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MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

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CONTINUITY OF ATTRACTORS

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(Joint work with Jack K. HALE)

Abstract — *For* $0 \leqslant \varepsilon \leqslant \varepsilon_0$, *let* $T_\varepsilon(t)$, $t \geqslant 0$, *be a family of semigroups on a Banach space* X *with attractors* $\mathscr{A}_\varepsilon$. *Here we describe some results of upper-semicontinuity and lower-semicontinuity at* $\varepsilon = 0$ *of the family of attractors* $\mathscr{A}_\varepsilon$.

Résume — *Pour* $0 \leqslant \varepsilon \leqslant \varepsilon_0$, *considérons une famille de semi-groupes* $T_\varepsilon(t)$, $t \geqslant 0$, *sur un espace de Banach X, ayant chacun un attracteur* $\mathscr{A}_\varepsilon$. *Une manière simple de comparer les attracteurs* $\mathscr{A}_\varepsilon$, *quand* ε *tend vers* 0, *est d'estimer la distance de Hausdorff de* $\mathscr{A}_0$ *à* $\mathscr{A}_\varepsilon$, $\varepsilon \neq 0$ *Ceci nous conduit à définir les notions de semicontinuités supérieure et inférieure de la famille* $\mathscr{A}_\varepsilon$ *en* $\varepsilon = 0$ *La semicontinuité supérieure est une propriété en général satisfaite, comme de nombreux exemples l'attestent En revanche, pour le moment, nous ne connaissons qu'un cas général de semicontinuité inférieure c'est le cas des systèmes gradients dont tous les points d'équilibre sont hyperboliques*

1. INTRODUCTION

Let X be a Banach space. For any subsets A, B of X, we define

$$\delta_X(A, B) = \sup_{a \in A} \operatorname{dist}_X(a, B)$$

where

$$\operatorname{dist}_X(a, B) = \inf_{b \in B} \|a - b\|_X .$$

Let $T(t)$, $t \geqslant 0$, be a C^r-semigroup on X, $r \geqslant 0$; that is, $T(t)$, $t \geqslant 0$, is a semigroup and, for each $t \geqslant 0$, $x \in X$, $T(t)x$ together with all derivatives up through order r in x are continuous. A set B in X *attracts* a set

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C in X under $T(t)$ if $\delta_X(T(t)\, C, B) \to 0$ as $t \to +\infty$. A set B in X is *invariant* if $T(t)\, B = B$ for $t \geqslant 0$. A set $\mathscr{A}$ in X is the *attractor* of T if $\mathscr{A}$ is compact, invariant and attracts any bounded set B in X. A set $\tilde{\mathscr{A}}$ in X is called a *local attractor* if it is compact, invariant and there is an open neighbourhood $\mathscr{U}$ of $\tilde{\mathscr{A}}$ such that $\tilde{\mathscr{A}}$ attracts $\mathscr{U}$. The semigroup $T(t)$ is *asymptotically smooth* if, for any bounded set B in X for which $T(t)\, B \subset B$, $t \geqslant 0$, there is a compact set $J \subset B$ which attracts B. If $T(t)$ is asymptotically smooth, $\{T(t)\, B, t \geqslant 0\}$ is bounded if B is bounded, and there is a bounded set B_1 which attracts each point of X, then $T(t)$ has an attractor $\mathscr{A}$ (see [Hale (1), (2)] and the references therein).

Suppose now that $\mathscr{E}$ is a topological space and $\{T_\varepsilon(t), t \geqslant 0, \varepsilon \in \mathscr{E}\}$ is a family of semigroups on X for which each $T_\varepsilon(t)$ has an attractor $\mathscr{A}_\varepsilon$, for $\varepsilon \in \mathscr{E}$. It is important to understand how the set $\mathscr{A}_\varepsilon$ depends upon ε. It is also the simplest question that one can ask. We say $\mathscr{A}_\varepsilon$ is *upper-semicontinuous* at $\varepsilon = 0$ if $\delta_X(\mathscr{A}_\varepsilon, \mathscr{A}_0) \to 0$ as $\varepsilon \to 0$. We say $\mathscr{A}_\varepsilon$ is *lower-semicontinuous* at $\varepsilon = 0$ if $\delta_X(\mathscr{A}_0, \mathscr{A}_\varepsilon) \to 0$ as $\varepsilon \to 0$. We say $\mathscr{A}_\varepsilon$ is continuous at $\varepsilon = 0$ if it is upper- and lower-semicontinuous at $\varepsilon = 0$. The same definitions hold if we replace the family of attractors $\mathscr{A}_\varepsilon$ by a family of local attractors $\tilde{\mathscr{A}}_\varepsilon$.

The first general result of upper-semicontinuity has probably been given by [Cooperman] (see also [Hale (2), Sections 2.5, 3.5 and 4.10]). Assume that $T_0(t)$ has a local attractor $\tilde{\mathscr{A}}_0$ attracting an open neighbourhood $\mathscr{U}_0$ of $\tilde{\mathscr{A}}_0$, that each $T_\varepsilon(t)$, $\varepsilon \in \mathscr{E}$, is asymptotically smooth and that $T_\varepsilon(t)\, x$ is continuous in (t, x, ε), the continuity in ε being uniform with respect to (t, x) in bounded sets of $\mathbb{R} \times \mathscr{U}_0$. Then, for ε in a small neighbourhood of 0 in $\mathscr{E}$, $T_\varepsilon(t)$ admits a local attractor $\tilde{\mathscr{A}}_\varepsilon$ which attracts a bounded open neighbourhood $\mathscr{U}_1$ of $\tilde{\mathscr{A}}_0$ and which is upper-semicontinuous at $\varepsilon = 0$. Here the upper-semicontinuity in ε is an easy consequence of the continuity hypothesis and of the strong stability properties of the attractors $\tilde{\mathscr{A}}_\varepsilon$.

