M2AN. Mathematical modelling and numerical analysis - Modélisation mathématique et analyse numérique

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M2AN. Mathematical modelling and numerical analysis - Modélisation mathématique et analyse numérique, tome 23, n° 3 (1989), p. 541-561

<http://www.numdam.org/item?id=M2AN_1989__23_3_541_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE
(Vol. 23, n° 3, 1989, p 541-561)

INDUCED TRAJECTORIES AND APPROXIMATE INERTIAL MANIFOLDS

by Roger TEMAM (1)

INTRODUCTION

Inertial manifolds are new objects that have been recently introduced in relation with the study of large time behavior of dynamical systems (see [FST] [FNST] and the other references quoted therein and below). From the mathematical point of view, these are smooth (at least Lipschitz) finite dimensional manifolds that are invariant by the flow and attracts exponentially all the orbits. In particular, of course, they contain the universal attractor of the system and when they exist, they produce an imbedding of the attractor (which may be fractal), in a smooth dimensional manifold.

From the physical point of view, inertial manifolds can be viewed as a modeling of turbulence : indeed as it is recalled below the existence of an inertial manifold is equivalent to an interaction law between small and large structures in a turbulent flow. For an orbit lying on an inertial manifold, small and large eddies are related by the equation of the manifold and any orbit tends exponentially rapidly to the manifold. Hence the equation of the manifold is the governing law for the permanent regime.

The existence of inertial manifolds has been shown for a broad class of dissipative partial differential equations using the methods of [FST] [FNST] or other methods or generalizations that have been developed : see [CFNT1], [CFNT2], [MpS], [T3] and the references therein. Nevertheless there are stil several dissipative partial differential equations, including the two-dimensional Navier-Stokes equations for which no existence result of inertial manifold is yet available (2).

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(2) The difficulty here is the spectral gap condition the spectrum of the main linear operator must have sufficiently large gaps , see the references above

As a substitute to inertial manifolds when they are not available, a concept of approximate inertial manifold has been introduced [FMT] [FSTi], and our aim in this article is to provide a method for constructing a sequence of approximate inertial manifolds (AIM). AIM are manifolds that attract the orbits, in a small (thin) neighborhood, exponentially rapidly. They yield approximate laws of interaction between small and large structures, i.e. interaction laws satisfied up to a small error. The fact that these manifolds are only approximate one is compensated by the fact that their equation is rather simple. In this article we restrict ourselves to the two-dimensional Navier-Stokes equations but the methods are general and will be developed elsewhere for other equations (see already M. Marion [M1] [M2]). See also another totally different construction of AIM for the Navier-Stokes equations in [Ti].

The method leading to the construction of the approximate inertial manifolds that we present here is new and seems to have some intrinsic interest ; we call it the principle of the induced trajectory. It consists in associating with a given orbit a family of orbits (called the induced trajectories), that approximate the initial orbit at higher and higher order of accuracy. Furthermore induced orbits lie on a finite dimensional manifold or in a small neighborhood of such a manifold which plays the role of approximate inertial manifold.

The article is organized as follows. In Section 1 we recall the functional setting of the Navier-Stokes equations and survey a few relevant results. In Section 2 we construct sequences of approximation of a given orbit similar to an asymptotic expansion. In Section 3 we define the induced trajectories and study their properties. Finally in Section 4 we show how one can use the induced trajectories to construct approximate inertial manifolds. The results presented here were announced in [T4] ; the application of the concepts developed here to the numerical solution of the Navier-Stokes equations will be studied elsewhere.

CONTENTS

1. THE NAVIER-STOKES EQUATIONS (SURVEY)

1.1. The equation

In their functional setting the Navier-Stokes equations appear as a differential equation in an infinite dimensional Hilbert space H :

$$\frac{du}{dt} + \nu A u + B(u) = f \,, \tag{1.1}$$

$$u(0) = u_0 \,. \tag{1.2}$$

Here $u = u(t)$ is a function from $[0, +\infty)$ into H, representing the velocity vector field ; $\nu > 0$ is the kinematic viscosity, $f \in H$ represents volume forces. The operator A is an unbounded positive self-adjoint closed operator in H with domain $D(A) \subset H$ called the Stokes operator ; its inverse A^{-1} is compact in H ; finally $B(u) = B(u, u)$ where B is a bilinear continuous operator from $D(A) \times D(A)$ into H, that satisfies further continuity properties recalled below.

We denote by $(\cdot\,,\cdot)$ and $|\cdot|$, the scalar product and the norm in H. We know that we can define the power A^s of A for all $s \in \mathbb{R}$, and A^s maps $D(A^s)$ onto H ; $|A^s \cdot|$ is a Hilbert norm on $D(A^s)$. We set $V = D(A^{1/2})$ and denote the norm and the scalar product in V by $\|\cdot\|$, $((\cdot\,,\cdot))$. The particular interest for the norms $|\cdot|$, $\|\cdot\|$, is that $\frac{1}{2}|u|^2$ is the kinetic energy and $\|u\|^2$ the enstrophy of a flow with velocity field u.

Since A^{-1} is self-adjoint compact in H, there exists an orthonormal basis of H consisting of the eigenvectors w_j of A :

$$A w_m = \lambda_m w_m, \quad m \geq 1 \,,$$
$$0 < \lambda_1 \leq \lambda_2 \leq \cdots, \quad \lambda_m \to +\infty \text{ for } m \to +\infty \,.$$

Equation (1.1) is the evolution equation for the velocity u for a viscous incompressible fluid in a bounded domain ; depending on the choice of A and H, the boundary conditions are the no-slip condition, or a free boundary condition, or the space periodicity (see [T1] [T2]) ; (1.2) is of course the initial condition for the velocity. In the case of the space periodicity boundary condition the eigenvectors w_m are directly related to sine and cosine functions, e.g. in space dimension 2 :

$$\frac{\tilde{j}}{|j|} \sin \frac{2\,\Pi j x}{L}, \quad \frac{\tilde{j}}{|j|} \cos \frac{2\,\Pi j x}{L},$$

where

$$j = (j_1, j_2) \in \mathbb{N}^2 , \quad \tilde{j} = (j_2, -j_1) \quad \text{and} \quad \frac{jx}{L} = \frac{j_1 x_1}{L_1} + \frac{j_2 x_2}{L_2} ,$$

L_i the period in direction x_i.

