsto \tilde{z} .$$

The fact that $\tilde{b}$ is an extension of b can be written :

$$\forall y \in V, \forall z \in V : \quad \bar{b}(y, z) = \tilde{b}(y, iz)$$

or :

$$\langle \bar{B}y, z \rangle_{V, V} = \langle \tilde{B}y, iz \rangle_{H', H} .$$

Let i^* be the adjoint of i :

$$i^* : H' \to V' \simeq V .$$

As i is compact, i^* is also compact (ref. [9], p. 90), and :

$$\forall y, z \in V : \quad \langle \bar{B}y, z \rangle_{V,V} = \langle i^* \circ \bar{B}y, z \rangle_{V,V}$$

so :

$$\bar{B} = i^* \circ \tilde{B} .$$

$\bar{B}$ is the composed of $\tilde{B}$ which is continuous, with i^* which is compact. It is compact. ■

Using this lemma, to prove that $\bar{B}$ is compact, we only have to prove the following proposition :

PROPOSITION II.4 : *Let $H = L^2(]0,1[) \times H_0^1(]0,1[)$ in which V is included, with compact injection. The functional $\bar{b}(y,z)$ can be extended in*

$$\tilde{b} : V \times H \to \mathbb{R}$$

which is bilinear continuous.

Proof: It is well known (see ref. [10], theorem 1.4.3.2) that :

$$H^1(]0,1[) \hookrightarrow L^2(]0,1[)$$
$$H^2(]0,1[) \hookrightarrow H^1(]0,1[)$$

with compact injection. This implies that :

$$V \hookrightarrow H \qquad \text{with compact injection .}$$

The functional $\bar{b}$ could obviously be extended to $V \times H$ if the highest derivative of z_1 and z_2, which are $z_1^{\bullet}$ and $z_2^{\bullet\bullet}$, would not interfere. We can write

$$\bar{b}(y,z) = \bar{b}_1(y,z) + \bar{b}_2(y,z) + \bar{b}_3(y,z) + b_4(y,z)$$

with :

$$\bar{b}_1(y,z) = \int_0^1 C[\varepsilon(u)\,\theta(y)\,\theta(z)\,S](x)\,dx$$

$$\bar{b}_2(y,z) = \int_0^1 C[\theta(u)\,\varepsilon(y)\,\theta(z)\,S](x)\,dx$$

$$\bar{b}_3(y,z) = \int_0^1 C\left[\theta(u)\,\theta(y)\,z_2\,\frac{S}{R}\right](x)\,dx$$

$$\bar{b}_4(y,z) = \int_0^1 C[\theta(u)\,\theta(y)\,z_1^{\bullet}](x)\,dx .$$

The only term in which a highest derivative of z appears is $\bar{b}_4$. In order to deal with it, let us integrate by parts. This is possible because (see ref. [9], p. 131) :

i) $y, z \in H^1(]0,1[) \Rightarrow yz \in H^1(]0,1[)$

$$(yz)^\bullet = y^\bullet z + yz^\bullet$$

ii) $y, z \in H^1(]0,1[) \Rightarrow \int_0^1 y^\bullet z = y(1)\,z(1) - y(0)\,z(0) - \int_0^1 yz^\bullet .$

In our problem :

$$\text{i)} \quad \left.\begin{array}{l}\theta(u) \in H^1\\ \theta(y) \in H^1\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\theta(u)\,\theta(y) \in H^1\\ [\theta(u)\,\theta(y)]^\bullet = [\theta(u)]^\bullet\,\theta(y) + \theta(u)\,[\theta(y)]^\bullet\end{array}\right.$$

$$\text{ii)} \quad \left.\begin{array}{l}\theta(u)\,\theta(y) \in H^1\\ z_1 \in H_0^1\end{array}\right\} \Rightarrow \int_0^1 \theta(u)\,\theta(y)\,z_1^\bullet = -\int_0^1 [\theta(u)\,\theta(y)]^\bullet\,z_1$$

So :

$$\forall y, z \in V \text{ we have : } \bar{b}_4(y,z) = -\int_0^1 C\,\{[\theta(u)]^\bullet\,\theta(y) + \theta(u)\,[\theta(y)]^\bullet\}\,z_1 .$$

Now, we can define for $y \in V$, $\tilde{z} \in H$

$$\tilde{b}_4(y,\tilde{z}) = -\int_0^1 C\,\{[\theta(u)]^\bullet\,\theta(y) + \theta(u)\,[\theta(y)]^\bullet\}\,\tilde{z}_1(x)\,dx$$

which is of course and extension of $\bar{b}_4$.

The extensions of $\bar{b}_1$, $\bar{b}_2$, $\bar{b}_3$, are obvious :

$$\tilde{b}_1(y,\tilde{z}) = \int_0^1 C\,[\varepsilon(u)\,\theta(y)\,\theta(\tilde{z})\,S](x)\,dx$$

$$\tilde{b}_2(y,\tilde{z}) = \int_0^1 C\,[\theta(u)\,\varepsilon(y)\,\theta(\tilde{z})\,S](x)\,dx$$

$$\tilde{b}_3(y,\tilde{z}) = \int_0^1 C\left[\theta(u)\,\theta(y)\,\tilde{z}_2\,\frac{S}{R}\right](x)\,dx .$$

Now, we have to prove that $\tilde{b}_1, \tilde{b}_2, \tilde{b}_3, \tilde{b}_4$ are continuous on $V \times H$. For this, we use :

$$y \in V \Rightarrow \left\{\begin{array}{l}\theta(y) \in H^1\\ \theta(y) \in L^\infty\end{array}\right\} \text{ and } \begin{array}{l}\|\theta(y)\|_{L^\infty} \leqslant C\,\|\theta(y)\|_{H^1} \leqslant C\,\|y\|_V\\ \|\theta(y)\|_{L^2} \leqslant C\,\|y\|_H \leqslant C\,\|y\|_V\end{array}$$

$$y \in V \Rightarrow \varepsilon(y) \in L^2 \text{ and } \|\varepsilon(y)\|_{L^2} \leq C\|y\|_V$$
$$\tilde{z} \in H \Rightarrow \theta(\tilde{z}) \in L^2 \text{ and } \|\theta(\tilde{z})\|_{L^2} \leq C\|\tilde{z}\|_H .$$

We get :

$$\begin{aligned}
\left|\tilde{b}_1(y, \tilde{z})\right| &\leq C\|\varepsilon(u)\|_{L^2}\|\theta(y)\|_{L^\infty}\|\theta(\tilde{z})\|_{L^2}\|S\|_{L^\infty} \\
&\leq C(u)\|y\|_V\|\tilde{z}\|_H \\
\left|\tilde{b}_2(y, \tilde{z})\right| &\leq C\|\theta(u)\|_{L^\infty}\|\varepsilon(y)\|_{L^2}\|\theta(\tilde{z})\|_{L^2}\|S\|_{L^\infty} \\
&\leq C(u)\|y\|_V\|\tilde{z}\|_H \\
\left|\tilde{b}_3(y, \tilde{z})\right| &\leq C\|\theta(u)\|_{L^\infty}\|\theta(y)\|_{L^2}\left\|\frac{S}{R}\right\|_{L^\infty}\|\theta(\tilde{z}_2)\|_{L^2} \\
&\leq C(u)\|y\|_V\|\tilde{z}\|_H .
\end{aligned}$$

Now, we notice that

$$y \in V \Rightarrow [\theta(y)]^\bullet \in L^2 \text{ and } \|[\theta(y)]^\bullet\| \leq C\|y\|_V$$

and we get :

$$\begin{aligned}
&\left|\tilde{b}_4(y, \tilde{z})\right| \leq \\
&\leq \int_0^1 \left|C[\theta(u)]^\bullet\,\theta(y)\,\tilde{z}_1\right| + \int_0^1 \left|C\theta(u)\,[\theta(y)^\bullet]\,\tilde{z}_1\right| \\
&\leq C\|[\theta(u)]^\bullet\|_{L^2}\|\theta(y)\|_{L^\infty}\|\tilde{z}_1\|_{L^2} + C\|\theta(u)\|_{L^\infty}\|[\theta(y)]^\bullet\|_{L^2}\|\tilde{z}_1\|_{L^2} \\
&\leq C(u)\|y\|_V\|\tilde{z}\|_H .
\end{aligned}$$

This ends the proof of the proposition II.4. ■

Remark : In this proposition, u was the solution of the static problem :

$$a(u, v) = L(v) \qquad \forall v \in V .$$

As a matter of fact, we have only used the property $u \in V$. So, we have proved that if we denote :

$$\forall v \in V \qquad \langle B(v)\,y, z\rangle = b(v\,;y, z)$$

then $B(v)$ is compact. $\bar{B}$ is nothing but $B(u)$.

