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CRITICAL POINTS IN THE ENERGY OF HYPERELASTIC MATERIALS (*)

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Abstract. — *This paper shows that for a hyperelastic material, the positivity of the second derivative of the energy functional at the solution u given by the implicit function theorem, implies that u is a local minimizer with respect to the topology of the space $W^{1,\infty}$. A counterexample shows that this result is not generally valid with respect to the topologies of the spaces $W^{1,p}$, $3 < p < +\infty$. In addition, when the stored energy function is polyconvex, the trivial solution is a minimizer in any Sobolev space $W^{1,p}$, $3 \leq p \leq +\infty$.*

Résumé. — *Dans ce travail, on montre que la positivité de la dérivée seconde de la fonctionnelle d'énergie calculée en la solution u , trouvée par le théorème des fonctions implicites, implique que u est un minimum local dans l'espace $[W^{1,\infty}(\Omega)]^3$. Par ailleurs, un contre-exemple met en évidence que ce résultat n'est pas valable, en général, pour les topologies des espaces $W^{1,p}$, $3 < p < +\infty$. Par contre, on montre que, si l'énergie est polyconvexe, la solution triviale est un minimum de la fonctionnelle d'énergie dans tous les espaces de Sobolev $W^{1,p}$, $3 \leq p \leq +\infty$.*

1. INTRODUCTION

The main problem in the theory of non-linear stationary elasticity is to find a position of equilibrium of an elastic body subjected to a given system of applied forces. This problem admits two mathematical models: a boundary value problem associated with a system of partial differential equations, or a minimization problem of a functional (the « energy » of the system). It is shown that there exists a solution of the first model by means of the implicit function theorem, for some special boundary conditions. It is also possible to establish the existence of a minimizer of the energy using,

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among other things, the assumption of polyconvexity introduced by Ball [1977], even if this procedure provides solutions that are not smooth in general. Our aim is to show that for some hyperelastic materials (including Ogden and St. Venant-Kirchhoff materials), the solution given by the implicit function theorem is a local minimum of the energy functional with respect to the topology of the space $W^{1,\infty}$, while in $W^{1,p}$, $3 < p < +\infty$ it can only be said that it is a stationary point of the total energy. In paragraph 6 it is shown that if the stored energy function is polyconvex, then the trivial solution is a minimizer of the energy in any Sobolev space $W^{1,p}$, $3 \leq p \leq +\infty$.

In the following, we shall briefly describe the non-linear three-dimensional elasticity problem and the basic existence results that will be needed later (for a more precise description, see Ciarlet [1987], Marsden-Hughes [1983], Wang-Truesdell [1973]).

Let us first specify the various notations we shall use, concerning notably vectors, matrices and function spaces.

As a rule, Latin indices, i, j, \dots , take their values in the set $\{1, 2, 3\}$. The repeated index convention for summation is systematically used, in conjunction with the above rule.

The usual partial derivatives will be written $\partial_i = \frac{\partial}{\partial x_i}$, $\partial_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$. We write ∇ and div for the gradient and divergence operators in R^n : for a vector field u , ∇u is the tensor field with components $(\nabla u)_{ij} = \partial_j u_i$; for a tensor field S , $\text{div } S$ is the vector field with components $\partial_j S_{ij}$. Given any function $W: M^3 \rightarrow R$,

$$\frac{\partial W}{\partial F}(F) = \left(\frac{\partial W}{\partial F_{ij}}(F) \right) \in M^3 \quad \text{and} \quad \frac{\partial^2 W}{\partial F^2}(F) = \left(\frac{\partial^2 W}{\partial F_{ij} \partial F_{kl}}(F) \right)$$

denote, respectively, the gradient and the second derivative of W , where M^3 is the set of all matrices of order 3.

If Ω is an open subset of R^3 , the space of r -times continuously differentiable functions $u: \Omega \rightarrow R$ is denoted by $C^r(\Omega)$. The subspace of $C^r(\Omega)$ consisting of infinitely differentiable functions with compact support in Ω is denoted $D(\Omega)$. The standard Sobolev spaces are denoted $W^{m,p}(\Omega)$, $W_0^{m,p}(\Omega) = \overline{D(\Omega)}$. It is always understood that a product of normed spaces is equipped with the product norm. We write $B_p(x, r)$ for the open ball of radius r centered at x in the space $[W^{m,p}(\Omega)]^3$.

Let Ω be a bounded open connected subset of R^3 whose boundary Γ is « sufficiently smooth ». We will use $\nu = (\nu_i)$ to denote the unit outer normal vector along the boundary Γ . We consider an isotropic, homogeneous elastic body which in the absence of applied forces occupies the reference configuration $\bar{\Omega}$. We assume that the body is subjected to volumic

forces, of density $f = (f_i)$ on Ω , and to surface forces, of density $g = (g_i)$ on the portion Γ_1 of the boundary Γ . When subject to applied forces, the body occupies a deformed configuration $\phi(\bar{\Omega})$, characterized by a mapping $\phi : \bar{\Omega} \rightarrow R^3$ that must be, in particular, orientation-preserving in the set $\bar{\Omega}$ and injective on the set Ω , in order to be physically acceptable. Such mapping ϕ , unknown of the elasticity problem, is called deformation. We will assume that both kind of applied forces are dead loads, i.e., both densities are independent of the deformation ϕ to which the body is subjected.

Combining the equations of equilibrium in the reference configuration, expressed in terms of the first Piola-Kirchhoff stress tensor, with the definition of an elastic material, and assuming fixed the deformation ϕ on the portion $\Gamma_0 = \Gamma \setminus \Gamma_1$ of the boundary, we obtain the following non-linear boundary value problem :

$$(1.1) \quad - \operatorname{div} \hat{T}(\nabla\phi(x)) = f(x), \quad x \in \Omega,$$

$$(1.2) \quad \hat{T}(\nabla\phi(x)) \nu = g(x), \quad x \in \Gamma_1,$$

$$(1.3) \quad \phi(x) = \phi_0(x), \quad x \in \Gamma_0,$$

where $\hat{T} : M_+^3 \rightarrow M^3$ is the response function associated with the first Piola-Kirchhoff stress tensor, and $M_+^3 = \{F \in M^3; \det F > 0\}$. Equations (1.1), (1.2) are formally equivalents to the principle of virtual work in the reference configuration, expressed by the equation

$$(1.4) \quad \int_{\Omega} \hat{T}(\nabla\phi(x)) : \nabla\theta(x) \, dx = \int_{\Omega} f(x) \cdot \theta(x) \, dx + \int_{\Gamma_1} g(x) \cdot \theta(x) \, d\Gamma,$$

valid for all sufficiently regular vector fields $\theta : \bar{\Omega} \rightarrow R^3$ which vanish on Γ_0 . In (1.4) « : » denotes the matrix inner product $A : B = A_{ij} B_{ij}$, and « . » the Euclidean inner product.

If we assume that the material is hyperelastic, i.e., there exists a stored energy function $\hat{W} : M_+^3 \rightarrow R$ such that :

$$(1.5) \quad \hat{T}(F) = \frac{\partial \hat{W}}{\partial F}(F), \quad \text{for all } F \in M_+^3,$$

equation (1.4) is equivalent to the Euler-Lagrange condition applied to the functional $I = W - (F + G)$, where

$$(1.6) \quad W(\phi) = \int_{\Omega} \hat{W}(\nabla\phi) \, dx,$$

$$(1.7) \quad F(\phi) = \int_{\Omega} f(x) \cdot \phi(x) \, dx,$$

$$(1.8) \quad G(\phi) = \int_{\Gamma_1} g(x) \cdot \phi(x) \, d\Gamma.$$

The functional W is called the strain energy, while the functional I is called the total energy. In consequence, particular solutions may be obtained by looking for minimizers of the total energy, i.e., for particular deformations ϕ that satisfy

$$(1.9) \quad \phi = \phi_0 \text{ on } \Gamma_0 \quad \text{and} \quad I(\phi) = \inf_{\psi = \phi_0 \text{ on } \Gamma_0} I(\psi).$$

The constitutive law for elastic materials, expressed above in terms of the first Piola-Kirchhoff stress tensor $\hat{T}(F)$, can be also written in terms of the second Piola-Kirchhoff stress tensor $\hat{\Sigma}(F) = F^{-1} \hat{T}(F)$. The Rivlin-Ericksen theorem insures the existence of a function $\tilde{\Sigma} : S^3_> \rightarrow S^3$ (S^3 denotes the set of all symmetric matrices of order 3, and $S^3_>$ those positive definite) of the form

$$\tilde{\Sigma}(C) = \gamma_0(i_C) I + \gamma_1(i_C) C + \gamma_2(i_C) C^2, \quad C = F^T F$$

such that $\tilde{\Sigma}(C) = \hat{\Sigma}(F)$ for all $F \in M^3_+$, γ_α being real-valued functions of the three principal invariants of C . If the functions γ_α are differentiable at the point $i_I = (3, 3, 1)$ and the reference configuration is a natural state ($\bar{\Omega}$ is an unstressed state), then there exist constants λ, μ such that $\tilde{\Sigma}(C)$ admits the development :

$$\tilde{\Sigma}(C) = \lambda(\text{tr } E) I + 2 \mu E + o(|E|).$$

The matrix C is called the right Cauchy-Green strain tensor, while the matrix E is called the Green-St. Venant strain tensor.

As already noted, the tensor $\tilde{\Sigma}(E) = \tilde{\Sigma}(I + 2 E)$ is defined only for the tensors E that belong to the set

$$\mathcal{E} = \left\{ \frac{1}{2} (C - I) \in S^3, C \in S^3_> \right\}$$

which is a neighbourhood of the origin in the space S^3 . In order to simplify the presentation, we shall assume that the tensor $\tilde{\Sigma}$ can be extended with regularity to all M^3 , but, except for notational technicalities, all of the results in this paper are also valid for the constitutive laws $\tilde{\Sigma}$ defined only in \mathcal{E} (cf. Ciarlet [1987]).

The implicit function theorem allows us to establish the existence and uniqueness of small solutions to boundary value problems (1.1)-(1.3) with $\Gamma_1 = \emptyset$ and small body forces (a pure displacement problem).