In space dimension 2, it is well-known that for u_0 given in $D(A^{1/2})$, (1.1), (1.2) possesses a unique solution u bounded from $[0, \infty[$ into $D(A^{1/2})$. Furthermore u is analytic from $]0, \infty[$ into $D(A)$; the domain of analyticity of u in the complex plan $\mathbb{C}$ comprises the region $\Delta(\|u_0\|)$ defined by

$$(1.3) \quad \Delta(\|u_0\|) = \{\zeta \in \mathbb{C}, \operatorname{Re} \zeta > 0, \ |\operatorname{Im} \zeta| \leq T_0 \text{ if } \operatorname{Re} \zeta \geq T_0 \\ |\operatorname{Im} \zeta| \leq \operatorname{Re} \zeta, \text{ if } \operatorname{Re} \zeta \leq T_0\} \ ;$$

here $T_0 = T_0(\|u_0\|)$ is a bounded increasing function of ν^{-1}, $|f|$, λ_1^{-1} and $\|u_0\|$; see [T1] [T2]. If u is solution of (1.1), (1.2), then we set for $t_* \geq 0$ arbitrary (3)

$$(1.4) \qquad M_0(t_*) = \sup_{s \geq t_*} |u(s)| \ , \quad M_1(t_*) = \sup_{s \geq t_*} \|u(s)\| \ .$$

Finally, let us recall some well known continuity properties of the operator B that will be repeatedly used : there exist absolute constants c_1, c_2 such that for every u, v, $w \in D(A)$:

$$(1.5) \qquad |B(u, v)| \leq c_1 \begin{cases} |u|^{1/2} \|u\|^{1/2} \|v\|^{1/2} |Av|^{1/2} \\ |u|^{1/2} |Au|^{1/2} \|v\| \end{cases}$$

$$(1.6) \qquad |(B(u, v), w)| \leq c_2 |u|^{1/2} \|u\|^{1/2} \|v\| \, |w|^{1/2} \|w\|^{1/2} \, .$$

Like c_1, c_2 all the quantities c_i, c_i' that will appear subsequently are absolute constants. We recall also that

$$(1.7) \ |B(u, v)| \leq c_3 \|u\| \, \|v\| \left(1 + \log \frac{|Au|^2}{\lambda_1 \|u\|^2}\right)^{1/2} , \quad \forall u, v \in D(A) ,$$

$$(1.8) \ (B(u, v), v) = 0 \ , \quad \forall u \ , \ v \in D(A) \, .$$

As mentioned in the Introduction, the following results apply to more general equations. In particular we can consider an abstract equation (1.1) and the only properties used on B are (1.5)-(1.8). We could also assume slightly different hypotheses on B and obtain slightly different results.

(3) In the applications the time t_* can be either $t_* = 0$, in which case M_0, M_1 depend on u_0 Or t_* can be a time large enough, after the entrance of the orbit in the absorbing set, in which case M_0, M_1 are independent of u_0 , explicit values of M_0, M_1 in term of the other data are given in [FMT]

1.2. Projections of the equations

In the following we consider for $m \in \mathbb{N}$ fixed, the space spanned by $w_1, \ldots, w_m$ and we denote by P_m the orthogonal projector in H onto this space, $Q_m = I - P_m$. We recall that P_m and Q_m are also orthogonal projectors in all the spaces $D(A^s)$ and that they commute with A and its powers. When u is solution of (1.1), (1.2) we write $p_m = P_m u$, $q_m = Q_m u$ and projecting (1.1) on $P_m H$ and $Q_m H$ we find a coupled system of equations satisfied by p_m and q_m

$$\frac{dp_m}{dt} + \nu A p_m + P_m B(p_m + q_m) = P_m f, \tag{1.9}$$

$$\frac{dq_m}{dt} + \nu A q_m + Q_m B(p_m + q_m) = Q_m f \tag{1.10}$$

It is clear that p_m which corresponds to the eigenfrequencies $\lambda_1^{-1}, \ldots, \lambda_m^{-1}$ represents the superposition of large structures in the flow, while q_m, corresponding to eigenfrequencies $\leq \lambda_{m+1}^{-1}$ represents the superposition of small structures. Of course the choice of a cut-off value m is arbitrary but, necessary, since the λ_m constitute an unbounded increasing sequence.

When the index m is understood we write for the sake of simplicity

$$P = P_m, \quad Q = Q_m, \quad \lambda = \lambda_m, \quad \Lambda = \lambda_{m+1}$$

Some a priori-estimates on $q = q_m$ valid for large t, were derived in [FMT]. They show that the kinetic energy and the enstrophy carried by q_m, i.e. the small eddies, is small for large t (and large m), whatever the initial data.

Let us consider an initial datum u_0 in (1.2) satisfying

$$|u_0| \leq R_0, \quad \|u_0\| \leq R_1 \tag{1.11}$$

Then we know that there exists a time t_* that depends on R_0, R_1, and the other data, ν, $|f|$, λ_1, such that for $t \geq t_*$,

$$|u(t)| \leq M_0, \quad \|u(t)\| \leq M_1, \tag{1.12}$$

where M_0, M_1 are independent of u_0, but depend on the other data (4).

We now recall, with some slight improvements, the estimates on q_m in [FMT] (5),

(4) This is related to the existence of an absorbing set in H and V for the dynamical system (1.1), t_* is the entrance time in these absorbing sets, see [T3].

(5) In [FMT] the estimates on q_m are valid for large m, those given here are valid for all m.

(1.13) *For any orbit of* (1.1), *after a time* t_1 *which depends only on the data* ν, $|f|$, λ_1 *and on* u_0 *through* R_0, *the small eddies component of* u, $q_m = Q_m u$ *is small in the following sense*

$$|q_m(t)| \leqslant K_0 L^{1/2} \delta , \quad \|q_m(t)\| \leqslant K_1 L^{1/2} \delta^{1/2} ,$$
$$|q_m'(t)| \leqslant K_0' L^{1/2} \delta , \quad |Aq_m(t)| \leqslant K_2 L^{1/2} .$$

Here

$$\delta = \frac{\lambda_1}{\Lambda} = \frac{\lambda_1}{\lambda_{m+1}} , \quad L = 1 + \log \frac{\lambda_{m+1}}{\lambda_1} , \tag{1.14}$$

and K_0, K_1, K_2, K_0', depend only on the data ν, $|f|$, λ_1.

We briefly recall the proof of (1.13) in order to introduce some necessary notations. Upon taking the scalar product of (1.10) with $q = q_m$ and using (1.8) we find

$$\frac{1}{2}\frac{d}{dt}|q|^2 + \nu\|q\|^2 = (Qf, q) - (B(p,p), q) - (B(q,p), q) .$$

We use (1.6) and (1.7) and observe that

$$|A\varphi|^2 \leqslant \lambda_m \|\varphi\|^2, \quad 1 + \log \frac{|A\varphi|^2}{\lambda_1 \|\varphi\|^2} \leqslant L , \quad \forall \varphi \in P_m H . \tag{1.15}$$

This yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|q|^2 + \nu\|q\|^2 &\leqslant |Qf|\,|q| + c_3 L^{1/2}\|p\|^2 |q| + c_2 \|p\|\,\|q\|\,|q| \\ &\leqslant (|Qf| + c_3 L^{1/2} M_1^2 + c_2 M_1^2)|q| \\ &\leqslant c_1' \lambda_{m+1}^{-1/2}(|Qf| + L^{1/2} M_1^2)\|q\| \\ &\leqslant \frac{\nu}{2}\|q\|^2 + \frac{c_2'}{\nu\lambda_{m+1}}(|Qf|^2 + LM_1^4)\end{aligned}$$

$$\frac{d}{dt}|q|^2 + \nu\|q\|^2 \leqslant \frac{c_4}{\nu\lambda_{m+1}}(|Qf|^2 + M_1^4) L \tag{1.16}$$

$$\frac{d}{dt}|q|^2 + \nu\lambda_{m+1}|q|^2 \leqslant \frac{c_4}{\nu\lambda_{m+1}}(|Qf|^2 + M_1^4) L .$$

By integration in time we find

$$|q_m(t)|^2 \leqslant |q_m(t_*)|^2 \exp(-\nu\lambda_{m+1}(t - t_*)) + \frac{c_4}{\nu^2 \lambda_{m+1}^2}(|Qf|^2 + M_1^4) L . \tag{1.17}$$