We have now seen that

i) The functional a is bilinear symmetric continuous, coercive.

ii) $\forall v \in V$, the functional

$$y, z \mapsto b(v\,;y, z) : V \times V \to \mathbb{R}$$

is bilinear symmetric continuous.

iii) The operator $\bar{B}$ associated to $b(u\,;y, z) = \bar{b}(y, z)$ is compact.

So, with reference [7], we know that the set of eigenvalues of the problem :

$$a(y, z) = \lambda b(u\,;y, z) \quad \forall z \in V$$

is a sequence of real non zero values, which goes to infinity. There is one smallest eigenvalue in modulus, which is the buckling value of the arch.

III. DIFFERENTIABILITY WITH RESPECT TO THE MIDSURFACE

Now we would like to change the shape of the midsurface ω of the arch, and follow the variations of the buckling value.

More precisely, ω is known as the graph of a function $\varphi : [0, 1] \to \mathbb{R}$. φ has been chosen in the space $W^{3,\infty}(]0, 1[)$, which from now on will be denoted W. φ is going to be a variable, belonging to an open subset $\Phi \subset W$. φ will be called the shape of the arch. The buckling value is then a function of φ. We are going to prove that it is Fréchet differentiable if it is simple, differentiable in a weaker sense that we will precise later if not.

In order to study the dependence in φ, we denote now :

$$\begin{array}{lll} \varepsilon(\varphi\,;v) & \text{for} & \varepsilon(v) \\ \theta(\varphi\,;v) & \text{for} & \theta(v) \\ K(\varphi\,;v) & \text{for} & K(v)\,. \end{array}$$

When $v \in V$ is fixed, ε, K and θ depend on φ through $S, \frac{1}{S}, \frac{1}{R}$ and their first derivatives with respect to x. Moreover, we denote :

$$\begin{array}{lll} a(\varphi\,;u, v) & \text{for} & a(u, v) \\ L(\varphi\,;v) & \text{for} & L(v)\,. \end{array}$$

For a given shape φ, the displacement field at the equilibrium is the unique solution $u_\varphi \in V$ of :

$$a(\varphi\,;u_\varphi, v) = L(\varphi\,;v) \quad \forall v \in V\,.$$

In reference [1], it has been proved that the mapping :

$$\varphi \mapsto u_\varphi : \Phi \subset W \to V$$

is Fréchet differentiable.

Let us also denote $b(\varphi, u_\varphi ; y, z)$ instead of $b(u ; y, z)$ and :

$$\bar{b}(\varphi ; y, z) = b(\varphi, u_\varphi ; y, z) .$$

The buckling value $\lambda(\varphi)$ for the shape φ is the smallest eigenvalue in modulus of the problem.

$$\begin{aligned} a(\varphi ; y_\varphi, z) &= \lambda(\varphi)\,\bar{b}(\varphi ; y_\varphi, z) \\ &= \lambda(\varphi)\, b(\varphi, u_\varphi ; y_\varphi , z) \quad \forall z \in V . \end{aligned}$$

The differentiability result we are going to give is an application of a general result given in reference [7] that we recall now.

III.1. Recall of general results

III.1.1. *Definitions and differential notation*

First, for a function $f : \Phi \subset W \to \mathscr{B}$ (Banach space) which is Fréchet-differentiable, we denote :

$\dfrac{df}{d\varphi}(\varphi) \cdot \psi$ the Fréchet differential in the direction ψ

$\dfrac{df}{d\varphi}(\varphi)$ is the linear continuous mapping from W into V such that :

$$\|\delta_\psi^2 f(\varphi)\|_{\mathscr{B}} \leqslant \underline{\varepsilon}(\psi)\|\psi\|_W$$

where $\underline{\varepsilon}(\psi) \to 0$ when $\psi \to 0$

$$\delta_\varphi^2 f(\varphi) = f(\varphi + \psi) - f(\varphi) - \frac{df}{d\varphi}(\varphi) \cdot \psi .$$

For a function $g : \Phi \times V \to \mathscr{B}$ if for each $y \in V$ the mapping $\varphi \mapsto g(\varphi, y)$ is Fréchet differentiable, we denote $\dfrac{\partial}{\partial\varphi} g(\varphi, y) \cdot \psi$ its Fréchet differential with respect to φ in the direction ψ. It verifies :

$$\forall y \in V : \quad \left\| \frac{\partial}{\partial\varphi} g(\varphi, y) \cdot \psi \right\|_{\mathscr{B}} \leqslant C(y)\|\psi\|_W$$

$$\|\delta_\psi^2 g(\varphi, y)\| = \left\| g(\varphi + \psi, y) - g(\varphi, y) - \frac{\partial}{\partial\varphi} g(\varphi, y) \cdot \psi \right\|_{\mathscr{B}} \leqslant \underline{\varepsilon}(\psi, y)\|\psi\|$$

where $\quad \forall y \in V \quad \underline{\varepsilon}(\psi, y) \to 0 \quad$ when $\quad \psi \to 0 .$

Then we will use the following directional derivatives :

DEFINITION III.1 (ref. [11]) : *Let* $J : \Phi \subset W \to \mathbb{R}$ *be given.*
1. *J is semi-differentiable with respect to* φ *if:*

$$\forall \varphi \in \Phi\,,\quad \forall \psi \in W\,,\quad \exists J'(\varphi, \psi) \in \mathbb{R} \text{ s.t. :}$$
$$\frac{J(\varphi + t\psi) - J(\varphi)}{t} \underset{\substack{t>0 \\ t \to 0}}{\to} J'(\varphi, \psi)\,.$$

2. *J is uniformly semi-differentiable if it is semi-differentiable and:* $\forall \psi_1 \in W$

$$\left.\begin{matrix} \forall \varepsilon > 0 & \exists \delta_0 > 0 \\ & \delta_1 > 0 \end{matrix}\right\} \text{ s.t. } \left\{\begin{matrix} 0 < t < \delta_0 \\ \|\psi - \psi_1\| < \delta_1 \end{matrix}\right\} \Rightarrow$$
$$\Rightarrow \left| \frac{J(\varphi + t\psi) - J(\varphi)}{t} - J'(\varphi, \psi_1) \right| < \varepsilon$$

3. *J is locally convex (resp. concave) if it is semi-differentiable, and if the mapping* $\psi \mapsto J'(\varphi, \psi)$ *is convex (resp. concave).*
4. *J is regularly locally convex (resp. concave) if it is uniformly semi-differentiable and locally convex (resp. concave).*

This is a notion of differentiability which is weaker than Fréchet-differentiable, and which is useful in optimization, because one can derive necessary optimality conditions from this derivative.

Now, we recall the results of reference [7] that we will use :

III.1.2. *Hypothesis*

V is a Hilbert space, W is Banach space, Φ is an open subset of W. Let be given :

$$\begin{aligned} a :\ & \Phi \times V \times V \to \mathbb{R} \\ & \varphi, y, z \quad \mapsto a(\varphi\,; y, z) \end{aligned}$$

bilinear, symmetric, continuous, coercive in y, z for each $\varphi \in \Phi$

$$\begin{aligned} \bar{b} :\ & \Phi \times V \times V \to \mathbb{R} \\ & \varphi, y, z \quad \mapsto \bar{b}(\varphi\,; y, z) \end{aligned}$$

bilinear, symmetric, continuous in y, z for each $\varphi \in \Phi$ (not necessarily positive).

