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INERTIAL MANIFOLDS OF DAMPED SEMILINEAR WAVE EQUATIONS (*)

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In the present communication we give an account of the results obtained by the authors in [8], [9], and [10], with some slight improvements in what refers to [9]. We are concerned with the qualitative dynamics of a one-dimensional semilinear damped wave equation and its dependence with respect to the coefficient of the second-order time derivative, hereafter denoted by ε^2, this parameter being considered to vary right up to the limiting value $\varepsilon^2 = 0$, in which case the equation turns into a semilinear diffusion one.

Our work has been motivated mostly by the question whether, for ε small, the dynamics of the damped wave equation is or not equivalent in some reasonable sense to that of the limiting diffusion equation. A strong evidence in this direction is the remarkable fact established by Henry (1985) [7] and Angenent (1986) [1] that the limiting diffusion equation is automatically Morse-Smale as soon as the stationary states are all hyperbolic. As it is well-known, this would immediately imply equivalence if our system had been finite-dimensional and we had been dealing with a regular perturbation problem instead of a singular one.

As it has become clear in the recent years, when dealing with infinite-dimensional systems like the diffusion equation considered here, the notion of equivalence based on a comparison of all orbits is easily too severe, so that it becomes convenient to restrict the attention to some smaller, usually finite-dimensional, invariant set which still contains the essential elements of the dynamics. The smallest such invariant set is the so-called global attractor. Of course, a very natural choice consists in asking oneself about equivalence restricted to the global attractor itself. This is indeed the

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approach of Hale, Magalhães, Oliva (1984) [4], which in particular contains a Morse-Smale theory adapted to this situation. In fact, very recently, the particular singular perturbation problem described above has begun to be studied from this point of view by Hale, Raugel (1988) [5].

That problem is here studied from a different point of view, the difference lying in that instead of restricting our attention to the global attractor we consider a larger invariant set, namely a finite-dimensional (local) invariant manifold of class C^1 containing the global attractor, i.e. what in the current terminology is called an inertial manifold (of class C^1).

In this connection we show that there exist an integer n and a real number $\bar{\varepsilon} > 0$ such that, for every ε belonging to the interval $[0, \bar{\varepsilon})$, the global attractor of the corresponding dynamical system is contained in an invariant manifold of class C^1 and dimension n, and for $\varepsilon \to 0$ both this manifold and the vector field on it converge in the C^1 topology towards the ones corresponding to $\varepsilon = 0$. By combining this result with that of Henry and Angenent mentioned above, and applying the standard theory of Morse-Smale, we can conclude that if the stationary states are all hyperbolic, then, for ε small enough, the dynamical system generated by the damped wave equation and the one generated by the diffusion equation are equivalent restricted to the inertial manifolds above. These results are the object of our paper [10], whose contents are abridged in § 2 below.

Of course, the existence of an inertial manifold is not so general a fact as the existence of a compact global attractor. In particular, in order to obtain the manifolds above we need to restrict ourselves to the one-dimensional case as well as to values of ε sufficiently small. In fact, we can give an example where, for large values of ε, the global attractor is not contained in any finite-dimensional manifold of class C^1 (whether invariant or not), and in fact this situation is generic with respect to a special class of perturbations. This example is described in § 3. The result given here is slightly improved with respect to [9], where we observed only that the global attractor was not contained in any finite-dimensional manifold of class C^1 invariant by the flow.

Some independent results about this problem have been obtained recently by Chow, Lu (1988) [3], whose paper contains a general study of the existence of smooth invariant manifolds containing the global attractor for a class of problems which in particular includes the one considered here. Concerning the limiting behaviour of such manifolds in the singular limit $\varepsilon \to 0$, they obtain a result of convergence essentially in the C^0 topology. Another related work is that of Hale, Raugel (1988) [5], who centre their attention directly on the global attractor and show that for $\varepsilon \to 0$ this set converges in the Hausdorff topology towards the one corresponding to $\varepsilon = 0$. Although this property is weaker than the one obtained here, their result applies to the more general case of several space variables. Also, a different result on the same problem has been announced recently by Babin, Vishik (1987) [2].

1. THE EQUATIONS AND SOME PRELIMINARIES

Our results apply specifically to the following problem, where u is a function of $x \in (0, L)$ and $t \in \mathbb{R}$ with values in $\mathbb{R}$:

$$\varepsilon^2 u_{tt} + 2\alpha u_t = \beta u_{xx} + f(x, u) + q(x) \tag{1.1}$$

$$u|_{x=0} = \rho_0 , \qquad u|_{x=L} = \rho_L \qquad (1.2)_D$$

$$u|_{t=0} = u_0 , \qquad \varepsilon u_t|_{t=0} = \varepsilon v_0 \tag{1.3}$$

or the analogous one where $(1.2)_D$ is replaced by

$$u_x|_{x=0} = \sigma_0 , \qquad u_x|_{x=L} = \sigma_L . \qquad (1.2)_N$$

Henceforth, the boundary conditions $(1.2)_D$ or $(1.2)_N$ will be referred to as $(1.2)_B$, where B stands for either D or N. In the preceeding equations, ε is a real parameter which we consider to vary right up to the value $\varepsilon = 0$, α and β are fixed positive real parameters, f is a function $(0, L) \times \mathbb{R} \to \mathbb{R}$, q is a fixed function of $x \in (0, L)$, ρ_0, ρ_L, σ_0, σ_L are real numbers, and the initial data u_0 and v_0 are given functions of $x \in (0, L)$. The function f is assumed to satisfy the following conditions :

