M2AN. Mathematical modelling and numerical analysis - Modélisation mathématique et analyse numérique

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M2AN. Mathematical modelling and numerical analysis - Modélisation mathématique et analyse numérique, tome 23, nº 2 (1989), p. 261-282
<http://www.numdam.org/item?id=M2AN_1989__23_2_261_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

(Vol 23, n° 2, 1989, p. 261-282)

ON A CLASS OF IMPLICIT AND EXPLICIT SCHEMES OF VAN-LEER TYPE FOR SCALAR CONSERVATION LAWS (*)

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Abstract. — *The convergence of second order accurate schemes towards the entropy solution of scalar conservation laws is studied We make use of the Van-Leer method to get an affine approximation of the flux The construction procedure leads to Total Variation Diminishing (TVD) schemes in the implicit and in the explicit cases*

The proposed schemes can be presented as corrected upwind schemes The physical problems where the flux is the physical variable, motivated this study

Résumé. — *On étudie ici une extension des schémas MUSCL proposés par Van-Leer pour l'approximation des lois de conservation La méthode utilise une représentation affine par maille des flux numériques Cette technique de correction des flux est motivée par des problèmes physiques pour lesquels il est difficile de représenter simplement la relation de dépendance entre le flux conservatif et les autres variables physiques*

Des résultats de convergence sont établis pour les schémas explicites et implicites, du second ordre en espace et du premier ou du second ordre en temps appliqués à une équation scalaire

INTRODUCTION

Recently much effort has been made for obtaining second order accurate schemes for conservation laws, which do not exhibit spurious oscillations, near discontinuities of the solution, and lead to sharp discrete shock solutions. In general the first order accurate schemes, give poor resolution to discontinuities, because of their important numerical diffusion. First the construction procedure of nonoscillatory second order schemes was the Flux Corrected Transport (FCT) of Book-Boris-Hain [1]. This approach is based of the limitation of the numerical flux. Some years ago Van-Leer [9] derived

(*) Received in December 1987.

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a scheme using a flux limiter based on the affine approximation of the solution in each cell of the grid. More recently other authors have delt with the high order schemes : Harten [3], Le Roux [4], Majda and Osher [5], Mock [6], Osher and Chakravarthy [7], Vila [10].

For all the proposed schemes, we prove that these schemes are TVD together with the convergence of the approximate solution towards the entropy satisfying solution. This paper is built as follows : section 1 is devoted to the statement of the problem, where we review the relevent theory of weak solution of scalar conservation laws. In § 2, we do the same for the theory of approximate solutions. Section 3 is concerned with the construction of second order accurate explicit schemes and the proof of convergence results related to the used limiters. In section 4 we construct a high order implicit scheme, for which we prove an entropy inequality. Finally § 5 gives results of numerical experiments.

1. STATEMENT OF THE PROBLEM

We consider the numerical solution of the following problem : find a bounded u satisfying the quasi-linear equation :

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0 \tag{1.1}$$

for $(x, t) \in \mathbb{R} \times]0, T[$; $T > 0$ and

$$u(x, 0) = u_0(x) \tag{1.2}$$

with $\qquad f \in C^1(\mathbb{R}) \quad \text{and} \quad u_0 \in L^\infty(\mathbb{R}) \cap BV_{\text{loc}}(\mathbb{R})$

where $BV_{\text{loc}}(\mathbb{R})$ denotes the space of the locally bounded variation functions.

We seek a weak solution to the problem (1.1)-(1.2), i.e. a bounded function $u \in L^\infty(\mathbb{R} \times]0, T[)$ satisfying :

$$\int_{\mathbb{R}} \int_{]0, T[} \left(u \frac{\partial \phi}{\partial t} + f(u) \frac{\partial \phi}{\partial x} \right) \mathrm{d}x \, \mathrm{d}t + \int_{\mathbb{R}} u_0(x) \, \phi(x, 0) \, \mathrm{d}x = 0 \tag{1.3}$$

for all test functions $\phi \in C^\infty(\mathbb{R} \times [0, T[)$ with compact support in $\mathbb{R} \times [0, T[$.

The solution to (1.1)-(1.2) is not necessarily unique and the physical one is characterized by the following entropy condition :

$$\int_{\mathbb{R}} \int_{]0, T[} \left(\eta(u) \frac{\partial \phi}{\partial t} + F(u) \frac{\partial \phi}{\partial x} \right) \mathrm{d}x \, \mathrm{d}t \geqslant 0 \tag{1.4}$$

for all $\phi \in C^{\infty}(\mathbb{R} \times \mathbb{R}^+)$ with compact support in $\mathbb{R} \times \mathbb{R}^+$, and $\phi \geq 0$, where η is the entropy function and F the entropy flux associated with the entropy function η.

Usually, when we deal with the numerical solution to the problem (1.1)-(1.2), we assume that the solution u is constant or affine in each cell of the grid. However some physical problems like reservoirs simulation use as data the flux $f(u)$ but only discrete values of f may be found. Because of this the proposed schemes are based on the affine approximation of the flux $f(u)$. The constructed explicit and implicit schemes are TVD second order accurate and the approximate solution given by these schemes converges towards the entropy solution of (1.1)-(1.2). The studied schemes can be presented as corrected upwind schemes.

2. PRELIMINARIES

Let h be the spatial grid size and k be the time grid size related to h by the fixed positive number r through

$$r = \frac{k}{h}.$$

A weak solution u to the problem (1.1)-(1.2) is approximated by a function u_h defined on $\mathbb{R} \times]0, T[$ by

$$u_h(x, t) = u_j^n \quad \text{for} \quad (x, t) \in I_j \times J_n \tag{2.1}$$

with

$$I_j \times J_n = \left]\left(j - \frac{1}{2}\right) h, \left(j + - \frac{1}{2}\right) h\right[\times \left]\left(n - \frac{1}{2}\right) k, \left(n + \frac{1}{2}\right) k\right[\quad \forall j \in \mathbb{Z}, \ \forall n \in \mathbb{N}$$

such that $n \leq N = [T/k] + 1$; where $[\alpha]$ denotes the integer part of α.

The initial condition (1.2) is projected into the space of piecewise constant functions by:

$$u_j^0 = \frac{1}{h} \int_{I_j} u_0(x)\, dx \quad \forall j \in \mathbb{Z} \tag{2.2}$$

the studied schemes are written in conservation form

$$u_j^{n+1} = u_j^n - r(G_{j+1/2} - G_{j-1/2}) \tag{2.3}$$

where $G \in C^1(\mathbb{R}^2 \times \mathbb{R})$ and is given by:

$$G_{j+1/2} = G(u_{j-p+1}^n, \ldots, u_{j+p}^n ; u_{j-q+1}^{n+1}, \ldots, u_{j+q}^{n+1}) \tag{2.4}$$

G is the numerical flux, with the consistency condition :

$$G(u, \dots, u ; u, \dots, u) = f(u) \tag{2.5}$$

the scheme (2.4) contains the explicit and implicit cases.