THEOREM 1.1 : *If the response function associated with the second Piola-Kirchhoff stress tensor verifies :*

- i) $\check{\Sigma}(E) \in C^2(M^3; M^3),$
- ii) $\check{\Sigma}(E) = \lambda(\text{tr } E) I + 2 \mu E + o(|E|), \quad \lambda, \mu > 0,$

then, for each number $p > 3$ there exist a neighbourhood F^p of the origin in the space $[L^p(\Omega)]^3$ and a neighbourhood $U^p \subset B_p(0, 1)$ of the origin in the space

$$(1.10) \quad V^p(\Omega) = \{v \in [W^{2,p}(\Omega)]^3, v = 0 \text{ on } \Gamma\},$$

such that, for all function $f \in F^p$, the boundary value problem (1.1)-(1.3) with $\Gamma_1 = \emptyset$ and $\phi_0 = Id$ admits an unique solution $u(f)$ in U^p , where $u(f)$ is the displacement associated with the deformation $\phi(f)$ ($= Id + u(f)$). \square

The deformation $\phi(f)$ given by theorem 1.1, is acceptable physically in the sense that it preserves the orientation and it is injective.

This analysis can be also applied to pure traction problems ($\Gamma_0 = \emptyset$), but in this case it is necessary to introduce certain compatibility conditions with regard to the applied forces (cf. Chillingworth, Marsden and Wan [1982, 1983], Le Dret [1985]). The need for a $W^{2,p}, p > 3$ -regularity for the solutions of the corresponding linearized system limits the domain of application of this result and it is, for example, impossible to apply it to mixed problems of displacement-traction, except in some very particular cases. The extension of the theorem 1.1 to constitutive laws $\check{\Sigma}$ defined only in a neighbourhood of the origin (in order to take into consideration singular behavior when $\det F \rightarrow 0^+$) offers no difficulty (cf. Ciarlet [1987]).

If the material is hyperelastic, particular solutions may be obtained as minimizers of the energy over an appropriate set of functions. However, the non-convexity of the stored energy function \hat{W} (convexity is incompatible with the axiom of the material frame-indifference and with the singular behavior as $\det F \rightarrow 0^+$; cf. Ciarlet [1987, theorem 4.8-1]) is an obstacle to the application of classical existence theorems in convex analysis. In order to resolve this difficulty, Ball [1977] introduced the concept of polyconvexity. A function $\hat{W}: M_+^3 \rightarrow R$ is polyconvex if

$$(1.11) \quad \hat{W}(F) = G(F, \text{cof } F, \det F), \quad \text{for all } F \in M_+^3$$

where $G : M^3 \times M^3 \times (0, +\infty) \rightarrow R$ is a convex function and $\text{cof } F$ is the matrix of cofactors of F . For this class of stored energy functions, which does not include the St. Venant-Kirchhoff material, Ball has obtained the existence of a weak minimizer to the energy.

THEOREM 1.2 : *Let $\hat{W} : M_+^3 \rightarrow R$ be a stored energy function verifying the following hypotheses :*

- i) *Polyconvexity*
- ii) *Behavior as $\det F \rightarrow 0^+$: for almost all $x \in \Omega$,*

$$(1.12) \quad \lim_{\det F \rightarrow 0^+} \hat{W}(x, F) = +\infty$$

iii) *Coercivity : there exist $a \in R$, $b > 0$, $p \geq 2$, $q \geq \frac{p}{p-1}$ and $r > 1$ such that :*

$$(1.13) \quad \hat{W}(F) \geq a + b(\|F\|^p + \|\text{cof } F\|^q + (\det F)^r), \quad \text{for all } F \in M_+^3.$$

Let $f \in [L^p(\Omega)]^3$ and $g \in [L^\sigma(\Gamma_1)]^3$ be such that the linear forms :

$$(1.14) \quad \psi \in [W^{1,p}(\Omega)]^3 \rightarrow \int_{\Omega} f \cdot \psi \, dx,$$

$$(1.15) \quad \psi \in [W^{1,p}(\Omega)]^3 \rightarrow \int_{\Gamma_1} g \cdot \psi \, d\Gamma,$$

are continuous. We assume that the area $\Gamma_0 > 0$. Let $\phi_0 \in [W^{1,p}(\Omega)]^3$ be given in such a way that the set

$$A = \{\psi \in [W^{1,p}(\Omega)]^3, \text{cof } \nabla \psi \in [L^q(\Omega)]^9, \det \nabla \psi \in L^r(\Omega),$$

$$\det \nabla \psi > 0 \text{ a.e. in } \Omega, \psi = \phi_0 \text{ on } \Gamma_0\},$$

is not empty. Consider finally the total energy functional defined in (1.9) and assume that :

$$(1.17) \quad \inf_{\psi \in A} I(\psi) < +\infty.$$

Then the problem : Find $\phi \in A$ such that

$$(1.18) \quad I(\phi) = \inf_{\psi \in A} I(\psi)$$

has at least one solution. \square

A restriction of this existence result is that it is not known whether the solutions of the associated minimization problem verify, even in a weak sense, the equilibrium equations. A search of the literature reveals that in general the theorems guaranteeing this make stronger growth assumptions on the stored energy than are necessary to prove existence. Furthermore, even if the solutions obtained verify $\det \nabla \phi > 0$ a.e. in Ω , additional conditions are needed to insure the global injectivity of ϕ (see Ball [1981] and Ciarlet-Nečas [1985]).

Contrary to the existence result obtained by the implicit function theorem, here are no restrictions on the magnitude of the applied forces nor on the types of boundary conditions: the mixed displacement-traction problem, pure displacement and pure traction. In the last case, it is necessary to include an additional condition into the definition of set A (1.16), such as

$$\int_{\Omega} \psi \, dx = \vec{e}$$

\vec{e} being a constant vector in R^3 , in order to preserve the coercivity of the total energy (Ball [1977]).

2. POSITIVITY OF THE SECOND DERIVATIVE OF THE TOTAL ENERGY AT THE SOLUTION FOUND BY THE IMPLICIT FUNCTION THEOREM

Let us consider a hyperelastic, homogeneous and isotropic material, with stored energy function \hat{W} , whose reference configuration $\bar{\Omega}$ is a natural state. It is shown here that if the response function associated with the Cauchy stress tensor satisfies some particular assumptions, which are satisfied by Ogden and St. Venant-Kirchhoff materials, the second derivative of the energy functional is positive definite at the solution found by the implicit function theorem.

Throughout this article, the stored energy function will be expressed in terms of the three principal invariants of the right Cauchy-Green strain tensor $C = F^T F$, or equivalently, in terms of the Green-St. Venant strain tensor $E = \frac{1}{2} (C - I)$. We use the following notational devices to indicate which functional dependence is considered :

$$\hat{W} : F \in M_+^3 \rightarrow \hat{W}(F) \in R ,$$

$$\tilde{W} : i_C \in i(S_+^3) \rightarrow \tilde{W}(i_C) \in R ,$$

$$\check{W} : E \in \mathcal{E} \rightarrow \check{W}(E) \in R ,$$

where

$$i(S_{>}^3) = \{i_A \in R^3; A \in S_{>}^3\},$$

$i : S_{>}^3 \rightarrow (0, +\infty)^3$ being the mapping assigning to a matrix its principal invariants, and where the set \mathcal{E} is defined by :

$$(2.1) \quad \mathcal{E} = \{E : 2E + I \in S_{>}^3\}.$$

The axiom of material frame-indefference implies that there exists a function \tilde{W} such that :

$$(2.2) \quad \hat{W}(F) = \tilde{W}(i_C), \quad \text{for all } F \in M_+^3 \text{ and } C = F^T F.$$

Thus, \tilde{W} and be defined by the formula :

$$(2.3) \quad \check{W}(E) = \tilde{W}(i_{2E+I}), \quad E \in \mathcal{E},$$

(cf. Ciarlet [1986]).

In the following, we shall assume that $\check{\Sigma}(E) \in C^2(M^3, M^3)$ and that the Lamé constants λ, μ are > 0 . Then, if the applied body forces f belong to the neighbourhood $F^p(p > 3)$ given by theorem 1.1, there exists an unique solution $u(f) \in U^p \subset V^p$ of the boundary value problem (1.1)-(1.3) with $\Gamma_1 = \emptyset$ and $\phi_0 = Id$ (see remark 3.1).

Let I be the energy functional defined in $[W_0^{1,p}(\Omega)]^3$ by :

$$(2.4) \quad I(u) = \int_{\Omega} \check{W}(I + \nabla u) \, dx - \int_{\Omega} f \cdot (Id + u) \, dx,$$

where $3 < p_0 \leq p \leq +\infty$, with p_0 depending on the constitutive law of the material, it simply has to be large enough so that the energy is finite on the set A^p of admissible displacements given by :

$$(2.5) \quad A^p = \{u \in [W_0^{1,p}(\Omega)]^3, \det(I + \nabla u) > 0 \text{ a.e. in } \Omega\},$$

(since \hat{W} is continuous, then at least $p_0 = +\infty$).

In particular, the total energy functional for a St. Venant-Kirchhoff material,

$$\begin{aligned} I : u \in [W_0^{1,p}(\Omega)]^3 &\rightarrow I(u) = \\ &= \int_{\Omega} \left[\frac{\lambda}{2} (\text{tr } E)^2 + \mu \text{tr } E^2 \right] dx - \int_{\Omega} f \cdot (Id + u) \, dx, \end{aligned}$$

is well defined for all p satisfying $4 \leq p \leq +\infty$.

THEOREM 2.1: *With the same assumptions as in theorem 1.1, let $u(f)$ denote the unique solution in $U^p \subset V^p$ of the boundary value problem :*

$$(2.6) \quad A(u) = - \operatorname{div} [(I + \nabla u) \check{\Sigma}(E(u))] = f, \quad f \in F^p.$$

Then, $u(f)$ is a critical point of the energy functional I defined in (2.4), in the sense that

$$(2.7) \quad I'(u(f)) \theta = 0,$$

for all vector fields $\theta \in [W_0^{1,p}(\Omega)]^3$, with $p_0 \leq p \leq +\infty$.

