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**NON-HOMOGENEOUS NEUMANN PROBLEMS  
IN DOMAINS WITH SMALL HOLES (\*)**

by C. CONCA <sup>(1)</sup> and P. DONATO <sup>(2)</sup>

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Résumé. — Dans cet article on étudie le comportement limite des solutions de problèmes de Neumann non homogènes dans des ouverts finement perforés. Plus précisément, on considère, pour tout  $\varepsilon > 0$  ( $\varepsilon \rightarrow 0$ ), l'ouvert  $\Omega_\varepsilon$  obtenu en retirant d'un ouvert borné fixe  $\Omega$  un ensemble  $T_\varepsilon$  de trous, distribués périodiquement, de périodicité  $\varepsilon$ , chacun de taille  $r(\varepsilon)$  et on suppose que la taille des trous est petite par rapport à la période ( $r(\varepsilon)/\varepsilon \rightarrow 0$ ). On étudie le comportement asymptotique des solutions  $u_\varepsilon$  de l'équation  $-\Delta u_\varepsilon = f$  dans  $\Omega_\varepsilon$ , avec des conditions de Neumann non homogènes sur le bord des trous.

Des estimations a priori détaillées, exprimées en fonction des paramètres  $\varepsilon$  et  $r(\varepsilon)$ , donnent l'ordre de grandeur exact de la norme  $H^1$  des solutions. Cet ordre de grandeur est différent suivant que la donnée de Neumann est ou non à moyenne nulle sur le bord des trous. On montre que, après normalisation, les solutions convergent vers la solution d'un problème limite que l'on caractérise explicitement. Pour certaines tailles des trous, un terme constant apparaît au deuxième membre de l'équation limite. Pour les autres tailles on a convergence vers zéro des solutions. On présente également des résultats concernant les correcteurs pour ce type de problèmes.

Abstract. — The limit behaviour of the solutions of non-homogeneous Neumann problems in open domains with small holes is studied. More precisely, for each  $\varepsilon > 0$  ( $\varepsilon \rightarrow 0$ ), an open domain  $\Omega_\varepsilon$  is obtained by removing from a given open set  $\Omega$  a set  $T_\varepsilon$  of periodically distributed holes, with period  $\varepsilon$ . The size of each hole is  $r(\varepsilon)$  and it is assumed to be smaller than the period (i.e.  $r(\varepsilon)/\varepsilon \rightarrow 0$ ). The asymptotic behaviour of the solutions  $u_\varepsilon$  of the equation  $-\Delta u_\varepsilon = f$  in  $\Omega_\varepsilon$ , with a non-homogeneous Neumann boundary condition on the boundary of the holes is studied.

Sharp a priori estimates, expressed in terms of the parameters  $\varepsilon$  and  $r(\varepsilon)$ , provide the exact order of magnitude of the  $H^1$ -norm of the solutions. This order of magnitude changes depending if the mean value of the Neumann data on the boundary of the holes is either zero or not. After normalisation the solution are proved to converge to the solution of a boundary value problem which is explicitly given. For some sizes of the holes, a constant right-hand side term appears in the limit problem. In the other cases, the solutions converge to zero. Some results concerning corrector terms for this kind of problems are also presented.

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## INTRODUCTION

In this paper we study an elliptic boundary-value problem in a perforated domain of  $\mathbb{R}^N$ . The region where the differential problem is formulated consists of a (fixed) bounded subset of  $\mathbb{R}^N$  in which perforations (or holes) are made. The holes are assumed to be identical and periodically distributed in the domain (see *fig.* 1.2). Let  $\varepsilon$  be a small parameter representing the distance between two adjacent holes, and let  $r(\varepsilon)$  denote the size of each hole. Assuming that  $(r(\varepsilon)/\varepsilon)$  tends to zero, as  $\varepsilon \rightarrow 0$ , our goal in this paper is to study the asymptotic behaviour, as  $\varepsilon \rightarrow 0$ , of the solution of Poisson equation in this domain, with a non-homogeneous Neumann boundary condition on the boundary of the holes, and with a homogeneous Dirichlet condition on the external boundary of the domain.

The results concerning the limit behaviour of the solution of this problem depend on the behaviour (as  $\varepsilon \rightarrow 0$ ) of the size  $r(\varepsilon)$  of the holes. For our study, we shall decompose the solution of the problem into three components. The first one is the solution of Laplace equation with a non-homogeneous constant Neumann boundary condition on the holes. The second component is also the solution of Laplace equation, but with a non-homogeneous Neumann boundary condition with zero mean-value on the boundary of each hole. Finally, the third component corresponds to the solution of Poisson equation with a homogeneous Neumann boundary condition on the holes. Our study of the problem consists in investigating the asymptotic behaviour of each one of these components, separately, and in collecting together the results. As it will be seen, the first of these components plays a leading role with respect to the other ones. Therefore, in this introduction we shall limit ourselves to describe the results concerning this case : Laplace equation with a non-homogeneous constant Neumann boundary condition.

The first result is that there exists a « critical size » of the holes that separates different limit behaviours of the solution, as  $\varepsilon \rightarrow 0$ . We derive this property by obtaining accurate upper and lower bounds of the  $H^1$ -norm of the solution. These a priori estimates depend on the two small parameters of the problem, i.e., the period  $\varepsilon$  and the size  $r(\varepsilon)$  of the holes. The solution remains bounded in  $H^1$  for the *critical size*, as  $\varepsilon \rightarrow 0$ . If the size of the holes is *lower* than this critical size, then the solution converges strongly to zero in  $H^1$ . It diverges in  $H^1$  if the size of the holes is *bigger* than the critical one. This critical size is  $r(\varepsilon) \sim \varepsilon^{N/(N-1)}$ . It is the size of the holes for which the  $\mathbb{R}^{N-1}$ -Lebesgue measure of the boundaries of the holes remains bounded (from below and from above) by strictly positive constants. Note that in this case, the total flux on the boundary of the holes (i.e., the integral of the constant Neumann boundary data) remains bounded (from below and from

above), as  $\varepsilon$  tends to zero. The proof of the upper bounds of the  $H^1$ -norm of the solution is based on an accurate estimate of the  $L^2$ -norm of the trace of the solution (on the boundaries of the holes) in terms of its  $H^1$ -norm. On the other hand, the lower bounds are obtained by using suitable test functions in the variational formulation of the problem.

The upper and the lower bounds we obtain for the  $H^1$ -norm of the solution are exactly of the same order. Therefore, the a priori estimates suggest to study the asymptotic behaviour of the solution after renormalization by the order of its  $H^1$ -norm. Passing to the limit we show that there exists a «second critical size». If the size  $r(\varepsilon)$  of the holes is *bigger* (or *equal*) than this second critical size, then the renormalized solution has a weak-limit in  $H^1$ , which is characterized as the (unique) solution of an elliptic boundary-value problem in the whole domain. This limit problem consists in Poisson equation with a constant non-zero right-hand side, and with a homogeneous Dirichlet boundary condition. The non-homogeneous right-hand side of the limit equation is obtained as a weak-limit of a sequence of Radon measures concentrated on the boundaries of the holes. On the other hand, if the size  $r(\varepsilon)$  of the holes is *lower* than the second critical size, then the renormalized solution weakly converges to zero, as  $\varepsilon \rightarrow 0$ . This means that under the second critical size, the non-homogeneous (constant) boundary condition on the boundaries of the holes can be completely neglected at the limit (even after renormalization of the solution). The second critical size is smaller than the first critical size. If  $N \geq 3$ , this size is  $r(\varepsilon) \sim \varepsilon^{N/(N-2)}$ , and if  $N = 2$ , the size  $r(\varepsilon)$  is such that the sequence  $\varepsilon^{-2}(\log(\varepsilon/r(\varepsilon)))^{-1}$  has a strictly positive limit, as  $\varepsilon$  goes to zero. It is interesting to remark that the second critical size coincides with the critical size that appears in the study of Poisson equation in a perforated domain with a homogeneous Dirichlet condition on the holes and on the external boundary of the domain (for a complete study of this problem we refer to D. Cioranescu & F. Murat [4]).

To obtain a more precise description of the (weak) convergences of the renormalized solution, we also present correcting terms for these convergences. In case of spherical holes, an explicit formula of the correctors is exhibited. The proofs of the results concerning the correcting terms are based on a general pattern developed by L. Tartar [11]. In case of spherical holes, we follow the same approach as in D. Cioranescu & F. Murat [4].

As it has been already mentioned, this paper is only concerned with the case where the size  $r(\varepsilon)$  of the holes verifies:  $(r(\varepsilon)/\varepsilon) \rightarrow 0$ , as  $\varepsilon \rightarrow 0$ . For the study of the problem in the periodic case (i.e., in case the size of the holes is of the same order than the distance  $\varepsilon$  between adjacent holes), we refer to D. Cioranescu & P. Donato [3].

Problems close to ours consist in studying Poisson equation (or a more general elliptic equation) in a perforated domain with a homogeneous

Neumann or Dirichlet boundary condition on the holes. They have been studied by several authors. Using homogenization techniques, in the periodic case, by D. Cioranescu & J. Saint Jean Paulin [5], using the  $\Gamma$ -convergence notion (introduced by E. De Giorgi & T. Franzoni [6]) by S. Mortola & A. Profeti [8]. For a general treatment of homogenization problems in the periodic case we refer to the books by A. Bensoussan, J. L. Lions & G. Papanicolaou [2], J. L. Lions [7], E. Sanchez-Palencia [10], and to L. Tartar [11]. When the size of the holes is very small compared to the distance between them, the Dirichlet problem is extensively studied in D. Cioranescu & F. Murat [4] by using the energy method. For the study of this problem, and several other homogenization problems in the framework of  $\Gamma$ -convergence (or epi-convergence) theory we refer to the book by H. Attouch [1], and the references therein.