For each $\varphi \in \Phi$, 2 operators $A(\varphi)$ and $\bar{B}(\varphi) \in \mathscr{L}(V)$ are associated to a and $\bar{b}$, such that

$$a(\varphi\,; y, z) = \langle A(\varphi) \,.\, y, z \rangle$$
$$\bar{b}(\varphi\,; y, z) = \langle \bar{B}(\varphi) \,.\, y, z \rangle\,.$$

Like in paragraph II, $\bar{B}(\varphi)$ is supposed to be compact.

Then the bilinear forms a and $\bar{b}$ are supposed to be differentiable with respect to φ in the following sense :

Let $BL(V)$ denote the space of bilinear continuous forms defined on V, equipped with its usual norm :

$$\forall L \in BL(V): \quad \|L\| = \sup_{\substack{y \in V \\ z \in V \\ y, z \neq 0}} \frac{|L(y, z)|}{\|y\| \|z\|}.$$

Then, for each $\varphi \in \Phi$, the mappings $y, z \mapsto a(\varphi ; y, z)$ and $y, z \mapsto \bar{b}(\varphi ; y, z)$ belong to $BL(V)$, so $a(\varphi ; y, z)$ and $\bar{b}(\varphi ; y, z)$ define two mappings :

$$\varphi \mapsto a(\varphi ; ., .): \Phi \subset W \to BL(V)$$
$$\varphi \mapsto \bar{b}(\varphi ; ., .): \Phi \subset W \to BL(V).$$

These two mappings are supposed to be Fréchet-differentiable. Writing the definitions, this can be written explicitely in the following way :

$\forall \varphi \in \Phi$, $\forall y, z \in V$, $\forall \psi \in W$, there exists $\frac{\partial a}{\partial \varphi}(\varphi ; y, z) . \psi$ and $\frac{\partial \bar{b}}{\partial \varphi}(\varphi ; y, z) . \psi$ depending linearly and continuously on ψ, satisfying :

(H1a) $$\left| \frac{\partial a}{\partial \varphi}(\varphi ; y, z) . \psi \right| \leq C(\varphi) \|\psi\|_W \|y\|_V \|z\|_V$$

(H2a) $$|\delta_\psi^2 a(\varphi ; y, z)| \leq \underline{\varepsilon}(\psi) \|\psi\|_W \|y\|_V \|z\|_V$$

where $$\underline{\varepsilon}(\psi) \to 0 \quad \text{when} \quad \psi \to 0.$$

(H1$\bar{b}$) similar to (H1a)

(H2$\bar{b}$) similar to (H2a).

Recalling that $\mathscr{L}(V)$ is classically equipped with the norm :

$$\forall \ell \in \mathscr{L}(V) \quad \|\ell\| = \sup_{\substack{y \in V \\ y \neq 0}} \frac{\|\ell . y\|}{\|y\|}$$

writing the definitions, one can get :

LEMMA III.2 :

$$\left.\begin{array}{l}\varphi \mapsto a(\varphi ; ., .): \Phi \to BL(V) \\ \textit{is Fréchet differentiable}\end{array}\right\} \Leftrightarrow \left\{\begin{array}{l}\varphi \mapsto A(\varphi): \Phi \to \mathscr{L}(V) \\ \textit{is Fréchet differentiable}\end{array}\right.$$

and, for every $y, z \in V$:

$$\left\langle \left[\frac{\partial A}{\partial \varphi}(\varphi) . \psi \right] . y, z \right\rangle = \frac{\partial a}{\partial \varphi}(\varphi ; y, z) . \psi .$$

So we also know that $A(\varphi)$ and $\bar{B}(\varphi)$ are Fréchet-differentiable with respect to φ.

III.1.3. *The results*

What follows has been proved in reference [7] :

THEOREM III.3 : *Let $\lambda_1^+(\varphi)$ be the smallest positive eigenvalue of the problem :*

$$\exists y_\varphi \in V\ ,\quad y_\varphi \neq 0 \text{ s.t. } a(\varphi\,; y_\varphi, z) = \lambda(\varphi)\,\bar{b}(\varphi\,; y_\varphi, z) \quad \forall z \in V$$

and $\lambda_1^-(\varphi)$ the biggest negative one. Then, $\lambda_1^+(\varphi)$ is regularly locally concave, $\lambda_1^-(\varphi)$ is regularly locally convexe. Moreover :

$$(\lambda_1^+)'(\varphi, \psi) = \operatorname{Inf}\left\{\frac{\partial a}{\partial\varphi}(\varphi\,; y_\varphi, y_\varphi)\,.\,\psi - \lambda_1^+(\varphi)\frac{\partial \bar{b}}{\partial\varphi}(\varphi\,; y_\varphi, y_\varphi)\,.\,\psi /\right.$$
$$\left. A(\varphi)\,.\,y_\varphi = \lambda_1^+(\varphi)\,\bar{B}(\varphi)\,y_\varphi,\ \bar{b}(\varphi\,; y_\varphi, y_\varphi) = 1\right\}$$

$$(\lambda_1^-)'(\varphi, \psi) = -\operatorname{Inf}\left\{\frac{\partial a}{\partial\varphi}(\varphi\,; y_\varphi, y_\varphi)\,.\,\psi - \lambda_1^-(\varphi)\frac{\partial \bar{b}}{\partial\varphi}(\varphi\,; y_\varphi, y_\varphi)\,.\,\psi /\right.$$
$$\left. A(\varphi)\,.\,y_\varphi = \lambda_1^-(\varphi)\,\bar{B}(\varphi)\,y_\varphi,\ \bar{b}(\varphi\,; y_\varphi, y_\varphi) = -1\right\}.$$

COROLLARY III.4 : *If $\lambda_1^+(\varphi)$ (resp. $\lambda_1^-(\varphi)$) is a simple eigenvalue, then it is Fréchet-differentiable and :*

1. $$\frac{d}{d\varphi}\lambda_1^+(\varphi)\,.\,\psi = \frac{\partial a}{\partial\varphi}(\varphi\,; y_\varphi, y_\varphi)\,.\,\psi - \lambda_1^+(\varphi)\frac{\partial \bar{b}}{\partial\varphi}(\varphi\,; y_\varphi, y_\varphi)\,.\,\psi$$

where

$$\begin{cases} A(\varphi)\,.\,y_\varphi = \lambda_1^+(\varphi)\,\bar{B}(\varphi)\,.\,y_\varphi \\ \bar{b}(\varphi\,; y_\varphi, y_\varphi) = 1 \end{cases}$$

2. $$\frac{d}{d\varphi}\lambda_1^-(\varphi)\,.\,\psi = -\frac{\partial a}{\partial\varphi}(\varphi\,; y_\varphi, y_\varphi)\,.\,\psi + \lambda_1^-(\varphi)\frac{\partial \bar{b}}{\partial\varphi}(\varphi\,; y_\varphi, y_\varphi)\,.\,\psi$$

where

$$\begin{cases} A(\varphi)\,.\,y_\varphi = \lambda_1^+(\varphi)\,\bar{B}(\varphi)\,.\,y_\varphi \\ \bar{b}(\varphi\,; y_\varphi, y_\varphi) = -1. \end{cases}$$

III.2. Application to the arch problem

III.2.1. *Generalities*

We know that the bilinear forms a and $\bar{b}$ of the arch problem satisfy the hypothesis required in III.1 except for the differentiability conditions with respect to φ. We notice that the linear operator $B(\varphi) \in \mathscr{L}(V)$ associated to $\bar{b}$ by :

$$\bar{b}(\varphi ; y, z) = b(\varphi, u_\varphi ; y, z) = \langle \bar{B}(\varphi) . y, z \rangle \quad \forall y, z \in V$$

is nothing but the operator $\bar{B}$ studied in II.2. So it is compact.

Thus, we now have to check that the bilinear forms a and $\bar{b}$ are differentiable with respect to φ.

The bilinear form a has been studied in details in reference [1]. A differentiability proof as well as a way to compute numerically $\frac{\partial a}{\partial \varphi}(\varphi ; y, z) . \psi$ is given in the paper. So, we now concentrate on the differentiability of $\bar{b}(\varphi ; . , .)$.