Proof: Let $u \in [W^{2,p}(\Omega)]^3 \cap A^p$ and $\theta \in [W_0^{1,p}(\Omega)]^3$. We show that :

$$(2.8) \quad I'(u) \theta = \int_{\Omega} \frac{\partial \check{W}}{\partial F} (I + \nabla u) : \nabla \theta \, dx - \int_{\Omega} f \cdot \theta \, dx.$$

The mapping

$$u \in [W^{2,p}(\Omega)]^3 \rightarrow F \check{\Sigma}(E) \in [W^{1,p}(\Omega)]^9$$

is well defined, because the Sobolev space $W^{1,p}(\Omega)$ is an algebra for $p > 3$ (see e.g. Adams [1975]) and furthermore it is of class C^1 when $\check{\Sigma} \in C^2(M^3, M^3)$ (cf. Valent [1979]). Moreover,

$$\frac{\partial \check{W}}{\partial F} (I + \nabla u) = (I + \nabla u) \check{\Sigma}(E(u)) \in [W^{1,p}(\Omega)]^9,$$

$$\text{for all } u \in [W^{2,p}(\Omega)]^3 \cap A^p.$$

From this relation, we deduce that the mapping :

$$(2.9) \quad u \in [W^{2,p}(\Omega)]^3 \cap A^p \rightarrow \frac{\partial \check{W}}{\partial F} (I + \nabla u) \in [W^{1,p}(\Omega)]^9$$

is of class C^1 . Expression (2.8) is then follows by a simple computation.

On the other hand, the deformation associated with the displacement $u(f) \in [W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)]^3$ preserves the orientation. Then, $u(f) \in A^p$ and, moreover,

$$(2.10) \quad \frac{\partial \check{W}}{\partial F} (I + \nabla_x u(f)) = \hat{T}(I + \nabla_x u(f)),$$

where $\nabla_x u(f)$ denotes the gradient of $u(f)$ with respect to x .

If we substitute (2.10) in (2.8) we obtain the principle of virtual work corresponding to the boundary value problem (2.6). Hence the theorem is proved. \square

In order to establish the positivity of the second derivative of the total energy at $u(f)$, we need the following lemmas.

LEMMA 2.1 : Let $\tilde{W} : i(S_{>}^3) \rightarrow R$ be of class C^1 . Then, the response associated with the first Piola-Kirchhoff stress tensor is of the form :

$$(2.11) \quad \hat{T}(F) = 2[\delta_1 F + \delta_2(i_1 I - FF^T) F + \delta_3 i_3 F^{-T}] ,$$

for all $F \in M_+^3$, δ_k being the functions defined by :

$$(2.12) \quad \delta_k = \frac{\partial \tilde{W}}{\partial i_k}(i_C) , \quad i_k = i_k(C) .$$

(see Ciarlet [1987], theorem 4.4-2). \square

LEMMA 2.2 : Let $\tilde{W} : i(S_{>}^3) \rightarrow R$ be of class C^2 , then the bilinear mapping $\frac{\partial^2 \tilde{W}}{\partial F^2}(I) : M^3 \times M^3 \rightarrow R$ is given by :

$$(2.13) \quad \frac{\partial^2 \tilde{W}}{\partial F^2}(I)(H, K) = 2(\delta_1 + \delta_2)|_{i_i}(H : K) + 2(-\delta_2 - \delta_3)|_{i_i}(H^T : K) + \\ + 4(\delta_2 + \delta_3 + \delta_{11} + 4\delta_{12} + 2\delta_{13} + 2\delta_{22} + 4\delta_{23} + \delta_{33})|_{i_i}(I : H)(I : K) ,$$

for all $H, K \in M^3$, δ_{jk} being the functions defined by :

$$(2.14) \quad \delta_{jk} = \frac{\partial^2 \tilde{W}}{\partial i_j \partial i_k}(i_C) , \quad i_j = i_j(C) .$$

Proof: Let $F \in M_+^3$. We may apply lemma 2.1 to obtain the following expression for the linear mapping $\frac{\partial \hat{W}}{\partial F}(F) : M^3 \rightarrow R$:

$$(2.15) \quad \left(\frac{\partial \hat{W}}{\partial F}(F) : G \right) = 2\delta_1(F : G) + 2\delta_2[(i_1 I - FF^T) F : G] + \\ + 2\delta_3 i_3(F^{-T} : G) ,$$

for all $G \in M^3$, δ_i being the functions defined in (2.12). From this expression, it follows by a simple computation that the linear mapping

$$(2.16) \quad \frac{\partial^2 \hat{W}}{\partial F^2}(F) : H \in M^3 \rightarrow \frac{\partial^2 \hat{W}}{\partial F^2}(F) H \in L(M^3, R)$$

is of the form :

$$(2.17) \quad \frac{\partial^2 \hat{W}}{\partial F^2}(F) H = 2 \delta_1 H + 2 \delta_2 (i_1 H - H F^T F - F H^T F - F F^T H) + \\ + 2 \delta_3 i_3 (-F^{-T} H^T F^{-T}) + 2 \delta_{1k} \left(\frac{\partial i_k(C)}{\partial F} H \right) F \\ + 2 \delta_{2k} \left(\frac{\partial i_k(C)}{\partial F} H \right) (i_1 F - F F^T F) + 2 i_3 \delta_{3k} \left(\frac{\partial i_k(C)}{\partial F} H \right) F^{-T} \\ + 2 \delta_2 \left(\frac{\partial i_1(C)}{\partial F} H \right) F + 2 \delta_3 \left(\frac{\partial i_3(C)}{\partial F} H \right) F^{-T},$$

for all $F \in M_+^3$, $H \in M^3$. On the other hand, we have (cf. Ciarlet [1987]) :

$$(2.18) \quad \frac{\partial i_1(C)}{\partial F} G = (2 F : G),$$

$$(2.19) \quad \frac{\partial i_2(C)}{\partial F} G = 2 (i_1(C) F - F F^T F : G),$$

$$(2.20) \quad \frac{\partial i_3(C)}{\partial F} G = 2 (i_3 F^{-T} : G).$$

Replacing (2.18)-(2.20) in expression (2.17) and letting $F = I$, we obtain the result. \square

LEMMA 2.3 : Let $\tilde{W} : i(S_>^3) \rightarrow R$ be of class C^2 , such that the following inequalities :

$$(2.21) \quad (\delta_2 + \delta_3)|_{i_l} < 0,$$

$$(2.22) \quad (\delta_2 + \delta_3 + \delta_{11} + 4 \delta_{12} + 2 \delta_{13} + 2 \delta_{22} + 4 \delta_{23} + \delta_{33})|_{i_l} \geq 0,$$

hold. Then, there exists a constant $\alpha > 0$ such that :

$$(2.23) \quad \frac{\partial^2 \hat{W}}{\partial F^2}(I)(\nabla \theta, \nabla \theta) \geq \alpha \varepsilon_{ij}^2(\theta),$$

for all $\theta \in [W_0^{1,p}(\Omega)]^3$, where the functions ε_{ij} are the components of the linearized strain tensor : $2 \varepsilon_{ij}(\theta) = \partial_i \theta_j + \partial_j \theta_i$.

Proof: From relation

$$(2.24) \quad 2 \varepsilon_{ij}^2(\theta) = \partial_i \theta_j \partial_i \theta_j + \partial_i \theta_j \partial_j \theta_i = (\nabla \theta^T : \nabla \theta) + (\nabla \theta : \nabla \theta),$$

we deduce that

$$(2.25) \quad (\nabla \theta^T : \nabla \theta) = 2 \varepsilon_{ij}^2(\theta) - (\nabla \theta : \nabla \theta).$$

On the other hand, the reference configuration $\bar{\Omega}$ is a natural state. This means that $\hat{T}(I) = 0$ and that, by expression (2.11),

$$(2.26) \quad (\delta_1 + 2\delta_2 + \delta_3)|_{i_l} = 0.$$

Substituting relations (2.25) (2.26) in expression (2.13) combined with hypotheses (2.21) (2.22), we conclude the proof of this lemma. \square

The preceding lemmas allow us to obtain a result which shows the positivity of the second derivative of the total energy at the solution found by the implicit function theorem.

THEOREM 2.2: *Assume that the stored energy function \tilde{W} is of class C^2 and that it verifies relations (2.21) (2.22). If $\tilde{\Sigma}(E) \in C^2(M^3, M^3)$ then, for each p , $p_0 \leq p \leq +\infty$, there exist a neighbourhood of the origin \tilde{F}^p in the space $[L^p(\Omega)]^3$ and a constant $\beta > 0$ such that, for each function $f \in \tilde{F}^p$ the solution $u(f)$ of the boundary value problem (2.6) verifies :*

$$(2.27) \quad I''(u(f))(\theta, \theta) \geq \beta \|\theta\|_{1,2,\Omega}^2,$$

for all vector fields $\theta \in [W_0^{1,p}(\Omega)]^3$.

Proof: We first compute the second Gâteaux derivative of the functional I . Let $u \in [W^{2,p}(\Omega)]^3 \cap A^p$ and $v, \theta \in [W_0^{1,p}(\Omega)]^3$, with $p_0 \leq p \leq +\infty$, be given. From relation (2.8) it follows that :

$$(2.28) \quad I''(u)(v, \theta) = \int_{\Omega} \frac{\partial^2 \hat{W}}{\partial F^2} (I + \nabla u)(\nabla v, \nabla \theta) dx.$$

If $f = 0$, the function $u(0) = 0$ is the unique solution of the boundary value problem (2.6) in the neighbourhood $U^p \subset V^p$ of the origin. The associated deformation gradient is the identity ; hence from lemma 2.3, it can be deduced that there exists $\alpha > 0$ such that

$$(2.29) \quad I''(0)(\theta, \theta) = \int_{\Omega} \frac{\partial^2 \hat{W}}{\partial F^2} (I)(\nabla \theta, \nabla \theta) dx \geq \int_{\Omega} \alpha \varepsilon_{ij}^2(\theta) dx,$$

ε_{ij} being the linealized strain tensor defined in lemma 2.3. Next, as a consequence of Korn inequality, the seminorm $|\varepsilon(\cdot)|_{0,\Omega}$ is a norm on the space $[H_0^1(\Omega)]^3$ equivalent to the norm $\|\cdot\|_{1,2,\Omega}$; therefore there exists a constant $\bar{\alpha} > 0$ so that :

$$(2.30) \quad I''(0)(\theta, \theta) \geq \bar{\alpha} \|\theta\|_{1,2,\Omega}^2,$$

for all vector fields $\theta \in [W_0^{1,p}(\Omega)]^3$, $p_0 \leq p \leq +\infty$.