In many of the encountered problems, $T_\varepsilon(t)\, x$ has not the strong continuity property mentioned above. For instance, $T_\varepsilon(t)$, for $\varepsilon \neq 0$, can correspond to a Galerkin approximation or a time discretization of $T_0(t)$ (see the examples 2.1 and 2.2 below). However the upper-semicontinuity property still holds in this case, because actually, the semigroups $T_\varepsilon(t)$ need only « approximate » $T_0(t)$ on bounded sets of X, in a fairly general sense (see [Hale, Lin and Raugel] for this approximation condition).

In other cases, it may be possible to assert that $\mathscr{A}_\varepsilon$ is upper-semicontinuous at $\varepsilon = 0$, even if $T_\varepsilon(t)$ does not approximate $T_0(t)$ on bounded neighbourhoods of $\mathscr{A}_0$ in X. For instance, the sets $\mathscr{A}_\varepsilon$ may satisfy some additional smoothness properties and lie in a smoother subspace Y of X so that the upper-semicontinuity property still holds if the semigroups $T_\varepsilon(t)$ approximate $T_0(t)$ only on bounded neighbourhoods of $\mathcal{A}_0$ in Y. The restriction of the discussion to the *compact* attractors (instead of a comparison of the semigroups on arbitrary bounded sets) plays a crucial role. At this time, there is no general theorem of upper-semicontinuity which takes into account the smoothness properties of the attractors $\mathcal{A}_\varepsilon$. But, in the problems where the dependence in the parameter ε is not too « bad », one should be able to prove the upper-semicontinuity of the attractors $\mathcal{A}_\varepsilon$, by exploiting their additional specific properties. Two such examples are described in Section 2 (see examples 2.3 and 2.4).

Without some further hypotheses on the flow restricted to the attractor $\mathcal{A}_0$, there will be no lower-semicontinuity of the sets $\mathcal{A}_\varepsilon$ at $\varepsilon = 0$. Let us consider the following ordinary differential equation depending on the real parameter ε :

$$\dot{x} = -(x+1)(x^2 - \varepsilon).$$

If $\varepsilon < 0$, $\mathcal{A}_\varepsilon = \{(-1)\}$; for $\varepsilon = 0$, $\mathcal{A}_0 = [-1, 0]$ and if $0 < \varepsilon \leq 1$, $\mathcal{A}_\varepsilon = [-1, \sqrt{\varepsilon}]$. Clearly $\mathcal{A}_\varepsilon$ is not lower-semicontinuous at $\varepsilon = 0$, for $\varepsilon \leq 0$. This drastic change in the size of the attractor $\mathcal{A}_\varepsilon$ when ε passes through zero in this example is caused by the fact that zero is not a hyperbolic equilibrium. If $T_0(t)$ is a Morse-Smale system (that is, the non-wandering set is a finite set consisting only of hyperbolic equilibria and hyperbolic periodic orbits, with the stable and unstable manifolds transversal), then the attractors $\mathcal{A}_\varepsilon$ are continuous at $\varepsilon = 0$ and the corresponding flows restricted to the attractors are shown to be topologically equivalent (see [Hale, Magalhães and Oliva, chapter 10]). This result contains much more information than lower-semicontinuity. For lower-semicontinuity, the requirement of hyperbolicity is natural, as shown by the above example. From an intuitive point of view, the condition of transversality should be unnecessary. Moreover, transversality is a global property for which no general procedure for verification is available. Here we present a class of semigroups $T_\varepsilon(t)$ for which one has the lower-semicontinuity property (see Theorem 3.1 in Section 3 and [Hale, Raugel (2)]). Roughly speaking, the lower-semicontinuity property holds for systems $T_\varepsilon(t)$ which approximate $T_0(t)$ in an appropriate sense and whose limit at $\varepsilon = 0$ is a gradient system, the equilibrium points of which are hyperbolic. We remark that all of the required conditions are local except the condition (4) in the definition 3.1 of gradient systems ; but this condition (4) is often easy to verify in applications.

Although the property of hyperbolicity of the equilibrium points is a strong hypothesis, it is generic in many examples. For instance, for scalar parabolic or hyperbolic equations in one space variable with the nonlinearity $f(u)$ depending only on the dependent variable u and not on its derivatives or the spatial variable, generic hyperbolicity has been proved by [Brunovsky and Chow], [Smoller and Wasserman], [Henry (2)] (for a related result, see [Rocha]) For the same situation with several space variables, generic hyperbolicity with respect to the domain has been shown by [Henry (3)] In the case of several space variables, with $f(u, x) = h(u) - g(x)$, generic hyperbolicity with respect to g has been shown by [Babin and Vishik (1)]

Our result on lower-semicontinuity is general enough to be applied to numerical approximations of parabolic equations or to singularly perturbed problems Finally, let us emphasize that this lower-semicontinuity property should hold for more general systems than gradient ones