We write

$$K_0 = \frac{2\,c_4}{\nu^2\,\lambda_1^2}\,(|f|^2 + M_1^4) \tag{1.18}$$

and, with the notation (1.4) :

$$|q_m(t_*)| \leqslant |u(t_*)| \leqslant M_0\,.$$

Hence

$$|q_m(t)|^2 \leqslant M_0^2 \exp(-\nu\lambda_1(t-t_*)) + \frac{1}{2} K_0\, L\delta^2 \tag{1.19}$$

and for $t \geqslant t_0$,

$$t_0 = t_* + \frac{1}{\nu\lambda_1} \log \frac{K_0}{2\,M_0^2}\,, \tag{1.20}$$

we obtain

$$|q_m(t)|^2 \leqslant K_0\, L\delta^2\,. \tag{1.21}$$

Then we want to estimate the norm of q_m in $D(A^{1/2})$ in a similar manner. Upon taking the scalar product of (1.10) with Aq_m in H, we find after some similar computations (for the details see [FMT]) :

$$\frac{d}{dt}\,\|q_m\|^2 + \nu\,|Aq_m|^2 \leqslant \frac{c_5}{\nu}\left(|Qf|^2 + M_1^4 + \frac{M_0^2\,M_1^4}{\nu^2}\right) L \tag{1.22}$$

$$\frac{d}{dt}\,\|q_m\|^2 + \nu\lambda_{m+1}\,\|q_m\|^2 \leqslant \frac{c_5}{\nu}\left(|Qf|^2 + M_1^4 + \frac{M_0^2\,M_1^4}{\nu^2}\right) L\,.$$

We set

$$K_1 = \frac{2\,c_5}{\nu^2\,\lambda_1}\left(|f|^2 + M_1^4 + \frac{M_0^2\,M_1^4}{\nu^2}\right)$$

and for $t \geqslant t_0$ we write

$$\begin{aligned} \|q_m(t)\|^2 &\leqslant \|q_m(t_0)\|^2 \exp(-\nu\lambda_{m+1}(t-t_0)) + \frac{1}{2} K_1\, L\delta \\ &\leqslant M_1^2 \exp(-\nu\lambda_{m+1}(t-t_0)) + \frac{1}{2} K_1\, L\delta\,. \end{aligned} \tag{1.23}$$

If $t \geqslant t_1$,

$$t_1 = t_0 + \frac{1}{\nu\lambda_1} \log \frac{K_1\, L\delta}{2\,M_1^2} \tag{1.24}$$

we find as announced :

$$\|q_m(t)\|^2 \leq K_1 L\delta . \tag{1.25}$$

The estimate on q_m' in (1.13) is derived from that on q_m by observing that q_m is analytic in time in $\Delta(\|u_0\|)$ like u, and by using Cauchy formula in the strip $\Delta(\|u_0\|)$. Finally the estimate on Aq_m in (1.13) follows promptly from (1.8), the previous estimates and (1.5)-(1.7).

2. ASYMPTOTIC EXPANSIONS

From now on we assume that the data are fixed, in particular ν, f, A and to some extent u_0. To a given solution u of (1.1), (1.2) we want to associate a sequence of functions that approximate u at higher and higher levels of accuracy, like an asymptotic expansion.

The order of the truncation m is temporarily fixed ($P = P_m$, $Q = Q_m$), although our aim is eventually to study asymptotic expansions valid for m large. For each m we decompose the solution of (1.1), (1.2) as

$$\begin{cases} u(t) = p_m(t) + q_m(t) , \\ p_m(t) = P_m u(t) , \quad q_m(t) = Q_m u(t) , \end{cases} \tag{2.1}$$

and we define a sequence of functions $q_{jm} = q_{jm}(t)$, $j \in \mathbb{N}$, approximating q_m. Each q_{jm} is of the form

$$q_{jm} = k_{0m} + \cdots + k_{jm} \tag{2.2}$$

where the k_{jm} are recursively defined as follows.

For $j = 0, 1$,

$$\nu A k_{0m} + Q_m B(p_m) = Q_m f , \tag{2.3}$$

$$\nu A k_{1m} + Q_m B(p_m, k_{0m}) + Q_m B(k_{0m}, p_m) = 0 . \tag{2.4}$$

Then for $j \geq 2$

$$(2.5)_j \quad k_{j-2,m}' + \nu A k_{jm} + Q_m B(p_m, k_{j-1,m}) + Q_m B(k_{j-1,m}, p_m) + \\ + \sum_{\substack{r,s=0 \\ r \text{ or } s = j-2}}^{j-2} Q_m B(k_{r,m}, k_{s,m}) = 0 .$$

Of course, everywhere in (2.3)-(2.5), the time variable t is understood, $k_{jm} = k_{jm}(t)$. For each j and t the existence and uniqueness of $k_{jm}(t)$ is easy. Since the quantities $k_{0m}, \ldots, k_{j-1,m}$ are known when we determine k_{jm}, it suffices to observe that k_{jm} is solution of an equation

$$\nu A k_{jm}(t) = \varphi(t) ,$$

and this is equivalent to the inversion of A, i.e. the solution of a (linear) *Stokes problem.*

By adding the relations (2.3)-(2.5) we refer relations satisfied by the q_{jm} (see (2.2)) :

$$\nu A q_{0m} + Q_m B(p_m) = Q_m f, \tag{2.6}$$

$$\nu A q_{1,m} + Q_m B(p_m, q_{0m}) + Q_m B(q_{0m}, p_m) = Q_m f, \tag{2.7}$$

and for $j \geqslant 2$

$$(2.8)_j \quad q'_{j-2,m} + \nu A q_{jm} + Q_m B(p_m) + Q_m B(p_m, q_{j-1,m}) + \\ + Q_m B(q_{j-1,m}, p_m) + Q_m B(q_{j-2,m}) = Q_m f .$$

For (2.8) we have used

$$B(q_{j-2,m}, q_{j-2,m}) = \sum_{r,s=0}^{j-2} B(k_{r,m}, k_{s,m}) . \tag{2.9}$$

In Section 3 we shall compare the q_{jm} and the q_m but, at this point, we derive some a priori estimates on the quantities q_{jm}, k_{jm}.

THEOREM 2.1 : *There exist constants κ_j that are independent of m but depend on j and on the data ν, $|f|$, λ_1 ; there exists $t_2(\geqslant t_1)$ depending only on ν, $|f|$, λ_1, such that for each m, each j and each $t \geqslant t_2$ the following estimates are valid*

$$(2.10)_j \quad \begin{cases} |k_{jm}(t)| & \leqslant \kappa_j\, \delta^{1+j/2} L^{1/2+j/2} \\ \|k_{jm}(t)\| & \leqslant \kappa_j\, \delta^{1/2+j/2} L^{1/2+j/2} \\ |Ak_{jm}(t)| & \leqslant \kappa_j\, \delta^{j/2} L^{1/2+j/2} \\ |k'_{jm}(t)| & \leqslant \kappa_j\, \delta^{1+j/2} L^{1/2+j/2} \end{cases}$$

$$(2.11)_j \quad \begin{cases} |q_{jm}(t)| & \leqslant \kappa_j\, \delta L^{1/2} \\ \|q_{jm}(t)\| & \leqslant \kappa_j\, \delta^{1/2} L^{1/2} \\ |Aq_{jm}(t)| & \leqslant \kappa_j\, L^{1/2} \\ |q'_{jm}(t)| & \leqslant \kappa_j\, \delta L^{1/2} . \end{cases}$$

Proof: Because of (2.2), inequalities (2.11) follow readily from (2.10) by summation, observing that $\delta L \leqslant 1$. We only prove inequalities (3.2).