It is well known that a crucial estimate required in convergence proofs of difference schemes is a bound on the total variation of the solution.

The total variation, $TV(u^{n+1})$ is defined by :

$$TV(u^{n+1}) = \sum_{j \in \mathbb{Z}} \left| u_{j+1}^{n+1} - u_j^{n+1} \right| \tag{2.6}$$

an important class of difference schemes is those which are Total Variation Diminishing (TVD), that is :

$$TV(u^{n+1}) \leqslant TV(u^n) \tag{2.7}$$

we recall some useful properties of the TVD character. Let $\Delta_+ v_j = v_{j+1} - v_j$, in the explicit case the scheme (2.3) may be written in an incremental form :

$$u_j^{n+1} = u_j^n - C_{j-1/2}^n \Delta_+ u_{j-1}^n + D_{j+1/2}^n \Delta_+ u_j^n \tag{2.8}$$

sufficient conditions for the scheme (2.8) to be TVD are (see [3]) :

$$0 \leqslant C_{j+1/2}^n , 0 \leqslant D_{j+1/2}^n , C_{j+1/2}^n + D_{j+1/2}^n \leqslant 1 \tag{2.9}$$

the incremental form of the implicit scheme is :

$$u_j^n = u_j^{n+1} + C_{j-1/2}^{n+1} \Delta_+ u_{j-1}^{n+1} - D_{j+1/2}^{n+1} \Delta_+ u_j^{n+1} \tag{2.10}$$

it is shown in [3] that sufficient conditions for the scheme (2.10) to be TVD are :

$$-\infty < C \leqslant C_{j-1/2}^{n+1} \leqslant 0 \quad \text{and} \quad -\infty < C \leqslant D_{j+1/2}^{n+1} \leqslant 0 \tag{2.11}$$

where C is a constant.

Let f be the flux of the scalar conservation law, we suppose that

$$f = f_1 + f_2 \quad \text{such that} \quad f_1' \geqslant 0 \quad \text{and} \quad f_2' \leqslant 0$$

this decomposition exists always for a general f.

The general form of the proposed schemes in explicit or implicit cases is :

$$u_j^{n+1} = u_j^n - r\,\Delta_+ G_{j-1/2} \,. \tag{2.12}$$

To get second order accuracy in space and first order in time we take :

$$G_{j-1/2} = g_{j-1/2} + \frac{h}{2}\,\delta_{1,j-1} - \frac{h}{2}\,\delta_{2,j} \tag{2.13}$$

where

$$g_{j-1/2} = f_1(u_{j-1}) + f_2(u_j)$$

is an upwind numerical flux associated with the decomposition of f, and

$$\delta_{i,j} = \begin{cases} \sigma \operatorname{Min}\left\{\left|\hat{\delta}_{i,j}\right|, \frac{a}{h}\,|f_i(u_{j+1}) - f_i(u_j)|, \frac{a}{h}\,|f_i(u_j) - f_i(u_{j-1})|\right\} \\ \text{if } \sigma = \operatorname{sgn}\,(\hat{\delta}_{i,j}) = \operatorname{sgn}\,[f_i(u_{j+1}) - f_i(u_j)] = \\ \qquad = \operatorname{sgn}\,[f_i(u_j) - f_i(u_{j-1})] \\ 0 \text{ elsewhere} \end{cases} \tag{2.14}$$

where

$$\hat{\delta}_{i,j} = \frac{f_i(u_{j+1}) - f_i(u_{j-1})}{2\,h}$$

$i = 1, 2$ and a is a positive constant depending on the type of limitations such that : $0 < a \leqslant 2$.

Remark 2.1 : If the flux f is convex, the upwind flux associated with the decomposition of f, coincides with the E.-O. numerical flux.

To obtain second order accuracy in space and in time, the numerical flux is taken to be :

$$G_{j-1/2} = g_{j-1/2} + \frac{h}{2}\,\delta_{1,j-1} - \frac{h}{2}\,\delta_{2,j} - \frac{\varepsilon k}{2}\,[f_1'(u_{j-1})\,\delta_{1,j-1} + f_2'(u_j)\,\delta_{2,j}] \tag{2.15}$$

$\varepsilon = -1$ in the implicit scheme and $\varepsilon = +1$ in the explicit scheme.

From the formula (2.14), there exists $\alpha_{i,j+1/2}$ and $\beta_{i,j-1/2}$ such that

$$0 \leqslant \alpha_{i,j+1/2} \leqslant a \quad \text{and} \quad 0 \leqslant \beta_{i,j-1/2} \leqslant a$$

$$\delta_{i,j} = \alpha_{i,j+1/2}\,\frac{f_i(u_{j+1}) - f_i(u_j)}{h} = \beta_{i,j-1/2}\,\frac{f_i(u_j) - f_i(u_{j-1})}{h}\,. \tag{2.16}$$

Remark 2.2 : The correction of the slope δ is given by (2.14) or by (2.16) once α and β are known.

The following proposition shows that there is a relation between the second order accuracy and the choice of α and β.

PROPOSITION 2.1 : *If* $\alpha = 1 + O(h)$ *and* $\beta = 1 + O(h)$ *in* (2.16) ; *then the scheme* (2.12)-(2.13) *in second order accurate in space ; and the scheme* (2.12)-(2.15) *is second order accurate in space and in time.*

Proof: Since for $\alpha = \beta = 1$, the scheme (2.12)-(2.13) is second order accurate in space (by a Taylor expansion). It suffices to check that :

$$O(h)[f(u_{j+1}) - f(u_j)] = O(h^2)$$

we have:

$$\begin{aligned} O(h)[f(u_{j+1}) - f(u_j)] &= O(h)\, f'(\xi_{j+1/2})(u_{j+1} - u_j) \\ &= O(h)\, hf'(\xi_{j+1/2})\, u_x(\zeta_{j+1/2}) \\ &= O(h^2) \end{aligned}$$

hence the scheme (2.12)-(2.13) is second order accurate in space. With similar step we show that the scheme (2.12)-(2.15) is second order in space and in time.

3. APPROXIMATION OF (1.1)-(1.2) BY SECOND ORDER EXPLICIT SCHEMES

In this section, we consider explicit schemes of the form :

$$u_j^{n+1} = u_j^n - r\,\Delta_+ G_{j-1/2}^n \tag{3.1}$$

the numerical flux G is defined according to the order of accuracy.