2. EXAMPLES OF UPPER-SEMICONTINUITY

We will not state the general upper-semicontinuity result contained in [Hale, Lin and Raugel, Section 2], because the precise hypotheses are a little technical Let $\varepsilon > 0$ be a parameter which will tend to zero and, for $0 < \varepsilon \leq \varepsilon_0$, let X_ε be a family of subspaces of the Banach space $X \equiv X_0$ For $0 \leq \varepsilon \leq \varepsilon_0$, let $T_\varepsilon(t)$, $t \geq 0$, be a C^0-semigroup on X_ε which is asymptotically smooth Assume that $T_0(t)$ has a local attractor $\mathscr{A}_0$ and that there are an open neighbourhood $\mathscr{U}_0$ of $\mathscr{A}_0$ and, for any positive numbers t_0, t_1, with $0 < t_0 < t_1$, a positive function $\eta(t_0, t_1, \varepsilon)$ such that

$$\lim_{\varepsilon \to 0} \eta(t_0, t_1, \varepsilon) = 0 , \tag{2 1}$$

and, if $u_0 \in \mathscr{U}_0 \cap X_\varepsilon$ has the property that $T_0(t)\, u_0$, $T_\varepsilon(t)\, u_0$ belong to $\mathscr{U}_0$ for $t \in [t_0, t_2]$ where $t_0 \leq t_2 \leq t_1$, then,

$$\| T_0(t)\, u_0 - T_\varepsilon(t)\, u_0 \|_X \leq \eta(t_0, t_1, \varepsilon) \quad \text{for} \quad t_0 \leq t \leq t_2 \tag{2 2}$$

Then, $T_\varepsilon(t)$ has a local attractor $\mathscr{A}_\varepsilon \subset \mathscr{U}_0$ and $\delta_X(\mathscr{A}_\varepsilon, \mathscr{A}_0) \to 0$ as $\varepsilon \to 0$ provided that $T_\varepsilon(t)$ satisfies some additional hypotheses These hypotheses are satisfied, for example, if there are positive constants δ_0, t_0, t_1 and three open sets $N_1 \subset N_2 \subset N_3$ (with $\mathscr{A}_0 \subset N_1$, $\mathscr{A}_0$ attracting N_1 under $T_0(t)$ and $T_0(t) N_1 \subset N_2$ for $t \geq 0$) such that, for $0 < \varepsilon \leq \varepsilon_0$, $T_\varepsilon(t)(N_1 \cap X_\varepsilon) \subset N_2$ for $0 \leq t \leq t_0$, and for any $x_\varepsilon \in \mathscr{N}(N_2, \delta_0) \cap X_\varepsilon$, there exists $t(x_\varepsilon) > 0$ such that $T_\varepsilon(t)\, x_\varepsilon \in N_3$ for $0 \leq t \leq t(x_\varepsilon)$, where $\mathscr{N}(N_2, \delta_0)$ denotes the δ_0-neighbourhood of N_2

The two first examples will illustrate this theorem

2.1. Semidiscretization in space of a parabolic equation

Here we describe a simple situation (A more general case is given in [Hale, Lin and Raugel]) Let V and H be two (real) Hilbert spaces such that

V is included in H with a continuous and dense embedding ; the space H is identified with its dual space and the inner product of H, as well as the duality pairing between V and its dual space V', is denoted by $(. , .)$. We introduce a continuous, symmetric bilinear form on $V \times V : (u, v) \in V \times V \to a(u, v)$, which is V-elliptic, and we denote by $A \in \mathscr{L}(V ; V')$ the corresponding operator defined by

$$\forall u, v \in V , \quad a(u, v) = (Au, v) .$$

Now we consider the nonlinear equation

$$\begin{cases} \dfrac{du}{dt} + Au = f(u) , \\ u(0) = u_0 , \end{cases} \tag{2.3}$$

where u_0 belongs to V and $f : V \to H$ is locally Lipschitz continuous.

We set

$$D(A) \equiv \{v \in V ; Av \in H\} .$$

By [Henry (1), Chapter 2], we know that under the above hypotheses on A, f and u_0, there is a unique solution in V of equation (2.3) on a maximal interval of existence $(0, \tau(u_0))$. Here we assume that all solutions are defined for $t \geq 0$, so that we can introduce the C^0-semigroup $T_0(t) : V \to V$, $t \geq 0$, defined by $T_0(t) u_0 = u(t, u_0)$. We also suppose that $T_0(t)$ has a (local) attractor $\tilde{\mathscr{A}}_0$ attracting a bounded open neighbourhood $\mathscr{U}_0$.

Now, let us turn to a finite-dimensional approximation of equation (2.3). Let $\varepsilon > 0$ be a real parameter which will tend to 0 and $(V_\varepsilon)_\varepsilon$ be a family a finite-dimensional subspaces of V. We introduce the operator $\mathscr{A}_\varepsilon \in \mathscr{L}(V_\varepsilon ; V_\varepsilon)$ defined by

$$\forall v_\varepsilon \in V_\varepsilon , \quad (A_\varepsilon w_\varepsilon, v_\varepsilon) = a(w_\varepsilon, v_\varepsilon) \quad \text{for} \quad w_\varepsilon \quad \text{in} \quad V_\varepsilon . \tag{2.4}$$

Let $Q_\varepsilon \in \mathscr{L}(H ; V_\varepsilon)$ be the projector on V_ε in the space H, i.e.,

$$\forall v \in H , \quad \forall v_\varepsilon \in V_\varepsilon , \quad (v - Q_\varepsilon v, v_\varepsilon) = 0 \tag{2.5}$$

and let $P_\varepsilon \in \mathscr{L}(V ; V_\varepsilon)$ be the projector on V_ε in the space V, i.e.,

$$\forall v \in V , \quad \forall v_\varepsilon \in V_\varepsilon , \quad a(v - P_\varepsilon v, v_\varepsilon) = 0 . \tag{2.6}$$

Now, consider the following equation in V_ε :

$$\begin{cases} \dfrac{du_\varepsilon}{dt} + A_\varepsilon u_\varepsilon = Q_\varepsilon f(u_\varepsilon) , \\ u_\varepsilon(0) = u_{0\varepsilon} , \end{cases} \tag{2.3$_\varepsilon$}$$

where $u_{0\varepsilon} \in V_\varepsilon$. We introduce the map $T_\varepsilon(t) : V_\varepsilon \to V_\varepsilon$, defined by $T_\varepsilon(t)\, u_{0\varepsilon} = u_\varepsilon(t, u_{0\varepsilon})$ as long as it exists.