We examine separately the cases $j = 0$, 1 and then we proceed by induction for $j \geqslant 2$. We rely of course on (1.13).

For a purely technical reason the inequalities (2.10) will be proved for t in an appropriate domain of the complex plan. We recall (see (1.12), (1.20), (1.24)), that

$$\|u(t)\| \leqslant M_1 \quad \text{for} \quad t \geqslant t_1 . \tag{2.12}$$

Hence by (1 3), where $t = 0$ is replaced by $t = t_1$, we see that u is analytic in the region $t_1 + \Delta(M_1)$, $T_0 = T_0(M_1)$ It follows also from the proof of (1 3) that

$$\|u(t)\| \leqslant 2(1 + M_1), \quad \text{for} \quad t \in t_1 + \Delta(M_1) \tag{2 13}$$

In fact (2 10) will be proved in a sequence of slightly decreasing regions, namely

$$\left\{t_1 + \frac{3}{2} T_0(M_1) + \frac{1}{2}\left(1 + \frac{1}{2^{j+1}}\right)(- T_0(M_1) + \Delta(M_1))\right\} \tag{2 14$_j$}$$

i) *Case* $j = 0$

For $j = 0$ and $t \geqslant t_1$, t_1 as in (1 13), we write

$$A k_{0m} = \frac{1}{\nu}(Q_m f - QB(p_m)) \tag{2 15}$$

$$\begin{aligned} |A k_{0m}| &\leqslant \frac{1}{\nu}|Q_m f| + \frac{1}{\nu}|B(p_m)| \\ &\leqslant \text{(thanks to (1 7), (1 15))} \\ &\leqslant \frac{1}{\nu}|Q_{mf}| + \frac{c_3}{\nu} L^{1/2}\|p_m\|^2 \\ &\leqslant \frac{1}{\nu}|Q_m f| + \frac{c_3}{\nu} M_1^2 L^{1/2} \\ &\leqslant \frac{1}{\nu}(|f| + c_3 M_1^2) L^{1/2} \end{aligned} \tag{2 16}$$

This shows the third inequality in (2 10) for $j = 0$, the first and second inequalities are proved by observing that the norm of A^{-1} in $\mathscr{L}(Q_m H)$ is bounded by λ_{m+1}^{-1}

$$|A^{-1}|_{\mathscr{L}(Q_m H)} \leqslant \lambda_{m+1}^{-1} \tag{2 17}$$

For proving the fourth inequality in (2 10) we observe that k_{0m} is analytic in the same region $t_1 + \Delta(M_i)$ as u Also the estimates (2 16) are valid in that region of $\mathbb{C}$ as well Hence for $t \in \frac{3}{4}(T_0(M_1) + \Delta(M_1))$ (see (2 14)), we find by application of Cauchy formula to a circle centered at t of radius $\frac{T_0(M_1)}{4\sqrt{2}}$ that

$$|k'_{0m}(t)| \leqslant \frac{4\sqrt{2}}{T_0(M_1)} \sup_{s \in t_1 + \Delta(M_1)} |k_{0m}(s)|, \tag{2 18}$$

(2 10) follows

ii) *Case* $j = 1$

Thanks to (2.4), the estimates on B and the previous estimates on q_m and q_{0m}, we have

$$
\begin{aligned}
(2.19)\quad |Ak_{1m}| &\leqslant \\
&\leqslant \frac{1}{\nu}\,|B(p_m, k_{0m})| + \frac{1}{\nu}\,|B(k_{0m}, p_m)| \\
&\leqslant \frac{c_3}{\nu} L^{1/2}\|p_m\|\,\|k_{0m}\| + \frac{c_1}{\nu}\,|k_{0m}|^{1/2}\,\|k_{0m}\|^{1/2}\,\|p_m\|^{1/2}\,|Ap_m|^{1/2} \\
&\leqslant \text{(with (1.15))} \\
&\leqslant \frac{c_3}{\nu} L^{1/2} M_1 \|k_{0m}\| + \frac{c_1}{\nu}\,\|k_{0m}\|\,\|p_m\| \\
&\leqslant \left(\frac{c_3}{\nu} L^{1/2} M_1 + \frac{c_1}{\nu} M_1\right)\|k_{0m}\|\,.
\end{aligned}
$$

Using the estimate on $\|k_{0m}\|$ we obtain the third inequality (2.10) for $j = 1$ and then the first two follow from (2.17). The fourth inequality is proved like (2.18).

iii) *Case* $j \geqslant 2$

We proceed by induction and assume that inequalities (2.10) have been proved at order $0, \ldots, j-1$, in the regions (2.14). We want to prove them at order j.

We infer from $(2.5)_j$ that

$$
\begin{aligned}
(2.20)\qquad |Ak_{j,m}| \leqslant \frac{1}{\nu}\,|B(p_m, k_{j-1,m})| + \frac{1}{\nu}\,|B(k_{j-1,m}, p_m)| + \\
+ \frac{1}{\nu} \sum_{\substack{r,s=0 \\ r \text{ or } s = j-2}}^{j-2} |B(k_{r,m}, k_{s,m})| + \frac{1}{\nu}\,|k'_{j-2,m}|\,.
\end{aligned}
$$

We now use the estimates (1.5)-(1.7) on B and (1.12), (1.15) :

$$
\begin{aligned}
|Ak_{j,m}| \leqslant \frac{c_3}{\nu} L^{1/2}\|p_m\|\,\|k_{j-1,m}\| + \frac{c_1}{\nu}\,|k_{j-1,m}|^{1/2}\,|Ak_{j-1,m}|^{1/2}\,\|p_m\| + \\
+ \frac{c_1}{\nu} \sum_{\substack{r,s=0 \\ r \text{ or } s = j-2}}^{j-2} |k_{r,m}|^{1/2}\,|Ak_{r,m}|^{1/2}\,\|k_{s,m}\| + \frac{1}{\nu}\,|k'_{j-2,m}|\,.
\end{aligned}
$$

Thanks to the induction hypothesis, we obtain the following estimate valid in the region $(2.14)_{j-1}$:

$$
\begin{aligned}
(2.21)\qquad |Ak_{j,m}| &\leqslant \kappa\delta^{j/2} L^{j/2}(1 + L^{1/2}) + \kappa\delta^{j/2} L^{j/2-1/2} \\
&\leqslant \kappa\delta^{j/2} L^{1/2+j/2}\,.
\end{aligned}
$$

Like the κ_j, κ is a constant depending on ν, $|f|$, λ_1 and j that may be different at different places in the text. This proves the third inequality (2.10) at order j in the region $(2.14)_{j-1}$. Thanks to (2.17) we obtain the first and second inequality (2.10) at order j in the same region. Finally due to Cauchy formula we obtain the fourth inequality (2.10) in the region $(2.14)_j$ which is slightly smaller than $(2.14)_{j-1}$.

Theorem 2.1 is proved $(t_2 = t_1 + T_0(M_1))$.