($f1$) $f(\,.\,, u)$ belongs to the Sobolev space $H^1(0, L)$ for every $u \in \mathbb{R}$, and in the case $B = D$ f satisfies the condition $f(0, \rho_0) = f(L, \rho_L) = 0$; $f(x, .\,)$ is of class $C^{2+\eta}$ for every $x \in (0, L)$, $f_x(x, .\,)$ is of class $C^{1+\eta}$ for almost every $x \in (0, L)$, and for every bounded open interval $J \subset \mathbb{R}$, the quantities

$$\sup_{x \in (0, L)} \|f(x, .\,)|_J\|_{C^{2+\eta}} , \qquad \int_0^L \|f_x(x, .\,)|_J\|^2_{C^{1+\eta}}\, dx$$

are both finite.

$$(f^*) \quad c := \limsup_{|u| \to \infty} \sup_{x \in (0, L)} \frac{f(x, u)}{u} < \begin{cases} \beta\pi^2/L^2 , & \text{if } B = D \\ 0 , & \text{if } B = N . \end{cases} \tag{1.4}$$

Concerning the function q, we assume simply that it belongs to $L_2(0, L)$.

Remark : The case $B = D$ with $f(0, \rho_0)$ or $f(L, \rho_L)$ not equal to zero can be reduced to the preceeding one by letting $\rho : (0, L) \to \mathbb{R}$ be any smooth function satisfying $\rho(0) = \rho_0$, and $\rho(L) = \rho_L$, and changing $f(x, u)$ and $q(x)$ respectively by $f(x, u) - f(x, \rho(x))$ and $q(x) + f(x, \rho(x))$.

Let u^* be the solution in $H^2(0, L)$ of the equation $\beta u_{xx} + q(x) = 0$ with the non-homogeneous boundary conditions $(1.2)_B$. By switching over to the new variable $\bar{u} := u - u^*$, the problem reduces to the homogeneous case $q = 0$, $\rho_0 = \rho_L = 0$, $\sigma_0 = \sigma_L = 0$; the role of f is now played by the function $\bar{f}(x, \bar{u}) := f(x, u^*(x) + \bar{u})$, which can be verified to inherit properties $(f1)$ and (f^*) from f. Furthermore, by suitably rescaling time and space and dividing equation (1.1) by a constant, the problem can be normalized to $2\alpha = 1$, $\beta = 1$, $L = \pi$. Henceforth, problem (1.1), $(1.2)_B$, (1.3) will always be considered in this particular normalized homogeneous form.

In the following this problem will be considered as a second order evolution problem on the Hilbert space $E = L_2 := L_2(0, L)$, namely

$$\varepsilon^2 \ddot{u} + \dot{u} + Au = Fu \tag{1.5}$$

$$u(0) = u_0, \qquad \varepsilon \dot{u}(0) = \varepsilon v_0 \tag{1.6}$$

where A and F are the operators given by

$$Au = -u_{xx} \tag{1.7}$$

$$Fu = f(\,.\,, u(\,.\,)) \tag{1.8}$$

with domains E^1 and $E^{1/2}$ respectively equal to H_B^2 and H_B^1. Here H_B^k $(k = 1, 2)$ denote the closures in the Sobolev spaces $H^k := H^k(0, L)$ of the set $\{u : (0, L) \to \mathbb{R} \mid u \in C^\infty([0, L])$ and satisfies the boundary conditions $(1.2)_B\}$.

In general, our results apply to an abstract evolution problem of the form (1.5), (1.6) where u takes values in a general Hilbert space E, A is a self-adjoint operator on E having numerical range bounded from below and compact resolvent, and F is a nonlinear operator with the properties of belonging to $C_{bdd}^{1+\eta}(E^{1/2}, E^{1/2})$ (which in our case follows from condition $(f1)$), representing the gradient of some functional on $E^{1/2}$, and satisfying an abstract version of condition (f^*) (see [10] for a more precise description). Here and in the following, E^α denote the usual power spaces associated with the operator A. Hereafter, the greatest lower bound of the numerical range of A will be denoted by λ_1.

The properties of $-A$ being the generator of an analytic semigroup on E and F belonging to $C_{bdd}^{1+\eta}(E^{1/2}, E^{1/2})$ determine that problem (1.5), (1.6) with $\varepsilon = 0$ fits in the standard theory of semilinear evolution equations of parabolic type, which ensures that it generates a semidynamical system of class $C^{1+\eta}$ on $E^{1/2}$.

As usual, for $\varepsilon \neq 0$ we shall take as state variable the pair $(u, \dot{u}) =: (u, v) =: U$, whose values will be considered in the space $E^{1/2} \times E =: \mathbb{E}$. In terms of this variable, problem (1.5), (1.6) takes the first-order form

$$\dot{U} + \mathbb{A}_\varepsilon U = \mathbb{F}_\varepsilon U \tag{1.9}$$

$$U(0) = U_0 \tag{1.10}$$

where $\mathbb{A}_\varepsilon$ and $\mathbb{F}_\varepsilon$ denote the operators on $E^{1/2} \times E$ given by

$$\mathbb{A}_\varepsilon(u, v) = (-v, \varepsilon^{-2}(Au + v)) \tag{1.11}$$

$$\mathbb{F}_\varepsilon(u, v) = (0, \varepsilon^{-2} Fu) \tag{1.12}$$

with domains respectively equal to $E^1 \times E^{1/2}$ and $E^{1/2} \times E$. It is a standard fact that $-\mathbb{A}_\varepsilon$ is the generator of a group on $E^{1/2} \times E$. On the other hand, the properties of F imply that $\mathbb{F}_\varepsilon$ maps $E^{1/2} \times E$ to itself and this mapping belongs to $C^{1+\eta}(E^{1/2} \times E,\ E^{1/2} \times E)$. With this, the problem fits in the standard theory of semilinear evolution equations of hyperbolic type, which ensures that it generates a dynamical system of class $C^{1+\eta}$ on $E^{1/2} \times E$.