#### ACKNOWLEDGMENTS

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Appendix A.

Appendix B.

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## 1. MAIN CONVERGENCE THEOREMS AND A PRIORI ESTIMATES

### 1.1. Formulation of the problem

Let  $T$  be an open bounded subset of  $\mathbb{R}^N$  ( $N \geq 2$ ) with a smooth boundary  $\partial T$ . We assume that  $0$  belongs to  $T$ , and that  $T$  is star-shaped with respect to  $0$ . Since  $T$  is bounded, we shall assume that  $T$  is strictly contained in a cube  $] -L, L[^N$  of  $\mathbb{R}^N$ ,  $L$  being a (strictly) positive real number (see *fig. 1.1*).

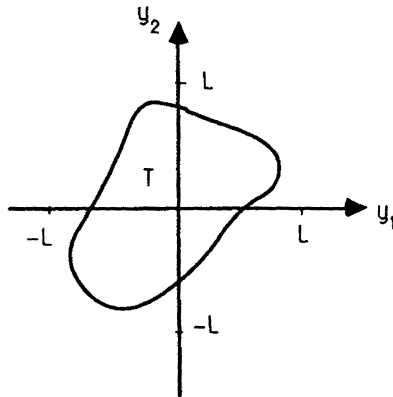


Figure 1.1. — The representative hole  $T$  ( $N = 2$ ).

Let  $\Omega$  be an open bounded subset of  $\mathbb{R}^N$  such that the  $\mathbb{R}^N$ -Lebesgue measure of its boundary  $\partial\Omega$  is zero, and let  $\varepsilon$  be a real parameter taking values in a sequence of (positive) numbers converging to zero. Besides, let  $r: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a continuous map verifying the following conditions :

$$(1.1a) \quad \lim_{s \rightarrow 0^+} r(s)/s = 0$$

$$(1.1b) \quad r(s) < (s/4L) \quad \forall s > 0.$$

For each  $\varepsilon$ , and for any integer vector  $\mathbf{k}$  in  $\mathbb{Z}^N$ , we shall denote by  $T(\varepsilon, \mathbf{k})$  the translated image of  $r(\varepsilon) T$  by the vector  $\varepsilon \mathbf{k}$ , i.e.,

$$T(\varepsilon, \mathbf{k}) = \varepsilon \mathbf{k} + r(\varepsilon) T .$$

According to this definition we introduce the region  $T_\varepsilon$  of  $\Omega$ , defined by :

$$T_\varepsilon = \cup \{ T(\varepsilon, \mathbf{k}) \mid \bar{T}(\varepsilon, \mathbf{k}) \subset \Omega \} ,$$

and we set :

$$\Omega_\varepsilon = \Omega \setminus \bar{T}_\varepsilon .$$

Let us observe that  $\Omega_\varepsilon$  represents the subregion of  $\Omega$  consisting of the whole domain  $\Omega$  in which we have removed a finite number of « small » holes. All of them have the same shape  $r(\varepsilon) T$ , and they are periodically distributed in  $\Omega$ , with period  $\varepsilon$  in each axis-direction. The distance between two adjacent holes is of the order of  $\varepsilon$ , and the diameter of each hole is  $r(\varepsilon)$  times the diameter of  $T$ . It can be remarked that the size of the holes with respect to the distance between two adjacent holes goes to zero, as  $\varepsilon \rightarrow 0$ , since the function  $r(\cdot)$  verifies (1.1a). On the other hand, condition (1.1b) implies that the holes do not overlap (see fig. 1.2).

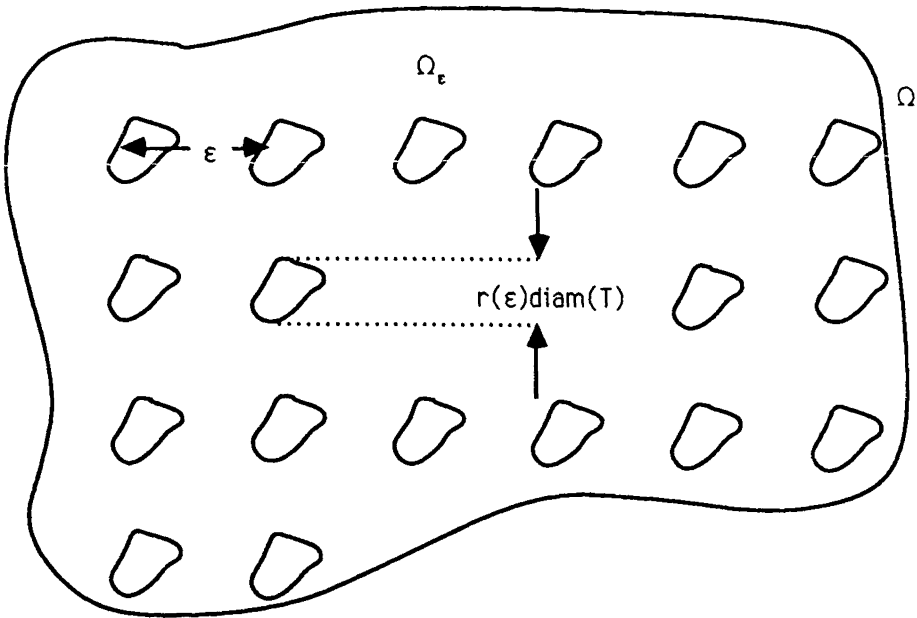


Figure 1.2. — The region  $\Omega_\varepsilon$  in the two-dimensional case.

Let  $f$  be a given function in  $L^2(\Omega)$ , and let  $g$  be given in  $L^2(\partial T)$ . For each  $\varepsilon$  we define  $g_\varepsilon$  in  $L^2(\partial T_\varepsilon)$  by :

$$(1.2) \quad g_\varepsilon(x) = g((x - \varepsilon \vec{k})/r(\varepsilon)), \quad \text{for } x \in \partial T(\varepsilon, \vec{k}).$$

Our aim in this paper is to study the asymptotic behaviour, as  $\varepsilon \rightarrow 0$ , of the solution  $u_\varepsilon$  of the following non-homogeneous Neumann boundary-value problem :

$$(1.3a) \quad -\Delta u_\varepsilon = f \quad \text{in } \Omega_\varepsilon$$

$$(1.3b) \quad \partial u_\varepsilon / \partial n = g_\varepsilon \quad \text{on } \partial T_\varepsilon$$

$$(1.3c) \quad u_\varepsilon = 0 \quad \text{on } \partial \Omega,$$

where, in (1.3b),  $\partial/\partial n$  denotes the external normal derivative with respect to  $\Omega_\varepsilon$ .

## 1.2. Variational formulation of the problem

In order to establish the variational formulation of problem (1.3), let us introduce the following space :

$$V_\varepsilon = \{ \varphi \in H^1(\Omega_\varepsilon) \mid \varphi = 0 \text{ on } \partial \Omega \}$$

equipped with the norm :

$$\| \varphi \|_{1, \Omega_\varepsilon} = \left( \int_{\Omega_\varepsilon} |\varphi(x)|^2 dx + \int_{\Omega_\varepsilon} |\nabla \varphi(x)|^2 dx \right)^{1/2}.$$

Multiplying (1.3a) by any (smooth) function in  $V_\varepsilon$  and integrating by parts in  $\Omega_\varepsilon$ , it is elementary to check using (1.3b), (1.3c) (and density arguments) that the variational formulation of (1.3) is :

(1.4a) Find  $u_\varepsilon \in V_\varepsilon$ , such that :

$$(1.4b) \quad \int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla \varphi dx = \int_{\Omega_\varepsilon} f \varphi dx + \int_{\partial T_\varepsilon} g_\varepsilon \varphi ds \quad \forall \varphi \in V_\varepsilon.$$

The left-hand side in (1.4b) defines a continuous bilinear form in  $V_\varepsilon$ , which is coercive, for each  $\varepsilon$ . Moreover, since  $f \in L^2(\Omega)$ , and  $g_\varepsilon \in L^2(\partial T_\varepsilon)$ , the right-hand side in (1.4b) defines a linear continuous form in  $V_\varepsilon$ . Therefore, for each  $\varepsilon$ , problem (1.4) has a unique solution  $u_\varepsilon$  in  $V_\varepsilon$ . We shall refer to  $u_\varepsilon$  as the (unique) weak-solution of (1.3), and our goal in what follows is to study the limit behaviour of the sequence  $\{u_\varepsilon\}$ , as  $\varepsilon \rightarrow 0$ .



### 1.3. Decomposition of the solution of (1.3)

The starting point for the study of our problem consists in decomposing the solution  $u_\varepsilon$  of (1.3) (or (1.4)) as follows :

$$(1.5) \quad u_\varepsilon = v_\varepsilon + t_\varepsilon + z_\varepsilon$$

where  $v_\varepsilon$ ,  $t_\varepsilon$ ,  $z_\varepsilon$  are respectively the unique weak-solutions of the following boundary-value problems :

$$(1.6a) \quad -\Delta v_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon$$

$$(1.6b) \quad \partial v_\varepsilon / \partial n = \bar{g} \quad \text{on } \partial T_\varepsilon$$

$$(1.6c) \quad v_\varepsilon = 0 \quad \text{on } \partial \Omega,$$

$$(1.7a) \quad -\Delta t_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon$$

$$(1.7b) \quad \partial t_\varepsilon / \partial n = g_\varepsilon^0 \quad \text{on } \partial T_\varepsilon$$

$$(1.7c) \quad t_\varepsilon = 0 \quad \text{on } \partial \Omega,$$

and

$$(1.8a) \quad -\Delta z_\varepsilon = f \quad \text{in } \Omega_\varepsilon$$

$$(1.8b) \quad \partial z_\varepsilon / \partial n = 0 \quad \text{on } \partial T_\varepsilon$$

$$(1.8c) \quad z_\varepsilon = 0 \quad \text{on } \partial \Omega,$$

where, in (1.6b), (1.7b),  $\bar{g}$ ,  $g_\varepsilon^0$  are defined by :

$$(1.9a) \quad \bar{g} = (1/|\partial T|) \int_{\partial T} g \, ds$$

$$(1.9b) \quad g_\varepsilon^0 = g_\varepsilon - \bar{g}$$

where, in (1.9a),  $|\partial T|$  denotes the  $\mathbb{R}^{N-1}$ -Lebesgue measure of  $\partial T$ .