3. THE INDUCED TRAJECTORIES

We call *induced trajectories* associated to a trajectory

$$u(t) = p_m(t) + q_m(t)\,, \tag{3.1}$$

the trajectories $u_{jm} = u_{jm}(t)$ defined by

$$u_{j,m}(t) = p_m(t) + q_{j,m}(t)\,. \tag{3.2}$$

Since

$$\chi_{j,m} = u_{j,m} - u = q_{j,m} - q_m\,, \tag{3.3}$$

the comparison of the induced trajectories $u_{j,m}$ to u is reduced to the comparison of $q_{j,m}$ and q_m. Upon subtracting (1.10) from (2.6), (2.7) or $(2.8)_j$ we obtain

$$\nu A\chi_{0,m} + Q_m B(p_m, q_m) + Q_m B(q_m, p_m) = Q_m B(q_m) + q_m'\,, \tag{3.4}$$

$$\nu A\chi_{j,m} + Q_m B(p_m, \chi_{0,m}) + Q_m B(\chi_{0,m}, p_m) = Q_m B(q_m)\,, \tag{3.5}$$

and for $j \geqslant 2$,

$$\begin{aligned}\nu A\chi_{j,m} + Q_m B(p_m, \chi_{j-1,m}) + Q_m B(\chi_{j-1,m}, p_m) &= \\ = Q_m B(q_m) - Q_m B(q_{j-2,m}) - \chi_{j-2,m}'\,. \end{aligned} \tag{3.6_j}$$

Our aim in this section is to prove the following.

THEOREM 3.1 : *There exist constant $\bar{\kappa}_j$ that are independent of m but depend on j and on the data ν, $|f|$, λ_1 ; there exists t_3 depending only on ν, $|f|$, λ_1, such that for each m, each j and each $t \geqslant t_3$ the inequalities hereafter hold :*

$$\begin{cases} |\chi_{j,m}(t)| & \leqslant \bar{\kappa}_j\, \delta^{3/2+j/2} L^{1+j/2}\,, \\ \|\chi_{j,m}(t)\| & \leqslant \bar{\kappa}_j\, \delta^{1+j/2} L^{1+j/2}\,, \\ |A\chi_{j,m}(t)| & \leqslant \bar{\kappa}_j\, \delta^{1/2+j/2} L^{1+j/2}\,, \\ |\chi_{j,m}'(t)| & \leqslant \bar{\kappa}_j\, \delta^{3/2+j/2} L^{1+j/2}\,. \end{cases} \tag{3.7}$$

Remark 3.1 : Theorem 3.1 implies in particular that an orbit $u(\cdot)$ can be approximated for t large ($t \geq t_3$) at an arbitrary order of accuracy by an induced trajectory $u_{j,m}$, provided m is sufficiently large. The order of accuracy is given for each j by (3.7) ($\delta = \delta_m$, $L = L_m$).

Proof of Theorem 3.1 : The proof relies on (1.13) and Theorem 2.1. As in Theorem 2.1, we shall establish (3.7) for t in an appropriate region of $\mathbb{C}$, namely

$$(3.8)_j \qquad t_2 + \frac{3}{2} T_0(M_1) + \frac{1}{2}\left(1 + \frac{1}{3.2^j}\right)(- T_0(M_1) + \Delta(M_1)) .$$

Note that

$$(2.14)_j \subset (3.8)_j \subset (2.14)_{j-1} .$$

i) *Case* $j = 0$

We start from (3.4). We infer from (1.5), (1.7), (1.15) and (2.10) with $j = 0$, that for $t \geq t_2$:

$$(3.9) \quad |A\chi_{0m}| \leq \frac{c_3}{\nu} L^{1/2}\|p_m\|\,\|q_m\| + \frac{c_1}{\nu}|q_m|^{1/2}\,\|q_m\|^{1/2}\,\|p_m\|^{1/2}|Ap_m|^{1/2} + \frac{c_1}{\nu}|q_m|^{1/2}\,\|q_m\|\,|Aq_m|^{1/2} + \frac{\kappa}{\nu} L^{1/2}\delta .$$

We then write

$$\|p_m\| \leq \|u\| \leq M_1 , \quad |Ap_m| \leq \lambda_{m+1}^{1/2}\|p_m\| \leq \lambda_{m+1}^{1/2} M_1$$

and use the estimates (1.13) for q_m. We obtain

$$|A\chi_{0,m}| \leq \kappa L\delta^{1/2} + \kappa L^{1/2}\delta^{1/2} + \kappa L\delta + KL^{1/2}\delta^{1/2}$$

$$|A\chi_{0,m}| \leq \kappa L\delta^{1/2} .$$

This shows the third estimate (3.7) for $j = 0$; the first and second estimates follow readily using (2.17). Finally the estimate on $\chi'_{0,m}$ is proved like (2.18) using Cauchy formula.

ii) *Case* $j = 1$

We start from (3.5). As for (3.4), (3.9) we see that for $t \geq t_2$:

$$(3.10) \quad |A\chi_{1,m}| \leq \frac{1}{\nu}|B(p_m, \chi_{0m})| + \frac{1}{\nu}(B(\chi_{0,m}, p_m)| + \frac{1}{\nu}|B(q_m)| + \frac{1}{\nu}|q'_m|$$

$$\leq \frac{c_3}{\nu} L^{1/2} \|p_m\| \, \|\chi_{0,m}\|$$
$$+ \frac{c_1}{\nu} |\chi_{0,m}|^{1/2} \|\chi_{0,m}\|^{1/2} \|p_m\|^{1/2} |Ap_m|^{1/2}$$
$$+ \frac{c_1}{\nu} |q_m|^{1/2} \|q_m\| \, |Aq_m|^{1/2} + \frac{1}{\nu} |q_m'|$$
$$\leq \text{(with (1.12) and (1.15))}$$
$$\leq \frac{c_3}{\nu} L^{1/2} M_1 \|\chi_{0,m}\| + \frac{c_1}{\nu} M_1 \|\chi_{0,m}\|$$
$$+ \frac{c_1}{\nu} |q_m|^{1/2} \|q_m\| \, |Aq_m|^{1/2} + \frac{1}{\nu} |q_m'| \, .$$

Using the estimates on q_m and $\chi_{0,m}$ we obtain

$$|A\chi_{1,m}| \leq \kappa (L^{3/2} \delta + L\delta + L^{1/2} \delta)$$
$$|A\chi_{1,m}| \leq \kappa L^{3/2} \delta \, .$$

The third inequality (3.7) for $j = 1$ is proved ; the first two inequalities follow from (2.17) ; the fourth one is derived using Cauchy formula.

iii) *Case* $j > 2$

We start from $(3.6)_j$. We observe that

$$q_m = q_{j-2,m} - (q_m - q_{j-2,m}) = q_{j-2,m} - \chi_{j-2,m}$$
$$B(q_m) - B(q_{j-2,m}) = B(q_{j-2,m} - \chi_{j-2,m}) - B(q_{j-2,m}) =$$
$$= - B(\chi_{j-2,m}, q_{j-2,m}) - B(q_{j-2,m}, \chi_{j-2,m}) + B(\chi_{j-2,m}) \, .$$