We have defined :

$$\bar{b}(\varphi ; y, z) = b(\varphi, u_\varphi ; y, z) \quad \forall y, z \in V .$$

For any $\varphi \in \Phi$ and $v \in V$, there exists a linear operator $B(\varphi, v) \in \mathscr{L}(V)$ such that :

$$b(\varphi, v ; y, z) = \langle B(\varphi, v) . y, z \rangle \quad \forall y, z \in V$$

and of course :

$$\bar{B}(\varphi) = B(\varphi, u_\varphi) .$$

According to lemma III.2, studying the differentiability of $\bar{b}$ is equivalent to study the differentiability of :

$$\varphi \mapsto \bar{B}(\varphi) : \Phi \subset W \to \mathscr{L}(V) .$$

Also, $\bar{B}$ depends implicitely on φ through u_φ. But we know (ref. [1]) that the mapping :

$$\varphi \mapsto u_\varphi : \Phi \subset W \to V$$

is Fréchet differentiable. So we will study the differentiability of $\bar{B}(\varphi)$ using differentiability properties of the mapping.

$$\begin{array}{ll} \varphi, v \mapsto B(\varphi, v) & : \Phi \times V \to \mathscr{L}(V) \\ (\text{or}: \ \varphi, v \mapsto b(\varphi, v\ ; .\,, .\,) & : \Phi \times V \to BL(V)) \end{array}$$

which is explicite. More precisely, we will get the differentiability property of $\bar{B}(\varphi)$ from $B(\varphi, v)$ through the following lemma.

LEMMA III.5 : *If for any v, y, $z \in V$ the mapping*

$$\varphi \mapsto b(\varphi, v\ ; y, z) : \Phi \subset W \to \mathbb{R}$$

is Fréchet differentiable and satisfies :

$$\text{(H1}b\text{)} \qquad \left| \frac{\partial b}{\partial \varphi}(\varphi, v\ ; y, z) \,.\, \psi \right| \leqslant C(\varphi) \|\psi\|_W \|v\|_V \|y\|_V \|z\|_V$$

$$\text{(H2}b\text{)} \qquad |\delta_\psi^2 b(\varphi, v\ ; y, z)| \leqslant \varepsilon(\psi) \|\psi\|_W \|v\|_V \|y\|_V \|z\|_V$$

then the mapping $\varphi \mapsto \bar{B}(\varphi) : \Phi \subset W \to \mathscr{L}(V)$ is Fréchet differentiable and :

$$\frac{\partial B}{\partial \varphi}(\varphi) \,.\, \psi = \frac{\partial \bar{B}}{\partial \varphi}(\varphi, u_\varphi) \,.\, \psi + B(\varphi, u'_{\varphi, \psi})$$

or equivalent, $\forall y, z \in V$:

$$\frac{\partial}{\partial \varphi} b(\varphi, v\ ; y, z) \,.\, \psi = \frac{\partial}{\partial \varphi} b(\varphi, u_\varphi\ ; y, z) \,.\, \psi + b(\varphi, u'_{\varphi, \psi}\ ; y, z)$$

where

$$u'_{\varphi, \psi} = \frac{d}{d\varphi} u_\varphi \,.\, \psi\,.$$

Remark : Hypothesis (H1b) and (H2b) mean that the trilinear form $b(\varphi, .\,; .\,, .\,)$ is differentiable for the classical norm :

$$\|b(\varphi, .\,; .\,, .\,)\| = \sup_{\substack{v, y, z \in V \\ v, y, z \neq 0}} \frac{|b(\varphi, v\ ; y, z)|}{\|v\|_V \|y\|_V \|z\|_V}\,.$$

Proof : We will differentiate $\bar{B}(\varphi) = B(\varphi, u_\varphi)$ using the composition of the mappings :

$$\varphi \mapsto u_\varphi : \Phi \to V$$

and :

$$\varphi, v \mapsto B(\varphi, v) : \Phi \times V \to \mathscr{L}(V)\,.$$

We know that the first one is differentiable. We have to check that the second one is differentiable with respect to the pair (φ, v).

We have seen in paragraph II.1 that $v, y, z \mapsto b(\varphi, v\ ; y, z)$ is trilinear continuous. This implies that $B(\varphi, v)$ depends on v in a linear continuous manner. Then, it is easy to see that if :

1) $\forall v \in V$, the mapping $\varphi \mapsto B(\varphi, v) : \Phi \to \mathscr{L}(V)$ is Fréchet-differentiable

2) $\|B(\varphi + \psi, v) - B(\varphi, v)\|_{\mathscr{L}(V)} \leq \underline{\varepsilon}(\psi)\|v\|_V \quad (\underline{\varepsilon}(\psi) \to 0 \text{ when } \psi \to 0)$

then B is Fréchet-differentiable with respect to the pair (φ, v). Condition 2) can be interpreted as the fact that the partial differential of B with respect to v depends continuously on φ.

Now, using lemma III.2, conditions 1) and 2) can be translated on the functional b.

Condition 1) is equivalent to :

$$\forall v, y, z \in V \quad \text{there exists} \quad \frac{\partial b}{\partial \varphi}(\varphi, v\ ; y, z) \,.\, \psi \text{ s.t.}$$

$$\text{(H1}b\text{)} \qquad \left|\frac{\partial}{\partial \varphi} b(\varphi, v\ ; y, z) \,.\, \psi\right| \leq C(\varphi)\|\psi\|_W \|y\|_V \|z\|_V$$

$$\text{(H2}b\text{)} \qquad |\delta_\psi^2 b(\varphi, v\ ; y, z)| \leq \underline{\varepsilon}(\psi)\|\psi\|_W \|y\|_V \|z\|_V \,.$$

Condition 2) is equivalent to :

$$\forall v \in V\ ,\ \forall y, z \in V\ , \text{ the mapping } \varphi \mapsto b\ (\varphi, v\ ; y, z)$$

depends continuously on φ and :

$$|b(\varphi + \psi, v\ ; y, z) - b(\varphi, v\ ; y, z)| \leq \varepsilon(\psi)\|v\|_V \|y\|_V \|z\|_V \,.$$

These new 2 conditions are of course implied by (H1b) and (H2b) given in this lemma.

We notice that these hypothesis are stronger than necessary.

Then $\bar{B}$ can be differentiated like a composed function, by the chain rule. ■

III.2.2. *Differentiation of* $\varphi \mapsto b(\varphi, v\ ; y, z)$

In this paragraph, we show that the mapping $\varphi \mapsto b(\varphi, .\ ; ., .)$ is differentiable in the space of trilinear continuous functionals, or, in other words, that :

$$\forall v, y, z \in V \text{ there exists } \frac{\partial b}{\partial \varphi}(\varphi, v\ ; y, z) \,.\, \psi \text{ s.t.}$$

$$\text{(H1}b\text{)} \qquad \left|\frac{\partial}{\partial \varphi} b(\varphi, v\ ; y, z) \,.\, \psi\right| \leq C(\varphi)\|\psi\|_W \|v\|_V \|y\|_V \|z\|_V$$

$$\text{(H2}b\text{)} \qquad |\delta_\psi^2 b(\varphi, v\ ; y, z)| \leq \underline{\varepsilon}(\psi)\|\psi\|_W \|v\|_V \|y\|_V \|z\|_V \,.$$

We will use several times the following lemma :

LEMMA III.6 : *Let F, G, H be 3 Banach spaces, and :*

$$\ell : F \times G \to H \text{ be a continuous bilinear function.}$$

Let : $$f : \Phi \subset W \to F \text{ and } g : \Phi \subset W \to G$$

be two Fréchet-differentiable functions.
Then :

$$L : \Phi \subset W \to H : \varphi \mapsto \ell\,(f(\varphi), g(\varphi))$$

is Fréchet-differentiable and :

$$(1) \qquad \frac{d}{d\varphi} L(\varphi) \,.\, \psi = \ell\,(f', g) + \ell\,(f, g')$$

where f' denotes $\frac{d}{d\varphi} f(\varphi) \,.\, \psi$

$$\left\| \frac{d}{d\varphi} L(\varphi) \,.\, \psi \right\| \leqslant [\|f'\|\,\|g\| + \|f\|\,\|g'\|]$$

$$(2) \quad \|\delta_\psi^2\, L(\varphi)\| \leqslant [\|f'\|\,\|g'\| + \|f'\|\,\|\delta^2 g\| + \|\delta^2 f\|\,\|g(\varphi + \psi)\| + \|f\|\,\|\delta^2 g\|]$$

where $\delta^2 g$ denotes $\delta_\psi^2\, g(\varphi)$.