As a first remark concerning this decomposition of  $u_\varepsilon$ , let us note that problems (1.6), -, (1.8) are particular cases of problem (1.3). For example, if  $f = 0$ , and the average of  $g$  on  $\partial T$  is zero, then problem (1.3) reduces to (1.7) (i.e., in this case  $v_\varepsilon = z_\varepsilon = 0$ ). To study the asymptotic behaviour of  $u_\varepsilon$ , we shall study separately the limit behaviours of the sequences  $\{v_\varepsilon\}$ ,  $\{t_\varepsilon\}$ , and  $\{z_\varepsilon\}$ , respectively. This decomposition of the problem, at first glance, may appear unexpected. It will however soon become apparent. In fact, as we shall see, the three components of  $u_\varepsilon$  in this decomposition have different limit behaviours, as  $\varepsilon \rightarrow 0$ . Therefore, the study of each component separately, will not only provide the limit behaviours of

the  $u_\varepsilon$ , but it will also allow us to obtain better information about the asymptotic behaviour of problem (1.3) when it reduces to (1.6), (1.7) or (1.8).

#### 1.4. Asymptotic behaviour of problem (1.6)

In this section we study the asymptotic behaviour of problem (1.6). In what follows we assume that :

$$(1.10) \quad \bar{g} \neq 0 .$$

First, let us observe that the variational formulation of problem (1.6) is :

(1.11a) Find  $v_\varepsilon \in V_\varepsilon$ , such that :

$$(1.11b) \quad \int_{\Omega_\varepsilon} \nabla v_\varepsilon \cdot \nabla \varphi \, dx = \bar{g} \int_{\partial T_\varepsilon} \varphi \, ds \quad \forall \varphi \in V_\varepsilon .$$

##### 1.4.a. *A priori estimates*

Our starting point for the study of this problem is Theorem 1.1, which gives detailed a priori  $H^1(\Omega_\varepsilon)$ -estimates of the solutions of (1.6) in terms of the size  $r(\varepsilon)$  of the holes, and the distance  $\varepsilon$  between them. The a priori estimates depend on the dimension  $N$  of the space and the diameter  $r(\varepsilon)$  of the holes. To establish this theorem, the following cases have to be distinguished :

(i) The size  $r(\varepsilon)$  of the holes is *exactly* of the order of  $\varepsilon^{N/(N-2)}$  if  $N \geq 3$ , i.e., the case in which there exists a strictly positive constant  $a$ , such that :

$$(1.12a) \quad \lim_{\varepsilon \rightarrow 0^+} r(\varepsilon) \varepsilon^{-N/(N-2)} = a, \quad \text{if } N \geq 3$$

and  $r(\varepsilon)$  verifies the following condition in the two-dimensional case :

$$(1.12b) \quad \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{-2} (\log (\varepsilon/r(\varepsilon)))^{-1} = a \quad \text{if } N = 2 .$$

(ii) The order of the size  $r(\varepsilon)$  of the holes is *bigger* than the size defined by (1.12), i.e.,

$$(1.13a) \quad \lim_{\varepsilon \rightarrow 0^+} r(\varepsilon) \varepsilon^{-N/(N-2)} = +\infty \quad \text{if } N \geq 3$$

$$(1.13b) \quad \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{-2} (\log (\varepsilon/r(\varepsilon)))^{-1} = +\infty \quad \text{if } N = 2 .$$

(iii) The order of the size  $r(\varepsilon)$  of the holes is *smaller* than the size defined by (1.12), i.e.,

$$(1.14a) \quad \lim_{\varepsilon \rightarrow 0^+} r(\varepsilon) \varepsilon^{-N/(N-2)} = 0 \quad \text{if } N \geq 3$$

$$(1.14b) \quad \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{-2} (\log(\varepsilon/r(\varepsilon)))^{-1} = 0 \quad \text{if } N = 2.$$

It can be observed that these three situations do not take into account all the possible behaviours, as  $\varepsilon \rightarrow 0$ , of the sequence  $\{r(\varepsilon)\}$ , with  $r(\cdot)$  verifying (1.1). However, by passing to a subsequence, all of them are included between these three cases. Therefore, in what follows we will mainly restrict our attention to these cases. On the other hand, it can also be remarked that in the two-dimensional case, condition (1.12b) does not define a unique behaviour of  $r(\varepsilon)$ , as  $\varepsilon \rightarrow 0$ . For example,  $r(\varepsilon) = \varepsilon \exp(-1/a\varepsilon^2)$  and  $r(\varepsilon) = \varepsilon^2 \exp(-1/a\varepsilon^2)$  are two sequences that have different behaviours as  $\varepsilon$  goes to zero, but both verify (1.12b).

**THEOREM 1.1 :** *Assume that (1.1) and (1.10) hold true. Let  $\{v_\varepsilon\}$  be the sequence of (unique) solutions of problem (1.6). Then there exist two (positive) constants  $m = m(\Omega, T, \bar{g}) \leq M = M(\Omega, T, \bar{g})$ , which are independent of  $\varepsilon$ , such that :*

$$(1.15) \quad \left\{ \begin{array}{l} \text{If } r(\cdot) \text{ verifies (1.12) or (1.13), then} \\ m r(\varepsilon)^{N-1} \varepsilon^{-N} \leq \|v_\varepsilon\|_{1, \Omega_\varepsilon} \leq M r(\varepsilon)^{N-1} \varepsilon^{-N} \end{array} \right.$$

$$(1.16) \quad \left\{ \begin{array}{l} \text{If } r(\cdot) \text{ verifies (1.14a), and } N \geq 3, \text{ then} \\ m (r(\varepsilon)/\varepsilon)^{N/2} \leq \|v_\varepsilon\|_{1, \Omega_\varepsilon} \leq M (r(\varepsilon)/\varepsilon)^{N/2} \end{array} \right.$$

$$(1.17) \quad \left\{ \begin{array}{l} \text{If } r(\cdot) \text{ verifies (1.14b), and } N = 2, \text{ then} \\ m (r(\varepsilon)/\varepsilon) (\log(\varepsilon/r(\varepsilon)))^{1/2} \leq \|v_\varepsilon\|_{1, \Omega_\varepsilon} \leq M (r(\varepsilon)/\varepsilon) (\log(\varepsilon/r(\varepsilon)))^{1/2} \end{array} \right.$$

for all  $\varepsilon$ . ■

This theorem provides exact estimates of the  $H^1(\Omega_\varepsilon)$ -norm of  $v_\varepsilon$ , for all the possible sizes  $r(\varepsilon)$  of the holes, and for all  $N$ . It can be remarked that the upper and lower estimates are continuous with respect to the size  $r(\varepsilon)$  of the holes. Continuous in the sense that if (1.12a) holds (and  $N \geq 3$ ), then the estimates (1.15) coincides with (1.16), and if (1.12b) holds (and  $N = 2$ ), then (1.15) and (1.17) provide the same estimate.

Let us now investigate how the asymptotic behaviour of  $v_\varepsilon$  (as  $\varepsilon \rightarrow 0$ ) depends on the size  $r(\varepsilon)$  of the holes. As a first step in this direction, let us first derive those cases in which the  $H^1(\Omega_\varepsilon)$ -norm of  $v_\varepsilon$  is bounded, divergent, or it goes to zero, as  $\varepsilon \rightarrow 0$ . A brief computation using Theorem 1.1 shows that the following three situations arise :

(j) The size  $r(\varepsilon)$  of the holes is *exactly* of the order of  $\varepsilon^{N/(N-1)}$ , i.e., the case in which there exists a strictly positive constant  $b$ , such that :

$$(1.18) \quad \lim_{\varepsilon \rightarrow 0^+} r(\varepsilon) \varepsilon^{-N/(N-1)} = b .$$

In this case the size  $r(\varepsilon)$  of the holes satisfies (1.13) and Theorem 1.1 (cf. (1.15)) states that the  $H^1(\Omega_\varepsilon)$ -norm of the sequence  $\{v_\varepsilon\}$  remains bounded, as  $\varepsilon \rightarrow 0$ .

(jj) The order of the size  $r(\varepsilon)$  of the holes is *bigger* than  $\varepsilon^{N/(N-1)}$ , i.e.,

$$(1.19) \quad \lim_{\varepsilon \rightarrow 0^+} r(\varepsilon) \varepsilon^{-N/(N-1)} = +\infty .$$

In this case, the size  $r(\varepsilon)$  of the holes satisfies (1.13), and from Theorem 1.1, it follows that the lower bound of  $v_\varepsilon$  goes to infinity as  $\varepsilon \rightarrow 0$ . It is therefore clear that in this case, if  $v_\varepsilon$  can be extended to all  $\Omega$  by means of a linear continuous operator from  $V_\varepsilon$  to  $H_0^1(\Omega)$ , then the extension of  $v_\varepsilon$  will diverge in  $H_0^1(\Omega)$ , as  $\varepsilon \rightarrow 0$ . It is however interesting to investigate in this case the asymptotic behaviour of the sequence obtained multiplying  $v_\varepsilon$  by  $r(\varepsilon)^{-(N-1)} \varepsilon^N$ .