As can be seen from the proof of theorem 2.1, the mapping :

$$(2.31) \quad u \in A^p \cap [W^{2,p}(\Omega)]^3 \rightarrow \frac{\partial \hat{W}}{\partial F} (I + \nabla u) \in [W^{1,p}(\Omega)]^9$$

is of class C^1 ; so, given $\varepsilon = \frac{\bar{\alpha}}{2}$, there exists $\delta > 0$ such that $\|u\|_{2,p,\Omega} \leq \delta$ implies :

$$(2.32) \quad \left\| \frac{\partial^2 \hat{W}}{\partial F^2} (I + \nabla u) - \frac{\partial^2 \hat{W}}{\partial F^2} (I) \right\| \leq \frac{\bar{\alpha}}{2}.$$

Then, there exists a neighbourhood of the origin $\tilde{F}^p \subset F^p$ in the space $[L^p(\Omega)]^3$ such that the inequality (2.32) holds for all $u(f)$, $f \in \tilde{F}^p$. To see this, it suffices to observe that the implicit function

$$(2.33) \quad f \in F^p \subset [L^p(\Omega)]^3 \rightarrow u(f) \in U^p \subset [W^{2,p}(\Omega)]^3,$$

is continuous.

From (2.28) we deduce :

$$(2.34) \quad \begin{aligned} I''(u(f))(\theta, \theta) &= \int_{\Omega} \frac{\partial^2 \hat{W}}{\partial F^2} (I + \nabla_x u(f))(\nabla \theta, \nabla \theta) dx \\ &= \int_{\Omega} \frac{\partial^2 \hat{W}}{\partial F^2} (I)(\nabla \theta, \nabla \theta) dx + \int_{\Omega} \left[\frac{\partial^2 \hat{W}}{\partial F^2} (I + \nabla_x u(f)) \right. \\ &\quad \left. - \frac{\partial^2 \hat{W}}{\partial F^2} (I) \right] (\nabla \theta, \nabla \theta) dx, \end{aligned}$$

and the conclusion follows from relations (2.30) and (2.32) with $\beta = \frac{\bar{\alpha}}{2}$. \square

By the Rivlin-Ericksen theorem, there exists a mapping $\bar{T} : S^3_{>} \rightarrow S^3$ of the form :

$$(2.35) \quad \bar{T} : B \in S^3_{>} \rightarrow \bar{T}(B) = \beta_0(i_B) I + \beta_1(i_B) B + \beta_2(i_B) B^2,$$

such that $\bar{T}(B)$ is the response function associated with the Cauchy stress tensor. $B = FF^T$ is the left Cauchy-Green strain tensor and β_i are real functions of the three principal invariants of the matrix B . In the following, we shall express conditions (2.21) (2.22) in terms of the functions β_i .

THEOREM 2.3 : *Consider an elastic, homogeneous and isotropic material whose response function associated with the Cauchy stress tensor $\bar{T}(B)$ is defined as in (2.35) and verifies the hypotheses :*

a) The functions $\beta_i : (0, +\infty)^3 \rightarrow R$ are of class C^2 .

b) The relations

$$(2.36) \quad \partial_1 \tilde{\beta}_2 + i_1 \partial_2 \tilde{\beta}_2 + \partial_2 \tilde{\beta}_1 = 0 ,$$

$$(2.37) \quad \partial_2 \tilde{\beta}_0 + i_3 \partial_3 \tilde{\beta}_2 = 0 ,$$

$$(2.38) \quad i_3 \partial_3 \tilde{\beta}_1 + i_1 i_3 \partial_3 \tilde{\beta}_2 - \partial_1 \tilde{\beta}_0 = 0 ,$$

hold with $\tilde{\beta}_\alpha = \frac{\beta_\alpha \sqrt{i_3}}{2}$.

Then, the material is hyperelastic.

If the hypotheses of theorem 1.1 are verified, and if the inequalities

$$(2.39) \quad (\beta_0 - \beta_2)|_{i_t} < 0 ,$$

$$(2.40) \quad (-\beta_0 + 2\beta_2 + 4\partial_1\beta_0 + 8\partial_2\beta_0 + 2\partial_3\beta_0 + 2\partial_1\beta_1 + 4\partial_1\beta_2 - 4\partial_2\beta_2)|_{i_t} \geq 0 ,$$

are also verified, then there exist a neighbourhood $\tilde{F}^p \subset F^p$ of the origin in the space $[L^p(\Omega)]^3$ and a constant $\beta > 0$ such that for each function $f \in \tilde{F}^p$, the solution $u(f)$ of the boundary value problem (2.6) verifies :

$$(2.41) \quad I''(u(f))(\theta, \theta) \geq \beta \|\theta\|_{1,2,\Omega}^2 ,$$

for all vector fields $\theta \in [W_0^{1,p}(\Omega)]^3$.

Proof: Consider the functions $\delta_i : (0, +\infty)^3 \rightarrow R$ given by :

$$(2.42) \quad \delta_1 = \frac{1}{2} (\sqrt{i_3} \beta_1 + i_1 \sqrt{i_3} \beta_2) ,$$

$$(2.43) \quad \delta_2 = -\frac{1}{2} \sqrt{i_3} \beta_2 ,$$

$$(2.44) \quad \delta_3 = \frac{1}{2\sqrt{i_3}} \beta_0 .$$

The rotational of the field $(\delta_1(i_j), \delta_2(i_j), \delta_3(i_j))$ is zero thanks to hypotheses (2.36)-(2.38). It follows from Poincaré lemma that there exists a function $\tilde{W} : i(S_>^3) \rightarrow R$ such that :

$$\frac{\partial \tilde{W}}{\partial i_k}(i_C) = \delta_k(i_C) ,$$

the functions δ_k being those defined in (2.42)-(2.44). Thus, the material is hyperelastic and its stored energy function is $\hat{W}: M_+^3 \rightarrow R$ defined by:

$$\hat{W}(F) = \tilde{W}(i_C), \quad F \in M_+^3.$$

Furthermore, it can be deduced from condition a) that $\tilde{W}: i(S_>^3) \rightarrow R$ is of class C^3 .

If $u(f)$ is the solution of the boundary value problem (2.6) given by theorem 1.1, the conclusion follows by theorem 2.2, since relations (2.21) (2.22) are verified for the functions δ_i defined by (2.42)-(2.44) as a consequence of conditions (2.39) (2.40). \square

3. RELATION BETWEEN ENERGY MINIMIZERS IN $[W_0^{1,\infty}(\Omega)]^3$ AND THE SOLUTIONS FOUND BY THE IMPLICIT FUNCTION THEOREM

In paragraph 2 it has been shown that the solution $u(f)$ given by the implicit function theorem verifies:

$$(3.1) \quad I'(u(f))\theta = 0, \quad \text{for all } \theta \in [W_0^{1,p}(\Omega)]^3, \quad 3 < p_0 \leq p \leq +\infty,$$

and that there exists a constant $\beta > 0$ such that

$$(3.2) \quad I''(u(f))(\theta, \theta) \geq \beta \|\theta\|_{1,2,\Omega}^2, \quad \text{for all } \theta \in [W_0^{1,p}(\Omega)]^3,$$

We will now prove that conditions (3.1) (3.2) imply that $u(f)$ is a local minimizer in the space $[W_0^{1,\infty}(\Omega)]^3$. In paragraph 5, a counterexample will show that this result does not hold in general with respect to the topologies of the spaces $W^{1,p}$, $3 < p < \infty$ ($p > 3$ is not a restriction, it is to insure the existence of $u(f)$).

THEOREM 3.1: *Assume that the energy density $\tilde{W}: i(S_>^3) \rightarrow R$ is of class C^2 and that it satisfies relations (2.21) (2.22). If the mapping $\tilde{\Sigma} \in C^2(M^3, M^3)$ verifies:*

$$(3.3) \quad \tilde{\Sigma}(E) = \lambda(\text{tr } E)I + 2\mu E + 0(|E|^2),$$

with $\lambda > 0$, $\mu > 0$, then for each function $f \in \tilde{F}^\infty$, the solution $u(f)$ of the boundary value problem given by theorem 2.2 is a strict local minimum, in the space $[W_0^{1,\infty}(\Omega)]^3$, of the energy functional defined in (2.4).