In order to prove that $T_\varepsilon(t)$ also admits a local attractor $\tilde{\mathscr{A}}_\varepsilon$, for ε small enough, we make some additional hypotheses on the spaces V_ε, $\varepsilon > 0$: there exist an integer $m > 0$ and, for any β, $\frac{1}{2} \leq \beta \leq 1$, a constant $C(\beta) > 0$ such that, for all w in $X^\beta \equiv D(A^\beta)$,

$$\text{(i)} \qquad \|w - P_\varepsilon w\|_V + \|w - Q_\varepsilon w\|_V \leq C(\beta)\, \varepsilon^{m(2\beta - 1)} \|w\|_{X^\beta}\,, \qquad (2.7)$$

and

$$\text{(ii)} \qquad \|w - P_\varepsilon w\|_H + \|w - Q_\varepsilon w\|_H \leq C(\beta)\, \varepsilon^{2m\beta}\, \|w\|_{X^\beta}\,. \qquad (2.7)$$

Note that the hypotheses (2.7) are realistic (see [Hale, Lin and Raugel, Section 3]). In this paper, it has been shown that if $T_\varepsilon(t)\, u_{0\varepsilon}$ and $T_0(t)\, u_{0\varepsilon}$ belong to $\mathscr{U}_0$ for $0 \leq t \leq t_1$, then, for $0 < t_0 \leq t \leq t_1$, we have

$$\|T_0(t)\, u_{0\varepsilon} - T_\varepsilon(t)\, u_{0\varepsilon}\|_V \leq C_0\, e^{c_1 t_1} \frac{\varepsilon^m}{t_0}\,, \qquad (2.8)$$

which is similar to (2.2). Then the general result of [Hale, Lin and Raugel, Theorem 2.4] implies that there are a positive constant ε_0 and an open neighbourhood $\mathscr{U}_1$ of $\tilde{\mathscr{A}}_0$ such that, for $0 < \varepsilon \leq \varepsilon_0$, $T_\varepsilon(t)$ has a local attractor $\tilde{\mathscr{A}}_\varepsilon$ attracting $\mathscr{U}_1$. Moreover $\tilde{\mathscr{A}}_\varepsilon$ is upper-semicontinuous at $\varepsilon = 0$.

2.2. Semidiscretization in time of a parabolic equation

We now turn to a semidiscretization in time of equation (2.3) by a one-step method. Here we assume moreover that A has a compact resolvent and that f belongs to $C^2(V\,;H)$. Let k be a positive time increment which will tend to 0 and let $t_n = nk$, $n \in \mathbf{N}$, and define an approximation u^n of the solution of (2.3) at time t_n by the recursion formula

$$\begin{cases} u^{n+1} = (1 - (1-\theta)\, kA)(1 + \theta kA)^{-1}\, u^n + k(1 + \theta kA)^{-1}\, f(u^n)\,, \\ u^0 = u_0\,, \end{cases} \qquad (2.3)_k$$

where $\frac{1}{2} < \theta \leq 1$.

We introduce the mapping $T_k \in \mathscr{L}(V\,;V)$ defined by $T_k\, u_0 = u^1$ where u^1 is given by the formula $(2.3)_k$. For any integer $n \geq 1$, $T_k^n\, u_0 = u^n$. We remark that $T_k^n : \mathbf{N} \to C^0(V\,;V)$ is a discrete semigroup and that all the definitions given in the introduction can be extended to this case. Let

α_0, α_1 be two positive constants with $\alpha_0 < \alpha_1$. In [Hale, Lin and Raugel, Section 4], it has been proved that if $T_k^n u_0$ and $T(nk)\, u_0$ belong to $\mathcal{U}_0$ for $0 \leq n \leq m$ and $0 \leq nk \leq mk + k_0$ respectively, where $\frac{\alpha_0}{k} < m \leq \frac{\alpha_1}{k}$, then

$$\max_{\frac{\alpha_0}{k} \leq n \leq m} \left\| T(nk)\, u_0 - T_k^n u_0 \right\|_V \leq C_0\, e^{c_1 \alpha_1} \frac{\sqrt{k}}{\alpha_0}, \tag{2.9}$$

which is similar to (2.2). Then Theorem 2.4 of [Hale, Lin and Raugel] implies that there are a positive constant k_0 and an open neighbourhood $\mathcal{U}_1$ of $\tilde{\mathcal{A}}_0$ such that, for $0 < k \leq k_0$, T_k has a local attractor $\tilde{\mathcal{A}}_k$ attracting $\mathcal{U}_1$. And $\tilde{\mathcal{A}}_k$ is upper-semicontinuous at $k = 0$.

The two following examples of upper-semicontinuity fully exploit the additional smoothness properties of attractors.

2.3. A singularly perturbed hyperbolic equation

Consider now the hyperbolic equation

$$\begin{cases} \text{(i)} \quad \varepsilon \dfrac{d^2 u_\varepsilon}{dt^2} + \dfrac{du_\varepsilon}{dt} - \Delta u_\varepsilon = -\tilde{f}(u_\varepsilon) - h(x) & \text{in } \Omega \times (0, +\infty) \\ \text{(ii)} \quad u_\varepsilon = 0 & \text{on } \partial\Omega\,, \\ \text{(iii)} \quad u_\varepsilon(0, x) = u_0(x)\,, \quad \dfrac{du_\varepsilon}{dt}(0, x) = u_1(x)\,, & \end{cases} \tag{2.10}$$

