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MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
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ON THE OPTIMAL DESIGN OF ELASTIC SHAFTS (*)

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Abstract. — *In order to study the design of hollow shafts with maximal torsional rigidity, we define a functional associated with the shape of the shaft and investigate its minimization Introducing a relaxation by means of a duality approach we are able to apply convex analysis techniques and prove the existence of the optimal design.*

Résumé. — *Afin d'étudier la forme de poutres creuses de rigidité maximale a la torsion, on définit une fonctionnelle associée à la forme de la section droite et on cherche à la minimiser. En introduisant une relaxation moyennant une certaine classe de multiplicateurs de Lagrange, on applique des techniques d'analyse convexe pour montrer l'existence de la section droite optimale*

1. INTRODUCTION

We consider the problem of torsion of a hollow elastic shaft. We denote by Ω the region occupied by the cross section in the $x - y$ plane, which we shall assume to be doubly connected. We denote by Γ_0 and Γ the interior and exterior boundary of the domain Ω. The direction of the applied torque coincides with the z-axis. We assume that the shaft material is homogeneous and isotropic. We express the nonzero components of the stress tensor in terms of the stress function :

$$\tau_{xz} = G\theta u_y , \quad \tau_{yz} = G\theta u_x$$

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where G is the shear modulus, θ is the angle of twist per unit length of the shaft and u is the stress function satisfying :

$$u_x = \partial u/\partial x\ ,\quad u_y = \partial u/\partial y\ .$$

It is well known that for this case, the torsion problem is reduced to finding the stress function u such that :

$$-\Delta u = 2 \quad \text{in} \quad \Omega \tag{1.1}$$

$$u = 0 \quad \text{on} \quad \Gamma \tag{1.2}$$

$$u = c \quad \text{on} \quad \Gamma_0 \tag{1.3}$$

$$-\int_{\Gamma_0} \partial u/\partial n \, d\sigma = 2\, A_0 \tag{1.4}$$

where A_0 is the area of the region bounded by the curve Γ_0 and c is an unknown quantity whose value can be determined using (1.4). The torsional rigidity K_Ω is given by

$$K_\Omega = 2\left(\int_\Omega u\, d\omega + CA_0\right)\ . \tag{1.5}$$

Let us assume that the boundary Γ_0 and the following isoperimetric condition are given :

$$meas \quad \Omega = A \tag{1.6}$$

where A is a positive constant. We look for the shape of Ω such that the rigidity K_Ω is maximized. Among others, this problem has been studied by N. Banichuk [3], see also other related papers by Cea [6] and Cea-Malanowski [8].

We remark that by minimizing the functional

$$v \to J(v) = 1/2 \int_\Omega |\nabla v|^2\, d\omega - 2 \int_\Omega v\, d\omega$$

on a suitable function space we obtain a solution for the problem (1.1)-(1.4). For the corresponding solution u_Ω we have :

$$J(u_\Omega) = -\,1/2 \int_\Omega |\nabla u_\Omega|^2\, d\omega = -\,1/2\, K_\Omega\ .$$

Thus, the domain Ω that maximizes K_Ω, minimizes $J(u_\Omega)$. By using this property, we shall define a new « relaxed » problem and applying some convex analysis techniques we shall prove the existence of the optimal domain Ω. In fact, the relaxed problem leads to the minimization of a concave functional on a convex set of functions. The concavity structure will allow us to prove that there exists a characteristic function where the minimum is attained. This approach is similar to the one used by Gonzalez de Paz [13] for the study of the existence of a domain with minimal capacity when the interior boundary is unknown. In the appendix we show how our results can be applied to study capacity problems where the exterior boundary is unknown a priori.

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2. THE RELAXED PROBLEM

Let Ω_0 be a star-shaped, connected, bounded domain in $\mathbb{R}^2$ with Lebesgue measure A_0 and boundary Γ_0 which is Lipschitz continuous. Let B_R be an open disc with center at some point in the interior of Ω_0 ; in order to allow for the feasible domains to be contained in the disc, we choose the radius R large enough so that for $d = \text{dist } (\partial B_R, \Gamma_0)$, the annulus with outer boundary ∂B_R and width d has an area greater than the given constant A, we put $D_R = B_R \setminus \bar{\Omega}_0$ and denote by $\| \cdot \|$ the usual L^2-norm in B_R. Furthermore, let μ be a positive, bounded function such that :

$$0 \leq \mu \leq 1 \quad \text{almost everywhere in } B_R \tag{2.1}$$

$$\int_{B_R} \mu \, d\omega = A_0 + A \tag{2.2}$$

$$\int_{\Omega_0} \mu \, d\omega = A_0 \,. \tag{2.3}$$

Following the definitions introduced by Lanchon [16], we put

$$E_R = \{ v \mid v \in H_0^1(B_R), v = \text{const. on } \Omega_0 \} \;;$$

here $H_0^1(B_R)$ denotes the usual Sobolev space (see Neças [18]).