(jjj) The order of the size  $r(\varepsilon)$  of the holes is *smaller* than  $\varepsilon^{N/(N-1)}$ , i.e.

$$(1.20) \quad \lim_{\varepsilon \rightarrow 0^+} r(\varepsilon) \varepsilon^{-N/(N-1)} = 0 .$$

In this case, the size  $r(\varepsilon)$  of the holes can verify (1.12), (1.13) or (1.14). In any of these situations, Theorem 1.1 implies that the upper bound of the  $H^1(\Omega_\varepsilon)$ -norm of  $v_\varepsilon$  goes to zero, as  $\varepsilon \rightarrow 0$ . It is then clear that in this case the extension of  $v_\varepsilon$  will (strongly) converge to zero in  $H_0^1(\Omega)$ , as  $\varepsilon \rightarrow 0$ . This means that in this case the holes are so small that the non-homogeneous Neumann boundary data  $\bar{g}$  does not provide any contribution to the limit. In this case too, it is interesting to study the asymptotic behaviour of  $v_\varepsilon$  renormalized by its corresponding upper (or lower) bounds of the  $H^1(\Omega_\varepsilon)$ -norm estimate.

From the above remarks it seems natural to regard the size  $\varepsilon^{N/(N-1)}$  as a critical size of the holes (i.e., case (1.18)). It can be remarked that for this

(critical) size of the holes, the  $\mathbb{R}^{N-1}$ -Lebesgue measure of the boundary  $\partial T_\varepsilon$  of  $T_\varepsilon$  has a limit, as  $\varepsilon \rightarrow 0$ , and we have :

$$(1.21) \quad \lim_{\varepsilon \rightarrow 0^+} |\partial T_\varepsilon| = b^{N-1} |\Omega| |\partial T|$$

where  $|\Omega|$  denotes the  $\mathbb{R}^N$ -Lebesgue measure of  $\Omega$ , and  $|\partial T_\varepsilon|$ ,  $|\partial T|$  denote the  $\mathbb{R}^{N-1}$ -Lebesgue measures of  $\partial T_\varepsilon$ ,  $\partial T$ , respectively.

The proof of Theorem 1.1 will be given in Section 2.1. The proof of the upper bounds is based on a sharp estimate of the constant appearing in the (trace) embedding of  $H^1(\Omega_\varepsilon)$  into  $L^2(\partial T_\varepsilon)$  (cf. Lemma 2.1). The lower bounds are obtained using suitable test functions in the variational formulation (1.11) of problem (1.6). These functions depend on the size  $r(\varepsilon)$  of the holes. In case  $r(\cdot)$  verifies (1.14), the test functions that we use have been introduced in D. Cioranescu & F. Murat [4].

1.4.b. *The main theorem of convergence*

By using the a priori estimates established in Section 1.4.a, we can now proceed to describe the asymptotic behaviour, as  $\varepsilon \rightarrow 0$ , of the solution of problem (1.6). We begin by pointing out that the functions  $v_\varepsilon$ 's are a priori only defined in  $\Omega_\varepsilon$ , and not in all  $\Omega$ , as it should be desired for the study of their asymptotic behaviours. We shall therefore introduce a family  $\{P_\varepsilon\}$  of linear extension-operators,  $P_\varepsilon \in \mathcal{L}(V_\varepsilon, H_0^1(\Omega))$ , such that for all  $\varepsilon$  :

$$(1.22a) \quad (P_\varepsilon \varphi)(x) = \varphi(x) \quad \forall x \in \Omega_\varepsilon$$

$$(1.22b) \quad \|\nabla P_\varepsilon \varphi\|_{0, \Omega} \leq C \|\nabla \varphi\|_{0, \Omega_\varepsilon} \quad \forall \varphi \in V_\varepsilon$$

where C is a constant independent of  $\varepsilon$ . The proof of the existence of at least one such family will be given in appendix A (cf. Lemma A.1). This proof makes use of a similar extension result proved in D. Cioranescu & J. Saint Jean Paulin [5].

Let us recall that Theorem 1.1 provides the exact order of the  $H^1(\Omega_\varepsilon)$ -norm of  $v_\varepsilon$ . Therefore, if  $\{P_\varepsilon\}$  is any family of (linear continuous) extension-operators verifying (1.22), then Theorem 1.1 also provides the exact order of  $P_\varepsilon v_\varepsilon$ , and it follows that the sequence  $\{P_\varepsilon v_\varepsilon\}$  verifies in  $H_0^1(\Omega)$  the same a priori estimates than the sequence  $\{v_\varepsilon\}$  in  $H^1(\Omega_\varepsilon)$ . The following theorem is the central result of this section. It gives detailed information about the limit behaviour in  $H_0^1(\Omega)$  of the sequence obtained by a suitable renormalization of  $\{P_\varepsilon v_\varepsilon\}$ .

**THEOREM 1.2 :** *Assume that the function  $r(\cdot)$  verifies (1.1), and that (1.10) holds. Let  $\{v_\varepsilon\}$  in  $V_\varepsilon$  be the sequence of the unique solutions of (1.6). Then*

for any family  $\{P_\epsilon\}$  of (linear continuous) extension-operators verifying (1.22), we have :

$$(1.23) \quad \begin{cases} \text{If } r(\cdot) \text{ verifies (1.12) or (1.13), then} \\ r(\epsilon)^{-(N-1)} \epsilon^N P_\epsilon v_\epsilon \rightharpoonup v \text{ weakly in } H_0^1(\Omega), \text{ as } \epsilon \rightarrow 0, \end{cases}$$

where  $v$  in  $H_0^1(\Omega)$  is the (unique) solution of the following problem :

$$(1.24a) \quad -\Delta v = |\partial T| \bar{g} \quad \text{in } \Omega$$

$$(1.24b) \quad v = 0 \quad \text{on } \partial\Omega$$

$$(1.25) \quad \begin{cases} \text{If } r(\cdot) \text{ verifies (1.14a), and } N \geq 3, \text{ then} \\ (r(\epsilon)/\epsilon)^{-N/2} P_\epsilon v_\epsilon \rightharpoonup 0 \text{ weakly in } H_0^1(\Omega), \text{ as } \epsilon \rightarrow 0. \end{cases}$$

$$(1.26)$$

$$\begin{cases} \text{If } r(\cdot) \text{ verifies (1.14b), and } N = 2, \text{ then} \\ (r(\epsilon)/\epsilon)^{-1} (\log(\epsilon/r(\epsilon)))^{-1/2} P_\epsilon v_\epsilon \rightharpoonup 0 \text{ weakly in } H_0^1(\Omega), \text{ as } \epsilon \rightarrow 0. \quad \blacksquare \end{cases}$$

As a first remark concerning Theorem 1.2, let us observe that if the holes have the critical size (i.e. if  $r(\cdot)$  verifies (1.18), which implies that  $r(\cdot)$  also verifies (1.13)), then the convergence result (1.23) of Theorem 1.2 can also be rewritten as follows :

$$(1.27) \quad P_\epsilon v_\epsilon \rightharpoonup b^{N-1} v \text{ weakly in } H_0^1(\Omega), \text{ as } \epsilon \rightarrow 0,$$

where  $v$  is the unique solution of problem (1.24), and  $b$  is defined by (1.18).

From Theorem 1.2 we can point out that the limit behaviour of the sequence  $\{P_\epsilon v_\epsilon\}$  (renormalized by the bounds given by Theorem 1.1) is completely different if either the size  $r(\epsilon)$  of the holes verifies (1.12) or (1.13), or if it verifies (1.14). Therefore, the size  $r(\epsilon) \sim \epsilon^{N/(N-2)}$  if  $N \geq 3$  (or  $r(\epsilon)$  verifying (1.12b) if  $N = 2$ ) can be regarded as a « special » size (or a « second critical size ») of the holes, which is strictly smaller than the critical size defined below (i.e.,  $r(\epsilon) \sim \epsilon^{N/(N-1)}$ ) (see fig. 1.3).

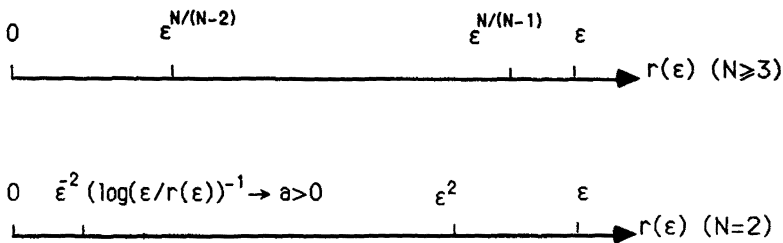


Figure 1.3.

If the size  $r(\varepsilon)$  of the holes is smaller than this « special » size, then even if the sequence  $\{P_\varepsilon v_\varepsilon\}$  is renormalized, it weakly converges to zero in  $H_0^1(\Omega)$ , as  $\varepsilon \rightarrow 0$  (cf. (1.25), (1.26)). This means that when the holes are « very very small » the non-homogeneous Neumann data  $\bar{g}$  can be completely neglected at the limit. As far as we know, this is the first example in this kind of problems where two different critical sizes of the holes arise in the study of the limit behaviour of the problem. It is interesting to remark that the « special » size (i.e.,  $r(\varepsilon) \sim \varepsilon^{N/(N-2)}$  if  $N \geq 3$ , and  $r(\varepsilon)$  verifying (1.12b) if  $N = 2$ ) coincides with the « critical » size that appears in the study of the non-homogeneous Laplace equation in  $\Omega_\varepsilon$  with a homogeneous Dirichlet boundary condition on the boundaries of the holes (see D. Cioranescu & F. Murat [4]).