where Ω is a bounded smooth domain or a convex polyhedral domain in $\mathbb{R}^n$, $n = 1, 2, 3$, ε is a positive parameter which will tend to zero, $h(x)$ is a given function in $L^2(\Omega)$ and (u_0, u_1) belongs to $X_0 = H_0^1(\Omega) \times L^2(\Omega)$. Suppose that $\tilde{f}$ belongs to $C^2(\mathbb{R}\,;\mathbb{R})$, that

$$\overline{\lim_{|y| \to +\infty}}\ \frac{-\tilde{f}(y)}{y} \leq 0 \tag{2.11}$$

and, for $n \geq 2$, there is a constant $c_0 > 0$ such that

$$|\tilde{f}''(y)| \leq c_0(|y|^\gamma + 1) \quad \text{for} \quad y \in \mathbb{R}\,, \tag{2.12}$$

where

$$\begin{cases} 0 \leq \gamma < +\infty & \text{if} \quad n = 2 \\ 0 \leq \gamma < 1 & \text{if} \quad n = 3\,. \end{cases} \tag{2.13}$$

Along with equation (2.10), we consider the limiting parabolic equation when $\varepsilon = 0$

$$\begin{cases} \text{(i)} & \dfrac{du}{dt} - \Delta u = - \tilde{f}(u) - h(x) \quad \text{in} \quad \Omega \times (0, +\infty) \\ \text{(ii)} & u = 0 \quad \text{on} \quad \partial\Omega , \\ \text{(iii)} & u(0, x) = u_0(x) . \end{cases} \tag{2.14}$$

Under the above hypotheses, there is an attractor $\tilde{\mathscr{A}}_0$ of (2.14) in $H_0^1(\Omega)$ (see [Hale (1)]). Moreover, $\tilde{\mathscr{A}}_0$ is in $H^2(\Omega) \cap H_0^1(\Omega)$. Also, for $\varepsilon > 0$, (2.10) admits an attractor $\mathscr{A}_\varepsilon$ in X_0, which belongs to the space $X_1 \equiv (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ (see [Hale (1)], [Haraux], [Ghidaglia and Témam]). How is $\mathscr{A}_\varepsilon$ related to $\tilde{\mathscr{A}}_0$ for ε small ? In order to make a comparison, we introduce the set

$$\mathscr{A}_0 = \left\{ (\varphi, \psi) : \varphi \in \tilde{\mathscr{A}}_0, \psi = \Delta\varphi - \tilde{f}(\varphi) - h, \varphi \in \tilde{\mathscr{A}}_0 \right\}$$

which is a natural embedding of the attractor $\tilde{\mathscr{A}}_0$ into X_0.

THEOREM 2.1 : *Under the above hypotheses, the sets $\mathscr{A}_\varepsilon$ are upper-semicontinuous at $\varepsilon = 0$, i.e.,*

$$\lim_{\varepsilon \to 0} \delta_{X_0}(\mathscr{A}_\varepsilon, \mathscr{A}_0) = 0 .$$

For a proof of this result, we refer the reader to [Hale, Raugel (1)]. In this proof, we widely use the fact that, for $\varepsilon \geq 0$, $\mathscr{A}_\varepsilon \subset X_1$. More precisely, let ε_0 be a fixed positive constant ; one shows that there is a positive constant c such that, for $0 \leq \varepsilon \leq \varepsilon_0$, if $(u_\varepsilon(t), \frac{\partial u_\varepsilon}{\partial t}(t))$ belongs to $\mathscr{A}_\varepsilon$, for $t \in \mathbb{R}$, then

$$\left\| \frac{\partial^2 u_\varepsilon}{\partial t^2}(t) \right\|_1 + \left\| \frac{\partial u_\varepsilon}{\partial t}(t) \right\|_1 + \| u_\varepsilon(t) \|_2 \leq c , \quad \text{for} \quad t \in \mathbb{R} , \tag{2.15}$$

where $\|\cdot\|_i$, $i = 0, 1, 2$, denotes the norm in $L^2(\Omega)$, $H_0^1(\Omega)$, $H^2(\Omega) \cap H_0^1(\Omega)$ respectively (see [Hale, Raugel (1), (3)]). This property has the two following important consequences. If $(u_\varepsilon(t), \frac{\partial u_\varepsilon}{\partial t}(t))$ is a solution of (2.10) in $\mathscr{A}_\varepsilon$, and if $\ell_\varepsilon(t) = \varepsilon \frac{\partial^2 u_\varepsilon}{\partial t^2}(t)$, then $\|\ell_\varepsilon(t)\|_0 \to 0$ as $\varepsilon \to 0$ uniformly in $t \in \mathbb{R}$ and $u_\varepsilon(t)$ is a solution of the *regularly* perturbed parabolic equation

$$\frac{\partial u_\varepsilon}{\partial t} = \Delta u_\varepsilon - \tilde{f}(u_\varepsilon) - h - \ell_\varepsilon(t) \quad \text{in} \quad \Omega , \tag{2.16}$$

with homogeneous Dirichlet conditions. Inequality (2.15) also implies that, for $0 \leqslant \varepsilon \leqslant \varepsilon_0$, $\mathscr{A}_\varepsilon \subset B$ where B is a bounded set in X_1 ; thus, for $0 \leqslant \varepsilon \leqslant \varepsilon_0$, $\mathscr{A}_\varepsilon$ belongs to a special type of compact set in X_0. These two properties allow us to show the upper-semicontinuity at $\varepsilon = 0$.

Finally, we remark that (2.10) and (2.14) are gradient systems. The proof of Theorem 2.1 given in [Hale, Raugel (1)] does not exploit this fact and therefore is also valid for more general systems, that are not gradient (like those described in [Ghidaglia, Témam, Section 5]).