Proof: A Taylor expansion of the functional I around $u(f)$ yields:

$$(3.4) \quad I(u(f) + v) - I(u(f)) = \int_0^1 (1-t) I''(u(f) + tv)(v, v) dt.$$

The function $\hat{W}: M_+^3 \rightarrow R$ is of class C^3 , since $\hat{W}(F) = F\hat{\Sigma}(E)$ and $\hat{\Sigma} \in C^2(M^3, M^3)$. Then

$$(3.5) \quad I''(u)(\theta, \theta) = \int_{\Omega} \frac{\partial^2 \hat{W}}{\partial F^2} (I + \nabla u)(\nabla \theta, \nabla \theta) dx,$$

for all $u \in A^\infty$ and for all vector fields $\theta \in [W_0^{1,\infty}(\Omega)]^3$. On the other hand, $u(f) \in B_\infty(0, 1) \subset V^\infty$. Then, if

$$(3.6) \quad \rho_0 = \text{dis}(u(f), \partial B_\infty(0, 1)) > 0,$$

it follows that

$$(3.7) \quad \sup_{x \in \Omega} \text{ess} |\nabla_x u(f) + \nabla v|(x) < 1,$$

$$\text{for all } v \in B_\infty\left(0, \frac{\rho_0}{2}\right) \subset [W_0^{1,\infty}(\Omega)]^3.$$

Consequently, the matrices $(I + \nabla_x u(f) + t(\nabla v))$, $0 \leq t \leq 1$, are invertible for almost all $x \in \Omega$, and thus:

$$(3.8) \quad u(f) + tv \in A^\infty, \quad 0 \leq t \leq 1.$$

In conclusion, relation (3.4) make sense for all $v \in B_\infty\left(0, \frac{\rho_0}{2}\right)$. Furthermore, the operator:

$$(3.9) \quad I'': A^\infty \rightarrow L([W_0^{1,\infty}(\Omega)]^3 \times [W_0^{1,\infty}(\Omega)]^3, R)$$

is continuous, hence there exists a positive number $\rho' \leq \rho_0/2$ such that for all vectors $v \in B_\infty(0, \rho')$:

$$(3.10) \quad \|I''(u(f) + tv) - I''(u(f))\| \leq \beta/2, \quad 0 \leq t \leq 1,$$

β being the positive constant given by theorem 2.2. From (3.4) and (3.10) it then follows that

$$(3.11) \quad I(u(f) + v) - I(u(f)) \geq \gamma \|v\|_{1,2,\Omega}^2, \quad v \in B_\infty(0, \rho'),$$

for some $\gamma > 0$. \square

COROLLARY 3.1: *Given $f \in \tilde{F}^\infty$, there exists $\varepsilon > 0$ so that:*

$$(3.12) \quad \inf_{\|v - u(f)\|_{1,\infty,\Omega} = \varepsilon} I(v) = I(u(f)).$$

Proof: From theorem 3.1 it can be concluded that for each $f \in \tilde{F}^\infty$, there exists $\varepsilon > 0$ such that :

$$(3.13) \quad 0 < \|v - u(f)\|_{1,\infty,\Omega} < \varepsilon \Rightarrow I(v) > I(u(f))$$

i.e., $u(f)$ is a strict local minimum of the total energy. Furthermore, for all $v \in B_\infty(u(f), \varepsilon)$

$$\begin{aligned} I(v) - I(u(f)) &= \int_0^1 (1-t) I''(u(f) + t(v - u(f))) \\ &\quad (v - u(f), v - u(f)) dt \\ &= \int_0^1 (1-t) \left(\int_\Omega \frac{\partial^2 \hat{W}}{\partial F^2} (I + t \nabla v + (1-t) \nabla_x u(f)) \right. \\ &\quad \left. (\nabla(v - u(f)), \nabla(v - u(f))) dx \right) dt, \end{aligned}$$

and, since $(I + t \nabla v + (1-t) \nabla_x u(f)) \in [L^\infty(\Omega)]^q$ and \hat{W} is of class C^3 , it follows that

$$(3.14) \quad I(v) - I(u(f)) \leq k \|\nabla v - \nabla_x u(f)\|_{0,2,\Omega}^2,$$

for some $k > 0$. The conclusion follows as a consequence of :

$$(3.15) \quad \inf_{\|v - u(f)\|_{1,\infty,\Omega} = \varepsilon} \|\nabla v - \nabla_x u(f)\|_{0,2,\Omega}^2 = 0$$

for an ε small enough. \square

Remark 3.1: The extension of the results obtained up to now to the following cases offers no difficulty :

1) Constitutive laws $\tilde{\Sigma}$ defined only in a neighbourhood of the origin. This allows us to take into consideration the singular behavior when $\det F \rightarrow 0^+$

$$\lim_{\det F \rightarrow 0^+} \hat{W}(F) = +\infty$$

of the stored energy density (see Ciarlet [1987]).

2) Pure traction problem : in this case, it is necessary to assume that the applied forces verify the compatibility condition,

$$\int_\Omega f \cdot v \, dx + \int_\Gamma g \cdot v \, d\Gamma = 0,$$

for all $v \in W = \{v \in [H^1(\Omega)]^3; \varepsilon(v) = 0\}$, where $\varepsilon(v)$ is the linearized strain tensor. In this case, the analysis must be carried out in the quotient space $\hat{V}^p(\Omega)$ defined by

$$\hat{V}^p(\Omega) = [W^{2,p}(\Omega)]^3 / W.$$

3) Mixed displacement-traction problems, when $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$.

4. EXAMPLES : ST. VENANT-KIRCHHOFF AND OGDEN MATERIALS

In this paragraph, we show that the constitutive equations for St. Venant-Kirchhoff and Ogden materials verify the hypotheses of theorems 2.2 and 3.1.

Example 1 : St. Venant-Kirchhoff material

The constitutive law of a St. Venant-Kirchhoff material is

$$(4.1) \quad \check{\Sigma}(E) = \lambda(\text{tr } E) I + 2 \mu E,$$

$\lambda > 0$ and $\mu > 0$ being the Lamé constants of the material. It is an elastic, homogeneous material and its reference configuration is a natural state :

$$\hat{T}(I) = I \check{\Sigma}(0) = 0.$$

The response function associated with the second Piola-Kirchhoff stress tensor is :

$$\check{\Sigma}(C) = \left[\frac{\lambda}{2} (i_1(C) - 3) \right] I + \mu C, \quad C = I + 2 E.$$

This shows that this material is isotropic and it verifies the axiom of material frame-indifference. Furthermore, it is hyperelastic, and its stored energy function is defined by :

$$(4.2) \quad \hat{W}(F) = \check{W}(E) = \frac{\lambda}{2} (\text{tr } E)^2 + \mu \text{tr } E^2,$$

or equivalently by :

$$(4.3) \quad \hat{W}(F) = \check{W}(i_C) = \left(\frac{\lambda + 2 \mu}{8} \right) (i_1(C) - 3)^2 + \mu (i_1(C) - 2) - \frac{\mu}{2} (i_2(C) - 3).$$

The function $\check{\Sigma} \in C^\infty(M^3, M^3)$ and it thus satisfies the hypotheses of theorem 1.1 ; in fact it is the simplest example of a constitutive law that

verifies the aforesaid theorem. The stored energy function \tilde{W} is of class $C^\infty(i(S^3_>), R)$ and the functions δ_j, δ_{jk} defined as in (2.12), (2.14) respectively, are given by :

$$(4.4) \quad \begin{cases} \delta_1(i_C) = \left(\frac{\lambda + 2\mu}{4} \right) (i_1(C) - 3) + \mu, \\ \delta_2(i_C) = -\mu/2, \\ \delta_3(i_C) = 0, \end{cases}$$

$$(4.5) \quad \begin{cases} \delta_{11}(i_C) = \frac{\lambda + 2\mu}{4}, \\ \delta_{jk}(i_C) = 0, \quad j, k \neq 1 \end{cases}$$

Thus, conditions (2.21) (2.22) are equivalent to :

$$\mu > 0, \lambda \geq 0.$$

On the other hand, the total energy functional is given by :

$$I : v \in [W_0^{1,p}(\Omega)]^3 \rightarrow I(v) = \int_{\Omega} \left[\frac{\lambda}{2} (\text{tr } E)^2 + \mu \text{tr } E^2 \right] dx - \int_{\Omega} f \cdot (Id + v) dx,$$

with $4 \leq p \leq +\infty$, where $2E = \nabla v + \nabla v^T + \nabla v^T \nabla v$. Then, there exists $\beta > 0$ such that the solution $u(f)$ given by the implicit function theorem verifies :

$$I'(u(f))\theta = 0, I''(u(f))(\theta, \theta) \geq \beta \|\theta\|_{1,2,\Omega}^2,$$

for all vector fields $\theta \in [W_0^{1,p}(\Omega)]^3$, and for every $f \in \tilde{F}^p, 4 \leq p \leq +\infty$. In addition, if $f \in \tilde{F}^\infty, u(f)$ is a strict local minimum of the energy in the space $[W_0^{1,\infty}(\Omega)]^3$.

Example 2 : Ogden materials

We next consider a family of Ogden materials whose constitutive laws satisfy

$$(4.6) \quad \tilde{\Sigma}(E) = \lambda(\text{tr } E)I + 2\mu E + 0(|E|^2),$$

for arbitrary Lamé constants $\lambda > 0$ and $\mu > 0$. The existence of the stored energy functions verifying (4.6) is guaranteed by the following result due to Ciarlet-Geymonat [1982].

THEOREM 4.1 : *Let $\lambda > 0, \mu > 0$ be two given Lamé constants. There exist stored energy functions of the form :*

$$(4.7) \quad F \in M_+^3 \rightarrow \hat{W}(F) = a \|F\|^2 + b \|\text{cof } F\|^2 + \Gamma(\det F) + e ,$$

with $a > 0, b > 0, \Gamma(\delta) = c\delta^2 - d \log \delta, c > 0, d > 0, e \in R$, which satisfy

$$\hat{W}(F) = \check{W}(E) = \frac{\lambda}{2} (\text{tr } E)^2 + \mu \text{tr } E^2 + o(|E|^2), I + 2 E = F^T F .$$

In addition, the constants a and b are given by :

$$(4.8) \quad \begin{aligned} a &= \mu + \frac{1}{2} \Gamma'(1) , \\ b &= -\frac{\mu}{2} - \frac{1}{2} \Gamma'(1) , \\ 2b &= \frac{\lambda}{2} - \frac{1}{2} \Gamma'(1) - \frac{1}{2} \Gamma''(1) . \quad \square \end{aligned}$$

This stored energy function \hat{W} clearly satisfies

$$\lim_{\det F \rightarrow 0^+} \hat{W}(F) = +\infty .$$

Now, in this case, we shall extend \hat{W} to all of M^3 , by assigning the value $+\infty$ to $\hat{W}(F)$ when $\det F \leq 0$. We further observe that the admissible deformations must be chosen in $[W_0^{1,p}(\Omega)]^3, p \geq 6$, in order that the energy functional be finite.