We shall prove Theorem 1.2 in Section 2.2. Its proof consists in passing to the limit in the variational formulation of problem (1.6) by using suitable test functions. It can be observed that the limit equation (1.24) cannot directly be obtained by passing to the limit in equation (1.6a). In fact, except in case of « very very small » holes (i.e.,  $r(\varepsilon)$  verifying (1.14)), a non-zero second member appears at the limit. The main difficulty when passing to the limit in the variational formulation (1.11) of (1.6) is the boundary term occurring in the right-hand side of (1.11b). As  $\varepsilon \rightarrow 0$ , this term can be regarded as a sequence of measures on  $\Omega$ , concentrated for each  $\varepsilon$  on  $\partial T_\varepsilon$ . The constant  $|\partial T| \bar{g}$  appears in the proof of the theorem as the limit (in the sense of the weak\* topology of the space of Radon measures on  $\Omega$ ) of this sequence of measures.

To conclude our study of problem (1.6), let us mention that in Section 2 we show how the sequence  $\{v_\varepsilon\}$  can be corrected in order to obtain a strong convergence in (1.23), (1.25) and (1.26). Indeed, in Section 2.3 (cf. Theorem 2.2), we construct a periodic correcting function for the sequence of  $v_\varepsilon$ , and we show that far off the external boundary of  $\Omega$  and when  $\varepsilon \rightarrow 0$ ,  $v_\varepsilon$  behaves like a periodic function of period  $\varepsilon$ .

### 1.5. Asymptotic behaviour of problem (1.7)

In this section we study the asymptotic behaviour of problem (1.7). This problem corresponds to the case of a non-homogeneous Neumann data on  $\partial T_\varepsilon$  given by means of a sequence of functions  $g_\varepsilon^0$  in  $L^2(\partial T_\varepsilon)$  verifying :

$$(1.28) \quad g_\varepsilon^0(x) = g^0((x - \varepsilon \mathbf{k})/r(\varepsilon)) \quad \text{for } x \in \partial T(\varepsilon, \mathbf{k})$$

where  $g^0 = g - \bar{g}$  verifies :

$$(1.29) \quad \int_{\partial T} g^0 ds = 0 .$$

The variational formulation of problem (1.7) is :

(1.30a) Find  $t_\varepsilon \in V_\varepsilon$ , such that :

(1.30b) 
$$\int_{\Omega_\varepsilon} \nabla t_\varepsilon \cdot \nabla \varphi \, dx = \int_{\partial T_\varepsilon} g_\varepsilon^0 \varphi \, ds \quad \forall \varphi \in V_\varepsilon .$$

We begin the study of problem (1.7) by the following theorem that gives detailed a priori  $H^1(\Omega_\varepsilon)$ -estimates of the solution  $t_\varepsilon$  of problem (1.7).

**THEOREM 1.3 :** *Assume that  $r(\cdot)$  verifies (1.1). Let  $g^0 \neq 0$  in  $L^2(\partial T)$  be a given function verifying (1.29), and let  $g_\varepsilon^0$  in  $L^2(\partial T_\varepsilon)$  be defined by (1.28). Then there exist two (positive) constants  $m = m(\Omega, T, g^0) \leq M = M(\Omega, T, g^0)$ , which are independent of  $\varepsilon$ , such that :*

(1.31) 
$$m(r(\varepsilon)/\varepsilon)^{N/2} \leq \|t_\varepsilon\|_{1, \Omega_\varepsilon} \leq M(r(\varepsilon)/\varepsilon)^{N/2}$$

for all  $\varepsilon$ . ■

As a first remark concerning this theorem, let us observe that the estimate (1.31) proves that in this case the  $H^1(\Omega_\varepsilon)$ -norm of the solution  $t_\varepsilon$  of (1.7) converges to zero, as  $\varepsilon \rightarrow 0$ , for any size  $r(\varepsilon)$  of the holes, and for all  $N \geq 2$ . It is however interesting to investigate the asymptotic behaviour of the sequence  $\{t_\varepsilon\}$  renormalized by  $(r(\varepsilon)/\varepsilon)^{N/2}$ . To this end, we have :

**THEOREM 1.4 :** *Assume that the hypothesis of Theorem 1.3 hold true. Then for any family  $\{P_\varepsilon\}$  of (linear continuous) extension-operators verifying (1.22), we have :*

(1.32) 
$$(r(\varepsilon)/\varepsilon)^{-N/2} P_\varepsilon t_\varepsilon \rightharpoonup 0 \text{ weakly in } H_0^1(\Omega), \text{ as } \varepsilon \rightarrow 0. \quad \blacksquare$$

From this theorem we can point out that in case the non-homogeneous data on  $\partial T_\varepsilon$  has a zero mean-value the Neumann boundary condition does not provides any contribution at the limit. For this case too, we show in Section 3.3, how the sequence  $\{t_\varepsilon\}$  can be corrected in order to obtain a (locally) strong convergence in (1.32).

### 1.6. Asymptotic behaviour of problem (1.8)

We begin by giving the variational formulation of problem (1.8), which is :

(1.33a) Find  $z_\varepsilon \in V_\varepsilon$ , such that :

(1.33b) 
$$\int_{\Omega_\varepsilon} \nabla z_\varepsilon \cdot \nabla \varphi \, dx = \int_{\Omega_\varepsilon} f \varphi \, dx \quad \forall \varphi \in V_\varepsilon .$$



This problem has been studied by D. Cioranescu & J. Saint Jean Paulin [5] in case the size of the holes is of the same order as the distance between two adjacent holes. If the size  $r(\varepsilon)$  of the holes verifies (1.1), this problem is studied in H. Attouch [1, chapter 1] using  $\Gamma$ -convergence techniques. The following theorem describes the asymptotic behaviour of  $z_\varepsilon$  in case of « small » holes in  $\Omega$  :

**THEOREM 1.5 :** *Let  $f$  in  $L^2(\Omega)$  be a given function. Assume that the size  $r(\cdot)$  of the holes verifies (1.1), and let  $\{z_\varepsilon\}$  in  $V_\varepsilon$  be the sequence of (unique) solutions of problem (1.8). Then for any family  $\{P_\varepsilon\}$  of linear continuous extension-operators verifying (1.22), we have :*

$$(1.34a) \quad P_\varepsilon z_\varepsilon \rightharpoonup z \text{ weakly in } H_0^1(\Omega), \text{ as } \varepsilon \rightarrow 0$$

$$(1.34b) \quad \|z_\varepsilon - z\|_{1, \Omega_\varepsilon} \rightarrow 0, \text{ as } \varepsilon \rightarrow 0,$$

where  $z$  is the (unique) solution of the following problem :

$$(1.35a) \quad -\Delta z = f \quad \text{in } \Omega$$

$$(1.35b) \quad z = 0 \quad \text{on } \partial\Omega. \quad \blacksquare$$

A proof of this theorem can be found in H. Attouch [1, Th. 1.1]. For the sake of completeness, we give in Section 4 an alternative proof of this result.

**1.7. Asymptotic behaviour of the general problem (problem (1.3))**

In this section we summarize the results stated in Sections 1.4, 1.5, and 1.6, in order to describe the asymptotic behaviour of the sequence  $\{u_\varepsilon\}$ , solutions of problem (1.3). It is clear that its limit behaviour depends on the size  $r(\varepsilon)$  of the holes. We shall distinguish three cases :

(j) The size  $r(\varepsilon)$  of the holes is the *critical size* (i.e.,  $r(\varepsilon)$  verifies (1.18)). In this case, by using (1.5), (1.27), (1.31), and (1.34a), we deduce that :

$$(1.36) \quad P_\varepsilon u_\varepsilon \rightharpoonup b^{N-1} v + z \text{ weakly in } H_0^1(\Omega), \text{ as } \varepsilon \rightarrow 0,$$

where  $v$  is the solution of (1.24), and  $z$  is the solution of problem (1.35).

(jj) The size  $r(\varepsilon)$  of the holes is *bigger* than the critical size (i.e.,  $r(\varepsilon)$  verifies (1.19)). In this case, assuming that (1.10) holds, it follows from (1.9), (1.22), (1.31), (1.34a) that the sequence  $\{P_\varepsilon u_\varepsilon\}$  verifies the following estimate :

$$(1.37) \quad C_1 r(\varepsilon)^{N-1} \varepsilon^{-N} \leq \|P_\varepsilon u_\varepsilon\|_{1, \Omega} \leq C_2 r(\varepsilon)^{N-1} \varepsilon^{-N}$$

where  $C_1 = C_1(\Omega, T, f, g) \leq C_2 = C_2(\Omega, T, f, g)$  are two (positive) constants, which are independent of  $\varepsilon$ . This implies that the  $H^1(\Omega)$ -norm of

$P_\varepsilon u_\varepsilon$  goes to infinity as  $\varepsilon \rightarrow 0$ . However, as in some of the particular cases of problem (1.3), we can renormalize the sequence  $\{P_\varepsilon u_\varepsilon\}$  by  $r(\varepsilon)^{(N-1)\varepsilon^{-N}}$  and study its limit behaviour. By using (1.5), (1.23), (1.32) and (1.34a), we obtain :

$$(1.38) \quad r(\varepsilon)^{-(N-1)\varepsilon^{-N}} P_\varepsilon u_\varepsilon \rightharpoonup v \text{ weakly in } H_0^1(\Omega), \text{ as } \varepsilon \rightarrow 0,$$

where  $v$  is the solution of problem (1.24).