Thus

$$\begin{aligned} \nu A\chi_{j,m} = & - Q_m B(p_m, \chi_{j-1,m}) - Q_m B(\chi_{j-1,m}, p_m) \\ & - Q_m B(\chi_{j-2,m}, q_{j-2,m}) - Q_m B(q_{j-2,m}, \chi_{j-2,m}) \\ & + Q_m B(\chi_{j-2,m}) - \chi_{j-2,m}' \, . \end{aligned}$$

We now use the estimates (1.5)-(1.7) on B and (1.12), (1.15) :

$$|A\chi_{j,m}| \leq \frac{c_3}{\nu} L^{1/2} \|p_m\| \, \|\chi_{j-1,m}\| + \frac{c_1}{\nu} |\chi_{j-1,m}|^{1/2} |A\chi_{j-1,m}|^{1/2} \|p_m\|$$
$$+ \frac{c_1}{\nu} |\chi_{j-2,m}|^{1/2} |A\chi_{j-2,m}|^{1/2} \|q_{j-2,m}\|$$
$$+ \frac{c_1}{\nu} |q_{j-2,m}|^{1/2} |Aq_{j-2,m}|^{1/2} \|\chi_{j-2,m}\|$$
$$+ \frac{c_1}{\nu} |\chi_{j-2,m}|^{1/2} \|\chi_{j-2,m}\| \, |A\chi_{j-2,m}|^{1/2} + |\chi_{j-2,m}'| \, .$$

Thanks to the induction hypothesis we obtain the following estimate valid in the region $(2.14)_{j-1}$

$$\begin{aligned}|A\chi_{j,m}| &\leq \kappa\delta^{1/2+j/2} L^{1+j/2} + \kappa\delta^{1/2+j/2} L^{1/2+j/2}(1+\delta^{1/2} L^{1/2}) \\ &\quad + \kappa\delta^{j} L^{j} + \kappa\delta^{1/2+j/2} L^{j/2} \\ &\leq (\text{since } \delta \leq 1 \leq L,\ \delta L \leq 1) \\ &\leq \kappa\delta^{1/2+j/2} L^{1+j/2}.\end{aligned}$$

This proves the third inequality (3.7) at order j in the region $(2.14)_{j-1}$. Thanks to (2.17) we obtain the first and second inequality (3.7) at order j in the same region ; then thanks to Cauchy formula we obtain the fourth inequality (3.7) in the region $(3.8)_j$ (which is smaller than the region $(2.14)_{j-1}$).

The proof of Theorem 3.1 is complete $(t_3 = t_2 + T_0(M_1))$.

4. APPROXIMATE INERTIAL MANIFOLDS

Our aim is to use the previous approximation results for the construction of approximate inertial manifolds for the two-dimensional Navier-Stokes equations.

A first simple remark is to reinterpret (1.13). Indeed (1.13) amounts to saying that the flat space $P_m H$ is an approximate inertial manifold for the Navier-Stokes equations. Each orbit enters after a finite time (namely t_1) in a thin neighborhood of $P_m H$ of thickness $\kappa_0 L^{1/2} \delta$ in H, or $\kappa_1 L^{1/2} \delta^{1/2}$ in V. Of course the universal attractor $\mathscr{A}$ for these equations lies in this neighborhood.

We are more interested in nonflat inertial manifolds and with that respect, less obvious results follow from inequalities (3.7) at the order 0 or 1. At order 0 we recover the approximate inertial manifold $\mathscr{M}_0$ in [FMT]. Indeed let $\mathscr{M}_0$ be the quadratic surface of H of equation

$$Q_m \varphi = (\nu A)^{-1} (Q_m f - Q_m B(P_m \varphi)) \tag{4.1}$$

or in a more elementary form, setting $X = P_m \varphi$, $Y = Q_m \varphi$,

$$Y = (\nu A)^{-1} (Q_m f - Q_m B(X)). \tag{4.2}$$

Note that $X \in P_m H$, of dimension m, while $Y \in Q_m H$ which has infinite dimension and that the right hand-side of (4.2) is quadratic in X. Then $(3.7)_0$ states that after a finite time (namely t_3), $u(\cdot)$ enters in a thin neighborhood of $\mathscr{M}_0$ of thickness $\kappa_0 \delta^{3/2} L$ in H or $\kappa_0 \delta L$ in V

$$\begin{cases}\operatorname{dist}_H (u(t), \mathscr{M}_0) \leq |\chi_{0,m}(t)| \leq \kappa_0 \delta^{3/2} L \\ \operatorname{dist}_V (u(t), \mathscr{M}_0) \leq \|\chi_{0,m}(t)\| \leq \kappa_0 \delta L, \quad \text{for} \quad t \geq t_3.\end{cases} \tag{4.3}$$

Note that this neighborhood is thinner than the neighborhood of $P_m H$ previously mentioned by an order $(\delta L)^{1/2}$; of course the universal attractor $\mathscr{A}$ lies in this neighborhood of $\mathscr{M}_0$ and thus in its intersection with the previous neighborhood of $P_m H$ (see *fig.* 4.1).

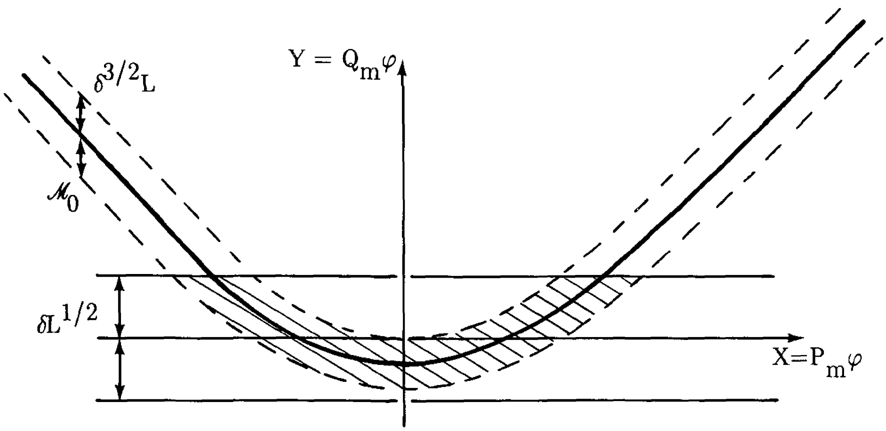

Figure 4.1. — Localization of the universal attractor $\mathscr{A}$ in H: $\mathscr{A}$ lies in the dashed region.

With inequalities (3.7) at order 1 we define another approximate inertial manifold $\mathscr{M}_1$ that attracts all the orbits in a finite time, in a still thinner neighborhood. Let $\Phi_0(X)$ denote the right hand-side of (4.1) and consider now the manifold $\mathscr{M}_1$ of equation

$$(4.4) \quad \begin{aligned} Y &= \Phi_1(X) \\ &= (\nu A)^{-1}\left(Q_m f - Q_m B(X, \Phi_0(X)) - Q_m B(\Phi_0(X), X)\right). \end{aligned}$$

According to $(3.7)_1$,

$$(4.5) \quad \begin{cases} \operatorname{dist}_H (u(t), \mathscr{M}_1) \leq |\chi_{1,m}(t)| \leq \kappa_1 \delta^2 L^{3/2} \\ \operatorname{dist}_V (u(t), \mathscr{M}_1) \leq \|\chi_{1,m}(t)\| \leq \kappa_1 \delta^{3/2} L^{3/2}, \quad \text{for} \quad t \geq t_3 . \end{cases}$$

Hence after a finite time $u(t)$ lies in a neighborhood of $\mathscr{M}_1$ of thickness $\kappa_1 \delta^2 L^{3/2}$ in H or $\kappa_1 \delta^{3/2} L^{3/2}$ in V ; this is thinner by an order $(\delta L)^{1/2}$ than the above neighborhood of $\mathscr{M}_0$ and by an order δL than the above neighborhood of $P_m H$.