2.4. A reaction diffusion equation on a thin domain

Suppose now that $\Omega \subset \mathbb{R}^n$, $n = 1, 2$, is a bounded open connected set with smooth boundary, $g : \bar{\Omega} \times [0, \varepsilon_0] \to \mathbb{R}$ is a C^3-function (where $\bar{\Omega}$ is the closure of Ω), satisfying

$$\begin{cases} g(x, 0) = 0\,, \quad g_0(x) \equiv \dfrac{\partial g}{\partial \varepsilon}(x, 0) > 0 \quad \text{for} \quad x \in \bar{\Omega}\,, \\ g(x, \varepsilon) > 0 \quad \text{for} \quad x \in \bar{\Omega}\,, \quad \varepsilon \in (0, \varepsilon_0]\,. \end{cases} \tag{2.17}$$

For α a positive constant and $\tilde{f}$ a C^2-function satisfying (2.11), (2.12) and (2.13), we consider the equation

$$\begin{cases} \text{(i)} & \dfrac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon + \alpha u_\varepsilon = -\tilde{f}(u_\varepsilon) \quad \text{in} \quad Q_\varepsilon \times (0, +\infty)\,, \\ \text{(ii)} & \dfrac{\partial u_\varepsilon}{\partial n_\varepsilon} = 0 \quad \text{on} \quad \partial Q_\varepsilon\,, \\ \text{(iii)} & u_\varepsilon(0, x, y) = u_0(x, y)\,, \end{cases} \tag{2.18}$$

where n_ε is the outer normal to ∂Q_ε, u_0 belongs to $H^1(Q_\varepsilon)$, with

$$Q_\varepsilon = \{(x, y) \in \mathbb{R}^{n+1} : 0 < y < g(x, \varepsilon), x \in \Omega\}\,, \quad \text{for} \quad 0 \leqslant \varepsilon \leqslant \varepsilon_0\,.$$

We want to relate the dynamics of (2.18) to the dynamics of the equation

$$\begin{cases} \text{(i)} & \dfrac{\partial u}{\partial t} - \dfrac{1}{g_0} \displaystyle\sum_{i=1}^{n} \dfrac{\partial}{\partial \xi_i}\left(g_0 \dfrac{\partial u}{\partial \xi_i}\right) + \alpha u = -\tilde{f}(u) \quad \text{in} \quad \Omega \times (0, +\infty)\,, \\ \text{(ii)} & \dfrac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega\,, \\ \text{(iii)} & u(0, x) = u_0(x)\,, \end{cases} \tag{2.19}$$

where u_0 belongs to $H^1(\Omega)$ and n is the outer normal to $\partial\Omega$.

Under the above hypotheses, in the scaled domain $Q = \Omega \times (0, 1)$ defined by the change of variables $x = \xi$, $y = g(\xi, \varepsilon)\,\eta$, the equation (2.18) has an attractor $\mathscr{A}_\varepsilon \subset H^1(Q)$. The equation (2.19) also has an attractor $\mathscr{A}_0 \subset H^1(\Omega)$ which is naturally embedded into $H^1(Q)$.

THEOREM 2 2 *Under the above hypotheses, the attractors $\mathcal{A}_\varepsilon \subset H^1(Q)$ are upper-semicontinuous at* $\varepsilon = 0$

For a complete proof of this result, we refer the reader to [Hale, Raugel (4)] Let $H^1_\varepsilon(Q)$ be the space $H^1(Q)$ endowed with the norm

$$\|\varphi\|_{H^1_\varepsilon(Q)} = \|\varphi\|_{H^1(Q)} + \frac{1}{\varepsilon}\left\|\frac{\partial\varphi}{\partial\eta}\right\|_{L^2(Q)}$$

For $\varepsilon > 0$, the attractors $\mathcal{A}_\varepsilon$ satisfy the following important a priori estimate there is a positive constant C such that, for $0 \leqslant \varepsilon \leqslant \varepsilon_0$,

$$\|\varphi\|_{H^1_\varepsilon(Q)} \leqslant C\,, \quad \text{for} \quad \varphi \in \mathcal{A}_\varepsilon \tag{2 20}$$

Let $T_\varepsilon(t)$, $\varepsilon > 0$, and $T_0(t)$ be the semigroups associated with (2 18) and (2 19) on the scaled domain $Q = \Omega \times (0, 1)$ Then, to prove Theorem 2 2, it is sufficient to obtain good estimates of the difference $\|T_\varepsilon(t)\varphi - T_0(t)\varphi_0\|_{H^1_\varepsilon(Q)}$ on finite time intervals $[t_0, \tau_0]$, with $0 < t_0 < \tau_0$, when $\|\varphi\|_{H^1_\varepsilon(Q)} \leqslant C$ and φ_0 is a « good » approximation of φ in $H^1(\Omega)$

Remark 2 1 If $g(x, \varepsilon) = \varepsilon$, then $\mathcal{A}_\varepsilon = \mathcal{A}_0$ for ε small This is proved by writing the solution u_ε of (2 18) on Q as $u_\varepsilon = v + w$ where $v = \int_0^1 u_\varepsilon(\xi, \eta)\,d\eta$ and by using the strong stability property of w

Another interesting example of upper-semicontinuity is contained in [Hale, Rocha]

3. A LOWER-SEMICONTINUITY RESULT

3.1. Let us recall the definition of a gradient system on a Banach space X

DEFINITION 3 1 *A C^r-semigroup $T(t)$, $t \geqslant 0$, $r \geqslant 0$, is said to be a* gradient system *if there exists a Lyapunov function for $T(t)$, that is, there is a continuous function $\mathcal{V} : X \to \mathbb{R}$ with the property that*

(1) *$\mathcal{V}(x)$ is bounded below,*
(2) *$\mathcal{V}(x) \to +\infty$ as $\|x\|_X \to +\infty$,*
(3) *$\mathcal{V}(T(t)x)$ is nonincreasing in t for each $x \in X$,*
(4) *if x is such that $\mathcal{V}(T(t)x) = \mathcal{V}(x)$ for all t in $\mathbb{R}$, then x is an equilibrium point, that is, $T(t)x = x$ for all t in $\mathbb{R}$*

We now state a particular case of the lower-semicontinuity result of [Hale, Raugel (2)] For $0 \leqslant \varepsilon \leqslant \varepsilon_0$, let X_ε be a family of subspaces of the Banach space X, endowed with the norm $\|\cdot\|_X$ and let $T_\varepsilon(t)$, $t \geqslant 0$, be a family of semigroups on X_ε We make the following hypotheses on $T_\varepsilon(t)$, for ε in $[0, \varepsilon_0]$.