(jjj) The size  $r(\varepsilon)$  of the holes is *smaller* than the critical size (i.e.,  $r(\varepsilon)$  verifies (1.20)). In this case, Theorems 1.1, 1.3, imply that the sequence  $\{P_\varepsilon(v_\varepsilon + t_\varepsilon)\}$  strongly converges to zero, as  $\varepsilon \rightarrow 0$ . Therefore, using (1.5), (1.34), it follows that :

$$(1.39a) \quad P_\varepsilon u_\varepsilon \rightharpoonup z \text{ weakly in } H_0^1(\Omega), \text{ as } \varepsilon \rightarrow 0,$$

and

$$(1.39b) \quad \|u_\varepsilon - z\|_{1, \Omega_\varepsilon} \rightarrow 0 \text{ in } \mathbb{R}, \text{ as } \varepsilon \rightarrow 0.$$

It can be remarked that the presence of  $f$  in equation (1.3a) implies that in this case the sequence  $\{P_\varepsilon u_\varepsilon\}$ , renormalized by means of the a priori bounds of  $\{v_\varepsilon\}$  or  $\{t_\varepsilon\}$ , is divergent in  $H_0^1(\Omega)$ . By this remark we complete our description of the (weak) limit behaviour of the solution of problem (1.3).

### 1.8. A correcting term in case of spherical holes

As it has been already mentioned, in Sections 2, 3, we introduce correcting terms for the solutions  $v_\varepsilon, t_\varepsilon$  of problems (1.6), (1.7), respectively (it can be remarked that the solution  $z_\varepsilon$  of problem (1.8) does not need to be corrected in order to obtain a strong convergence in (1.34a), because (1.34b) holds. Indeed, at least for suitable choices of the family  $\{P_\varepsilon\}$  of linear extension-operators, it can be easily checked that (1.34b) is equivalent to the fact that  $\{P_\varepsilon z_\varepsilon\}$  strongly converges to  $z$  in  $H_0^1(\Omega)$ , as  $\varepsilon \rightarrow 0$ ). As we shall see, the correctors that we study in Sections 2, 3, are defined as the solutions of some periodic boundary-value problems depending on  $\varepsilon$ . For its effective numerical computation it should be desirable to have an explicit formula for these terms. In case of spherical holes, we exhibit in this section an explicit formula that allows us to compute the corrector for the solution  $v_\varepsilon$  of problem (1.6).

In this section we shall assume that  $T$  is a sphere of  $\mathbb{R}^N$ , centered at the origin. For technical reasons, but without loss of generality, we shall assume that  $T$  satisfies :

$$(1.40) \quad |\partial T| = N$$

where  $N$  is the space dimension of  $\mathbb{R}^N$ . It can be remarked that  $T$  is contained in any of the cubes  $] -L, L[^N$  of  $\mathbb{R}^N$ , for all  $L$  greater than  $(N/S_N)^{1/(N-1)}$ , where  $S_N$  is the surface of the unit sphere of  $\mathbb{R}^N$ .

In order to define the correcting term for the sequence  $\{v_\varepsilon\}$ , let us introduce the functions  $\theta_\varepsilon$ , defined as follows :

If  $r(\cdot)$  verifies (1.12) or (1.13), then

$$(1.41a) \quad -\Delta\theta_\varepsilon = \mu_\varepsilon \quad \text{in } B(\varepsilon, \mathbf{k}) \setminus \bar{T}(\varepsilon, \mathbf{k})$$

$$(1.41b) \quad \theta_\varepsilon = 0 \quad \text{in } Y(\varepsilon, \mathbf{k}) \setminus B(\varepsilon, \mathbf{k})$$

$$(1.41c) \quad \partial\theta_\varepsilon/\partial n = \bar{g}r(\varepsilon)^{-(N-1)}\varepsilon^N \quad \text{on } \partial T(\varepsilon, \mathbf{k})$$

where  $Y(\varepsilon, \mathbf{k}) = ]k_1 - \varepsilon/2, k_1 + \varepsilon/2[ \times \dots \times ]k_N - \varepsilon/2, k_N + \varepsilon/2[$ ,  $B(\varepsilon, \mathbf{k})$  is the open ball of  $\mathbb{R}^N$  centered at  $\mathbf{k}$  of radius  $\varepsilon$  (see fig. 1.4), and

$$(1.42) \quad \mu_\varepsilon = -\bar{g}N \frac{\varepsilon^N}{\varepsilon^N - r(\varepsilon)^N}.$$

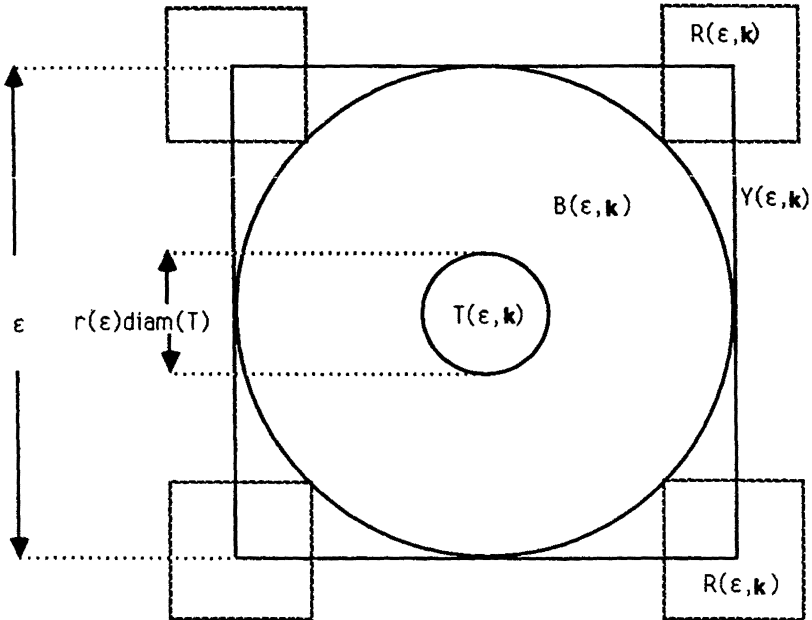


Figure 1.4.

If  $r(\cdot)$  verifies (1.14a), and  $N \geq 3$ , then  $\theta_\varepsilon$  is defined by :

$$(1.43a) \quad -\Delta\theta_\varepsilon = \mu_\varepsilon \quad \text{in } B(\varepsilon, \mathbf{k}) \setminus \bar{T}(\varepsilon, \mathbf{k})$$

$$(1.43b) \quad \theta_\varepsilon = 0 \quad \text{in } Y(\varepsilon, \mathbf{k}) \setminus B(\varepsilon, \mathbf{k})$$

$$(1.43c) \quad \partial\theta_\varepsilon/\partial n = \bar{g}(r(\varepsilon)/\varepsilon)^{-N/2} \quad \text{on } \partial T(\varepsilon, \mathbf{k})$$

and

$$(1.44) \quad \mu_\varepsilon = -\bar{g}N \frac{\varepsilon^{N/2} r(\varepsilon)^{(N-2)/2}}{\varepsilon^N - r(\varepsilon)^N}.$$

Finally, if  $r(\cdot)$  verifies (1.14b), and  $N = 2$ , then  $\theta_\varepsilon$  is defined by :

$$(1.45a) \quad -\Delta\theta_\varepsilon = \mu_\varepsilon \quad \text{in } B(\varepsilon, \mathbf{k}) \setminus \bar{T}(\varepsilon, \mathbf{k})$$

$$(1.45b) \quad \theta_\varepsilon = 0 \quad \text{in } Y(\varepsilon, \mathbf{k}) \setminus B(\varepsilon, \mathbf{k})$$

$$(1.45c) \quad \partial\theta_\varepsilon/\partial n = \bar{g}(r(\varepsilon)/\varepsilon)^{-1}(\log(\varepsilon/r(\varepsilon)))^{-1/2} \quad \text{on } \partial T(\varepsilon, \mathbf{k})$$

and

$$(1.46) \quad \mu_\varepsilon = -2\bar{g} \frac{\varepsilon}{\varepsilon^2 - r(\varepsilon)^2} (\log(\varepsilon/r(\varepsilon)))^{-1/2}.$$

When  $\mathbf{k}$  varies in  $\mathbb{Z}^N$ , the formulae (1.41), (1.43) or (1.45) define  $\theta_\varepsilon$  in  $\mathbb{R}^N \setminus \cup \{\bar{T}(\varepsilon, \mathbf{k}) \mid \mathbf{k} \in \mathbb{Z}^N\}$ . A brief computation using spherical coordinates provides an explicit expression for  $\theta_\varepsilon$  in the ring  $B(\varepsilon, \mathbf{k}) \setminus \bar{T}(\varepsilon, \mathbf{k})$ . Indeed, if  $r(\cdot)$  verifies (1.12) or (1.13), then we have :

$$(1.47a) \quad \theta_\varepsilon(x) = \mu_\varepsilon \left[ \frac{r(\varepsilon)^N}{N(N-2)} \left( \frac{1}{\varepsilon^{N-2}} - \frac{1}{r^{N-2}} \right) + \frac{\varepsilon^2 - r^2}{2N} \right] + \frac{\bar{g}\varepsilon^N}{N-2} \left( \frac{1}{r^{N-2}} - \frac{1}{\varepsilon^{N-2}} \right) \quad \text{if } N \geq 3$$

$$(1.47b) \quad \theta_\varepsilon(x) = \mu_\varepsilon \left[ -\frac{r(\varepsilon)^2}{2} \log\left(\frac{\varepsilon}{r}\right) + \frac{\varepsilon^2 - r^2}{4} \right] + \bar{g}\varepsilon^2 \log\left(\frac{\varepsilon}{r}\right) \quad \text{if } N = 2$$

where  $r = |x - \vec{\mathbf{k}}|$ , and  $\mu_\varepsilon$  is defined by (1.42).