3.1. Second order accuracy in space

To get a second order accurate scheme in space and first order in time ; we take the numerical flux given by :

$$G_{j-1/2}^n = g_{j-1/2}^n + \frac{h}{2}\,\delta_{1,j-1}^n - \frac{h}{2}\,\delta_{2,j}^n \tag{3.2}$$

where g is an upwind flux associated with the composition of f. The correction is made separately through f_1 and f_2 ; i.e. :

$$\delta_{1,j}^n = \frac{a}{h}\,\alpha_{1,j+1/2}^n[f_1(u_{j+1}^n) - f_1(u_j^n)] = \frac{a}{h}\,\beta_{1,j-1/2}^n[f_1(u_j^n) - f_1(u_{j-1}^n)]$$

$$\delta_{2,j}^n = \frac{a}{h}\,\alpha_{2,j+1/2}^n[f_2(u_{j+1}^n) - f_2(u_j^n)] = \frac{a}{h}\,\beta_{2,j-1/2}^n[f_2(u_j^n) - f_2(u_{j-1}^n)]$$

with $0 \leqslant \alpha_{i,k},\ \beta_{i,k} \leqslant 1$; $i = 1, 2$, and $k \in \mathbb{Q}$.

This scheme contains as a particular case the approximation of conservation laws with decreasing or increasing flux as in the case of the reservoir simulation problems. The scheme (3.2) is written as :

$$u_j^{n+1} = u_j^n - C_{j-1/2}^n \Delta_+ u_{j-1}^n + D_{j+1/2}^n \Delta_+ u_j^n \tag{3.3}$$

with

$$\begin{cases} C_{j-1/2}^n = \left(1 + \frac{a}{2}\beta_{1,j-1/2}^n - \frac{a}{2}\alpha_{1,j-1/2}^n\right) r \dfrac{\Delta_+ f_1(u_{j-1}^n)}{\Delta_+ u_{j-1}^n} \\ D_{j+1/2}^n = \left(1 - \frac{a}{2}\beta_{2,j+1/2}^n + \frac{a}{2}\alpha_{2,j+1/2}^n\right) r \dfrac{\Delta_+ f_2(u_j^n)}{\Delta_+ u_j^n} \end{cases} \tag{3.4}$$

and we choose

$$\nu_{1,j-1/2}^n = r \frac{\Delta_+ f_1(u_{j-1}^n)}{\Delta_+ u_{j-1}^n} ; \quad \nu_{2,j+1/2}^n = r \frac{\Delta_+ f_2(u_j^n)}{\Delta_+ u_j^n}$$

since f_1 is an increasing function and f_2 is a decreasing function then $\nu_1 \geq 0$ and $\nu_2 \leq 0$.

PROPOSITION 3.1 : *Suppose that the CFL condition :*

$$\nu = r\left(\sup_{|x| \leq \|u_0\|_{L^\infty(\mathbb{R})}} |f_1'(x)| + \sup_{|x| \leq \|u_0\|_{L^\infty(\mathbb{R})}} |f_2'(x)|\right) \leq 2/(2+a) \tag{3.5}$$

is satisfied, then the scheme (3.3) *is TVD and*

$$\|u^{n+1}\|_\infty \leq \|u^n\|_\infty \quad \forall n \in \mathbb{N} . \tag{3.6}$$

Proof: From (3.4) and using $0 < a \leq 2$ we deduce :

$$\begin{aligned} C_{j-1/2}^n &\geq \left(1 - \frac{a}{2}\alpha_{1,j-1/2}^n\right) \nu_{1,j-1/2}^n \\ &\geq \frac{2-a}{2} \nu_{1,j-1/2}^n \\ &\geq 0 \end{aligned}$$

and

$$\begin{aligned} D_{j+1/2}^n &\geq -\left(1 - \frac{a}{2}\beta_{2,j+1/2}^n\right) \nu_{2,j+1/2}^n \\ &\geq -\left(\frac{2-a}{2}\right) \nu_{2,j+1/2}^n \\ &\geq 0 \end{aligned}$$

we now show :

$$C_{j+1/2}^n + D_{j+1/2}^n \leqslant 1 .$$

Using C and D given in (3.4) we get :

$$C_{j+1/2}^n + D_{j+1/2}^n = \nu_{1,j+1/2}^n - \nu_{2,j+1/2}^n + \frac{a}{2} (\beta_{1,j+1/2}^n - \alpha_{1,j+1/2}^n) \times$$
$$\times \nu_{1,j+1/2}^n + \frac{a}{2} (\beta_{2,j+1/2}^n - \alpha_{2,j+1/2}^n) \nu_{2,j+1/2}^n \leqslant \left(\frac{2+a}{2} \right) \nu_{j+1/2}^n$$

where

$$\nu_{j+1/2}^n = \nu_{1,j+1/2}^n - \nu_{2,j+1/2}^n .$$

The CFL condition (3.5) yields

$$C_{j+1/2}^n + D_{j+1/2}^n \leqslant 1$$

to show (3.6), it suffices to prove that under (3.5) we get

$$C_{j-1/2}^n + D_{j+1/2}^n \leqslant 1$$

the use of expressions C and D gives

$$\begin{aligned} C_{j-1/2}^n + D_{j+1/2}^n &= \nu_{1,j-1/2}^n - \nu_{2,j+1/2}^n + \frac{a}{2} (\beta_{1,j-1/2}^n - \alpha_{1,j-1/2}^n) \nu_{1,j-1/2}^n \\ &\quad + \frac{a}{2} (\beta_{2,j+1/2}^n - \alpha_{2,j+1/2}^n) \nu_{2,j+1/2}^n \\ &\leqslant \left(\frac{2+a}{2} \right) \nu \\ &\leqslant 1 \end{aligned}$$

where condition (3.5) leads to

$$\nu \leqslant \frac{2}{2+a}$$

we thus obtain

$$\|u^{n+1}\|_\infty \leqslant \|u^n\|_\infty \quad \forall n \in \mathbb{N} .$$

We note that for $a = 1$; the CFL condition becomes :

$$\nu \leqslant \frac{2}{3}$$

and for $a = 2$; we have

$$\nu \leqslant \frac{1}{2} .$$

Using proposition 3.1 and applying Helly's theorem, we show that there exists a subsequence $\{u_m\}$ of (u_h) ; which converges towards a weak solution of (1.1)-(1.2).

To prove that the whole family (u_h) converges towards the entropy solution of (1.1)-(1.2), we slightly modify the definition of δ given in (2.14), as follows :

$$\delta_{i,j} = \begin{cases} \sigma \operatorname{Min} \left\{ |\hat{\delta}_{i,j}|, \frac{a}{h} |f_i(u_{j+1}) - f_i(u_j)|, \right. \\ \qquad \left. \frac{a}{h} |f_i(u_j) - f_i(u_{j-1})|, ch^{\alpha-1} \right\} \\ \text{if} \quad \sigma = \operatorname{sgn}(\hat{\delta}_{i,j}) = \operatorname{sgn}[f_i(u_{j+1}) - f_i(u_j)] = \\ \qquad = \operatorname{sgn}[f_i(u_j) - f_i(u_{j-1})] \\ 0 \text{ elsewhere} \end{cases} \tag{3.7}$$

where $i = 1, 2$; $c > 0$ and $\alpha \in]0, 1[$.

Before stating the convergence of the whole family (u_h) towards the entropy solution of the problem (1.1)-(1.2), we recall a result of VILA [10].