(H.1) $T_0(t)$, $t \geq 0$, is a C^1-gradient system which is asymptotically smooth,

(H.2) the set E_0 of equilibrium points of $T_0(t)$ is bounded,

(H.3) each element $\varphi_{j,0}$ of E_0 is hyperbolic.

Then, E_0 is a finite set of, say, N_0 elements. And since $T_0(t)$ satisfies (H.1) and (H.2), it admits an attractor $\mathscr{A}_0$, given by

$$\mathscr{A}_0 = \bigcup_{\varphi_{j,0} \in E_0} W^u(\varphi_{j,0}) ,$$

where $W^u(\varphi_{j,0})$ is the unstable manifold of $\varphi_{j,0}$.

(H.4) For $\varepsilon > 0$, $T_\varepsilon(t)$ is a C^1-semigroup and has a *local* attractor $\mathscr{A}_\varepsilon$ attracting $U_0 \cap X_\varepsilon$ where U_0 is a fixed open neighbourhood of $\mathscr{A}_0$ in X,

(H.5) If E_ε is the set of equilibrium points of $T_\varepsilon(t)$, there exists an open neighbourhood W_0 of E_0 in X such that $W_0 \cap E_\varepsilon = \{\varphi_{1,\varepsilon}, \dots, \varphi_{N_0,\varepsilon}\}$ where each $\varphi_{j,\varepsilon}$ is hyperbolic and $\varphi_{j,\varepsilon} \to \varphi_{j,0}$ as $\varepsilon \to 0$.

We define the local unstable sets

$$W^u_{\text{loc},\varepsilon}(\varphi_{j,\varepsilon}) \equiv \{y_\varepsilon \in U_j \cap X_\varepsilon : T_\varepsilon(-t)\, y_\varepsilon \in U_j \cap X_\varepsilon\,, \quad t \geq 0\,, \\ T_\varepsilon(-t)\, y_\varepsilon \to \varphi_{j,\varepsilon} \text{ as } t \to +\infty\}\,,$$

where U_j is a neighbourhood of $\varphi_{j,0}$ in X (and therefore of $\varphi_{j,\varepsilon}$ for ε small enough).

We furthermore assume that there are positive constants C_0, p and α such that

(H.6)
$$\delta_X(W^u_{\text{loc},0}(\varphi_{j,0}), W^u_{\text{loc},\varepsilon}(\varphi_{j,\varepsilon})) \leq C_0\, \varepsilon^p\,,$$

(H.7) For any x, y belonging to $\bigcup_{0 \leq \varepsilon \leq \varepsilon_0} \mathscr{A}_\varepsilon$,

$$\|T_0(t)\, x - T_0(t)\, y\|_X \leq C_0 \|x - y\|_X \exp \alpha t\,,$$

(H.8) For any $t_0^* > 0$, there is a positive number $C_0^* \equiv C_0^*(t_0)$ such that, for $y_\varepsilon \in \mathscr{A}_\varepsilon$,

$$\|T_0(t)\, y_\varepsilon - T_\varepsilon(t)\, y_\varepsilon\|_X \leq C_0^*\, \varepsilon^p \exp \alpha t \quad \text{for} \quad t \geq t_0^*\,.$$

THEOREM 3.1 : *Under the hypotheses* (H.1) *to* (H.8), *there are two positive constants C and q, with $q \leq p$, such that*

$$\delta_X(\mathscr{A}_0, \mathscr{A}_\varepsilon) \leq C\varepsilon^q\,. \tag{3.1}$$

Remark 3 1 If moreover, we assume that

$$\delta_X(W^u_{\text{loc}\ \varepsilon}(\varphi_{j\ \varepsilon}), W^u_{\text{loc}\ 0}(\varphi_{j\ 0})) \leqslant C_0\, \varepsilon^p \tag{3 2}$$

then, under some additional hypothesis, we have

$$\delta_X(\mathscr{A}_\varepsilon, \mathscr{A}_0) \leqslant C\varepsilon^q \tag{3 3}$$

In the proof of Theorem 3 1, we widely use the facts that there exists a Morse decomposition for $\mathscr{A}_0$ and that the local unstable manifolds $W^u_{\text{loc}\ \varepsilon}(\varphi_{j\ \varepsilon})$ have a « good » dependence in ε A similar weaker result had also been announced in [Babin, Vishik (2)]