By using the standard Lyapunov functional of problem (1.1), $(1.2)_B$, (1.3), and certain a priori estimates coming from condition (f^*), one obtains that for $\varepsilon = 0$ the semidynamical system is global, and for $\varepsilon \neq 0$ the dynamical system is global both in positive and negative time.

It is a well-known fact that, both for $\varepsilon = 0$ and for $\varepsilon \neq 0$, the preceeding problem has a compact global attractor. In the following this set will be denoted by $\mathscr{A}_\varepsilon$. We recall that $\mathscr{A}_\varepsilon$ consists of all initial states for which the solution is defined and bounded on $(-\infty, 0]$.

In order to deal with second order evolution problems of the form (1.5), (1.6) with $\varepsilon \neq 0$, we take the inner product on $\mathbb{E} = E^{1/2} \times E$ in a particular way specially adapted to the linear part of the equation. This inner product, which depends on ε, is the one associated with the norm given by

$$\|U\|_{\mathbb{E}}^2 := \begin{cases} \left\| \left(A - \dfrac{1}{4\varepsilon^2} I\right)^{1/2} u \right\|^2 + \left\| \dfrac{1}{2\varepsilon} u + \varepsilon v \right\|^2, & \text{if } 4\varepsilon^2 \lambda_1 > 1\,, \\ \left\| \left(A - \left(2\lambda_1 - \dfrac{1}{4\varepsilon^2}\right) I\right)^{1/2} u \right\|^2 + \left\| \dfrac{1}{2\varepsilon} u + \varepsilon v \right\|^2, & \text{if } 4\varepsilon^2 \lambda_1 < 1\,. \end{cases} \tag{1.13}$$

In particular, this inner product has the virtue of making the numerical range of $\mathbb{A}_\varepsilon$ to be contained in a vertical strip as narrow as possible. This is expressed by the following result, whose proof will be found in [9].

THEOREM 1.1 : *For* $4\varepsilon^2 \lambda_1 > 1$, *the numerical range of* $\mathbb{A}_\varepsilon$ *is contained in the vertical line* $\operatorname{Re} z = \dfrac{1}{2\varepsilon^2}$. *For* $4\varepsilon^2 \lambda_1 < 1$, *it is contained in the strip*

$$\left| \operatorname{Re} z - \frac{1}{2\varepsilon^2} \right| \leqslant \frac{1}{\varepsilon} \sqrt{\frac{1}{4\varepsilon^2} - \lambda_1}\,.$$

COROLLARY 1.2 : *For every* $\varepsilon > 0$, $-\mathbb{A}_\varepsilon$ *is the generator of a group of the form*

$$e^{-\mathbb{A}_\varepsilon t} = e^{-t/(2\varepsilon^2)} J_\varepsilon(t) \quad (t \in \mathbb{R}), \tag{1.14}$$

where, for $4\varepsilon^2 \lambda_1 > 1$, $J_\varepsilon(t)$ $(t \in \mathbb{R})$ *is a group of unitary operators, and for* $4\varepsilon^2 \lambda_1 < 1$, *it satisfies the bound*

$$\|J_\varepsilon(t)\|_{L(\mathbb{E},\mathbb{E})} \leq \exp\left(\frac{1}{\varepsilon}\sqrt{\frac{1}{4\varepsilon^2} - \lambda_1}\,|t|\right), \quad \forall t \in \mathbb{R}. \tag{1.15}$$

2. EXISTENCE AND CONVERGENCE IN THE PARABOLIC LIMIT

2.1. The setting and main results

By following the practice which is common in similar cases, in order to look for attracting invariant manifolds of (1.5), we shall decompose the state variable into fast and slow components, and we shall consider (1.5) as a (finite) perturbation of a linear system where the fast and slow components are mutually decoupled. The desired attracting invariant manifolds will then be sought for as graphs of mappings giving the fast components as a function of the slow ones. In Mora (1987) [8], this was done for $\varepsilon \neq 0$ by working on the first-order system (1.9) as a perturbation of the one corresponding to $F = 0$, and decomposing the variable according to the spectrum of $\mathbb{A}_\varepsilon$. Here, we adopt a somewhat different approach, the differences lying both in the way of decomposing the variable and of choosing the « unperturbed » linear system. In particular, here we consider U as decomposed into u and $\dot{u}$, which in its turn are decomposed according to the spectrum of A ; this has the advantage that the decomposition does not depend on ε.

Let λ_k $(k = 1, 2, \dots)$ denote the eigenvalues of A arranged in a non-decreasing sequence. Let us now take a positive integer n such that $\lambda_n < \lambda_{n+1}$, and consider the orthogonal decomposition invariant by A, $E = E_1 \oplus E_2$, where E_1 and E_2 denote the closed linear subspaces of E generated respectively by the first n eigenfunctions and the rest of them. In the following, the orthogonal projections of E onto E_i $(i = 1, 2)$ will be denoted as P_i, and the corresponding parts of A will be denoted as A_i. Parallel to this decomposition of E, the spaces E^α $\left(\alpha = \frac{1}{2}, 1\right)$ decompose also orthogonally as $E^\alpha = E_1^\alpha \oplus E_2^\alpha$, where $E_i^\alpha = P_i E^\alpha$. The spaces E_1, $E_1^{1/2}$, E_1^1 consist all in the same n-dimensional vector space provided with different but equivalent inner products. According to this fact, in the future the spaces $E_1^{1/2}$ and E_1^1 will be distinguished of E_1 only when the specific inner product plays a significant role.