THEOREM 3.1 [10] : *Let us consider the following prediction-correction scheme :*

$$\bar{U}_j^{n+1} = u_j^n - r\,\Delta_+ f_{j-1/2}^n \tag{3.8}$$

$$u_j^{n+1} = \bar{U}_j^{n+1} - \Delta_+ \bar{a}_{j-1/2}^{n+1} . \tag{3.9}$$

If we suppose that :

(i) *the scheme* (3.8) *is consistent with the conservation law* (1.1) *and its associated entropy condition* [10].

(ii) $|\bar{a}_{j-1/2}^{n+1}| \leqslant e(h) - \forall j \in \mathbb{Z}$; $\lim e(h) = 0$ when h tends to 0.

Then if the approximate solution u_h *constructed by* (3.1) *is bounded in* $BV \cap L^\infty$; u_h *converges towards* u *in* L^1_{loc} *and* u *is the entropy solution of* (1.1)-(1.2).

THEOREM 3.2 : *Let* $u_0 \in L^\infty(\mathbb{R}) \cap BV_{\text{loc}}(\mathbb{R})$; *then under the CFL condition* (3.5), *the approximate solution* u_h *constructed by* (2.1)-(3.1)-(3.2)-(3.7) *converges towards the entropy solution of the problem* (1.1)-(1.2).

Proof: We write the scheme (3.1) in prediction-correction form as the following :

$$\bar{U}_j^{n+1} = u_j^n - r[(f_1(u_j^n) - f_1(u_{j-1}^n)) + (f_2(u_{j+1}^n) - f_2(u_j^n))] \tag{3.10}$$

$$u_j^{n+1} = \bar{U}_j^{n+1} - \Delta_+ \bar{a}_{j-1/2}^n \tag{3.11}$$

with

$$\bar{a}_{j-1/2}^{n+1} = \frac{rh}{2}\,(\delta_{1,j-1}^{n} - \delta_{2,j}^{n})$$

(3.10) is a monotone scheme, then it is consistent with the associated entropy condition for the problem (1.1)-(1.2). Using the bounded variation in space of the discrete solution since the scheme (3.3)-(3.4) is TVD and from formula (3.7) we have :

$$|\bar{a}_{j-1/2}^{n+1}| \leqslant rch^{\alpha}\,; \quad c > 0, \quad 1 \geqslant \alpha > 0\,.$$

Then by theorem 3.1, the approximate solution u_h converges towards the entropy solution of the problem (1.1)-(1.2) as h tends to zero.

3.2. Second order accuracy in space and in time

By construction the scheme (3.1)-(3.2) is of quasi-order two in space and of first order in time. Now, we propose a scheme of quasi-order two in space and in time. For that we introduce the numerical flux G at the time step $(n + 1/2)$, that is :

$$G_{j-1/2}^{n+1/2} = G_{j-1/2}^{n} - \frac{k}{2}\,[f_1'(u_{j-1}^n)\,\delta_{1,j-1}^{n} + f_2'(u_j^n)\,\delta_{2,j}^{n}] \qquad (3.12)$$

in incremental form, the new scheme will be :

$$u_j^{n+1} = u_j^n - C_{j-1/2}^n\,\Delta_+ u_{j-1}^n + D_{j+1/2}^n\,\Delta_+ u_j^n \qquad (3.13)$$

with :

$$\begin{cases} C_{j-1/2}^n = \nu_{1,j-1/2}^n\left[1 + \frac{1}{2}\,\beta_{1,j-1/2}^n(1 - \nu_{1,j}^n) - \frac{1}{2}\,\alpha_{1,j-1/2}^n(1 - \nu_{1,j-1}^n)\right] \\ D_{j+1/2}^n = -\,\nu_{2,j}^n\left[1 - \frac{1}{2}\,\beta_{2,j+1/2}^n(1 - \nu_{2,j+1}^n) + \frac{1}{2}\,\alpha_{2,j+1/2}^n(1 - \nu_{2j}^n)\right] \end{cases} \qquad (3.14)$$

α and β are given by (2.16)

PROPOSITION 3.2 : *Under the CFL condition :*

$$\nu = \mathrm{r}[\sup\,|f_1'(x)| + \sup\,|f_2'(x)|\,] \leqslant$$
$$\leqslant \frac{1}{a}\left[-\left(1 + \frac{a}{2}\right) + \sqrt{\left(1 + \frac{a}{2}\right)^2 + 2\,a}\,\right] \qquad (3.15)$$

the sup is taken over the set $\{x\,;\,|x| \leqslant \|u_0\|_{\infty}\}$ *the scheme* (3.13)-(3.14) *is TVD and*

$$\|u^{n+1}\|_{L^\infty} \leqslant \|u^n\|_{L^\infty} \quad \forall n \in \mathbb{N}\,. \qquad (3.16)$$

Proof: From (3.14) we have

$$C^n_{j-1/2} \geqslant \nu_{1,j-1/2}\left(1 - \frac{a}{2}(1+\nu)\right)$$

$$C^n_{j-1/2} \geqslant 0 \quad \text{for} \quad \nu \leqslant \frac{2}{a} - 1$$

therefore using (3.15) we get

$$C^n_{j-1/2} \geqslant 0$$

similarly we obtain

$$D^n_{j+1/2} \geqslant 0$$

to prove the TVD character, we have to show :

$$C^n_{j+1/2} + D^n_{j+1/2} \leqslant 1$$

using (3.14) we get

$$C^n_{j+1/2} + D^n_{j+1/2} \leqslant \nu + \frac{a}{2}\nu + \frac{a}{2}(\nu^2_{1,j+1/2} + \nu^2_{2,j+1/2})$$

$$\leqslant \left(1 + \frac{a}{2}\right)\nu + \frac{a}{2}\nu^2$$

the condition (3.15) then gives :

$$C^n_{j+1/2} + D^n_{j+1/2} \leqslant 1$$

hence the scheme (3.13)-(3.14) is TVD.

Similarly we can prove that if (3.15) holds, then we have :

$$C^n_{j-1/2} + D^n_{j+1/2} \leqslant 1$$

hence

$$\|u^{n+1}\|_\infty \leqslant \|u^n\|_\infty \quad \forall n \in \mathbb{N}\,.$$

Remark 3.1 : In this case the constant a must satisfy $2 > a$ and for $a = 1$ the CFL condition becomes :

$$\nu \leqslant \frac{\sqrt{17} - 3}{2}$$

using proposition 3.2 and with the same step in the proof of theorem 3.2, we prove :

THEOREM 3.3 : *Let* $u_0 \in L^{\infty}(\mathbb{R}) \cap BV_{\mathrm{loc}}(\mathbb{R})$, *if the CFL condition* (3.15) *holds, then the approximate family* $\{u_h\}$ *obtained by the scheme* (2.1)-(3.7)-(3.13)-(3.14) *converges towards the entropy solution of the problem* (1.1)-(1.2) *as h tends to zero.*

4. APPROXIMATION OF (1.1)-(1.2) BY SECOND ORDER IMPLICIT SCHEMES

Next we consider the approximation of the problem (1.1)-(1.2) by the following implicit scheme

$$u_j^{n+1} = u_j^n - r\,\Delta_+ G_{j-1/2}^{n+1}\,. \tag{4.1}$$

4.1. Second order accuracy in space

In the case of second order accuracy in space and first order in time the numerical flux is given by :

$$G_{j-1/2}^{n+1} = g_{j-1/2}^{n+1} + \frac{h}{2}\,\delta_{1,j-1}^{n+1} - \frac{h}{2}\,\delta_{2,j}^{n+1} \tag{4.2}$$

as in § 3, g is the upwind flux and δ is given by (2.14).