We shall now verify that for stored energy functions of the form (4.6), the hypotheses of theorem 2.2 hold. The stored energy function is given in terms of the three principal invariants of the right Cauchy-Green strain tensor by

$$(4.9) \quad \tilde{W}(i_C) = ai_1(C) + bi_2(C) + \Gamma(\sqrt{i_3(C)}) + e .$$

for all $i_C \in i(S_+^3)$. This expression allows us to calculate the functions $\delta_j(i_C), \delta_{jk}(i_C)$:

$$(4.10) \quad \left\{ \begin{aligned} \delta_1(i_C) &= a, \delta_2(i_C) = b, \delta_3(i_C) = \Gamma'(\sqrt{i_3}) \frac{1}{2\sqrt{i_3}} , \\ \delta_{\alpha j}(i_C) &= 0, 1 \leq \alpha \leq 2, \delta_{33}(i_C) = \Gamma''(\sqrt{i_3}) \frac{1}{4i_3} - \\ &\quad - \Gamma'(\sqrt{i_3}) \frac{1}{4i_3\sqrt{i_3}} , \end{aligned} \right.$$

with $i_k = i_k(C)$. Therefore,

- i) The reference configuration is a natural state : $\hat{T}(I) = I\check{\Sigma}(0) = 0$.
- ii) The hypotheses (2.21) (2.22) of theorem 2.2 are verified :

$$1. \quad (\delta_2 + \delta_3)|_{i_1} < 0 \Leftrightarrow b + \frac{\Gamma'(1)}{2} < 0 \Leftrightarrow \mu > 0 .$$

$$2. \quad (\delta_2 + \delta_3 + \delta_{11} + 4 \delta_{12} + 2 \delta_{13} + 2 \delta_{22} + 4 \delta_{23} + \delta_{33})|_{i_1} \geq 0 \Leftrightarrow$$

$$b + \frac{\Gamma'(1)}{4} + \frac{\Gamma''(1)}{4} \geq 0 \Leftrightarrow \lambda \geq 0 \quad \text{by (4.8) .}$$

Consequently, for each $p \geq 6$, there exists a neighbourhood of the origin \tilde{F}^p such that if $f \in \tilde{F}^p$, the solution $u(f)$ given by the implicit function theorem verifies :

$$I'(u(f))\theta = 0, I''(u(f))(\theta, \theta) \geq \beta \|\theta\|_{1,2,\Omega}^2$$

for all vector fields $\theta \in [W_0^{1,p}(\Omega)]^3$. Furthermore, if $f \in \tilde{F}^\infty$ then $u(f)$ is a strict local minimum of the total energy in the space $[W_0^{1,\infty}(\Omega)]^3$.

5. A COUNTEREXAMPLE IN HYPERELASTICITY

It has been shown in paragraph 2 that, given a hyperelastic material satisfying the restrictions (2.21) (2.22), the second derivative of the total energy functional at the solution $u(f)$ given by the implicit function theorem, is positive definite in $[W_0^{1,p}(\Omega)]^3$, $p_0 \leq p \leq +\infty$ (p_0 being dependent on the constitutive law of the material, it simply has to be large enough so that the energy is finite). Nevertheless, this condition only allows us to state that u is a local minimum of the energy in $[W_0^{1,\infty}(\Omega)]^3$, as this result is generally false in the topologies $W^{1,p}(\Omega)$, $p_0 \leq p < +\infty$.

Let $\tilde{W} : i(S_>) \rightarrow R$ be stored energy functions of general form :

$$(5.1) \quad \tilde{W}(i_C) = k[(i_2(C) - 2\sqrt{i_3(C)} - i_3(C)) + (i_1(C) - 3)^2 - 2(i_2(C) - 2\sqrt{i_3(C)} - i_3(C))^2],$$

where the constant k is > 0 . We first observe that the reference configuration is a natural state, as is easily verified from the following relations :

$$(5.2) \quad \delta_1(i_C) = 2k(i_1(C) - 3),$$

$$(5.3) \quad \delta_2(i_C) = k[1 - 4(i_2(C) - 2\sqrt{i_3(C)} - i_3(C))],$$

$$(5.4) \quad \delta_3(i_C) = k \left[\left(1 + \frac{1}{\sqrt{i_3(C)}} (-1 + 4(i_2(C) - 2\sqrt{i_3(C)} - i_3(C))) \right) \right].$$

On the other hand, conditions (2.21) (2.22) are verified :

1. $(\delta_2 + \delta_3)|_{i_i} = -k < 0$, since $k > 0$ by assumption.
2. Using (5.2)-(5.4) we can compute the functions

$$\delta_{kj}(i_C) = \frac{\partial^2 \tilde{W}}{\partial i_j \partial i_k}(i_C).$$

This gives :

$$(5.5) \quad \delta_{11}(i_C) = 2k, \quad \delta_{12}(i_C) = 0, \quad \delta_{13}(i_C) = 0,$$

$$(5.6) \quad \delta_{22}(i_C) = -4k, \quad \delta_{23}(i_C) = 4k \left(1 + \frac{1}{\sqrt{i_3(C)}} \right),$$

$$(5.7) \quad \delta_{33}(i_C) = k \left[\frac{1}{2\sqrt{i_3(C)}i_3(C)} (1 - 4(i_2(C) - 2\sqrt{i_3(C)} - i_3(C))) - 4 \left(1 + \frac{1}{\sqrt{i_3(C)}} \right)^2 \right].$$

Thus, the relation

$$(\delta_2 + \delta_3 + \delta_{11} + 4\delta_{12} + 2\delta_{13} + 2\delta_{22} + 4\delta_{23} + \delta_{33})|_{i_i} = \frac{19}{2}k > 0,$$

holds, and this proves that condition (2.22) is verified.

If we furthermore take into consideration the relations :

$$(5.8) \quad i_1(C) = 3 + 2 \operatorname{tr} E,$$

$$(5.9) \quad i_2(C) = 3 + 4 \operatorname{tr} E + 2(\operatorname{tr} E)^2 - 2 \operatorname{tr} E^2,$$

$$(5.10) \quad \Gamma(\sqrt{i_3(C)}) = \Gamma(1) + \Gamma'(1) \left[\operatorname{tr} E + \frac{1}{2}(\operatorname{tr} E)^2 - \operatorname{tr} E^2 \right] + \frac{1}{2} \Gamma''(1)(\operatorname{tr} E)^2 + o(|E|^2),$$

where $\Gamma : (0, +\infty) \rightarrow \mathcal{R}$ is a function twice differentiable at $\delta = 1$, we obtain that the function \tilde{W} can be expanded as :

$$\tilde{W}(E) = \frac{\lambda}{2} (\operatorname{tr} E)^2 + \mu \operatorname{tr} E^2 + o(|E|^2),$$

with $\lambda = 6k > 0$ and $\mu = 2k > 0$. Consequently, the response function associated with the second Piola-Kirchhoff stress tensor satisfies :

$$\tilde{\Sigma}(E) = \lambda(\operatorname{tr} E)I + 2\mu E + o(|E|^2).$$

Theorem 1.1 allows us to state that, for each $p > 3$, there exists a neighbourhood of the origin F^p in the space $[L^p(\Omega)]^3$ and a neighbourhood of the origin U^p in the space

$$(5.11) \quad V^p(\Omega) = \{v \in [W^{2,p}(\Omega)]^3, v = 0 \text{ on } \Gamma\}$$

such that for all functions $f \in F^p$, the boundary value problem (1.1)-(1.3) with $\Gamma_1 = \emptyset$ and $\phi_0 = Id$, admits an unique solution $u(f) \in U^p$.

Let us consider the energy functional corresponding to zero body forces :

$$(5.12) \quad I : v \in [W_0^{1,p}(\Omega)]^3 \rightarrow I(v) = \int_{\Omega} \hat{W}(I + \nabla v) dx$$

where $\bar{\Omega}$ is the reference configuration and \hat{W} is extended to all of M^3 by assigning the value $+\infty$ to $\hat{W}(F)$ when $\det F \leq 0$. In order that the functional I is well defined on A^p , it is necessary to assume that $12 \leq p \leq +\infty$. The function $u = 0$ is clearly the unique solution of the boundary value problem (1.1)-(1.3) (with $\Gamma_1 = \emptyset$ and $\phi_0 = Id$) in the neighbourhood U^p of the origin. From theorem 2.2 we deduce that the second derivative of the functional I satisfies :

$$(5.13) \quad I''(0)(\theta, \theta) \geq \beta \|\theta\|_{1,2,\Omega}^2,$$

for all vector fields $\theta \in [W_0^{1,p}(\Omega)]^3$ with $12 \leq p \leq +\infty$. Furthermore, from theorem 3.1 it can be concluded that $u = 0$ is a strict local minimum of the energy in the space $[W_0^{1,\infty}(\Omega)]^3$.

Now, we shall show that $u = 0$ is not a local minimum of the energy in $[W_0^{1,p}(\Omega)]^3$, $12 \leq p < +\infty$. To do this, we shall verify that in any neighbourhood of the origin there exists a function \bar{u} such that $I(\bar{u}) < 0 = I(0)$. Hence, consider the function \bar{u} given by :

$$(5.14) \quad \bar{u}_1(x) = n^{-\alpha} [1 - \exp(-n^{\alpha+\gamma} \delta)], \quad \bar{u}_2(x) = \bar{u}_3(x) = 0,$$

where the constants α, γ are > 0 , δ represents the distance from $x \in \Omega$ to the boundary of Ω , and n is an integer.