This is standard in classical analysis.
The functional b depends on φ through,

$$S(\varphi) = (1 + \varphi^{\bullet 2})^{1/2}, \frac{1}{S(\varphi)}, \frac{1}{R(\varphi)} = \frac{-\varphi^{\bullet\bullet}}{S^3}.$$

The differentiability of these 3 functions from $\Phi \subset W = W^{3,\infty}$ into L^∞ has been studied in details in reference [1]. It has been proved that they are Fréchet-differentiable and the computation of their derivatives is given. We notice that these 3 functions require φ, $\varphi^\bullet$, $\varphi^{\bullet\bullet}$ but not $\varphi^{\bullet\bullet\bullet}$. So each of them, as well as their derivatives with respect to φ belong to $W^{1,\infty}$.

In order to differentiate b, we first differentiate $\varepsilon(\varphi, y)$ and $\theta(\varphi, y)$.

LEMMA III.7 : *$\forall y \in V$ the mapping $\varphi \mapsto \varepsilon(\varphi, y) : \Phi \to L^2(]0, 1[)$ is Fréchet-differentiable and :*

$$\left\| \frac{\partial \varepsilon}{\partial \varphi}(\varphi, y) \,.\, \psi \right\|_{L^2} \leqslant C\,\|\psi\|_W\,\|y\|_V$$

$$\|\delta_\psi^2\, \varepsilon(\varphi, y)\|_{L^2} \leqslant \varepsilon(\psi)\|\psi\|_W\,\|y\|_V\,.$$

Proof:

$$\varepsilon(\varphi, y) = \frac{1}{S(\varphi)} y_1^{\bullet} + \frac{1}{R(\varphi)} y_2 .$$

The mapping :

$$f, g \mapsto fg : L^{\infty} \times L^2 \to L^2$$

is bilinear continuous. $\frac{1}{S}$ and $\frac{1}{R}$ are differentiable with respect to φ in L^{∞}. With lemma III.6 we get :

1) $$\frac{\partial \varepsilon}{\partial \varphi}(\varphi, y) \cdot \psi = \left(\frac{1}{S}\right)' y_1^{\bullet} + \left(\frac{1}{R}\right)' y_2$$

where $\left(\frac{1}{S}\right)'$ denotes $\frac{d}{d\varphi}\left(\frac{1}{S}\right)(\varphi) \cdot \psi$

$\left(\frac{1}{R}\right)'$ denotes $\frac{d}{d\varphi}\left(\frac{1}{R}\right)(\varphi) \cdot \psi$

and :

$$\left\| \frac{\partial \varepsilon}{\partial \varphi}(\varphi, y) \cdot \psi \right\|_{L^2} \leqslant C \left\| \left(\frac{1}{S}\right)' \right\|_{L^{\infty}} \|y_1^{\bullet}\|_{L^2} + \left\| \left(\frac{1}{R}\right)' \right\|_{L^{\infty}} \|y_2\|_{L^2} .$$

But :

$$\left\| \left(\frac{1}{S}\right)' \right\|_{L^{\infty}} \leqslant C \|\psi\|_W , \quad \left\| \left(\frac{1}{R}\right)' \right\|_{L^{\infty}} \leqslant C \|\psi\|_W$$

so :

$$\left\| \frac{\partial \varepsilon}{\partial \varphi}(\varphi, y) \cdot \psi \right\|_{L^2} \leqslant C \|\psi\|_W \|y\|_V .$$

2) $$\begin{aligned} \|\delta^2 \varepsilon\|_{L^2} &\leqslant \left\| \delta^2\left(\frac{1}{S}\right) \right\|_{L^{\infty}} \|y_1\|_{L^2} + \left\| \delta^2\left(\frac{1}{R}\right) \right\|_{L^{\infty}} \|y_2\|_{L^2} \\ &\leqslant \underline{\varepsilon}(\psi) \|\psi\|_W \|y\|_V . \quad \blacksquare \end{aligned}$$

LEMMA III.8 : *$\forall y \in V$, the mapping $\varphi \mapsto \theta(\varphi, y) : \Phi \to L^2(]0, 1[)$ is Fréchet-differentiable and :*

$$\left\| \frac{\partial \theta}{\partial \varphi}(\varphi, y) \cdot \psi \right\|_{L^2} \leqslant C \|\psi\|_W \|y\|_V$$
$$\|\delta_{\psi}^2 \theta(\varphi, y)\|_{L^2} \leqslant \underline{\varepsilon}(\psi) \|\psi\|_W \|y\|_V .$$

The proof is the same as lemma III.7.

LEMMA III.9 : *$\forall y \in V$, the mapping $\varphi \mapsto \theta(\varphi, y) : \to L^{\infty}(]0,1[)$ is Fréchet-differentiable and :*

$$\left\| \frac{\partial \theta}{\partial \varphi}(\varphi, y) \cdot \psi \right\|_{L^{\infty}} \leqslant C \|\psi\|_W \|y\|_V$$
$$\|\delta_{\psi}^2 \theta(\varphi, y)\|_{L^{\infty}} \leqslant \underline{\varepsilon}(\psi) \|\psi\|_W \|y\|_V .$$

Proof :

$$\theta(\varphi, y) = \frac{1}{R} y_1 - \frac{1}{S} y_2^{\bullet} .$$

As $y_1, y_2^{\bullet}$ belong to $H^1(]0,1[)$, they also belong to $L^{\infty}(]0,1[)$. Then the mapping :

$$f, g \to fg : L^{\infty} \times L^{\infty} \to L^{\infty}$$

is bilinear continuous. So lemma III.6 tells us that the mapping :

$$\varphi \mapsto \theta(\varphi, y) : \Phi \subset W \to L^{\infty}$$

is Fréchet-differentiable and :

$$\frac{\partial \theta}{\partial \varphi}(\varphi, y) \cdot \psi = \left(\frac{1}{R}\right)' y_1 - \left(\frac{1}{S}\right)' y_2^{\bullet}$$
$$\left\| \frac{\partial \theta}{\partial \varphi}(\varphi, y) \cdot \psi \right\|_{L^{\infty}} \leqslant \left\| \left(\frac{1}{R}\right)' \right\|_{L^{\infty}} \|y_1\|_{L^{\infty}} + \left\| \left(\frac{1}{S}\right)' \right\|_{L^{\infty}} \|y_2^{\bullet}\|_{L^{\infty}} .$$

Then we know that

$$\forall f \in H_1 \quad \|f\|_{L^{\infty}} \leqslant C \|f\|_{H^1} .$$

This gives :

$$\left\| \frac{\partial \theta}{\partial \varphi}(\varphi, y) \cdot \psi \right\|_{L^{\infty}} \leqslant C \|\psi\|_W \|y\|_V .$$

The same estimation can be done on $\delta_{\varphi,\psi}^2 \theta(\varphi, y)$. ■

Remark : As a matter of fact, one can see that

$$\varphi \mapsto \theta(\varphi, y) : \Phi \subset W \to H^1(]0,1[)$$

is Fréchet-differentiable with :

$$\left\| \frac{\partial \theta}{\partial \varphi}(\varphi, y) \cdot \psi \right\|_{H^1} \leqslant C \|\psi\|_W \|y\|_V$$
$$\|\delta_{\psi}^2 \theta(\varphi, y)\|_{H^1} \leqslant \underline{\varepsilon}(\psi) \|\psi\|_W \|y\|_V .$$