Let us now introduce the preceeding decomposition in equation (1.5). Henceforth, the components of u in E_1 and E_2 will be denoted respectively as u_1 and u_2. By applying the projections P_1 and P_2, equation (1.5) transforms itself into a system for u_1 and u_2, which we shall write as follows :

$$\varepsilon^2 \ddot{u}_1 + \dot{u}_1 = P_1 F(u_1 + u_2) - A_1 u_1 \tag{2.1}$$

$$\varepsilon^2 \ddot{u}_2 + \dot{u}_2 + A_2 u_2 = P_2 F(u_1 + u_2) \tag{2.2}$$

where the term $A_1 u_1$ has been moved to the right-hand side because in fact we consider this system as a perturbation of the one which is obtained when its right-hand sides are set to zero. For $\varepsilon = 0$, the state variable is thus decomposed into the two components u_1 and u_2, which are to be considered as taking values respectively in $E_1^{1/2}$ and $E_2^{1/2}$. For $\varepsilon \neq 0$, the state variable will be considered as decomposed into the four components u_1, u_2, $\dot{u}_1$, $\dot{u}_2$, with values respectively in $E_1^{1/2}$, $E_2^{1/2}$, E_1, E_2.

It is known that if the gap between λ_{n+1} and $\max(\lambda_n, 0)$ is large enough, then the global attractor of the parabolic system (1.5) ($\varepsilon = 0$) is contained in a local invariant manifold of class C^1 and dimension n, M_0, which is given by a relation of the type

$$u_2 = h_0(u_1) \tag{2.3}$$

with h_0 belonging to $C^1(W_1, E_2^{1/2})$ and in fact to $C^1(W_1, E_2^1)$, where W_1 is a certain bounded domain in E_1. Here, this result will be accompanied by an extension to small non-zero values of ε. Specifically, it will be shown that, under the same gap condition there exists an $\bar{\varepsilon} > 0$ such that, for $\varepsilon \in (0, \bar{\varepsilon})$, the global attractor of the hyperbolic system (1.5) is also contained in a local invariant manifold of class C^1 and dimension n, M_ε, which is described by a set of relations giving u_2, $\dot{u}_1$, and $\dot{u}_2$ as functions of u_1 :

$$u_2 = h_\varepsilon(u_1) \tag{2.4}$$

$$\dot{u}_1 = k_\varepsilon(u_1) \tag{2.5}$$

$$\dot{u}_2 = \ell_\varepsilon(u_1) \tag{2.6}$$

where h_ε, k_ε, and ℓ_ε will belong respectively to $C^1(W_1, E_2^{1/2})$, $C^1(W_1, E_1)$, and $C^1(W_1, E_2)$, and in fact h_ε belongs to $C^1(W_1, E_2^1)$.

Although possibly these manifolds M_ε are normally hyperbolic, we shall not enter into this question, which on the other hand does not play any role in the development below.

Our main result consists in showing that, as $\varepsilon \to 0$, both the manifold M_ε and the vector field on it converge in the C^1 topology towards their analogous for $\varepsilon = 0$ Certainly, according to the preceeding paragraph, M_ε ($\varepsilon \neq 0$) are submanifolds of $E^{1/2} \times E$, while M_0 is a submanifold of $E^{1/2}$ In order that the problem of comparing M_ε ($\varepsilon \neq 0$) with M_0 be correctly posed, this last manifold is considered as embedded in $E^{1/2} \times E$ by taking u_1 and u_2 as determined by equations (2 1), (2 2) (with $\varepsilon = 0$) together with (2 3)

$$u_1 = P_1\, F(u_1 + h_0(u_1)) - A_1\, u_1 \qquad = k_0(u_1) \tag{2 7}$$

$$u_2 = P_2\, F(u_1 + h_0(u_1)) - A_2\, h_0(u_1) = \ell_0(u_1) \tag{2 8}$$

Notice that (2 8) has indeed a meaning since h_0 takes values in E_2^1 In fact, from the properties of h_0 stated above, it is obvious that k_0 and ℓ_0 belong respectively to $C^1(W_1, E_1)$ and $C^1(W_1, E_2)$ On the other hand, one should notice also that relations (2 5) and (2 7), besides being part of the specification of M_ε, they give also the evolution equation for the flow on M_ε In other words, k_ε is the projection on E_1 of the vector field on M_ε Thus, concerning the vector field on M_ε, our objective is to prove that, as $\varepsilon \to 0$, k_ε converges towards k_0 in the space $C^1(W_1, E_1)$

Our main result is contained in the following

THEOREM 2 1 *Let us consider problem* (1 5), (1 6) *with A and F satisfying the hypotheses stated in* § 1 *There exists a constant ℓ such that if λ_n and λ_{n+1} satisfy the conditions*

$$\lambda_{n+1} - \lambda_n > 4\,\ell \tag{2 9}$$

$$\lambda_{n+1} \qquad > 2\,\ell \tag{2 10}$$

then, there exist $\bar\varepsilon > 0$ and a bounded domain W_1 in E_1 such that

(a) *For $\varepsilon = 0$, the global attractor $\mathcal{A}_0$ is contained in M_0, an inflowing local invariant submanifold of $E^{1/2}$ of class C^1 and dimension n, which has the from* (2 3) *with $h_0 \in C^1(W_1, E_2^1)$*

(b) *For every $\varepsilon \in (0, \bar\varepsilon)$, the global attractor $\mathcal{A}_\varepsilon$ is contained in M_ε, an inflowing local invariant submanifold of $E^{1/2} \times E$ of class C^1 and dimension n, which has the form* (2 4)-(2 6), *with $h_\varepsilon \in C^1(W_1, E_2^1)$, $k_\varepsilon \in C^1(W_1, E_1)$, and $\ell_\varepsilon \in C^1(W_1, E_2) \cap C^0(W_1, E_2^1)$*

(c) *Let k_0* and *ℓ_0 be defined by* (2 7) *and* (2 8) *Then, as $\varepsilon \to 0$, h_ε converges towards h_0 in the space $C^1(W_1, E_2^1)$, k_ε converges towards k_0 in the space $C^1(W_1, E_1)$, and ℓ_ε converges towards ℓ_0 in both spaces $C^1(W_1, E_2)$ and $C^0(W_1, E_2^1)$*

(d) *For any* $\varepsilon \in [0, \bar{\varepsilon})$, *the solutions lying in* M_ε *are twice continuously differentiable with respect to time, with* $\ddot{u}$ *given by a relation of the type*

$$\ddot{u} = m_\varepsilon(u_1), \tag{2.11}$$

where $m_\varepsilon \leqslant C^0(W_1, E)$. *As* $\varepsilon \to 0$, m_ε *converges towards* m_0 *in the space* $C^0(W_1, E)$.