We now define on the Sobolev space the functional

$$v \to J_\mu(v) = 1/2 \int_{B_R} |\nabla v|^2 \, d\omega - \int_{B_R} \mu f v \, d\omega \,. \tag{2.4}$$

We remark that for the special case of the elastic torsion, the function f is a given positive constant. This special case is contained in our framework if we suppose that f is strictly positive and bounded.

The problem $P(\mu)$: The minimization of $v \to J_\mu(v)$ on E_R was treated by H. Lanchon [16]. This functional is convex and weakly lower semicontinuous, so that for each $\mu \in L^\infty(B_R, \mathbb{R}^+)$ there exists a $u_\mu \in E_R$ such that the functional is minimized (*cf.* Ekeland-Temam [10], Moreau [17]) and u_μ is the weak solution of the following boundary value problem :

$$-\Delta u_\mu = \mu f \quad \text{in} \quad D_R = B_R/\Omega_0 \tag{2.5}$$

$$u_\mu = c \quad \text{on} \quad \Omega_0 \tag{2.6}$$

$$u_\mu = 0 \quad \text{on} \quad \partial B_R \tag{2.7}$$

$$\int_{\Gamma_0} \partial u_\mu/\partial n \, d\sigma = -\int_{\Omega_0} \mu f \, d\omega . \tag{2.8}$$

In (2.8) n denotes the unit normal exterior to Ω_0 at each point of Γ_0. In the case f is a constant and $\mu = 1$ on Ω_0, this is the classical integral constraint (1.4).

Remark 2.1 : The element u_μ is a non-negative function. In fact define

$$u_\mu^+ = \max(u_\mu, 0) .$$

This is an element of $H_0^1(B_R)$ (*cf.* Kinderlehrer-Stampacchia [15]). Moreover, because of the extremality property of u_μ we have : $u_\mu^+ \in E_R$ and

$$\int_{B_R} \mu f u_\mu^+ \, d\omega = \int_{B_R} \mu f u_\mu \, d\omega .$$

If u_μ were strictly negative on a set of positive measure, then

$$\|\nabla u_\mu^+\| < \|\nabla u_\mu\|$$

so that

$$J_\mu(u_\mu^+) < J_\mu(u_\mu) .$$

This is a contradiction, so $u_\mu^+ = u_\mu$.

Remark 2.2 : The function u_μ is an element of $C^{1,1}(\bar{D}_R)$. First we recall that $u_\mu \in C^{1,K}(\bar{D}_R)$ for $0 \leq \alpha < 1$ (see Kinderlehrer-Stampacchia [13]). It follows that u_μ is a Lipschitz function. Besides, $\Delta u_\mu \in L^\infty(D_R)$. From these results and the boundary conditions (2.6) and (2.7) it follows :

$$u_\mu \in W^{2,\infty}(D_R)$$

(see C. Gebhardt [11] and R. Jensen [24]), this implies that ∇u_μ is a Lipschitz function (for the definition of $W^{2,\infty}(D_R)$, see J. Neças [18]).

The optimization problem related to μ : We now define the functional Φ on $L^\infty(B_R, \mathbb{R}^+)$ as follows :

$$\Phi(\mu) = J_\mu(u_\mu) = \min_{u \in E_R} J_\mu(u) .$$

We study the problem of minimization of Φ in $C \subset L^\infty(B_R, \mathbb{R}^+)$ where C denotes the convex set defined by the constraints (2.1), (2.2) and (2.3). The convex set C is compact for the topology $\sigma(L^\infty, L^1)$. We shall prove that the functional Φ is continuous for the same topology in order to show the existence of the minimizing element.

THEOREM 2.1 : *The functional* Φ *is* $\sigma(L^\infty, l^1)$*-continuous on* C.