Let $\bar{C} = (I + \nabla \bar{u})^T (I + \nabla \bar{u})$ denote the right Cauchy-Green strain tensor. The principal invariants of the matrix \bar{C} are given by :

$$(5.15) \quad i_1(\bar{C}) = (\nabla \bar{u} : \nabla \bar{u}) + 2 \partial_1 \bar{u}_1 + 3,$$

$$(5.16) \quad i_2(\bar{C}) = (\nabla \bar{u} : \nabla \bar{u}) + 2(1 + \partial_1 \bar{u}_1) + (1 + \partial_1 \bar{u}_1)^2$$

$$(5.17) \quad i_3(\bar{C}) = (1 + \partial_1 \bar{u}_1)^2.$$

These relations allow us to express the functional I at the point \bar{u} in the form :

$$(5.18) \quad I(\bar{u}) = k \int_{\Omega} \{ (\nabla \bar{u} : \nabla \bar{u}) + [(\nabla \bar{u} : \nabla \bar{u}) + 2 \partial_1 \bar{u}_1]^2 - 2(\nabla \bar{u} : \nabla \bar{u})^2 \} dx .$$

In this equation, if we replace \bar{u} by its value given by (5.14), we obtain :

$$(5.19) \quad I(\bar{u}) = kn^{2\gamma} \int_0^d [\exp(-2n^{\alpha+\gamma}\delta) - 2n^{2\gamma}\exp(-4n^{\alpha+\gamma}\delta)] S(\delta) d\delta + \\ + kn^{2\gamma} \int_0^d \left[n^\gamma \exp(-2n^{\alpha+\gamma}\delta) + 2 \exp(-n^{\alpha+\gamma}\delta) \frac{\partial \delta}{\partial x_1} \right]^2 S(\delta) d\delta ,$$

where d is the greatest distance to the boundary from points in Ω and $S(\delta)$ is the area of a surface parallel to, and at a distance δ from, the boundary of Ω . Developing expression (5.19) :

$$I(\bar{u}) = kn^{2\gamma} \left\{ \int_0^d [\exp(-2n^{\alpha+\gamma}\delta) - 2n^{2\gamma}\exp(-4n^{\alpha+\gamma}\delta)] S(\delta) d\delta + \right. \\ \left. + \int_0^d \left[n^{2\gamma}\exp(-4n^{\alpha+\gamma}\delta) + 4n^\gamma \exp(-3n^{\alpha+\gamma}\delta) \frac{\partial \delta}{\partial x_1} \right. \right. \\ \left. \left. + 4 \exp(-2n^{\alpha+\gamma}\delta) \left| \frac{\partial \delta}{\partial x_1} \right|^2 \right] S(\delta) d\delta \right\} \\ \leq kn^{2\gamma} \int_0^d [5 \exp(-2n^{\alpha+\gamma}\delta) + 4n^\gamma \exp(-3n^{\alpha+\gamma}\delta) \\ - n^{2\gamma}\exp(-4n^{\alpha+\gamma}\delta)] S(\delta) d\delta .$$

This integral may now be written as the sum of two integrals over the intervals $[0, d^*]$, $[d^*, d]$ with negative and positive integrand respectively, where :

$$d^* = \left(\log \frac{n^\gamma}{5} \right) n^{-(\alpha+\gamma)} .$$

An upper bound for both integrals is obtained by replacing $S(\delta)$ by $S^* = S(d^*)$, so that

$$I(\bar{u}) \leq kS^* n^{2\gamma} \int_0^d [5 \exp(-2n^{\alpha+\gamma}\delta) + \\ + 4n^\gamma \exp(-3n^{\alpha+\gamma}\delta) - n^{2\gamma}\exp(-4n^{\alpha+\gamma}\delta)] d\delta \\ = kS^* n^{\gamma-\alpha} \left[-\frac{n^{2\gamma}}{4} (1 - \exp(-4n^{\alpha+\gamma}d)) + \frac{5}{2} (1 - \exp(-2n^{\alpha+\gamma}d)) + \right. \\ \left. + \frac{4}{3} n^\gamma (1 - \exp(-3n^{\alpha+\gamma}d)) \right] .$$

This shows that $I(\bar{u})$ is < 0 for $\gamma > 0$ and n sufficiently large. On the other hand, for $12 \leq p < +\infty$, we have :

$$\begin{aligned} |\bar{u}|_{1,p,\Omega}^p &= \sum_{i=1}^3 \int_0^d n^{\gamma p} \left[\left(\frac{\partial \delta}{\partial x_i} \right)^p \exp(-pn^{\alpha+\gamma}\delta) \right] S(\delta) d\delta \\ &\leq n^{\gamma p} S(0) \int_0^d \exp(-pn^{\alpha+\gamma}\delta) d\delta \\ &= n^{(p-1)\gamma-\alpha} p^{-1} S(0) [1 - \exp(-pn^{\alpha+\gamma}d)] . \end{aligned}$$

and hence on setting $\alpha > \gamma(p-1) \geq 0$, we see that $|\bar{u}|_{1,p,\Omega} \leq \varepsilon$ may be made arbitrarily small with increasing n .

We observe, finally, that each choice of \bar{u} has positive determinant of the deformation gradient, hence it is an admissible deformation.

Remark 5.1 : The stored energy function \hat{W} given by (5.1) is not polyconvex, basically because there is a minus sign in front of the function $2(i_2(C) - 2\sqrt{i_3(C)} - i_3(C))^2$. We shall prove this by contradiction.

Assume that the stored energy function $\hat{W} : M_+^3 \rightarrow R$ is polyconvex. By using a characterization of polyconvex functions given by Ball [1977, theorem 4.4], the following condition holds :

For each $F \in M_+^3$ there exist numbers $a_i^j(F)$, $b_i^j(F)$, $c(F)$ such that :

$$\begin{aligned} \hat{W}(\bar{F}) &\geq \hat{W}(F) + a_i^j(F)(\bar{F}_{ij} - F_{ij}) + b_i^j(F)((\text{cof } \bar{F})_{ij} - (\text{cof } F)_{ij}) + \\ (5.20) \quad &+ c(F)(\det \bar{F} - \det F) \quad \text{for all } \bar{F} \in M_+^3 . \end{aligned}$$

Let us consider a displacement v given by $v_1 = v_1(x_2, x_3)$, $v_2 = v_3 = 0$; then, the deformation gradient $F = I + \nabla v$ has positive determinant. In addition, for each $\mu > 0$, the matrices

$$F_\mu := I + \mu \nabla v$$

belong to the set M_+^3 . On the other hand, from relations (5.15)-(5.17), the stored energy function given by (5.1) at F_μ can be written :

$$\hat{W}(F_\mu) = k[\mu^2(\nabla v : \nabla v) - 3\mu^4(\nabla v : \nabla v)^2], \quad \mu > 0 .$$

Then, condition (5.20) implies that

$$\begin{aligned} k[(\nabla v : \nabla v) - 3(\nabla v : \nabla v)^2] &\geq k[\mu^2(\nabla v : \nabla v) - 3\mu^4(\nabla v : \nabla v)^2] + \\ &+ a_i^j(I + \nabla v)(\mu - 1) \partial_i v_j + b_i^j(I + \nabla v)[(\mu^2 - 1)(\text{cof } \nabla v)_{ij} + (\mu - 1) \partial_j v_i] . \end{aligned}$$

This inequality is equivalent to the inequality :

$$0 \geq k[(\mu^2 - 1)(\nabla v : \nabla v) - 3(\mu^4 - 1)(\nabla v : \nabla v)^2] + \mu a(\nabla v) + \mu^2 b(\nabla v) + c(\nabla v)$$

where the numbers $a(\nabla v)$, $b(\nabla v)$, $c(\nabla v)$ are independent of μ . But, this does not hold for μ sufficiently large if k is > 0 . This contradiction proves the result. \square

6. POLYCONVEXITY AND MINIMIZERS OF THE ENERGY

Consider a homogeneous hyperelastic body which occupies the reference configuration $\bar{\Omega}$ in the absence of any applied forces. We assume that $\bar{\Omega}$ is a natural state, i.e., that there is no stress in the absence of deformation, and that the stored energy function is polyconvex, i.e., there exists a convex function $G : M^3 \times M^3 \times (0, +\infty) \rightarrow R$ such that

$$\hat{W}(F) = G(F, \text{cof } F, \det F).$$

In addition, let us suppose that $G(F, H, \delta)$ is of class C^1 .

With the assumption that body forces and surface traction are absent, while the boundary condition of place is of the form $\phi = Id$ on $\delta\Omega$, the function $u = 0$ is clearly a solution of the boundary value problem (1.1)-(1.3). In this paragraph our aim is to show that the trivial solution is a minimizer of the energy in any Sobolev space $[W^{1,p}(\Omega)]^3$, $3 \leq p \leq +\infty$ (cf. remark 6.1).

We attack this problem by using the characterization of polyconvex functions given by (5.20). Since G is of class C^1 , the coefficients on the right-side of (5.20) are given by the derivatives of G with respect to its arguments (cf. Ball [1977, theorem 4.4]). Then, for each matrix $F \in M_+^3$, the relation

$$(6.1) \quad \hat{W}(\bar{F}) \geq \hat{W}(F) + \left(\frac{\partial G}{\partial F}(F, \text{cof } F, \det F) : \bar{F} - F \right) + \\ + \left(\frac{\partial G}{\partial(\text{cof } F)}(F, \text{cof } F, \det F) : \text{cof } \bar{F} - \text{cof } F \right) \\ + \frac{\partial G}{\partial(\det F)}(F, \text{cof } F, \det F) (\det \bar{F} - \det F)$$

holds for all $\bar{F} \in M_+^3$.