In particular this result implies that

COROLLARY 2.2: *Under the hypotheses of Theorem* 2.1, *then, for* $\varepsilon \in [0, \bar{\varepsilon})$, *the solutions lying in the global attractor* $\mathcal{A}_\varepsilon$ *have* u, $\dot{u}$, *and* $\ddot{u}$ *bounded independently of* ε *respectively in the spaces* E^1, E^1, *and* E.

Remark : The constant ℓ depends on the bounds on F and DF in a certain ball containing the global attractor.

In the application to problem (1.1), $(1.2)_B$, (1.3), conditions (2.9), (2.10) reduce to

$$\begin{aligned} 2n+1 &> 4\ell, \quad \text{if} \quad B = D \\ 2n-1 &> 4\ell, \quad \text{if} \quad B = N \end{aligned}$$

which will allways be satisfied if n is taken large enough. Since it is known (Henry (1985) [7], Angenent (1986) [1]) that for $\varepsilon = 0$ this system is Morse-Smale whenever the stationary states are all of them hyperbolic, the standard theory of Morse-Smale allows to conclude that

COROLLARY 2.3: *Let us consider problem* (1.1), $(1.2)_B$, (1.3) *with* f *satisfying the hypotheses stated in* § 1, *and assume that the stationary states are all of them hyperbolic. Then, for* ε *small enough, the flow on* M_ε *is equivalent to that on* M_0. *In particular, the flow on* $\mathcal{A}_\varepsilon$ *is equivalent to that on* $\mathcal{A}_0$.

2.2. Idea of the proof

As it is usual in similar circumstances, our proof begins by modifying the equation far from the attractor so that we can deal with global invariant manifolds instead of local ones (i.e. the domain W_1 of h_ε, k_ε, ℓ_ε equals the whole of E_1). For every ε, the corresponding manifold M_ε should contain all (mild) solutions which stay defined and bounded as $t \to -\infty$. In order to obtain such manifolds we use the classical method of Lyapunov and Perron in the special form as it appears for instance in Vanderbauwhede, Van Gils (1987) [11]. The main idea consists in looking for M_ε as consisting not only of all solutions which stay bounded as $t \to -\infty$, but more generally all solutions which satisfy an exponential growth condition of the form

$$\|u(t)\|_{1/2} = O(e^{-\mu t}) \qquad \text{as} \qquad t \to -\infty \tag{2 12}$$

where μ will be a positive real number belonging to the interval $(\lambda_n, \lambda_{n+1})$ The admitting of these extra solutions will result in the fact of the set M_ε being really a differentiable manifold

The solutions which satisfy the growth condition (2 12) will be obtained as fixed points of certain mappings $u^0 \mapsto u$ which result of solving a pair of non-homogeneous linear equations of the form

$$\varepsilon^2 u_1 + u_1 \qquad = G_1(u^0) = f_1 \tag{2 13}$$

$$\varepsilon^2 u_2 + u_2 + A_2 u_2 = G_2(u^0) = f_2 \tag{2 14}$$

with the additional condition that $u = u_1 + u_2$ satisfies (2 12) It turns out that, for $\mu \in (0, \lambda_{n+1})$, and ε small enough, the set of solutions of (2 13), (2 14) which satisfy (2 12) is parametrized by $u_1(0) \in E_1$ Thus, by adding an initial condition of the form $u_1(0) = x$, we obtain a different mapping $u^0 \mapsto u$ for every $x \in E_1$ By applying a suitable version of the parametrized contraction theorem, we obtain that, under conditions (2 9) and (2 10) together with

$$\max (0, \lambda_n + 2\ell) < \mu < \lambda_{n+1} - 2\ell , \tag{2 15}$$

each of these mappings has a unique fixed point The totality of these fixed points will give us the set M_ε we are looking for, which in fact will be a manifold parametrized by $x \in E_1$ Finally, the behaviour of M_ε as $\varepsilon \to 0$ is also taken care of by our specific version of the parametrized contraction theorem on the basis of a previous detailed study of the behaviour as $\varepsilon \to 0$ of the solutions of the non-homogeneous linear equations (2 13), (2 14) with the additional conditions mentioned above

3. NON-EXISTENCE FAR FROM THE PARABOLIC LIMIT

In this Section we provide an example where for large values of ε the global attractor is not contained in any finite-dimensional manifold of class C^1 (whether invariant or not), and in fact this situation is generic with respect to a special class of perturbations

The reason why large values of ε make difficult that the global attractor be contained in a finite-dimensional manifold of class C^1 is mainly linear For large values of ε, the linear part of the equation at a stationary point easily has all the eigenvalues on the same vertical line of the complex plane Under these conditions, one can show that there exists a countable family of finite-dimensional manifolds of class C^1 through that point such that if a positive semiorbit tending to that point is contained in some finite-dimensional manifold of class C^1 then it is contained also in one of that countable family In § 3 1 this crucial fact is established for the linear problem By using a suitable C^1 linearization theorem, which we have developed specifically to apply to this problem and which we give in § 3 2, we can then translate this situation to the neighbourhood of a stationary point of a nonlinear problem