Proof: Firstly we establish the following assertion : there exists a ball B_ρ in $H_0^1(B_R)$ of radius ρ such that for every $\mu \in C$:

$$\min_{u \in E_R} J_\mu(u) = \min_{u \in E_R \cap B_\rho} J_\mu(u) .$$

Let μ be given, and let u_μ be the corresponding minimizing element of J_μ in E_R, then for every $v \in E_R$ we have :

$$(\nabla u_\mu, \nabla v) = (\mu f, v) .$$

Here the parentheses denote the usual scalar product in $L^2(B_R)$. For the special case $v = u_\mu$:

$$\|\nabla u_\mu\|^2 = (\mu f, u_\mu) \leq \|\mu f\|_L \, \|u_\mu\|_L$$

and by using the Cauchy-Schwarz and the Poincaré inequality :

$$\|u_\mu\|_L \leq \alpha \|u_\mu\| \leq \alpha' \|\nabla u_\mu\|$$

where α and α' are constants depending on the ball B_R, then we obtain for every $\mu \in C$:

$$\|\mu f\|_L \leq \|f\|_L$$

and finally,

$$\|\nabla u_\mu\| \leq \alpha' \|f\|_L$$

so the expected ball has radius $\rho = \alpha' \|f\|_L$.

Because of the Rellich-Kondrasov injection theorem, the set $K = E_R \cap B_\rho$ is compact in $L^1(B_R)$ (*cf.* Neças [16]). Besides, it is well known that if a family of affine functions is equicontinuous, the lower enveloppe of this family is a continuous function. So if we define the family $\{J_u : \mu \to J_\mu(u),\ u \in K\}$, we see that

$$\Phi(\mu) = \inf_{u \in K} J_\mu(\mu) = J_\mu(u_\mu) .$$

Let $\varepsilon > 0$ be given, and let $\mu \in C$ be such that $\mu - \mu_0 \in (1/\varepsilon K)^0$, the polar set of $(1/\varepsilon) K$. The latter is strongly compact in L^1, so μ is in a neighborhood of μ_0 for the topology of the uniform convergence of compact sets of L^1, so that we have for every $u \in K$:

$$\left| \langle \mu - \mu_0, u \rangle_{L^\infty, L^1} \right| \leqslant \varepsilon$$

where the brackets denote the (L^∞, L^1) duality. Then, for every $u \in K$:

$$|J_\mu(u) - J_\mu(u)| \leqslant \varepsilon$$

which stablishes the equicontinuity. We need only to remark that the topology used above is equivalent to the weak topology $\sigma(L^\infty, L^1)$ on the unit ball of $L^\infty(B_R)$, so that the functional is continuous for this topology on C (*cf.* Bourbaki [4]). This gives our next result :

THEOREM 2.2 : *There exists an element $\mu_R \in C$ such that*

$$\Phi(\mu_R) = \min_{\mu \in C} \Phi(\mu) .$$

Remark 2.3 : The functional Φ is the lower enveloppe of affine linear functions, so that it is concave. This implies that among the minimizing elements there are extremal points of C, and these are characteristic functions of sets with measure $A + A_0$ (*cf.* Castaing-Valadier [5]). So there exists $\mu_R = \chi_{\tilde{\Omega}}$ with $\tilde{\Omega} = \Omega_0 \cup \Omega_R$ with Ω_R an optimal set. We shall study the necessary conditions of optimality in order to obtain a description of the optimal domain as the solution of a free boundary value problem.

3. NECESSARY CONDITIONS OF OPTIMALITY AND THEIR CONSEQUENCES

THEOREM 3.1 : *The functional Φ has a weak derivative in the sense of Gateaux for every $\mu \in L^\infty(B_R, \mathbb{R}^+)$.*

Proof: Being Φ the lower enveloppe of a family of affine functions, it follows from a theorem of Valadier [21] :

$$\Phi'(\mu\, ; \alpha) = - \langle f u_\mu, \alpha \rangle_{L^1, L^\infty} \tag{3.1}$$

for every $\alpha = \gamma - \mu$ with $\gamma \in L^\infty(B_R, \mathbb{R}^+)$.

Remark 3.1 : Φ is concave and $\sigma(L^\infty, L^1)$-continuous, so it follows that its derivative is a Frechet-derivative also (*cf.* Valadier [21]).