The scheme (4.2) can be written :

$$u_j^{n+1} = u_j^n - C_{j-1/2}^{n+1}\,\Delta_+ u_{j-1}^{n+1} + D_{j+1/2}^{n+1}\,\Delta_+ u_j^{n+1} \tag{4.3}$$

with

$$\begin{cases} C_{j-1/2}^{n+1} = \left(1 + \dfrac{1}{2}\,\beta_{1,j-1/2}^{n+1} - \dfrac{1}{2}\,\alpha_{1,j-1/2}^{n+1}\right) \dfrac{r\,\Delta_+ f_1(u_{j-1}^{n+1})}{\Delta_+ u_{j-1}^{n+1}} \\ D_{j+1/2}^{n+1} = \left(1 - \dfrac{1}{2}\,\beta_{2,j+1/2}^{n+1} + \dfrac{1}{2}\,\alpha_{2,j+1/2}^{n+1}\right) \dfrac{r\,\Delta_+ f_2(u_j^{n+1})}{\Delta_+ u_j^{n+1}} \end{cases} \tag{4.4}$$

and (α, β) given (2.16) (at the time step $(n+1)$).

Let us set

$$\begin{cases} \nu_{1,j-1/2} = \dfrac{r\,\Delta_+ f_1(u_{j-1}^{n+1})}{\Delta_+ u_{j-1}^{n+1}} \\ \nu_{2,j+1/2} = \dfrac{r\,\Delta_+ f_2(u_j^{n+1})}{\Delta_+ u_j^{n+1}} \end{cases}$$

since f_1 is an increasing function and f_2 is a decreasing function then

$$\nu_{1,j-1/2} \geqslant 0 \quad \text{and} \quad \nu_{2,j+1/2} \leqslant 0\,.$$

PROPOSITION 4.1 : *The implicit scheme* (4.3)-(4.4) *is unconditionally* TVD *and*

$$\|u^{n+1}\|_\infty \leqslant \|u^n\|_\infty \quad \forall n \in \mathbb{N} .$$

Proof: According to lemma 3.2 [3], to prove the TVD character (4.3)-(4.4) it suffices to show that there exists $c > 0$ such that :

$$c \geqslant C_{j-1/2}^{n+1} \geqslant 0 \quad \text{and} \quad c \geqslant D_{j+1/2}^{n+1} \geqslant 0$$

from (4.4) we have :

$$\begin{aligned} C_{j-1/2}^{n+1} &= \left(1 + \frac{1}{2}\beta_{1,j-1/2}^{n+1} - \frac{1}{2}\alpha_{1,j-1/2}^{n+1}\right)\nu_{1,j-1/2} \\ &\leqslant \left(1 + \frac{1}{2}\beta_{1,j-1/2}^{n+1}\right)\nu_{1,j-1/2} \\ &\leqslant \frac{3}{2}\nu \end{aligned}$$

$$\begin{aligned} D_{j-1/2}^{n+1} &= -\left(1 - \frac{1}{2}\beta_{2,j+1/2}^{n+1} + \frac{1}{2}\alpha_{2,j+1/2}^{n+1}\right)\nu_{2,j+1/2} \\ &\leqslant -\left(1 + \frac{1}{2}\alpha_{2,j+1/2}^{n+1}\right)\nu_{2,j+1/2} \\ &\leqslant \frac{3}{2}\nu \end{aligned}$$

hence there exists $c = 3/2\,\nu$ which satisfies the required condition. From the third section we have

$$C_{j-1/2}^{n+1} \geqslant 0 \quad \text{and} \quad D_{j+1/2}^{n+1} \geqslant 0$$

then the scheme (4.3)-(4.4) is TVD.

(4.3) gives :

$$u_j^n = (1 + C_{j-1/2}^{n+1} + D_{j+1/2}^{n+1})\,u_j^{n+1} - C_{j-1/2}^{n+1}\,u_{j-1}^{n+1} - D_{j+1/2}^{n+1}\,u_{j+1}^{n+1}$$

thus

$$|u_j^n| \geqslant (1 + C_{j-1/2}^{n+1} + D_{j+1/2}^{n+1})\,|u_j^{n+1}| - C_{j-1/2}^{n+1}\,|u_{j-1}^{n+1}| - D_{j+1/2}^{n+1}\,|u_{j+1}^{n+1}|$$

hence

$$\|u^{n+1}\|_\infty \leqslant \|u^n\|_\infty .$$

To prove the convergence towards the entropy solution we can use similar technics as in § 2 (use of Ch^α), however when the flux f is convex we can make use of OSHER technics for the time continuous schemes, to get an entropy inequality, that is

$$\int_{u_j}^{u_{j+1}} \eta''(w)[G_{j+1/2} - f(w)]\, dw \leq 0$$

with η an entropy function and G the numerical flux given in (4.2). In the last inequality we dropped the superscript $n+1$ on u and G.

In practice it suffices to get this inequality for :

$$\eta(w) = \frac{w^2}{2}$$

then the entropy inequality becomes :

$$\int_{u_j}^{u_{j+1}} [G_{j+1/2} - f(w)]\, dw \leq 0 .$$

LEMMA 4.1 : *Let*

$$I = \int_{u_j}^{u_{j+1}} [G_{j+1/2} - f(w)]\, dw \tag{4.5}$$

and

$$f'_{i,j+1/2} = \frac{\Delta_+ f_i(u_j)}{\Delta_+ u_j}\ ; \quad i = 1, 2$$

then there existes $u_{j+1/2}$ between u_j and u_{j+1} such that :

$$I = \frac{1}{12} f''(u_{j+1/2})(\Delta_+ u_j)^3 - \frac{1}{2}(\Delta_+ u_j)^2 \times \\ \times \left\{(1 - \alpha_{1,j+1/2})\, f'_{1,j+1/2} - (1 - \beta_{2,j+1/2})\, f'_{2,j+1/2}\right\} . \tag{4.6}$$