Finally, in § 3 3 we consider an example where the global attractor contains an heteroclinic orbit from ϕ to ψ, with ψ being a stationary state with a linearization of the type described above, and we show that the function f can be perturbed in such a way that the connecting orbit avoids each of the countably many finite-dimensional manifolds mentioned above Therefore, one can conclude that the global attractor is not contained in any finite-dimensional manifold of class C^1 In fact, in our construction this situation is generic with respect to the considered class of perturbations of the function f

3.1. The linear case

In this paragraph we deal with a second order linear evolution problem of the form (1 5), (1 6) with $F = 0$, or equivalently the first order evolution problem (1 9), (1 10) with $\mathbb{F}_\varepsilon = 0$ In the following, e_k and λ_k ($k = 1, 2, \dots$) denote respectively a complete orthonormal system of eigenfunctions of A and the corresponding sequence of eigenvalues, which sequence is assumed to be non-decreasing Finally, E_k will denote the one-dimensional space generated by e_k We have the orthogonal decomposition invariant by A $E = \bigoplus_{k=1}^{\infty} E_k$ Correspondingly, the space $\mathbb{E} = E^{1/2} \times E$ has the orthogonal decomposition invariant by $\mathbb{A}_\varepsilon$ $\mathbb{E} = \bigoplus_{k=1}^{\infty} \mathbb{E}_k$, where $\mathbb{E}_k$ denotes the two-dimensional subspace of $\mathbb{E}$ given by $\mathbb{E}_k = E_k \times E_k$

Let us now assume that $4\,\varepsilon^2 \lambda_1 > 1$, and consider $e^{-\mathbb{A}_\varepsilon t}$ decomposed according to the formula (1 14) On each of the two-dimensional invariant subspaces $\mathbb{E}_k$, the effect of the group $J_\varepsilon(t)$ consists in a rotation of frequency $\omega_k = \frac{1}{\varepsilon}\sqrt{\lambda_k - \frac{1}{4\,\varepsilon^2}}$ From this fact it follows that, for every $U \in \mathbb{E}$, the function $\mathbb{R} \ni t \mapsto J_\varepsilon(t)\, U \in \mathbb{E}$ is almost periodic

The main result of this paragraph is the following

THEOREM 3 1 *Assume that A is self-adjoint with numerical range bounded from below and compact resolvent, and also that* $4\,\varepsilon^2 \lambda_1 > 1$ *If a* *positive semiorbit of* $e^{-\mathbb{A}_\varepsilon t}$ $(t \in \mathbb{R})$ *is contained in a finite-dimensional submanifold of* $\mathbb{E}$ *differentiable at the origin, then it is contained also in a linear subspace of the form* $\mathbb{E}_K = \bigoplus_{k \in K} \mathbb{E}_k$ *where* K *is a finite subset of* $\mathbb{N} \setminus \{0\}$.

The proof of this theorem reduces to the application of the two following lemmas.

LEMMA 3.2 : *Under the hypotheses of Theorem* 3.1, *if a positive semiorbit of* $e^{-\mathbb{A}_\varepsilon t}$ $(t \in \mathbb{R})$ *is contained in a submanifold* M *of* $\mathbb{E}$ *differentiable at the origin, then it is contained also in the tangent subspace of* M *at the origin.*

Proof: Let U be a point of the semiorbit which is assumed to be contained in M. Let F be the tangent subspace of M at the origin. Finally, let P denote the orthogonal projection of $\mathbb{E}$ onto F, and $Q := I - P$. The fact that F is tangent to M at the origin means that

$$\|QW\| = o(\|PW\|), \quad \text{as} \quad W \to 0 \text{ on } M .$$

In particular, this implies that

$$\|Q\, e^{-\mathbb{A}_\varepsilon t}\, U\| = o(\|P\, e^{-\mathbb{A}_\varepsilon t}\, U\|), \quad \text{as} \quad t \to +\infty ,$$

or, equivalently by (1.14),

$$\|QJ_\varepsilon(t)\, U\| = o(\|PJ_\varepsilon(t)\, U\|), \quad \text{as} \quad t \to +\infty .$$

Using the fact that $J_\varepsilon(t)\, U$ is an almost periodic function of t, one can then derive that $QU = 0$, i.e. $U \in F$. ■

LEMMA 3.3 : *Under the hypotheses of Theorem* 3.1, *if a positive semiorbit of* $e^{-\mathbb{A}_\varepsilon t}$ $(t \in \mathbb{R})$ *is contained in a finite-dimensional linear subspace of* $\mathbb{E}$, *then it is contained also in one of the particular from* $\mathbb{E}_K = \bigoplus_{k \in K} \mathbb{E}_k$ *where* K *is a finite subset of* $\mathbb{N} \setminus \{0\}$.

Proof: We first notice that the linear closure of a semiorbit of $e^{-\mathbb{A}_\varepsilon t}$ $(t \in \mathbb{R})$ coincides with the linear closure of the semiorbit of $J_\varepsilon(t)$ $(t \in \mathbb{R})$ starting at the same point. Now, the group $J_\varepsilon(t)$ $(t \in \mathbb{R})$ decomposes as a rotation of frequency ω_k on each of the two-dimensional subspaces $\mathbb{E}_k$, from which one sees easily that the dimension of the linear closure of a semiorbit of $J_\varepsilon(t)$ $(t \in \mathbb{R})$ is twice the number of frequencies involved. In particular, the linear closure of the orbit being finite-dimensional implies that only a finite number of frequencies are involved, and since all frequencies have finite multiplicity (because A has compact resolvent), this implies that only a finite number of proper modes are involved. ■

3.2. A C^1 linearization theorem

In this paragraph we give a C^1 linearization theorem which is applicable to certain stationary states of semilinear damped wave equations. In the finite-dimensional case, our result is included essentially in that of Hartman (1960) [6, Theorem (I)], which instead of our condition (3.2) requires only that L be a contraction. For the proofs of the following statements, the reader is referred to [9].