This implies the results of lemmas III.8 and III.9. But to prove this differentiability in H^1, one needs to differentiate $\frac{1}{S}$ and $\frac{1}{R}$ in the space $W^{1,\infty}$ instead of L^∞ which is used in the direct proof we have given. This differentiability of $\frac{1}{S}$ and $\frac{1}{R}$ in $W^{1,\infty}$ happens to be true. ■

Now, we have :

THEOREM III.10 : *$\forall v, y, z \in V$, the mapping $\varphi \mapsto b(\varphi, v\,; y, z)$ is Fréchet-differentiable and satisfies :*

$$\text{(H1}b\text{)} \qquad \left| \frac{\partial b}{\partial \varphi}(\varphi, v\,; y, z) \,.\, \psi \right| \leq C \|\psi\|_W \|v\|_V \|y\|_V \|z\|_V$$

$$\text{(H2}b\text{)} \qquad \left| \delta_\psi^2 b(\varphi, v\,; y, z) \right| \leq \underline{\varepsilon}(\psi) \|\psi\|_W \|v\|_V \|y\|_V \|z\|_V \,.$$

Moreover, one gets its derivative by the computation of derivatives of products under $\int$:

$$\frac{\partial b}{\partial \varphi}(\varphi, v\,; y, z) \,.\, \psi = \frac{\partial b_1}{\partial \varphi}(\varphi, v\,; y, z) \,.\, \psi + \frac{\partial b_2}{\partial \varphi}(\varphi, v\,; y, z) \,.\, \psi + \frac{\partial b_3}{\partial \varphi}(\varphi, v\,; y, z) \,.\, \psi$$

with :

$$b_1(\varphi, v\,; y, z) = C \int_0^1 [\varepsilon(v)\, \theta(y)\, \theta(z)](x)\, S(x)\, dx$$

$$\frac{\partial b_1}{\partial \varphi}(\varphi, v\,; y, z) \,.\, \psi = C \int_0^1 [\varepsilon'(v)\, \theta(y)\, \theta(z) + \varepsilon(v)\, \theta'(y)\, \theta(z) + \varepsilon(v)\, \theta(y)\, \theta'(z)]\, S\, dx + C \int_0^1 [\varepsilon(v)\, \theta(y)\, \theta(z)]\, S'\, dx$$

$\frac{\partial b_2}{\partial \varphi}, \frac{\partial b_3}{\partial \varphi}$ are obtained from this one by circular permutation on v, y, z.

We have denoted :

S' for $\frac{dS(\varphi)}{d\varphi} \cdot \psi$

$\varepsilon(v)$ for $\varepsilon(\varphi, v)$ (similar for θ)

$\varepsilon'(v)$ for $\frac{\partial \varepsilon}{\partial \varphi}(\varphi, v) \,.\, \psi$ (similar for θ) .

Proof: we have seen before that :

$$b(\varphi, v ; y, z) = b_1(\varphi, v ; y, z) + b_2(\varphi, v ; y, z) + b_3(\varphi, v ; y, z) .$$

By symmetry, we only need to work on one term.

We can write :

$$b_1(\varphi, v ; y, z) = C \langle \theta(\varphi, v)\, \varepsilon(\varphi, y), S(\varphi)\, \theta(\varphi, z) \rangle_{L^2, L^2}$$

with :

$$\begin{array}{ll} \theta(\varphi, v) \in L^\infty & S(\varphi) \in L^\infty \\ \varepsilon(\varphi, v) \in L^2 & \theta(\varphi, z) \in L^2 . \end{array}$$

We look at b_1 as a multilinear form and apply lemma III.6 several times. In order to avoid very heavy notations, we denote :

$$\begin{array}{ll} S' = \dfrac{dS}{d\varphi}(\varphi) \cdot \psi & \delta^2 S = \delta^2_{\varphi, \psi} S(\varphi) \\ \varepsilon'(v) = \dfrac{\partial \varepsilon}{\partial \varphi}(\varphi, v) \cdot \psi & \delta^2 \varepsilon = \delta^2_{\varphi, \psi} \varepsilon(\varphi, v) \\ \theta'(y) = \dfrac{\partial \theta}{\partial \varphi}(\varphi, y) \cdot \psi & \delta^2 \theta = \delta^2_{\varphi, \psi} \theta(\varphi, y) \end{array}$$

and remember that "prime functions" depend in a linear continuous manner on ψ.

a) The mapping $\varphi \mapsto S(\varphi)\, \theta(\varphi, z) : \Phi \to L^2$ is differentiable and (lemma III.6)

$$[S\theta(z)]' = S'\, \theta(z) + S\theta'(z)$$

$$\| [S\theta(z)]' \|_{L^2} \leq \| S' \|_{L^\infty} \| \theta(z) \|_{L^2} + \| S \|_{L^\infty} \| \theta'(z) \|_{L^2}$$

$$\begin{aligned} \| \delta^2 [S\theta(z)] \|_{L^2} \leq{} & C \| S' \|_{L^\infty} \| \theta'(z) \|_{L^2} + \| S' \|_{L^\infty} \| \delta^2 \theta(z) \|_{L^2} \\ & + \| \delta^2 S \|_{L^\infty} \| \theta(\varphi + \psi, z) \|_{L^2} + \| S \|_{L^2} \| \delta^2 \theta(z) \|_{L^2} . \end{aligned}$$

Then we use :

$$\begin{array}{ll} \| \theta(z) \|_{L^2} \leq C \| z \|_V & \text{(lemma II.1)} \\ \| \theta'(z) \|_{L^2} \leq C \| \psi \|_W \| z \|_V & \text{(lemma III.8)} \\ \| \delta^2 \theta(z) \|_{L^2} \leq \underline{\varepsilon}(\psi) \| \psi \|_W \| z \|_V & \text{(lemma III.8)} \end{array}$$

and we get :

$$\begin{aligned} \| [S\theta(z)]' \|_{L^2} &\leq C \| \psi \|_W \| z \|_V \\ \| \delta^2 [S\theta(z)] \|_{L^2} &\leq \underline{\varepsilon}(\psi) \| \psi \|_W \| z \|_V \end{aligned}$$

b) Similarly, $\varphi \mapsto \theta(\varphi, v)\,\varepsilon(\varphi, z) : \Phi \to L^2$ can be differentiated as the product of $\theta(\varphi, v) \in L^\infty$ and $\varepsilon(\varphi, y) \in L^2$. Using

$$\begin{array}{lll} \|\theta(\varphi, v)\|_{L^\infty} & \leq C\,\|v\|_V & \text{(lemma II.1)} \\ \|\varepsilon(\varphi, y)\|_{L^2} & \leq C\,\|y\|_V & \text{(lemma II.1)} \\ \|\theta'(\varphi, v)\|_{L^\infty} & \leq C\,\|\psi\|_W\,\|v\|_V & \text{(lemma III.9)} \\ \|\varepsilon'(\varphi, v)\|_{L^2} & \leq C\,\|\psi\|_W\,\|y\|_V & \text{(lemma III.7)} \\ \|\delta^2\theta(\varphi, v)\|_{L^\infty} & \leq \underline{\varepsilon}\,\|\psi\|\,\|\psi\|_W\,\|v\|_V & \text{(lemma III.9)} \\ \|\delta^2\varepsilon(\varphi, y)\|_{L^2} & \leq \underline{\varepsilon}(\psi)\,\|\psi\|_W\,\|y\|_V & \text{(lemma III.7)} \end{array}$$

and using lemma III.6 on bilinearity, we get :

$$\begin{gathered} [\theta(v)\,\varepsilon(y)]' = \theta'(v)\,\varepsilon(y) + \theta(v)\,\varepsilon'(y) \\ \|[\theta(v)\,\varepsilon(y)]'\|_{L^2} \leq C\,\|\psi\|_W\,\|v\|_V\,\|y\|_V \\ \|\delta^2[\theta(v)\,\varepsilon(y)]\|_{L^2} \leq \underline{\varepsilon}(\psi)\,\|\psi\|_W\,\|v\|_V\,\|y\|_V\,. \end{gathered}$$

c) Using again lemma III.6 on bilinearity, we get :

$$\frac{\partial b_1}{\partial \varphi}(\varphi, v\,; y, z)\,.\,\psi = C\,\langle [\theta(v)\,\varepsilon(y)]',\ S\theta(z)\rangle_{L^2, L^2} + \\ + C\,\langle \theta(v)\,\varepsilon(y),\ [S\theta(z)]'\rangle_{L^2, L^2}$$

$$\left|\frac{\partial b_1}{\partial \varphi}(\varphi, v\,; y, z)\,.\,\psi\right| \leq C\,\|\psi\|_W\,\|v\|_V\,\|y\|_V\,\|z\|_V$$

$$\left|\delta^2_\psi\, b_1(\varphi, v\,; y, z)\right| \leq \underline{\varepsilon}(\psi)\,\|\psi\|_W\,\|v\|_V\,\|y\|_V\,\|z_V\|\,.$$