We intend now to construct other (better) approximate inertial manifolds but the procedure will be more involved. The simplicity of the equation of $\mathscr{M}_0$, $\mathscr{M}_1$ resulted from the fact that the induced trajectories $u_{0,m}$,

$u_{1,m}$ lie in these manifolds but this is not the case anymore for $u_{2,m}$, etc. However we shall prove that $u_{2,m}$, $u_{3,m}$ are respectively very close from approximate inertial manifolds $\mathcal{M}_2$, $\mathcal{M}_3$.

The manifold $\mathcal{M}_2$

Equation (2.8) with $j = 2$ reads

$$\nu A q_{2,m} + Q_m B(p_m) + Q_m B(p_m, q_{1,m}) + + Q_m B(q_{1,m}, p_m) + Q_m B(q_{0,m}) = Q_m f - q'_{0,m} . \tag{4.6}$$

By differentiation of (2.6) we obtain

$$\begin{aligned} q'_{0,m} &= -(\nu A)^{-1}(Q_m B(p_m, p'_m) + Q_m B(p'_m, p_m)) \\ &= D\Phi_0(p_m) \cdot p'_m , \end{aligned} \tag{4.7}$$

where $D\Phi_0$ is the differential of Φ_0. On the other hand (1.9) yields

$$p'_m = \Psi(p_m, q_m) = -\nu A p_m - P_m B(p_m + q_m) + P_m f . \tag{4.8}$$

We now replace p'_m by an approximation $\bar{p}'_m$ and this yields an approximation $\bar{q}'_{0,m}$ of $q'_{0,m}$ and an approximation $\bar{q}_{2,m}$ of $q_{2,m}$:

$$\bar{p}'_m = \Psi(p_m, q_{0,m}) = -\nu A p_m - P_m B(p_m + q_{0,m}) + P_m f \tag{4.9}$$

$$\begin{aligned} \bar{q}'_{0,m} &= -(\nu A)^{-1}(Q_m B(p_m, \bar{p}'_m) + Q_m B(\bar{p}'_m, p_m)) \\ &= D\Phi_0(p_m) \cdot \bar{p}'_m \end{aligned} \tag{4.10}$$

$$\nu A \bar{q}_{2,m} + Q_m B(p_m) + Q_m B(p_m, q_{1,m}) + + Q_m B(q_{1,m}, p_m) + Q_m B(q_{0,m}) = Q_m f - \bar{q}'_{0,m} . \tag{4.11}$$

In this manner we obtain a trajectory

$$\bar{u}_{2,m} = p_m + \bar{q}_{2,m}$$

lying in the manifold $\mathcal{M}_2$ of equation

$$Y = \Phi_3(X)$$

where $Y = Q_m \varphi$, $X = P_m \varphi$ as before, and

$$\begin{aligned} \Phi_3(X) = (\nu A)^{-1} Q_m\{f - B(X) - B(X, \Phi_1(X)) - \\ - B(\Phi_1(X), X) - B(\Phi_0(X)) - D\Phi_0(X) \cdot \Psi(X, \Phi_0(X))\} . \end{aligned} \tag{4.12}$$

The distance in H or V of $u(t)$ to $\mathcal{M}_2$ is bounded by the corresponding norm of $\bar{u}_{2,m}(t) - u(t) = \bar{\chi}_{2,m}(t)$:

$$(4.13)\qquad \begin{cases} \operatorname{dist}_H\,(u(t),\mathcal{M}_2) \leqslant |\bar{u}_{2,m}(t)-u(t)| = |\bar{\chi}_{2,m}(t)| \\ \operatorname{dist}_V\,(u(t),\mathcal{M}_2) \leqslant \|\bar{u}_{2,m}(t)-u(t)\| \leqslant \|\bar{\chi}_{2,m}(t)\|\,. \end{cases}$$

But $\bar{\chi}_{2,m} = \chi_{2,m} + \bar{q}_{2,m} - q_{2,m}$

$$(4.14)\qquad \begin{aligned} \nu A(\bar{q}_{2,m}-q_{2,m}) &= -\bar{q}'_{0,m} + q'_{0,m} \\ |A(\bar{q}_{2,m}-q_{2,m})| &\leqslant \frac{1}{\nu}\,|\bar{q}'_{0,m}-q'_{0,m}| \\ &\leqslant \frac{1}{\nu}\,|(\nu A)^{-1}\,(Q_m\,B(p_m,p'_m-\bar{p}'_m) \\ &\quad + Q_m\,B(p'_m-\bar{p}'_m,p_m))|\,. \end{aligned}$$

Also

$$(4.15)\qquad \begin{aligned} \bar{p}'_m - p'_m &= P_m\,B(p_m+q_m) - P_m\,B(p_m+q_{0,m}) \\ &= -P_m\,B(p_m+q_m,\chi_{0,m}) - P_m\,B(\chi_{0,m},p_m+q_{0,m}) \end{aligned}$$

$$(4.16)\qquad \begin{aligned} |\bar{p}'_m - p'_m| &\leqslant \kappa L^{1/2}\|\chi_{0,m}\| \leqslant \kappa\delta L^{3/2} \\ \|\bar{p}'_m - p'_m\| &\leqslant \lambda_m^{1/2}|\bar{p}'_m - p'_m| \leqslant \kappa\delta^{1/2}\,L^{3/2}\,. \end{aligned}$$

Thus

$$|B(p_m,p'_m-\bar{p}'_m) + B(p'_m-\bar{p}'_m,p_m)| \leqslant cL^{1/2}\|p_m\|\,\|p'_m-\bar{p}'_m\| \leqslant \kappa\delta^{1/2}\,L^2$$

and because of (4.14) and (2.17)

$$(4.17)\qquad |A(\bar{q}_{2,m}-q_{2,m})| \leqslant \frac{1}{\nu}\,|\bar{q}'_{0,m}-q'_{0,m}| \leqslant \kappa\delta^{3/2}\,L^2\,.$$

Finally, for $t \geqslant t_3$:

$$(4.18)\qquad \begin{aligned} |A\bar{\chi}_{2,m}| &\leqslant |A\chi_{2,m}| + |A(\bar{q}_{2,m}-q_{2,m})| \\ &\leqslant \kappa\delta^{3/2}\,L^2\,. \end{aligned}$$

This bound on $|A\bar{\chi}_{2,m}|$ is of the same order as that on $A\chi_{2,m}$ and we conclude that for $t \geqslant t_3$:

$$(4.19)\qquad \begin{cases} \operatorname{dist}_H\,(u(t),\mathcal{M}_2) \leqslant |\bar{\chi}_{2,m}(t)| \leqslant \kappa\delta^{5/2}\,L^2 \\ \operatorname{dist}_V\,(u(t),\mathcal{M}_2) \leqslant \|\bar{\chi}_{2,m}(t)\| \leqslant \kappa\delta^2\,L^2\,. \end{cases}$$

By comparison with (4.5) we see that the orbits enter a neighborhood of $\mathcal{M}_2$ which is thinner than the corresponding neighborhood of $\mathcal{M}_1$ by an order $(\delta L)^{1/2}$.