Proof: By integration by parts of (4.5) we get

$$I = (\Delta_+ u_j)\left(G_{j+1/2} - \frac{1}{2}(f(u_j) + f(u_{j+1}))\right) + \\ + \frac{1}{2}\int_{u_j}^{u_{j+1}} f''(w)\left\{\left(\frac{1}{2}\Delta_+ u_j\right)^2 - \left(w - \frac{1}{2}(u_j + u_{j+1})\right)^2\right\} dw$$

since for all w between u_i and u_{i+1} we have

$$\left(\frac{1}{2}\Delta_+ u_j\right)^2 - \left(w - \frac{1}{2}(u_j + u_{j+1})\right)^2 \geq 0$$

then by the mean value theorem there existes $u_{j+1/2}$ between u_j and u_{j+1} such that

$$\int_{u_j}^{u_{j+1}} f''(w)\left\{\left(\frac{1}{2}\Delta_+ u_j\right)^2 - \left(w - \frac{1}{2}(u_j + u_{j+1})\right)^2\right\} dw =$$

$$= f''(u_{j+1/2}) \int_{u_j}^{u_{j+1}} \left\{\left(\frac{1}{2}\Delta_+ u_j\right)^2 - \left(w - \frac{1}{2}(u_j + u_{j+1})\right)^2\right\} dw$$

$$= \frac{1}{6} f''(u_{j+1/2})(\Delta_+ u_j)^3$$

and since

$$G_{j+1/2} = f_1(u_j) + f_2(u_{j+1}) + \frac{h}{2}\delta_{1,j} - \frac{h}{2}\delta_{2,j+1}$$

$$= f_1(u_j) + f_2(u_{j+1}) + \frac{1}{2}\alpha_{1,j+1/2}[f_1(u_{j+1}) - f_1(u_j)]$$

$$- \frac{1}{2}\beta_{2,j+1/2}[f_2(u_{j+1}) - f_2(u_j)]$$

hence

$$I = (\Delta_+ u_j)\left\{\left[f_1(u_j) - \frac{1}{2}(f_1(u_j) + f_1(u_{j+1}))\right]\right.$$

$$\left. + \left[f_2(u_{j+1}) - \frac{1}{2}(f_2(u_j) + f_2(u_{j+1}))\right]\right\}$$

$$+ (\Delta_+ u_j)^2\left(\frac{1}{2}\alpha_{1,j+1/2} f'_{1,j+1/2} - \frac{1}{2}\beta_{2,j+1/2} f'_{2,j+1/2}\right)$$

$$+ \frac{1}{12} f''(u_{j+1/2})(\Delta_+ u_j)^3$$

then

$$I = \frac{1}{12} f''(u_{j+1/2})(\Delta_+ u_j)^3 - \frac{1}{2}(\Delta_+ u_j)^2 \times$$

$$\times [(1 - \alpha_{1,j+1/2}) f'_{1,j+1/2} - (1 - \beta_{2,j+1/2}) f'_{2,j+1/2}] .$$

PROPOSITION 4.2 : *Suppose that f is convex and f' is bounded, if we put*

$$\gamma = \sup_{\|x\| \leqslant \|u_0\|_\infty} f''(x)$$

then the scheme (4.3)-(4.4) *defined by*

$$\alpha_{1,j+1/2} = \mathrm{Max}\left\{0, \mathrm{Min}\left\{1, \frac{\Delta_+ f_1(u_{j-1})}{(\Delta_+ f_1(u_j))}, 1 - \frac{\gamma}{f'_{1,j+1/2}}\Delta_+ u_j\right\}\right\} \quad (4.7)$$

$$\beta_{2,j+1/2} = \mathrm{Max}\left\{0, \mathrm{Min}\left\{1, \frac{\Delta_+ f_2(u_{j+1})}{(\Delta_+ f_2(u_j))}, 1 + \frac{\gamma}{f'_{2,j+1/2}}\Delta_+ u_j\right\}\right\} \quad (4.8)$$

satisfies the inequality

$$\int_{u_j}^{u_{j+1}} [G_{j+1/2} - f(w)]\, dw \leqslant 0 . \tag{4.9}$$

Proof: It is clear from (4.7) that if $\Delta_+ u_j \leqslant 0$ then $I \leqslant 0$. In the following let us consider the case where $\Delta_+ u_j > 0$. We will distinguish two cases :

a) if $\alpha_{1,j+1/2} = \beta_{2,j+1/2} = 0$, then we can check that

$$I = \int_{u_j}^{u_{j+1}} [f_1(u_j) + f_2(u_{j+1}) - f(w)]\, dw \leqslant 0 .$$

b) if $\alpha_{1,j+1/2} > 0$ we have $\beta_{2,j+1/2} > 0$, we have

$$\begin{aligned} I &\leqslant \frac{1}{12} f''(u_{j+1/2})(\Delta_+ u_j)^3 - \frac{1}{2} \gamma (\Delta_+ u_j)^3 \\ &\leqslant \frac{1}{2} (\Delta_+ u_j)^3 \left[\frac{1}{6} f''(u_{j+1/2}) - \gamma \right] \end{aligned}$$

from the definition of γ we get $I \leqslant 0$.

Such a scheme is TVD and satisfies the entropy condition. It is second order accurate in smooth regions outside of $f_i' = 0$, $i = 1, 2$.

Remark 4.1 : The scheme may be defined by giving α_1 and α_2, β_1 and β_2, α_1 and β_2 or β_1 and α_2.

From the second order accuracy preserving property stated in section 2, we observe that the choice of α in (4.7) and (4.8) preserves the second order accuracy of the used schemes.

We will make use of the following result due to VILA.

THEOREM 4.1 [10] : *If the numerical flux G is consistent with the conservative form of* (1.1) *and satisfies the condition*

$$\int_{u_j}^{u_{j+1}} \eta''(w)[G_{j+1/2} - f(w)]\, dw \leqslant 0$$

where η is an entropy function.

Then the limit solution u of the implicit scheme :

$$u_j^{n+1} = u_j^n - r\, \Delta_+ G_{j-1/2}^{n+1}$$

is the entropy solution of (1.1)-(1.2).

Now we state a convergence result for our implicit scheme.

THEOREM 4.2 : *Let $u_0 \in L^\infty(\mathbb{R}) \cap BV_{\mathrm{loc}}(\mathbb{R})$; then under the hypothesis of proposition* 4.2, *the approximate family $\{u_h\}$ obtained by the scheme*

(4.3)-(4.4) *converges towards the entropy solution of* (1.1)-(1.2) *as* h *tends to zero*.

Proof: The convergence of the family $\{u_h\}$ towards a weak solution of (1.1)-(1.2) is insured by proposition 4.1, and the convergence towards the entropy solution of (1.1)-(1.2) is a consequence of proposition 4.2 and theorem 4.1.