Remark 3.2 : The first order necessary conditions of optimality give for every $\alpha = \mu - \mu_R$, $\mu \in C$:

$$- \langle f u_R, \alpha \rangle \geqslant 0 \tag{3.2}$$

with u_R the corresponding solution for the boundary value problem $P(\mu_R)$.

If we restrict ourselves to characteristic functions, we obtain for every domain Ω in D_R with measure equal to A and such that Γ_0 is contained in $\partial\Omega$:

$$\int_{\Omega_R} f u_R \, d\omega \geqslant \int_{\Omega} f u_R \, d\omega . \tag{3.3}$$

The inequality (3.3) states that the integrand $f u_R$ must be « placed » in D_R so that the integral has a maximal value. We denote Γ the boundary of Ω_R related to D_R, Γ can be interpreted as a free boundary and we have :

THEOREM 3.2 : *Let f be a constant function, then there exists a positive number p_R such that*

$$\Omega_R = \{x \in D_R \mid u_R(x) > p_R\}$$

where the equality is understood to hold a.e., and

$$\Gamma = \{x \in D_R \mid u_R(x) = p_R\} .$$

Proof: The existence of a Lagrange multiplier related to the constraint (2.2) for the functional $\mu \to \int_{B_R} f u_R \, \mu \, d\omega$ is a classical fact (*cf.* Cea-Malanowski [8]). This means, there exists a constant p_R such that for all elements γ of the unit ball in $L^\infty(B_R, \mathbb{R}^+)$:

$$\int_{B_R} \mu u_R \, d\omega - p_R \int_{B_R} \mu \, d\omega \geqslant \int_{B_R} \gamma u_R \, d\omega - p_R \int_{B_R} \gamma \, d\omega .$$

Then we have for almost every $x \in B_R$:

$$u_R(x) > p_R \quad \text{implies} \quad \mu(x) = 1$$
$$u_R(x) < p_R \quad \text{implies} \quad \mu(x) = 0 .$$

For $\tilde{\Omega}$ as in Remark 2.3 we define $G = B_R \backslash \tilde{\Omega}$, and it follows :

$$\{x \in B_R \mid u_R(x) > p_R\} \subset \tilde{\Omega}$$
$$\{x \in B_R \mid u_R(x) < p_R\} \subset G .$$

Both inclusions must be understood in the sense almost everywhere.
Furthermore we have

$$\tilde{\Omega} \subset \{x \in B_R \mid u_R(x) \geqslant p_R\} \quad \text{a.e.}$$

which implies

$$\Omega_R \subset \{x \in D_R \mid u_R(x) \geqslant p_R\} \quad \text{a.e.}$$

From the definition of Ω_R it follows :

$$\{x \in D_R \mid u_R(x) < p_R\} \subset \Omega_R \quad \text{a.e.}$$

Besides, because of the regularity of $f : u_R \in H^2(D_R)$, so that the equation (2.5) is verified in the sense almost everywhere. This implies (see Zolesio [23]) : meas $(\{x \in D_R \mid u_R(x) = p_R\} \cap \Omega_R) = 0$ and the first assertion of the theorem is proved.

The characterization of Γ follows from the fact that the function u_R is continuous and superharmonic in D_R (see Gonzalez de Paz [13]).

COROLLARY 3.3 : *The support of the measure* $\mu\, d\omega$ *is the compact set :*

$$\tilde{\Omega} = \{x \in B_R \mid u_R(x) \geqslant p_R\} .$$

Remark 3.3 : For the boundary condition (2.6) we have :

$$u_R = c \geqslant p_R \quad \text{on} \quad \Omega_0 .$$

We shall omit for the rest of this paragraph the index R.

Remark 3.4 : The function $u \in H_0^1(B_R)$ is a solution of the following free boundary value problem :

$$-\Delta u = f \quad \text{in} \quad \Omega \tag{3.4}$$

$$\Delta u = 0 \quad \text{in} \quad D_R \backslash \bar{\Omega} \tag{3.5}$$

$$u = p \quad \text{on} \quad \Gamma \tag{3.6}$$

$$u = c \quad \text{on} \quad \Gamma_0 . \tag{3.7}$$

Remark 3.5 : We should point out that in the case Ω_0 is not star-shaped, $D_R \backslash \bar{\Omega}$ might have more than one connected component ; which should mean the existence of more « holes » in the cross section of the shaft.