On the other hand, from relation (2.1),

$$G(F, \text{cof } F, \det F) = \tilde{W}(\|F\|^2, \|\text{cof } F\|^2, (\det F)^2), F \in M_+^3$$

and we obtain that (6.1) is equivalent to the inequality :

$$(6.2) \quad \hat{W}(\bar{F}) \geq \hat{W}(F) + \left(\frac{\partial \tilde{W}}{\partial i_1}(i_C) \cdot \frac{\partial i_1}{\partial \bar{F}}(C) : \bar{F} - F \right) + \\ + \left(\frac{\partial \tilde{W}}{\partial i_2}(i_C) \cdot \frac{\partial i_2}{\partial(\text{cof } F)}(C) : \text{cof } \bar{F} - \text{cof } F \right) + \\ + \frac{\partial \tilde{W}}{\partial i_3}(i_C) \cdot \frac{\partial i_3}{\partial(\det F)}(C) (\det \bar{F} - \det F),$$

for all $\bar{F} \in M_+^3$. In particular, for $F = I$ we have

$$(6.3) \quad \hat{W}(I + \nabla v) \geq \hat{W}(I) + 2 \delta_1|_{i_I}(I : \nabla v) + 2 \delta_2|_{i_I}[(I : \text{cof } \nabla v) - (I : I)] + \\ + 2 \delta_3|_{i_I}[(I : \nabla v) + (I : \text{cof } \nabla v) + \det \nabla v],$$

for all admissible displacements $v \in A^p$, where A^p is the set of admissible deformations defined in (2.5), and for almost all $x \in \Omega$. Since the reference configuration is a natural state,

$$(\delta_1 + 2 \delta_2 + \delta_3)|_{i_I} = 0,$$

inequality (6.3) can be written in the form

$$(6.4) \quad \hat{W}(I + \nabla v) \geq \hat{W}(I) + 2 \delta_2|_{i_I}(I : \text{cof } \nabla v) + 2 \delta_3|_{i_I} \times \\ \times [(I : \text{cof } \nabla v) + \det \nabla v].$$

The integrals $\int_{\Omega} \hat{W}(I + \nabla v) dx$ are well defined for all $v \in A^p, p \geq 3$. We first note that for almost all $x \in \Omega$ the function $G : M^3 \times M^3 \times (0, +\infty) \rightarrow R$ is continuous (it is convex and real-valued on an open subset of a finite dimensional space); consequently the function

$$x \in \Omega \rightarrow G(I + \nabla v(x), \text{cof } (I + \nabla v(x)), \det (I + \nabla v(x)))$$

is measurable for each $v \in A^p$. In addition, if $p \geq 3$, $\text{cof } (I + \nabla v(x)) \in [L^1(\Omega)]^9$ and $\det (I + \nabla v) \in L^1(\Omega)$; then, from inequality (6.4) :

$$I(v) = \int_{\Omega} \hat{W}(I + \nabla v) dx \geq \int_{\Omega} \hat{W}(I) dx + 2(\delta_2 + \delta_3)|_{i_I} \times \\ \times \int_{\Omega} (I : \text{cof } \nabla v) dx + 2 \delta_3|_{i_I} \int_{\Omega} \det \nabla v dx$$

for all $v \in A^p, p \geq 3$.

For sufficiently smooth functions v , for instance in the space $[D(\Omega)]^3$, we can write

$$\begin{aligned} (\text{cof } \nabla v)_{ij} &= \partial_{i+2}(v_{j+2} \partial_{i+1} v_{j+1}) - \partial_{i+1}(v_{j+2} \partial_{i+2} v_{j+1}) \text{ (no summation)} \\ \det \nabla v &= \partial_j v_1 (\text{cof } \nabla v)_{1j}, \end{aligned}$$

and, consequently, an application of Green's formula shows that

$$\begin{aligned} \int_{\Omega} (I : \text{cof } \nabla v) \, dx &= 0 \\ \int_{\Omega} \det \nabla v \, dx &= \int_{\Gamma} (\text{cof } \nabla v) n \cdot v \, d\Gamma = 0 \end{aligned}$$

for all functions $v \in [D(\Omega)]^3$. Our aim is to show that these relations hold for all functions $v \in [W_0^{1,p}(\Omega)]^3$, $p \geq 3$.

Given an arbitrary function $v \in [W_0^{1,p}(\Omega)]^3$, let $(v_k) \in [D(\Omega)]^3$ be a sequence such that $v_k \rightarrow v$ in $[W^{1,p}(\Omega)]^3$, $p \geq 3$. Then, the sequence $(\text{cof } \nabla v_k, \det \nabla v_k)$ is bounded in the reflexive Banach space $[L^{p/2}(\Omega)]^9 \times L^{p/3}(\Omega)$ (each number $p/2$, $p/3$ is > 1). Therefore, we can extract a subsequence (v_ℓ) such that $(\text{cof } \nabla v_\ell, \det \nabla v_\ell)$ converges weakly to an element (H, δ) in the space $[L^{p/2}(\Omega)]^9 \times L^{p/3}(\Omega)$. Besides $H = \text{cof } \nabla v$ and $\delta = \det \nabla v$ (cf. Ciarlet [1987, theorems 7.5-1, 7.6-1]), so that the limits H and δ are unique and, therefore :

$$\begin{aligned} \text{cof } \nabla v_k &\rightharpoonup \text{cof } \nabla v \text{ in } [L^{p/2}(\Omega)]^9 \\ \det \nabla v_k &\rightharpoonup \det \nabla v \text{ in } L^{p/3}(\Omega) \end{aligned} \quad \text{(weak convergence)}$$

It then follows that

$$\int_{\Omega} (I : \text{cof } \nabla v) \, dx = \lim_{k \rightarrow +\infty} \int_{\Omega} (I : \text{cof } \nabla v_k) \, dx = 0$$

and

$$\int_{\Omega} \det \nabla v \, dx = \lim_{k \rightarrow +\infty} \int_{\Omega} \det \nabla v_k \, dx = 0$$

for all $v \in [W_0^{1,p}(\Omega)]^3$. Then, $I(v) \geq I(0)$ for all $v \in A^p$, $p \geq 3$, and, in consequence, $u = 0$ is a minimizer of the energy.

We summarize the result of the preceding considerations :

THEOREM 6.1 : *Let there be given a stored energy function $\hat{W} : M_+^3 \rightarrow R$ that satisfies the following assumptions :*

- i) *Polyconvexity*,
 ii) $G : M^3 \times M^3 \times (0, +\infty) \rightarrow R$ is of class C^1 , G being the convex function given by i).

Then, for each p , $3 \leq p \leq +\infty$, the trivial solution is a minimizer of the energy functional corresponding to zero body forces in the set of admissible deformations A^p defined as in (2.5). \square

Remark 6.1 : It is always possible to consider $p \geq 2$. But, in this case, we need to modify the set of admissible deformations in order to insure that the mapping $v \in [W^{1,p}(\Omega)]^3 \rightarrow \det \nabla v \in L^1(\Omega)$ be well defined and continuous (cf. Ciarlet [1987]).

CONCLUSIONS

1) The main conclusion is, of course, that for hyperelastic materials the positivity of the second variation of the energy functional I at the solution u_0 given by the implicit function theorem, implies u_0 locally minimizes I in a topology as strong as $W^{1,\infty}$; although

a) In $W^{1,p}$, $1 \leq p < +\infty$, one cannot necessarily conclude that u_0 is a local minimum.

b) In any topology as strong as $W^{1,\infty}$ we always have, for $\varepsilon > 0$ sufficiently small,

$$\inf_{\|u - u_0\| = \varepsilon} I(u) = I(u_0).$$

2) The space $W^{1,p}$ plays a basic role in the existence theory for minimizers in elasticity when the stored energy function is polyconvex (cf. Ball [1977]). An obvious question concerns when positivity of the second variation at u_0 holds under conditions of polyconvexity, is u_0 a strict local minimum of I in $W^{1,p}$? In this respect, we have studied this problem for $u_0 = 0$ and showed that $u_0 = 0$ is a minimizer of the energy functional corresponding to zero body forces in any Sobolev space $W^{1,p}$, $3 \leq p \leq +\infty$.

3) Examples of the stored energy functions \hat{W} satisfying positivity of the second variation of the energy, are the Ogden and St. Venant-Kirchhoff materials. However, for St. Venant-Kirchhoff materials it does not exist an existence result of a minimizer, since their stored energy functions are not polyconvex (cf. Raoult [1986]).

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REFERENCES

- R. A. ADAMS [1975] : *Sobolev Spaces*, Academic Press, New York.
- J. M. BALL [1977] : *Convexity conditions and existence theorems in nonlinear elasticity*. Arch. Rational Mech. Anal. 63, 337-403.
- J. M. BALL, J. C. CURRIE, P. J. OLVER [1981] : *Null Lagrangians, weak continuity and variational problems of arbitrary order*. J. Functional Analysis 41, 135-174.
- J. M. BALL, R. J. KNOPS, J. E. MARSDEN [1978] : *Two examples in nonlinear elasticity*. Proceedings, Conference on Nonlinear Analysis, Besançon 1977, pp. 41-49, Springer-Verlag, Berlin.
- P. G. CIARLET [1986] : *Élasticité Tridimensionnelle, Collection Recherches en Mathématiques Appliquées*, Masson.
- P. G. CIARLET [1987] : *Mathematical Elasticity*, vol. I. North Holland, Amsterdam.
- P. G. CIARLET, G. GEYMONAT [1982] : *Sur les lois de comportement en élasticité non linéaire compressible*. C.R. Acad. Sci. Paris. Ser. A 295, 423-426.
- P. G. CIARLET, J. NEČAS [1985] : *Global invertibility and self-contact conditions, in nonlinear, three-dimensional elasticity*, to appear.
- D. R. J. CHILLINGWORTH, J. E. MARSDEN, Y. H. WAN [1982] : *Symmetry and bifurcation in three dimensional elasticity*. Part. I. Arch. Rational Mech. Anal. 80, 295-331.
- D. R. J. CHILLINGWORTH, J. E. MARSDEN, Y. H. WAN [1983] : *Symmetry and bifurcation in three dimensional elasticity*. Part II. Arch. Rational Mech. Anal. 83, 363-395.
- R. J. KNOPS, L. E. PAYNE [1978] : *On potential wells and stability in nonlinear elasticity*. Math. Proc. Camb. Phil. Soc. 84, 177-190.
- LE DRET [1987] : *Structure of the set of equilibrated loads in nonlinear elasticity and applications to existence and nonexistence*. J. Elasticity 17, 123-141.
- J. E. MARSDEN, T. J. R. HUGHES [1983] : *Mathematical Foundations of Elasticity*. Prentice-Hall, Englewood Cliffs.
- A. RAOULT [1986] : *Non-polyconvexity of the stored energy function of a St. Venant-Kirchhoff material*. Aplikace Matematiky.
- T. VALENT [1979] : *Teoremi di esistenza e unicità in elastostatica finita*. Rend. Sem. Mat. Univ. Padova 60, 165-181.
- C.-C. WANG, C. TRUESDELL [1973] : *Introduction to Rational Elasticity*. Noordhoff, Groningen.