THEOREM 3.4 : *Let U be an open subset of a Banach space X, and T a C^1 map $U \to X$ with a fixed point p. Let L be the Fréchet derivative of T at p, i.e. $L := DT(p)$. Assume that L has a bounded inverse, and that the following properties are satisfied for some $\eta > 0$:*

$$DT(p+x) - L = o(\|x\|^{\eta}), \qquad as \qquad x \to 0, \tag{3.1}$$

$$\|L^{-1}\| \ \|L\|^{1+\eta} < 1 . \tag{3.2}$$

Then, there exist V, a neighbourhood of p in U, with $T(V) \subset V$, and $R : V \to X$, a C^1 diffeomorphism onto its image, with $R(p) = 0$, $DR(p) = I$, and

$$DR(p+x) - I = o(\|x\|^{\eta}), \qquad as \qquad x \to 0, \tag{3.3}$$

such that the following equation holds :

$$RT = LR . \tag{3.4}$$

Such a map is unique in the following sense : if V' and R' satisfy also the preceeding properties, then R and R' coincide in any ball centered at p and contained in $V \cap V'$.

Remarks :

(i) Condition (3.2) implies that L is a contraction.

(ii) The exponent η is by no means restricted to be less than 1 ; increasing η makes condition (3.2) less restrictive, but then condition (3.1) requires T to be closer to linear.

COROLLARY 3.5 : *Let X be a Banach space, and $T(t)$ $(t \in \mathbb{R})$ a group of diffeomorphisms of X with a fixed point p. Let $L(t)$ $(t \in \mathbb{R})$ be the group of bounded linear operators on X given by $L(t) := D(T(t))(p)$. Assume that,* *for some* $\tau \in \mathbb{R}$ *and some* $\eta > 0$, $T := T(\tau)$ *and* $L := L(\tau)$ *satisfy properties* (3.1) *and* (3.2). *Then there exist* V, *neighbourhood of* p, *and* $R : V \to X$, *a* C^1 *diffeomorphism onto its image, with* $R(p) = 0$, $DR(p) = I$, *and* (3.3), *such that, for every* $t \in \mathbb{R}$, *the equation*

$$RT(t) = L(t) R \tag{3.5}$$

holds in some ball centered at p *and contained in* V. *Such a ball can be chosen independently of* t *when* t *varies over any interval of the form* $[t_0, +\infty)$ *with* t_0 *finite.*

COROLLARY 3.6 : *Let us consider problem* (1.5), (1.6) *with the hypotheses stated in* § 1. *Let* u^* *be a stationary state, and let* λ_1 *be the lowest eigenvalue of the operator* $A - DF(u^*)$. *If* $4\,\varepsilon^2\,\lambda_1 < 1$, *then, near this stationary state, the flow is* C^1*-conjugate to its linearization.*

3.3. Exhibiting non-existence

Our example of non-existence belongs to problem (1.1), $(1.2)_N$, (1.3), which we consider in its normalized homogeneous form, i.e. with $q = 0$, $\sigma_0 = \sigma_L = 0$, $2\alpha = 1$, $\beta = 1$, $L = \pi$. The function f will have the form

$$f(x, u) = f_0(u) + g(x, u), \tag{3.6}$$

where f_0 will be a fixed function independent of x, and the perturbation g will be variable.

The fixed function f_0 is assumed to satisfy the general hypotheses $(f1), (f^*)$, which in this case reduce to requiring it to be of class $C^{1+\eta}$ and to satisfy $\limsup_{|u| \to \infty} (f_0(u)/u) < 0$. Besides this, we assume that f_0 satisfies also the following conditions :

$$f_0(0) = f_0(1) = 0\,; \qquad 1 \text{ is the only positive zero of } f_0\,, \tag{3.7}$$

$$0 < f_0'(0) < 1\,, \tag{3.8}$$

$$f_0'(1) < -\frac{1}{4\,\varepsilon^2}. \tag{3.9}$$

Since f_0 is independent of x, and the boundary conditions are of Neumann type, the dynamical system on $\mathbb{E} = H^1 \times L_2$ corresponding to $g = 0$ has a two-dimensional invariant linear subspace consisting of the states which are spatially homogeneous (i.e. constant with respect to x) ; on this subspace, (1.1) reduces to a second order ordinary differential equation. In the following, $\bar{0}$ and $\bar{1}$ denote the points of this subspace given respectively by $u = 0$, $\dot{u} = 0$ and $u = 1$, $\dot{u} = 0$. Conditions (3.7)-(3.9) imply the following facts :

Both $\bar{0}$ and $\bar{1}$ are hyperbolic stationary states . (3.10)

$\bar{0}$ has a one-dimensional unstable manifold ;

$\bar{1}$ is asymptotically stable . (3.11)

There is an heteroclinic orbit from $\bar{0}$ to $\bar{1}$. (3.12)

In fact, the heteroclinic orbit which connects $\bar{0}$ to $\bar{1}$ lies on the subspace of spatially homogeneous states.