Remark 3.6 : The gradient of u is continuous, so that

$$(\nabla u)^+ = (\nabla u)^- \quad \text{on} \quad \Gamma$$

where the plus sign denotes the limit at the boundary taken in the inward direction to Ω and the minus sign denotes the limit in the outward direction.

Because of the regularity of u, it is known that free boundaries of this type are locally Lipschitz (*cf.* Kinderlehrer-Stampacchia [15]). If we recall the fact that the free boundary Γ is a level set of u, we have in the neighborhoods of points where $|\nabla u| > 0$ on Γ:

$$\partial u/\partial n^+ = \partial u/\partial n^- \quad \text{on} \quad \Gamma .$$

Some analog free boundary value problems have been studied by Zolesio [22] using other optimal design techniques.

4. APPENDIX

The method described in this paper can be applied to prove the following result :

THEOREM : *Let Γ_0 be a given closed Lipschitz continuous curve, non intersecting itself so that the domain Ω_0 inclosed is star-shaped. Let W be the set of all doubly connected domains Ω with a given measure and with Γ_0 as inner boundary. Then there exists a domain $\Omega^* \in W$ such that for all $\Omega \in W$:*

$$\mathrm{Cap}_{\Omega^*}(\Omega_0) \leqslant \mathrm{Cap}_{\Omega}(\Omega_0) .$$

Proof: We need only to remark that for the case $f = \varepsilon$ with ε a positive constant, the stated results can be applied. Replacing the integral constraint (2.8) with the Dirichlet condition $u = 1$ on Ω_0 all the main results remain unchanged. For a given domain Ω, the corresponding solution u_ε of the boundary value problem has the form $u_\varepsilon = u_0 + u^\varepsilon$, where u_0 is the capacity potential of the domain Ω_0 related to Ω and u^ε the corresponding solution of the Poisson equation in Ω with homogeneous Dirichlet conditions. So we have :

$$\mathrm{En}_\varepsilon(\Omega) = 1/2\, \mathrm{Cap}_\Omega(\Omega_0) + (\nabla u_0, \nabla u^\varepsilon) + \\ + 1/2 \int_\Omega |\nabla u^\varepsilon|^2\, d\omega - \varepsilon \int_\Omega u_\varepsilon\, d\omega . \quad (4.1)$$

By applying theorem 2.2 for a given ε, we know there exists a domain $\Omega^* \in W$ such that for every $\Omega \in W$:

$$\mathrm{En}_\varepsilon(\Omega^*) \leqslant \mathrm{En}_\varepsilon(\Omega) .$$

Being u_0 harmonic in Ω, the second term of the right side in (4.1) vanishes. Besides, it is known that in the case $\varepsilon \to 0$, then $u^\varepsilon \to 0$ strongly in $H^1(\Omega)$, this implies for every $\Omega \in W$:

$$\mathrm{En}_\varepsilon(\Omega) \to 1/2\, \mathrm{Cap}_\Omega(\Omega_0)$$

which gives the result.

Other authors have proved that in the case the free boundary Γ is smooth enough :

$$|\nabla u| = \lambda \quad \text{on} \quad \Gamma$$

where λ is a positive constant which can be interpreted as a Lagrange multiplier for the functional $\Omega \to \int_\Omega |\nabla u_\Omega|^2 \, d\omega$ related to the measure constraint of the domain, here u_Ω denotes the corresponding potential (see for example Banichuk [3]).

It should be mentioned that Alt-Cafarelli [2] study the following related problem : find $v \in K$ which minimizes the functional

$$v \to J(v) = \int_\Omega |\nabla v|^2 \, d\omega + Q \int_\Omega \chi_{v>0} \, d\omega$$

where $K = \{v \in L^1_{\text{loc}}(\Omega) \mid \nabla v \in L^2(\Omega),\ v = u^0 \text{ on } S\}$, here $u^0 > 0$, $Q \geqslant 0$ and $S \subset \partial\Omega$ are given. For the case that Q and u^0 are constants, the solution of their problem solves ours for $A = \int_\Omega \chi_{u>0} \, d\omega$.

Besides, the stationary points of the functional J have the property

$$|\nabla u| = Q \quad \text{on} \quad \Gamma = \Omega \cap \partial \{u > 0\} \,.$$

In their case, Q is given and the constant A is a result ; in ours, A is given and the constant λ is a consequence of the necessary conditions of optimality (see another related results using different techniques in Acker [1]).

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