4.2. Second order accuracy in space and in time :

The new scheme will take the form :

$$u_j^{n+1} = u_j^n - r\,\Delta_+ G_{j-1/2}^{n+1/2} \tag{4.10}$$

with

$$G_{j-1/2}^{n+1/2} = G_{j-1/2}^{n+1} + \frac{1}{2} k[f_1'(u_{j-1}^{n+1})\,\delta_{1,j-1}^{n+1} + f_2'(u_j^{n+1})\,\delta_{2,j}^{n+1}] \tag{4.11}$$

this scheme can be written as

$$u_j^{n+1} = u_j^n - C_{j-1/2}^{n+1}\Delta_+ u_{j-1}^{n+1} + D_{j+1/2}^{n+1}\Delta_+ u_j^{n+1} \tag{4.12}$$

with

$$\begin{cases} C_{j-1/2}^{n+1} = \nu_{1,j-1/2}\left[1 + \frac{1}{2}\beta_{1,j-1/2}(1-\nu_{1,j}) - \frac{1}{2}\alpha_{1,j-1/2}(1-\nu_{1,j-1})\right] \\ D_{j+1/2}^{n+1} = -\nu_{2,j+\frac{1}{2}}\left[1 - \frac{1}{2}\beta_{2,j+1/2}(1-\nu_{2,j+1}) + \frac{1}{2}\alpha_{2,j+1/2}(1-\nu_{2,j})\right] \end{cases} \tag{4.13}$$

in (4.13) we omitted the subscript $(n+1)$ for α, β and ν.

$$\nu_{1,j} = f_1'(u_j^{n+1}), \quad \nu_{2,j} = f_2'(u_j^{n+1})$$

the scheme (4.12)-(4.13) is TVD if the coefficients C and D given in (4.13), are positive. To have

$$C_{j-1/2}^{n+1} \geq 0$$

it suffices that

$$1 \geq \frac{1}{2}(\alpha_{1,j-1/2} - \beta_{1,j-1/2})(1-\nu_{1,j}) + \frac{1}{2}\beta_{1,j-1/2}(\nu_{1,j} - \nu_{j-1})$$

this condition holds, if we choose α and β so that

$$\begin{cases} |\beta_{1,j-1/2}(\nu_{1,j} - \nu_{1,j-1})| \leq 1 \\ |(\alpha_{1,j-1/2} - \beta_{1,j-1/2})(1-\nu_{1,j-1})| \leq 1 \end{cases} \tag{4.14}$$

similar conditions can be obtained to get :

$$D^{n+1}_{j+1/2} \geqslant 0$$

we can proove by Taylor expansion the following :

PROPOSITION 4.3 : *If* $\alpha = 1 + O(h)$ *and* $\beta = 1 + O(h)$ *the condition* (4.14) *holds in the smoothness regions and the second order is preserved too.*

To prove the entropy character of the scheme (4.10)-(4.11), we will write it in the following form :

$$u^{n+1}_j = u^n_j - r\,\Delta_+ g^{n+1}_{j-1/2} + \Delta_+ a^{n+1}_{j-1/2} \tag{4.15}$$

with

$$g^{n+1}_{j-1/2} = f_1(u^{n+1}_{j-1}) + f_2(u^{n+1}_j)$$

and

$$a^{n+1}_{j-1/2} = -\frac{1}{2} rk[f_1'(u^{n+1}_{j-1})\,\delta^{n+1}_{1,j-1} + f_2'(u^{n+1}_j)\,\delta^{n+1}_{2,j}]$$

g is a numerical flux of an E-scheme ; it is then proved [7] that for all entropy flux F associated with the entropy η, we have :

$$-\eta'(u_j)\,\Delta_+ g_{j-1/2} + \Delta_+ H(u_j) = \\ = \int_{u_j}^{u_{j+1}} \eta''(w)[g_{j+1/2} - f(w)]\,dw \leqslant 0 \tag{4.16}$$

with

$$H(u_j) = F(u_j) + \eta'(u_j)[g_{j-1/2} - f(u_j)]$$

using (4.15) we write

$$\eta'(u^{n+1}_j)(u^{n+1}_j - u^n_j) = -r\eta'(u^{n+1}_j)\,\Delta_+ g^{n+1}_{j-1/2} + \Delta_+ a^{n+1}_{j-1/2}$$

since η is convex we have

$$\eta(u^{n+1}_j) - \eta(u^n_j) \leqslant \eta'(u^{n+1}_j)(u^{n+1}_j - u^n_j)$$

then

$$\eta(u^{n+1}_j) - \eta(u^n_j) + r\,\Delta_+ H(u^{n+1}_j) = \\ = \int_{u_j}^{u_{j+1}} \eta''(w)[g_{j+1/2} - f(w)]\,dw + \eta'(u^{n+1}_j)\,\Delta_+ a^{n+1}_{j-1/2}$$

thus

$$\eta(u_j^{n+1}) - \eta(u_j^n) + r\,\Delta_+ H(u_j^{n+1}) \leq \eta'(u_j^{n+1})\,\Delta_+ a_{j-1/2}^{n+1} \qquad (4.17)$$

if we add in the definition of δ the argument ch^α, we prove that the right side of (4.17) converges towards zero as h tends to zero. Then the solution given by the scheme (4.12)-(4.13) satisfies the entropy condition at the limit.

We thus have the following :

THEOREM 4.3 : *Let* $u_0 \in L^\infty(\mathbb{R}) \cap BV_{\text{loc}}(\mathbb{R})$, *then under the hypothesis of proposition* 4.2, *the family* $\{u_h\}$ *given by the implicit scheme* (4.12)-(4.13) *converges towards the entropy solution of the problem* (1.1)-(1.2) *as h tends to zero.*

Inversion property of the regular part of the nonlinear iteration : If the flux f is an increasing function, then the implicit scheme (4.12)-(4.13) may be written :

$$\psi(u)_j^{n+1} + \phi(u)_j^{n,n+1} = 0$$

where

$$\psi(u)_j^{n+1} = u_j^{n+1} + r[f(u_j^{n+1}) - f(u_{j-1}^{n+1})]$$

$$\phi(u)_j^{n,n+1} = \frac{rh}{2}\Delta_+ \delta_{j-1}^{n+1} + u_j^n .$$

PROPOSITION 4.3 : ψ *is inversible.*

Proof: It suffices that there exists $C > 0$ such that :

$$\|\psi'(u)\,w\|_{l^\infty} \geq C\,\|w\|_{l^\infty} \quad \forall w \in l^\infty$$

we have

$$\psi'(u)\,w = w_j + rf'(u_j)\,w_j - rf'(u_{j-1})\,w_{j-1}$$

let

$$\psi'(u)\,w = b$$

with

$$b_j = w_j(1 + a_j) - a_{j-1}\,w_{j-1}$$

and $a_j = rf'(u_j) > 0$; a_j is bounded since $\|u_j\|_\infty \leq$ constant thus

$$b_j + a_{j-1}\,w_{j-1} = w_j(1 + a_j)$$

hence

$$\sum_j b_j^2\left(1 + \frac{1}{\varepsilon}\right) + \sum_j a_j^2\,w_j^2(1 + \varepsilon) \geq \sum_j (1 + a_j)^2\,w_j^2 ; \quad \forall \varepsilon > 0 .$$