The manifold $\mathcal{M}_3$

The procedure is the same as for $\mathcal{M}_2$. We start from equation (2.8) with $j = 3$:

$$\nu A q_{3,m} + Q_m B(p_m) + Q_m B(p_m, q_{2,m}) + + Q_m B(q_{2,m}, p_m) + Q_m B(q_{1,m}) = Q_m f - q'_{1,m} . \tag{4.20}$$

By differentiation of (2.7) we obtain

$$\begin{aligned} q'_{1,m} &= -(\nu A)^{-1} Q_m(B(p_m, q'_{0,m}) + B(p'_m, q_{0,m}) + \\ &\quad + B(q_{0,m}, p'_m) + B(q'_{0,m}, p_m)) \\ &= D\Phi_1(p_m) \cdot p'_m , \end{aligned} \tag{4.21}$$

where $D\Phi_1$ is the differential of Φ_1. We now replace p'_m and $q'_{0,m}$ by their approximation $\bar{p}'_m$, $\bar{q}'_{0,m}$ above and this yields an approximation $\bar{q}'_{1,m}$ of $q'_{1,m}$ and an approximation $\bar{q}'_{3,m}$ of $q'_{3,m}$:

$$\begin{aligned} \bar{p}'_m &= -\nu A p_m - P_m B(p_m + q_{0,m}) + P_m f \\ \bar{q}'_{0,m} &= -(\nu A)^{-1} Q_m(B(p_m, \bar{p}'_m) + B(\bar{p}'_m, p_m)) \\ \bar{q}'_{1,m} &= -(\nu A)^{-1} Q_m(B(p_m, \bar{q}'_{0,m}) + B(\bar{p}'_m, q_{0m}) \\ &\quad + B(q_{0,m}, \bar{p}'_m) + B(\bar{q}'_{0,m}, p_m)) \end{aligned} \tag{4.22}$$

$$\nu A \bar{q}_{3,m} + Q_m(B(p_m) + B(p_m, q_{2,m}) + + B(q_{2,m}, p_m) + B(q_{1,m})) \doteq Q_m f - \bar{q}'_{1,m} . \tag{4.23}$$

Thus

$$\begin{aligned} A(\bar{q}_{3,m} - q_{3,m}) &= \frac{1}{\nu}(q'_{1,m} - \bar{q}'_{1,m}) \\ q'_{1,m} - \bar{q}'_{1,m} &= (\nu A)^{-1} Q_m(B(p_m, \bar{q}'_{0,m} - q'_{0,m}) + \\ &\quad + B(\bar{p}'_m - p'_m, q_{0,m}) + B(q_{0,m}, \bar{p}'_m - p'_m) \\ &\quad + B(\bar{q}'_{0,m} - q'_{0,m}, p_m)) \\ \bar{q}'_{0,m} - q'_{0,m} &= -(\nu A)^{-1} Q_m(B(p_m, \bar{p}'_m - p'_m) + B(\bar{p}'_m - p'_m, p_m) \\ \bar{p}'_m - p'_m &= -P_m(B(p_m + q_{1,m}) - B(p_m + q_m)) \\ &\quad - P_m B(p_m + q_m, \chi_{1,m}) + P_m B(\chi_{1,m}, p_m + q_{1,m}) . \end{aligned}$$

We recall that

$$\begin{aligned} |\bar{p}'_m - p'_m| &\leqslant \kappa \delta L^{3/2} \\ |A(\bar{q}'_{0,m} - q'_{0,m})| &\leqslant \kappa \delta^{1/2} L^2 . \end{aligned}$$

Then we find

$$(4.24)\qquad \begin{aligned} |A(\bar{q}'_{1,m} - q'_{1,m})| &\leq \kappa\delta L^{5/2} \\ |A(\bar{q}_{3,m} - q_{3,m})| &\leq \kappa\delta^2 L^{5/2}. \end{aligned}$$

We can conclude and state the desired result : the distance in H or V of $u(t)$ to $\mathcal{M}_3$ is bounded by the corresponding norm of

$$\bar{u}_{3,m}(t) - u(t) = \bar{\chi}_{3,m}(t)$$

where $\bar{u}_{3,m}(t) = p_m(t) + \bar{q}_{3,m}(t)$:

$$(4.25)\qquad \begin{cases} \text{dist}_H\,(u(t), \mathcal{M}_3) \leq |\bar{u}_{3,m}(t) - u(t)| = |\bar{\chi}_{3,m}(t)| \\ \text{dist}_V\,(u(t), \mathcal{M}_3) \leq \|\bar{u}_{3,m}(t) - u(t)\| = \|\bar{\chi}_{3,m}(t)\|. \end{cases}$$

Due to the estimates above, for $t \geq t_3$

$$\bar{\chi}_{3,m}(t) = \chi_{3,m}(t) + \bar{q}_{3,m}(t) - q_{3,m}(t)$$

$$|A\bar{\chi}_{3,m}(t)| \leq |A\chi_{3,m}(t)| + |A(\bar{q}_{3,m}(t) - q_{3,m}(t))|$$

$$(4.26)\qquad |A\bar{\chi}_{3,m}(t)| \leq \kappa\delta^2 L^{5/2}$$

and with (2.17) we obtain

$$(4.27)\qquad \begin{cases} \text{dist}_H\,(u(t), \mathcal{M}_3) \leq |\bar{\chi}_{3,m}(t)| \leq \kappa\delta^3 L^{5/2}, \\ \text{dist}_V\,(u(t), \mathcal{M}_3) \leq |\bar{\chi}_{3,m}(t)| \leq \kappa\delta^{5/2} L^{5/2}. \end{cases}$$

By comparison with (4.19) we see that the orbits enter a neighborhood of $\mathcal{M}_3$ which is thinner than the corresponding neighborhood of $\mathcal{M}_1$ by an order $(\delta L)^{1/2}$, and thinner than the neighborhood of $P_m H$ by an order $(\delta L)^2$. The equation of $\mathcal{M}_3$ is easily derived from (4.23), (4.22).

We can recapitulate our results in the following theorem.

THEOREM 4.1 : *There exist manifolds $\mathcal{M}_0, \ldots, \mathcal{M}_4$ explicitly defined above, such that after the time t_3 given by Theorem* 3.1, *each solution $u(\cdot)$ of* (1.1), (1.2) *belongs to a neighborhood of $\mathcal{M}_j$ of the form*

$$\begin{aligned} &\text{dist}_H\,(\varphi, \mathcal{M}_j) \leq \kappa\delta^{3/2 + j/2} L^{1 + j/2} \\ &\text{dist}_V\,(\varphi, \mathcal{M}_j) \leq \kappa\delta^{1 + j/2} L^{1 + j/2}, \quad j = 0, \ldots, 3 \end{aligned}$$

where κ depends on the data ν, $|f|$, λ_1 (and on j).

Remark 4.1 : If we want to approximate the Navier-Stokes equations for large times and wish to approach the attractor $\mathcal{A}$, it is probably better to construct Galerkin type approximations lying in these approximate inertial manifolds $\mathcal{M}_j$. This has already been successfully done for the manifold $\mathcal{M}_0$ of [FMT] (see [MT], [R]).

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