We now introduce a perturbation g which will break this special situation occurring for $g = 0$. This perturbation g will be allowed to vary within a ball in a certain Banach space $\mathcal{G}$ of functions $(0, \pi) \times \mathbb{R} \to \mathbb{R}$ supported in $(0, \pi) \times \bar{J}$, where $J = (a, b)$ is a fixed open interval with $0 < a < b < 1$, and for technical reasons a is restricted to be less than a certain quantity $a_0 > 0$ associated with the heteroclinic orbit of the case $g = 0$. The Banach space $\mathcal{G}$ consists of the linear space

$$\mathcal{G} := \{g : (0, \pi) \times \mathbb{R} \to \mathbb{R} \mid g \text{ satisfies } (f1), \text{ and } g(x, u) = 0, \forall u \notin \bar{J}\} \quad (3.13)$$

provided with the norm

$$\|g\|_{\mathcal{G}} := \sup_{x \in (0, \pi)} \|g(x, .)|_J\|_{C^{2+\eta}} + \left(\int_0^\pi \|f_x(x, .)|_J\|^2_{C^{1+\eta}} dx \right)^{1/2} . \quad (3.14)$$

In the following, the ball of radius δ within this Banach space will be denoted by $\mathcal{G}_\delta$. In order to indicate its dependence with respect to g, in the future the flow on $\mathbb{E} = E^{1/2} \times E$ corresponding to the function f given by (3.6) will be denoted by $T_g(t)$ $(t \in \mathbb{R})$. One can verify that, for every compact interval $[t_0, t_1] \subset \mathbb{R}$, the mapping $\mathbb{E} \times \mathcal{G} \ni (U, g) \to T_g(.) \in C([t_0, t_1], \mathbb{E})$ is of class $C^{1+\eta}_{bdd}$. Clearly, for every $g \in \mathcal{G}$, the corresponding flow still satisfies (3.10) and (3.11). In fact, these perturbed flows remain unchanged within the open set

$$\mathfrak{J} := \{U = (u, v) \in \mathbb{E} \mid u(x) \notin \bar{J},\ \forall x \in [0, \pi]\} .$$

Let us look at the flow in the neighbourhood of the stationary state $\bar{1}$, where we know that it does not depend on g. Condition (3.9) means that the linearization at $\bar{1}$ satisfies the hypothesis $4\,\varepsilon^2 \lambda_1 > 1$ of both Theorem 3.1 and Corollary 3.6. By applying those results, we can conclude that there exists a neighbourhood V of $\bar{1}$, which by the stability of $\bar{1}$ can be assumed positively invariant, and a countable family of finite-dimensional submanifolds of class C^1, which we shall denote by M_K (K varying among the finite subsets of $\mathbb{N}\setminus\{0\}$), such that if a positive semiorbit tending to $\bar{1}$ is contained in a finite-dimensional manifold of class C^1, then its restriction to V must be contained in one of the manifolds M_K

Let us now consider the only orbit that departs from $\bar{0}$ towards the positive u direction Before leaving $\mathfrak{J}$, this orbit will coincide with that corresponding to $g = 0$ Let us fix an arbitrary point U of this common initial arc Since $T_0(t)\, U \to \bar{1}$ as $t \to +\infty$, the continuity with respect to g ensures that, if δ is small enough, then the following property will hold

There exists $t_1 > 0$ such that, for every

$$g \in \mathcal{G}_\delta\,, \quad T_g(t_1)\, U \in V \tag{3 15}$$

In particular, this implies that, for $g \in \mathcal{G}_\delta$, $T_g(t) \to \bar{1}$ as $t \to +\infty$, i e the corresponding flow still satisfies (3 12)

Let Γ denote the C^1 mapping

$$\Gamma \quad \mathcal{G}_\delta \ni g \mapsto T_g(t_1)\, U \in V \subset \mathbb{E}\,, \tag{3 16}$$

where t_1 is the quantity appearing in (3 15) We claim that, for most $g \in \mathcal{G}_\delta$, $\Gamma(g) = T_g(t_1)\, U$ does not belong to any of the countably many manifolds M_K In fact, we can state the following result

THEOREM 3 7 *Consider problem* (1 1), (1 2)$_N$, (1 3), *with* $q = 0$, $\sigma_0 = \sigma_L = 0$, *and normalized to* $2\,\alpha = 1$, $\beta = 1$, $L = \pi$ *Let f have the form* (3 6), *with f_0 fixed satisfying the conditions mentioned above, and g varying in the Banach space $\mathcal{G}$ defined above There exists a $\delta > 0$ and a residual subset $\mathcal{R}$ of $\mathcal{G}_\delta$ such that if $g \in \mathcal{R}$ then there is no finite-dimensional manifold of class C^1 containing the global attractor*

The proof of Theorem 3 7 is based upon the following lemma, for whose rather technical proof the reader is referred to [9]

LEMMA 3 8 *Under the hypotheses of Theorem* 3 7, *there exists a* $\delta > 0$ *and a dense subset* $\mathcal{D}_\delta$ *of* $\mathcal{G}_\delta$ *such that if* $g \in \mathcal{G}_\delta$ *then* Range $D\Gamma(g)$ *is infinite-dimensional*

Proof of Theorem 3 7 It suffices to verify that, for every finite subset K of $\mathbb{N}\setminus\{0\}$, there is an open and dense subset $\mathcal{R}_K$ of $\mathcal{G}_\delta$ such that if $g \in \mathcal{R}_K$ then $\Gamma(g) \notin M_K$ From this the theorem will follow by a category argument The openness of $\mathcal{R}_K$ is an immediate consequence of the continuous dependence of solutions with respect to g The denseness of $\mathcal{R}_K$ follows from Lemma 3 8 Indeed, if $\mathcal{R}_K$ were not dense, there would be some open set $\mathcal{U} \subset \mathcal{G}_\delta$ such that $\Gamma(g) \in M_K$ for all $g \in \mathcal{U}$ But this would imply that, for every $g \in \mathcal{U}$, Range $D\Gamma(g) \subset T(M_K)_{\Gamma(g)}$, which contradicts Lemma 3 8 since $T(M_K)_{\Gamma(g)}$ is finite-dimensional ■

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