If

$$(1+\varepsilon)\sup \frac{a_j^2}{(1+a_j^2)} < 1$$

we get

$$\sum_j b_j^2 \geqslant \left(\frac{\varepsilon}{\varepsilon+1}\right)\sum_j w_j^2 \quad \text{thus} \quad \|\psi'(u)\,w\|_{l^\infty} \geqslant \left(\frac{\varepsilon}{\varepsilon+1}\right)\|w\|_{l^\infty}$$

then, by the local inversion theorem we obtain that ψ is inversible and from this property we deduce the existence and the uniqueness of the solution of the nonlinear iteration

$$\psi(u^{N+1}) + \phi(u^N) = 0 \quad N \in \mathbb{N}$$

5. NUMERICAL EXPERIMENTS

We compute the solution to the Cauchy problem, associated with the Buckeley-Leverett equation with a Riemann data

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}\left[\dfrac{4\,u^2}{4\,u^2+(1-u)^2}\right] = 0 \\ u(x,0) = \begin{cases} 1 & \text{if} \quad 3\,h \geqslant x \geqslant 0 \\ 0 & \text{if} \quad 1 \geqslant x > 3\,h \end{cases} \end{cases} \tag{5 1}$$

where h is the spatial step The exact solution contains a shoc and a rarefaction part and it is given by

$$u(x,t) = \begin{cases} 1 & \text{if} \quad x \leqslant 3\,h \\ v(\xi) & \text{if} \quad 3\,h \leqslant x \leqslant f'(u_*)\,t + 3\,h \\ 0 & \text{if} \quad x > f'(u_*)\,t + 3\,h \end{cases} \tag{5 2}$$

with

$$\xi = \frac{x-3\,h}{t}, \quad \text{and } v \text{ satisfies} \quad f'(v(\xi)) = \xi\,, \quad u_* = \frac{\sqrt{5}}{5}$$

in this example $f' = f_1'$, since $f' \geqslant 0$

1) We compute the solution at the time $t = 0\,24$, using the explicit schemes (3 1)-(3 2) and (3 1)-(3 12), we compare this computed solution with the one given by the first order scheme and the one given by the exact solution (*fig* 5 1) We observe that in this explicit second order accuracy in space, the first order and the second order accuracy in time give the same solution

2) The computed solution at the time $t = 0\,24$, by the implicit schemes (4 1)-(4 2) and (4 1)-(4 11) is presented in figure 5 2, the non linearity is

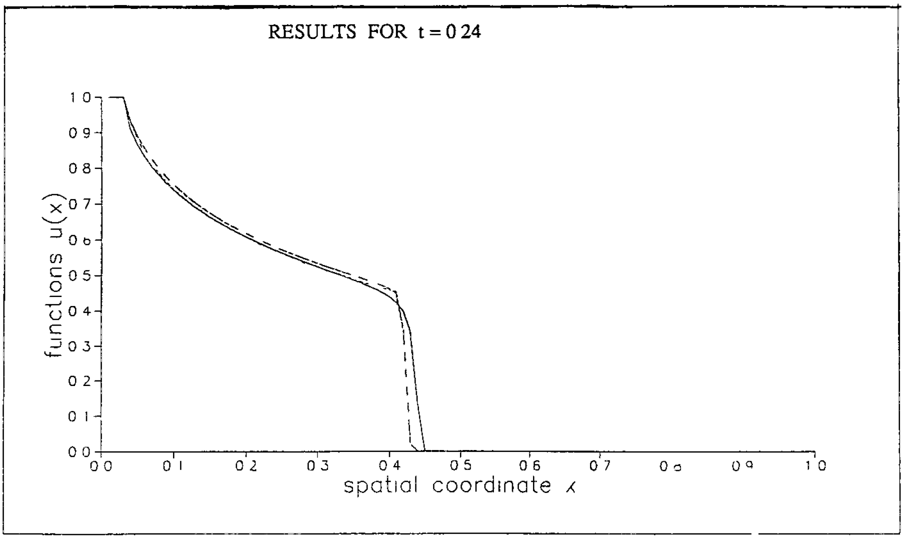

Figure 5.1. — Explicit schemes. *DT* = 0.03, C.F.L. number = 0.6 ; exact solution. / computed solution by first order scheme, / second order in space first order in time, / second order in space and in time.

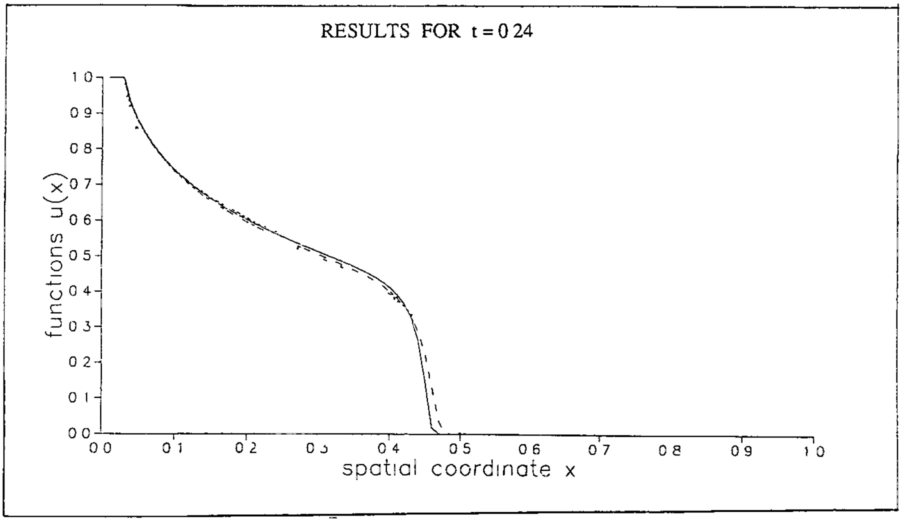

Figure 5.2. — Implicit schemes. *DT* = 0.008, C.F.L. number = 1.6 ; / exact solution, / computed solution by first order scheme, / second order in space first order in time, / second order in space and in time.

treated using the following iterative technic we apply Newton method to the operator ψ to get the iteration

$$u^{N+1} = u^N + \psi'^{-1}(u^N)\left[u^n + \frac{rh}{2}\Delta_+ \delta^N - \psi(u^N)\right] \qquad (5\ 3)$$

this iteration is made inside every step of time, u^n denotes the value of u at the time $n\,\Delta t$, u^N denotes the value of u at the N^{th} iteration , we recall that

$$\psi(u)_j^{n+1} = u_j^{n+1} + r[f(u_j^{n+1}) - f(u_{j-1}^{n+1})]$$

In this implicit scheme there is a difference between the first order and the second order accuracy in time

ACKNOWLEDGMENT

The authors are grateful to professor Thierry Gallouet for proposing the subject of this study and for a number of stimulating discussions

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