This ends the proof of theorem III.10. ■

Now, we are able to differentiate $\varphi \mapsto \bar{b}(\varphi\,; y, z)$:

COROLLARY III.11 : *The mapping* $\varphi \mapsto \bar{b}(\varphi\,; y, z) : \Phi \subset W \to \mathbb{R}$ *is differentiable for each* $y, z \in V$ *and satisfies :*

$$\left\|\frac{\partial \bar{b}}{\partial \varphi}(\varphi\,; y, z)\,.\,\psi\right\| \leq C\,\|\psi\|_W\,\|y\|_V\,\|z\|_V$$

$$\left\|\delta^2_\psi\,\bar{b}(\varphi\,; y, z)\right\| \leq \underline{\varepsilon}(\psi)\,\|\psi\|_W\,\|y\|_V\,\|z_V\|\,.$$

Moreover :

$$\frac{\partial \bar{b}}{\partial \varphi}(\varphi\,; y, z)\,.\,\psi = \frac{\partial b}{\partial \varphi}(\varphi, u_\varphi\,; y, z)\,.\,\psi + b(\varphi, u'_{\varphi,\psi}\,; y, z)$$

where $\frac{\partial b}{\partial \varphi}(\varphi, v\,; y, z)\,.\,\psi$ *is given in theorem III.10.*

Proof: One can get this result putting together lemma III.5 and theorem III.10. ■

III.2.3. *Differentiation of the buckling value. An analytical formula*

The differentiability of the buckling value comes from the general result of paragraph III.1 applied to the functionals a and $\bar{b}$ of the arch model. We have seen that these 2 functionals fulfill the hypothesis required for theorem III.3 and corollary III.4.

Before giving the complete result for the buckling problem, we notice that the differentiation of $\lambda(\varphi) = \lambda_1^+(\varphi)$ or $\lambda(\varphi) = \lambda_1^-(\varphi)$, would they be simple eigenvalues or multiple ones, requires the computation of :

$$\Lambda(\varphi, y_\varphi, \psi) = \frac{\partial a}{\partial \varphi}(\varphi ; y_\varphi, y_\varphi) \cdot \psi - \lambda(\varphi) \frac{\partial \bar{b}}{\partial \varphi}(\varphi ; y_\varphi, y_\varphi) \cdot \psi$$

where y_φ is an eigenvector associated to $\lambda(\varphi)$ (see theorem III.3). And as we have just seen in corollary III.11, this is also :

$$\Lambda(\varphi, y_\varphi, \psi) = \frac{\partial a}{\partial \varphi}(\varphi ; y_\varphi, y_\varphi) \cdot \psi - \\ - \lambda(\varphi)\left[\frac{\partial b}{\partial \psi}(\varphi, u_\varphi ; y_\varphi, y_\varphi) \cdot \psi + b(\varphi, u'_{\varphi, \psi} ; y_\varphi ; y_\varphi)\right].$$

In this expression, for the shape φ, u_φ which is the prebuckled equilibrium can be computed from a finite element program, and $\lambda(\varphi)$ and y_φ can be computed from any eigenvalue and eigenvector procedure. But it is convenient to avoid the computation of $u'_{\varphi, \psi}$ as it would need to be done for each $\psi \in W$. This can be done using the classical adjoint state technique :

PROPOSITION III.12 : *Let $p_\varphi \in V$ be the unique solution of:*

$$a(\varphi ; p_\varphi, w) = b(\varphi, w ; y_\varphi, y_\varphi) \quad \forall w \in V$$

Then : $$b(\varphi, u'_{\varphi, \psi} ; y_\varphi, y_\varphi) = -\frac{\partial a}{\partial \varphi}(\varphi ; u_\varphi, p_\varphi) \cdot \psi + \frac{\partial L}{\partial \varphi}(\varphi ; p_\varphi) \cdot \psi$$

(notice that p_φ depends on y_φ).

Proof: It is clear that :

$$b(\varphi, u'_{\varphi, \psi} ; y_\varphi, y_\varphi) = a(\varphi ; u'_{\varphi, \psi}, p_\varphi) .$$

Then, one can differentiate the equation

$$a(\varphi ; u_\varphi, v) = L(\varphi, v) \quad \forall v \in V$$

and then, choosing $v = p_\varphi$, get :

$$a(\varphi\,;u'_{\varphi,\psi}, p_\varphi) = -\frac{\partial a}{\partial \varphi}(\varphi\,;u_\varphi, p_\varphi)\,.\,\psi + \frac{\partial L}{\partial \varphi}(\varphi\,;p_\varphi)\,.\,\psi \quad \text{q.e.d.} \quad \blacksquare$$

Now we have the differentiability results for the buckling value :

THEOREM III.13 : *Suppose that the buckling value $\lambda(\varphi)$ is a multiple eigenvalue. Then :*

1) *a) If it is positive, it is regularly locally concave and :*

$$\lambda'(\varphi, \psi) = \operatorname*{Inf}_{y_\varphi \in Y_\varphi} \left\{ \frac{\partial a}{\partial \varphi}(\varphi\,;y_\varphi, y_\varphi)\,.\,\psi - \lambda(\varphi)\frac{\partial \bar{b}}{\partial \varphi}(\varphi\,;y_\varphi, y_\varphi)\,.\,\psi \right\}$$

with

$$Y_\varphi = \{ y_\varphi \in V\,;\,\forall z \in V : a(\varphi\,;y_\varphi, z) = \lambda(\varphi)\,\bar{b}(\varphi\,;y_\varphi, z)\,;\; \bar{b}(\varphi\,;y_\varphi, y_\varphi) = 1 \}$$

b) If it is negative, it is regularly locally convex and :

$$\lambda'(\varphi, \psi) = \operatorname*{Inf}_{y_\varphi \in Z_\varphi} \left\{ \frac{\partial a}{\partial \varphi}(\varphi\,;y_\varphi, y_\varphi)\,.\,\psi - \lambda(\varphi)\frac{\partial \bar{b}}{\partial \varphi}(\varphi\,;y_\varphi, y_\varphi)\,.\,\psi \right\}$$

with

$$Z_\varphi = \{ y_\varphi \in V\,;\,\forall z \in V : a(\varphi\,;y_\varphi, z) = \lambda(\varphi)\,\bar{b}(\varphi\,;y_\varphi, z)\,;\; \bar{b}(\varphi\,;y_\varphi, y_\varphi) = -1 \}$$

2. *$\forall y \in V$ we have :*

$$\frac{\partial \bar{b}}{\partial \varphi}(\varphi\,;y, y)\,.\,\psi = \frac{\partial b}{\partial \varphi}(\varphi, u_\varphi\,;y, y)\,.\,\psi - \frac{\partial a}{\partial \varphi}(\varphi\,;u_\varphi, p_\varphi)\,.\,\psi + \frac{\partial L}{\partial \varphi}(\varphi\,;p_\varphi)\,.\,\psi$$

where $p_\varphi \in V$ is the solution of:

$$a(\varphi\,;p_\varphi, w) = b(\varphi, w\,;y, y) \qquad \forall w \in V\,.$$

3. $\dfrac{\partial b}{\partial \varphi}(\varphi, u_\varphi\,;y, y)\,.\,\psi$ *is given in theorem III.10.*