On the other hand, if  $r(\cdot)$  verifies (1.14a), and  $N \geq 3$ , then

$$(1.48a) \quad \theta_\varepsilon(x) = \mu_\varepsilon \left[ \frac{r(\varepsilon)^N}{N(N-2)} \left( \frac{1}{\varepsilon^{N-2}} - \frac{1}{r^{N-2}} \right) + \frac{\varepsilon^2 - r^2}{2N} \right] + \frac{\bar{g}r(\varepsilon)^{(N-2)/2} \varepsilon^{N/2}}{N-2} \left( \frac{1}{r^{N-2}} - \frac{1}{\varepsilon^{N-2}} \right)$$

where  $\mu_\varepsilon$  is defined by (1.44).

Finally, if  $r(\cdot)$  verifies (1.14b), and  $N = 2$ , then we have :

$$(1.48b) \quad \theta_\varepsilon(x) = \mu_\varepsilon \left[ -\frac{r(\varepsilon)^2}{2} \log \left( \frac{\varepsilon}{r} \right) + \frac{\varepsilon^2 - r^2}{4} \right] + \bar{g}\varepsilon \left( \log \left( \frac{\varepsilon}{r(\varepsilon)} \right) \right)^{-1/2} \log \left( \frac{\varepsilon}{r} \right)$$

where  $\mu_\varepsilon$  is defined by (1.46), and  $r = |x - \mathbf{k}|$ .

By using the function  $\theta_\varepsilon$  defined above we can now establish the following result which provides correcting terms for the sequence  $\{v_\varepsilon\}$ , solutions of problem (1.6).

**THEOREM 1.6 :** *Assume that  $r(\cdot)$  verifies (1.1) and that (1.10) holds. Let  $\bar{\Omega}$  be any open (bounded) subset of  $\Omega$  such that  $\bar{\Omega} \subset \Omega$ . Then we have :*

*If  $r(\cdot)$  verifies (1.12) or (1.13), then there exists a (rest) function  $\sigma_\varepsilon$  in  $H^1(\Omega_\varepsilon)$ , such that :*

$$(1.49a) \quad r(\varepsilon)^{-(N-1)} \varepsilon^N v_\varepsilon = v + \theta_\varepsilon + \sigma_\varepsilon \quad \text{in } \Omega_\varepsilon$$

$$(1.49b) \quad \|\nabla \sigma_\varepsilon\|_{0, \bar{\Omega} \cap \Omega_\varepsilon} \rightarrow 0 \quad \text{in } \mathbb{R}, \text{ as } \varepsilon \rightarrow 0$$

where  $v$  is the solution of the (limit) problem (1.24), and  $\theta_\varepsilon$  is given by (1.47).

*If  $r(\cdot)$  verifies (1.14a), and  $N \geq 3$ , then there exists a (rest) function  $\sigma_\varepsilon$  in  $H^1(\Omega_\varepsilon)$ , such that :*

$$(1.50a) \quad (r(\varepsilon)/\varepsilon)^{-N/2} v_\varepsilon = \theta_\varepsilon + \sigma_\varepsilon \quad \text{in } \Omega_\varepsilon$$

$$(1.50b) \quad \|\nabla \sigma_\varepsilon\|_{0, \bar{\Omega} \cap \Omega_\varepsilon} \rightarrow 0 \quad \text{in } \mathbb{R}, \text{ as } \varepsilon \rightarrow 0$$

where  $\theta_\varepsilon$  is given by (1.48a).

*If  $r(\cdot)$  verifies (1.14b), and  $N = 2$ , then there exists a (rest) function  $\sigma_\varepsilon$  in  $H^1(\Omega_\varepsilon)$ , such that :*

$$(1.51a) \quad (r(\varepsilon)/\varepsilon)^{-1} (\log(\varepsilon/r(\varepsilon)))^{-1/2} v_\varepsilon = \theta_\varepsilon + \sigma_\varepsilon \quad \text{in } \Omega_\varepsilon$$

$$(1.51b) \quad \|\nabla \sigma_\varepsilon\|_{0, \bar{\Omega} \cap \Omega_\varepsilon} \rightarrow 0 \quad \text{in } \mathbb{R}, \text{ as } \varepsilon \rightarrow 0$$

where  $\theta_\varepsilon$  is given by (1.48b). ■

The proof of this theorem is given in Section 5. It makes essential use of the explicit formulae (1.47), (1.48) for the function  $\theta_\varepsilon$ . To conclude, we would like to remark that following just the same pattern as in this section one can also compute a correcting term  $\theta_\varepsilon$  (with explicit formulae) for the sequence  $\{t_\varepsilon\}$ , solutions of problem (1.7), in case of spherical holes. For brevity in our exposition, we shall omit here these computations.

**2. PROOFS OF THE RESULTS IN THE CASE OF A CONSTANT NON-HOMOGENEOUS DATA**

In this section we shall prove Theorems 1.1, and 1.2, stated in Sections 1.4a, 1.4b, respectively. Throughout this section,  $C$  will denote different constants independent of  $\epsilon$ , and  $\{P_\epsilon\}$  will be any family of linear continuous extension operators from  $V_\epsilon$  onto  $H_0^1(\Omega)$ , verifying (1.22).

**2.1. A priori estimates. Proof of Theorem 1.1**

We shall divide the proof of this theorem into two parts. First, we prove the upper estimates in (1.15), (1.16), (1.17), and next we prove the lower estimates. In order to prove the upper bounds, we shall use the following lemma that we prove later in this section.

LEMMA 2.1 : *Assume that the function  $r(\cdot)$  verifies (1.1). Then there exists a constant  $C = C(\Omega, T)$ , which is independent of  $\epsilon$ , such that :*

$$(2.1) \quad \begin{cases} \text{If } r(\cdot) \text{ verifies (1.12) or (1.13), then} \\ \|\varphi\|_{0, \partial T_\epsilon}^2 \leq Cr(\epsilon)^{N-1} \epsilon^{-N} \|\varphi\|_{1, \Omega_\epsilon}^2 \quad \forall \varphi_\epsilon \in V_\epsilon \end{cases}$$

$$(2.2) \quad \begin{cases} \text{If } r(\cdot) \text{ verifies (1.14a), and } N \geq 3, \text{ then} \\ \|\varphi\|_{0, \partial T_\epsilon}^2 \leq Cr(\epsilon) \|\varphi\|_{1, \Omega_\epsilon}^2 \quad \forall \varphi_\epsilon \in V_\epsilon \end{cases}$$

$$(2.3) \quad \begin{cases} \text{If } r(\cdot) \text{ verifies (1.14b), and } N = 2, \text{ then} \\ \|\varphi\|_{0, \partial T_\epsilon}^2 \leq Cr(\epsilon) \log(\epsilon/r(\epsilon)) \|\varphi\|_{1, \Omega_\epsilon}^2 \quad \forall \varphi_\epsilon \in V_\epsilon. \quad \blacksquare \end{cases}$$

(a) *Proof of the upper bounds in Theorem 1.1*

Assume that  $r(\cdot)$  verifies (1.12) or (1.13). Taking  $\varphi = v_\epsilon$  in the variational formulation of problem (1.6) (cf. (1.11b)), and using Cauchy-Schwarz inequality, it follows that :

$$(2.4) \quad \|\nabla v_\epsilon\|_{0, \Omega_\epsilon}^2 \leq |\partial T_\epsilon|^{1/2} |\bar{g}| \|v_\epsilon\|_{0, \partial T_\epsilon}.$$

Since all the holes have the same shape, and the number of holes in  $\Omega_\epsilon$  is of the order of  $|\Omega| \epsilon^{-N}$ , then there exist two (positive) constants  $C_1 \leq C_2$ , which are independent of  $\epsilon$ , such that :

$$(2.5) \quad C_1 r(\epsilon)^{N-1} \epsilon^{-N} \leq |\partial T_\epsilon| \leq C_2 r(\epsilon)^{N-1} \epsilon^{-N}.$$

Combining (2.4) with (2.5), we obtain :

$$\|\nabla v_\epsilon\|_{0, \Omega_\epsilon}^2 \leq Cr(\epsilon)^{(N-1)/2} \epsilon^{-N/2} \|v_\epsilon\|_{0, \partial T_\epsilon}$$

where  $C = |\bar{g}| C_2$ . Therefore, from (2.1) it follows that :

$$(2.6a) \quad \|\nabla v_\varepsilon\|_{0,\Omega_\varepsilon}^2 \leq Cr(\varepsilon)^{N-1} \varepsilon^{-N} \|v_\varepsilon\|_{1,\Omega_\varepsilon}.$$

Besides that, by using Poincaré's inequality in  $\Omega$ , and (1.22b), it follows that there exists a constant  $C$ , such that :

$$(2.6b) \quad \|P_\varepsilon v_\varepsilon\|_{1,\Omega}^2 \leq C \|\nabla P_\varepsilon v_\varepsilon\|_{0,\Omega}^2 \leq C \|\nabla v_\varepsilon\|_{0,\Omega_\varepsilon}^2.$$

Combining (2.6a) with (2.6b), and using the fact that  $\|v_\varepsilon\|_{1,\Omega_\varepsilon}^2 \leq \|P_\varepsilon v_\varepsilon\|_{1,\Omega}^2$ , we obtain :

$$\|v_\varepsilon\|_{1,\Omega_\varepsilon}^2 \leq Cr(\varepsilon)^{N-1} \varepsilon^{-N} \|v_\varepsilon\|_{1,\Omega_\varepsilon}$$

which proves the upper bound in (1.15), with  $M = C$ . The proof of the upper bounds in (1.16), (1.17) are similar to this one. We note that in these cases one needs to use (2.2) or (2.3) instead of (2.1).