3.2. Applications of Theorem 3.1

Example 3 1 We now consider the equation (2 14) where $\tilde{f}$, h and Ω satisfy the hypotheses of Section 2 3 It is a gradient system with Lyapunov function

$$\mathscr{V}_0(\varphi) = \int_\Omega \left(\frac{1}{2} |\nabla\varphi(x)|^2 + F(\varphi(x)) + h(x)\, \varphi(x) \right) dx$$

where $F(y) = \int_0^y f(s)\, ds$ We can write equation (2 14) in the form (2 3) if we set $V = H_0^1(\Omega)$, $H = L^2(\Omega)$, $f(u) = -\tilde{f}(u) - h$, $A = -\Delta$ with Dirichlet homogeneous boundary conditions Thus a finite-dimensional approximation of equation (2 14) is given by $(2\ 3)_\varepsilon$ which is still a C^1-gradient system, the associated Lyapunov functional on X_ε being

$$\mathscr{V}_\varepsilon(\varphi) = \int_\Omega \left(\frac{1}{2} |\nabla\varphi(x)|^2 + F_\varepsilon(\varphi(x)) + Q_\varepsilon\, h(x)\, \varphi(x) \right) dx$$

where $F_\varepsilon(y) = \int_0^y Q_\varepsilon\, f(s)\, ds$ and $\varphi \in X_\varepsilon$ Hence equation $(2\ 3)_\varepsilon$ has an attractor $\mathscr{A}_\varepsilon$, for $\varepsilon \geqslant 0$

Assume that all of the equilibrium points of (2 14) are hyperbolic Then one easily proves that the hypotheses of Theorem 3 1 and the inequality (3 2) are satisfied with $p = 1$ Therefore one can apply Theorem 3 1 and the estimates (3 1) and (3 3) hold for a real number q, $0 < q \leqslant 1$, for $0 < \varepsilon \leqslant \varepsilon_0$ (see [Hale, Raugel (2)])

Example 3 2 We now turn to the semidiscretization in time $(2\ 3)_k$ of equation (2 14) Here the continuous semigroup $T_\varepsilon(t)$, for $\varepsilon > 0$, is replaced by the discrete semigroup T_k, $k > 0$, defined in Section 2 2 Note that

Theorem 3.1 and Estimate (3.3) can be extended to this case (see [Hale, Raugel (2)]). Here, of course, T_k is no longer a gradient system. However it is gradient-like ; that is, the local attractor $\tilde{\mathscr{A}}_k$ is the union of the unstable manifolds of the equilibria. Assume that all of the equilibrium points of equation (2.14) are hyperbolic ; then one easily shows that T_k satisfies hypotheses similar to those of Theorem 3.1 and Remark 3.1 with $p = \frac{1}{2}$. Hence there are two positive constants C and q, with $q \leq \frac{1}{2}$, such that

$$\delta_{H_0^1(\Omega)}(\mathscr{A}_0, \tilde{\mathscr{A}}_k) + \delta_{H_0^1(\Omega)}(\tilde{\mathscr{A}}_k, \mathscr{A}_0) \leq C k^q .$$

Example 3.3 : Let us come back to the Example 2.3. By Theorem 2.1, the sets $\mathscr{A}_\varepsilon$ are upper-semicontinuous at $\varepsilon = 0$. Since the systems (2.10) and (2.14) are gradient, the sets $\mathscr{A}_\varepsilon$ are also lower-semicontinuous at $\varepsilon = 0$.

THEOREM 3.2 : *If* (2.11), (2.12), (2.13) *hold and if all of the equilibrium points are hyperbolic, there are positive constants* ε_0, C *and* p, *with* $p \leq \frac{1}{2}$ *such that, for* $0 < \varepsilon \leq \varepsilon_0$,

$$\delta_{X_0}(\mathscr{A}_0, \mathscr{A}_\varepsilon) + \delta_{X_0}(\mathscr{A}_\varepsilon, \mathscr{A}_0) \leq C \varepsilon^p . \tag{3.4}$$

The proof of Theorem 3.2 is quite similar to the one of Theorem 3.1. The following property is an important ingredient of the proof of Theorem 3.2. At first, note that φ_j, $1 \leq j \leq N_0$, is an equilibrium point of (2.14) if and only if $(\varphi_j, 0)$ is an equilibrium point of (2.10). Let $W^u_{\mathrm{loc},\varepsilon}((\varphi_j, 0))$, for $\varepsilon > 0$, and $\tilde{W}^u_{\mathrm{loc}}(\varphi_j)$ denote local unstable manifolds of $(\varphi_j, 0)$ and φ_j respectively, for $1 \leq j \leq N_0$. We then introduce the set

$$W^u_{\mathrm{loc},0}(\varphi_j) = \left\{(v, w) \in X_0 : w = -\tilde{f}(v) - h + \Delta v,\ v \in \tilde{W}^u_{\mathrm{loc}}(\varphi_j)\right\} ,$$

and we prove that, for $\varepsilon > 0$, and $1 \leq j \leq N_0$,

$$\delta_{X_0}(W^u_{\mathrm{loc},0}(\varphi_j), W^u_{\mathrm{loc},\varepsilon}((\varphi_j, 0))) + \\ + \delta_{X_0}(W^u_{\mathrm{loc},\varepsilon}((\varphi_j, 0)), W^u_{\mathrm{loc},0}(\varphi_j)) \leq C \varepsilon^{1/2} . \tag{3.5}$$

One even obtains a better estimate of the distance in $H_0^1(\Omega)$ between the sets $\tilde{W}^u_{\mathrm{loc}}(\varphi_j)$ and $P_1 W^u_{\mathrm{loc},\varepsilon}((\varphi_j, 0))$, where $P_1 \in \mathscr{L}(X_0 ; H_0^1(\Omega))$ is the projection onto the first component (cf. [Hale, Raugel (3)]).

If (2.10) and (2.14) are one-dimensional equations, then they admit inertial manifolds which have a continuous dependence in ε (see [Mora, Solà-Morales]). In higher dimensions one can show that if (2.14) is a Morse-

Smale system, (2 10) ıs stıll a Morse-Smale system for ε small enough (see [Hale, Raugel (5)])

Fınally let us poınt out that a sımılar result of lower-semıcontınuıty of attractors ıs true for the reactıon dıffusıon equatıon on a thın domaın (see [Hale, Raugel (4)])

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