COROLLARY III.14 : *Suppose that the buckling value is simple. Then it is Fréchet differentiable, and :*

a) If it is positive :

$$\frac{d}{d\varphi}\lambda(\varphi)\cdot\psi = \frac{\partial a}{\partial\varphi}(\varphi\,;y_\varphi, y_\varphi)\cdot\psi - \lambda(\varphi)\frac{\partial \bar{b}}{\partial\varphi}(\varphi\,;y_\varphi, y_\varphi)\cdot\psi$$

where $y_\varphi \in V$ is the only solution of:

$$\begin{cases} a(\varphi\,;y_\varphi, z) = \lambda(\varphi)\,\bar{b}(\varphi\,;y_\varphi, z) & \forall z \in v \\ \bar{b}(\varphi\,;y_\varphi, y_\varphi) = 1 \end{cases}$$

b) If it is negative :

$$\frac{d}{d\varphi}\lambda(\varphi)\cdot\psi = -\frac{\partial a}{\partial\varphi}(\varphi\,;y_\varphi, y_\varphi)\cdot\psi + \lambda(\varphi)\frac{\partial b}{\partial\varphi}(\varphi\,;y_\varphi, y_\varphi)\cdot\psi$$

where $y_\varphi \in V$ is the only solution of:

$$\begin{cases} a(\varphi\,;y_\varphi, z) = \lambda(\varphi)\,\bar{b}(\varphi\,;y_\varphi, z) & \forall z \in v \\ \bar{b}(\varphi\,;y_\varphi, y_\varphi) = -1 \end{cases}$$

$\frac{\partial b}{\partial\varphi}(\varphi\,;y_\varphi, y_\varphi)\cdot\psi$ *can be computed as in theorem III.13.*

This is theorem III.3, corollary III.4, theorem III.10 and proposition III.12 put together. ■

III.2.4. *Numerical computation of* $\lambda'(\varphi, \psi)$

We are interested in the derivative of $\lambda(\varphi)$ in order to optimize the shape φ so that $|\lambda(\varphi)|$ be as big as possible. This will have to be done using an algorithm adapted to regularly locally convex or concave functionals.

The computation of the derivative of $\lambda(\varphi)$ requires the computation of :

$$\Lambda(\varphi, y_\varphi, \psi) = \frac{\partial a}{\partial\varphi}(\varphi\,;y_\varphi, y_\varphi)\cdot\psi - \lambda(\varphi)\Big[\frac{\partial b}{\partial\varphi}(\varphi, u_\varphi\,;y_\varphi, y_\varphi)\cdot\psi - \\ - \frac{\partial a}{\partial\varphi}(\varphi\,;u_\varphi, p_\varphi)\cdot\psi + \frac{\partial L}{\partial\varphi}(\varphi\,;p_\varphi)\cdot\psi\Big]$$

where :

$$\begin{aligned} &u_\varphi \in V: \quad a(\varphi\,;u_\varphi, v) = L(\varphi\,;v) && \forall v \in V \\ &\lambda(\varphi), y_\varphi \in V: \quad a(\varphi\,;y_\varphi, z) = \lambda(\varphi)\,b(\varphi, u_\varphi\,;y_\varphi, z) && \forall z \in V \\ &p_\varphi \in V: \quad a(\varphi\,;p_\varphi, w) = b(\varphi, w\,;y_\varphi, y_\varphi) && \forall w \in V\,. \end{aligned}$$

For a given shape φ these are computed by a finite element program (solution of linear equation, and computation of eigen values and eigenvectors).

Then, for a given ψ, $\Lambda(\varphi, y_\varphi, \psi)$ can be computed. The only difficulty comes from the very heavy formulas. This problem has already been faced at in reference [1]. In this reference, we have neaded to compute $\frac{\partial a}{\partial \varphi}(\varphi ; u, v) \cdot \psi$ and $\frac{\partial L}{\partial \varphi}(\varphi ; v) \cdot \psi$ for given φ, u, v, ψ. We will use the same organization here. The basic idea is to use modular programming, in order to avoid to develop formulas.

The program is a sequence of subroutines, each one calling previous ones.

$\Lambda(\varphi, y_\varphi, \psi)$ is the integral of a complicated function $F(\varphi, u_\varphi, y_\varphi, p_\varphi, \psi)$. It is approximated by a quadrature formula :

$$\sum_{k=1}^{M} \omega_k F(\varphi, u_\varphi, y_\varphi, p_\varphi, \psi)(x_k)$$

and we need to compute $F(\varphi, u_\varphi, y_\varphi, p_\varphi, \psi)(x_k)$ numerically, for given x_k, φ, u_φ, y_φ, p_φ, ψ.

The detail of the computation of the parts concerning $\frac{\partial a}{\partial \varphi}$ and $\frac{\partial L}{\partial \varphi}$ is given in reference [1].

Here we give the detailed sequence of subroutines which is needed to compute the $\frac{\partial b}{\partial \varphi}(\varphi ; u_\varphi, y_\varphi, y_\varphi) \cdot \psi$ term.

Denoting $S'(\varphi)$ for $\frac{\partial S}{\partial \varphi}(\varphi) \cdot \psi$, we recall that (ref. [1]) :

$$S'(\varphi)(x) = \frac{\varphi^\bullet \psi^\bullet}{S(\varphi)}(x)$$

$$\left(\frac{1}{S}\right)'(\varphi)(x) = -\frac{S'}{S^2}(\varphi)(x)$$

$$\left(\frac{1}{R}\right)'(\varphi)(x) = -\left[\left(\frac{1}{S}\right)^3 \psi^{\bullet\bullet} - 3\left(\frac{1}{S}\right)^2\left(\frac{1}{S}\right)' \varphi^{\bullet\bullet}\right](x).$$

Then, denoting $\varepsilon'(v)$ for $\frac{\partial}{\partial \varphi}\varepsilon(\varphi, v) \cdot \psi$:

$$\varepsilon'(v) = \left(\frac{1}{S}\right)' v_1^\bullet + \left(\frac{1}{R}\right)' v_2$$

$$\theta'(v) = \left(\frac{1}{R}\right)' v_1 - \left(\frac{1}{S}\right)' v_2^\bullet.$$

For : $b_1(\varphi, v ; y, z) = C \int_0^1 [C\varepsilon(\varphi, v)\,\theta(\varphi, y)\,\theta(\varphi, z)]\, S(\varphi)(x)\, dx$

we have

$$\frac{\partial b_1}{\partial \varphi}(\varphi, v\,; y, z) \,.\, \psi = \sum_{k=1}^{M} [C\varepsilon'(\varphi, v)\,\theta(\varphi, y)\,\theta(\varphi, z)\,S(\varphi)](x_k) + \\ + [C\varepsilon(\varphi, v)\,\theta'(\varphi, y)\,\theta(\varphi, z)\,S(\varphi)](x_k) \\ + [C\varepsilon(\varphi, v)\,\theta(\varphi, y)\,\theta'(\varphi, z)\,S(\varphi)](x_k) \\ + [C\varepsilon(\varphi, v)\,\theta(\varphi, y)\,\theta(\varphi, z)\,S'(\varphi)](x_k)\,.$$

Then, using the symmetry of b_1, b_2, b_3, we have :

$$\frac{\partial b}{\partial \varphi}(\varphi, u_\varphi\,; y_\varphi, y_\varphi) \,.\, \psi = \frac{\partial b_1}{\partial \varphi}(\varphi, u_\varphi\,; y_\varphi, y_\varphi) \,.\, \psi + 2\,\frac{\partial b_1}{\partial \varphi}(\varphi, u_\varphi\,; u_\varphi, y_\varphi) \,.\, \psi\,.$$

This gives an approximation of $\Lambda(\varphi, y_\varphi, \psi)$, which can be used in an optimization procedure.

CONCLUSION

A rigorous proof of directional differentiability of buckling load has been given in a functional space setting. Then a method of numerical computation of derivative is given, which can be used in an appropriate optimization algorithm.

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