(b) *Proof of the lower bounds in Theorem 1.1*

To obtain the lower bounds we shall prove that for all function  $r(\cdot)$  verifying (1.1), the following (lower) estimates of the  $H^1(\Omega_\varepsilon)$ -norm of  $v_\varepsilon$  hold :

$$(2.7) \quad \|v_\varepsilon\|_{1,\Omega_\varepsilon} \geq mr(\varepsilon)^{N-1} \varepsilon^{-N}$$

$$(2.8a) \quad \|v_\varepsilon\|_{1,\Omega_\varepsilon} \geq m(r(\varepsilon)/\varepsilon)^{N/2} \quad \text{if } N \geq 3$$

$$(2.8b) \quad \|v_\varepsilon\|_{1,\Omega_\varepsilon} \geq m(r(\varepsilon)/\varepsilon)(\log(\varepsilon/r(\varepsilon)))^{1/2} \quad \text{if } N = 2$$

where  $m$  is a constant independent of  $\varepsilon$ . Therefore, if  $r(\cdot)$  verifies (1.12) or (1.13), then (2.7) gives a better (or greater) lower estimate than (2.8). This proves (1.15). On the other hand, if  $r(\cdot)$  verifies (1.14), then the right-hand side of (2.8) is greater than the right-hand side of (2.7), as  $\varepsilon \rightarrow 0$ . Then the lower bounds in (1.16), (1.17) hold true.

Let us first prove (2.7). Let  $\Omega'$  be an open (bounded) subset of  $\Omega$ , such that  $\bar{\Omega}' \subset \Omega$ , and let  $\omega$  in  $C_0^\infty(\bar{\Omega})$  be any given function verifying the following conditions :

$$(2.9a) \quad \omega(x) = 1 \quad \forall x \in \Omega'$$

$$(2.9b) \quad 0 \leq \omega(x) \leq 1 \quad \forall x \in \bar{\Omega}.$$

Taking  $\varphi = \omega|_{\Omega_\varepsilon}$  in the variational formulation of (1.6) (cf. (1.11b)), we have :

$$\int_{\Omega} \nabla v_\varepsilon \cdot \nabla \omega \, dx = \bar{g} \int_{\partial T_\varepsilon} \omega \, ds$$

which using Cauchy-Schwarz inequality implies that :

$$(2.10a) \quad |\bar{g}| \int_{\partial T_\epsilon} \omega \, ds \leq \| \nabla \omega \|_{0, \Omega} \| \nabla v_\epsilon \|_{0, \Omega_\epsilon} .$$

Since  $\omega$  verifies (2.9), it follows that :

$$(2.10b) \quad \int_{\partial T_\epsilon} \omega \, ds \geq \int_{\partial T_\epsilon \cap \Omega'} ds \geq C |\Omega'| |\partial T| r(\epsilon)^{N-1} \epsilon^{-N} .$$

Combining (2.10a) with (2.10b), we obtain :

$$\| \nabla v_\epsilon \|_{0, \Omega_\epsilon} \geq C \frac{|\Omega'| |\partial T|}{\| \nabla \omega \|_{0, \Omega}} r(\epsilon)^{N-1} \epsilon^{-N}$$

which proves (2.7) with  $m = C |\Omega'| |\partial T| / \| \nabla \omega \|_{0, \Omega}$ .

We pass now to the proof of (2.8). The proof consists in choosing suitable test functions in (1.11b). The test functions that we use were originally introduced by D. Cioranescu & F. Murat [4, Section 2, Examples 2.1, 2.6]. In their paper, these authors construct a sequence  $\{\omega_\epsilon\}$  of functions verifying the following properties :

$$(2.11) \quad \omega_\epsilon \in H^1(\Omega_\epsilon)$$

$$(2.12) \quad \omega_\epsilon = 1 \quad \text{on } \partial \Omega$$

$$(2.13) \quad \omega_\epsilon = 0 \quad \text{on } \partial T_\epsilon$$

$$(2.14a) \quad \| \nabla \omega_\epsilon \|_{0, \Omega_\epsilon}^2 \leq C \epsilon^{-N} \frac{1}{(1/r(\epsilon))^{N-2} - (1/\epsilon)^{N-2}} \quad \text{if } N \geq 3$$

$$(2.14b) \quad \| \nabla \omega_\epsilon \|_{0, \Omega_\epsilon}^2 \leq C \epsilon^{-2} (\log(\epsilon/r(\epsilon)))^{-1} \quad \text{if } N = 2$$

for all  $\epsilon$ .

Taking  $\varphi = 1 - \omega_\epsilon$  (1.11b) (which is possible, since  $(1 - \omega_\epsilon) \in V_\epsilon$ ), we obtain :

$$\int_{\Omega_\epsilon} \nabla v_\epsilon \cdot \nabla \omega_\epsilon \, dx = \bar{g} \int_{\partial T_\epsilon} ds$$

which using Cauchy-Schwarz inequality and (2.5), implies that :

$$(2.15) \quad C_1 |\bar{g}| r(\epsilon)^{N-1} \epsilon^{-N} \leq \| \nabla v_\epsilon \|_{0, \Omega_\epsilon} \| \nabla \omega_\epsilon \|_{0, \Omega_\epsilon} .$$



Combining (2.15) with (2.14a) (in case  $N \geq 3$ ), and (2.15) with (2.14b) (in case  $N = 2$ ), we obtain :

$$(2.16a) \quad C_1 |\bar{g}| r(\varepsilon)^{N-1} \varepsilon^{-N} \leq \varepsilon^{-N/2} \left( \frac{1}{(1/r(\varepsilon))^{N-2} - (1/\varepsilon)^{N-2}} \right)^{1/2} \times \\ \times \|\nabla v_\varepsilon\|_{0, \Omega_\varepsilon} \quad \text{if } N \geq 3$$

$$(2.16b) \quad C_1 |\bar{g}| r(\varepsilon) \varepsilon^{-2} \leq \varepsilon^{-1} (\log(\varepsilon/r(\varepsilon)))^{-1/2} \|\nabla v_\varepsilon\|_{0, \Omega_\varepsilon} \quad \text{if } N = 2$$

which implies (2.8) with  $m = C_1 |\bar{g}|$ , because  $\bar{g}$  verifies (1.10). This completes the proof of the lower bounds in Theorem 1.1.

To conclude the proof of Theorem 1.1, let us prove Lemma 2.1.

*Proof of Lemma 2.1 :* Let us define  $S_\varepsilon$  and  $\gamma_\varepsilon$  by :

$$S_\varepsilon = B(0, \varepsilon) \setminus \overline{r(\varepsilon) T} \\ \gamma_\varepsilon = \partial(r(\varepsilon) T)$$

where  $B(0, \varepsilon)$  is the open ball centered at the origin of radius  $\varepsilon$  (see fig. 2.1).

We begin the proof of the lemma by remarking that in order to prove (2.1), (2.2), (2.3), it suffices to prove that there exists a constant  $C$ , independent of  $\varepsilon$ , such that :

$$(2.17) \quad \left\{ \begin{array}{l} \text{If } r(\cdot) \text{ verifies (1.12) or (1.13), then} \\ \|\varphi\|_{0, \gamma_\varepsilon}^2 \leq Cr(\varepsilon)^{N-1} \varepsilon^{-N} \|\varphi\|_{1, S_\varepsilon}^2 \quad \forall \varphi \in H^1(S_\varepsilon) \end{array} \right.$$

$$(2.18) \quad \left\{ \begin{array}{l} \text{If } r(\cdot) \text{ verifies (1.14a), and } N \geq 3, \text{ then} \\ \|\varphi\|_{0, \gamma_\varepsilon}^2 \leq Cr(\varepsilon) \|\varphi\|_{1, S_\varepsilon}^2 \quad \forall \varphi \in H^1(S_\varepsilon) \end{array} \right.$$

$$(2.19) \quad \left\{ \begin{array}{l} \text{If } r(\cdot) \text{ verifies (1.14b), and } N = 2, \text{ then} \\ \|\varphi\|_{0, \gamma_\varepsilon}^2 \leq Cr(\varepsilon) \log(\varepsilon/r(\varepsilon)) \|\varphi\|_{1, S_\varepsilon}^2 \quad \forall \varphi \in H^1(S_\varepsilon). \end{array} \right.$$

Let  $\varphi$  be a given function in  $C^\infty(\bar{S}_\varepsilon)$ . We denote by  $(\rho, \theta)$ ,  $\theta = (\theta_1, \dots, \theta_{N-1})$ , the spherical coordinates in  $\mathbb{R}^N$ . Since  $T$  is star-shaped with respect to the origin, then the boundary  $\partial T$  of  $T$  can be (parametrically) represented as follows :

$$\partial T = \{(\rho, \theta) \mid \rho = \Phi(\theta), \theta \in Q\}$$

where  $Q = [0, 2\pi] \times [-\pi/2, \pi/2]^{N-2}$ , and  $\Phi$  is a given non-negative function from  $Q$  into  $\mathbb{R}_+$ . Moreover, since  $\partial T$  has been assumed to be

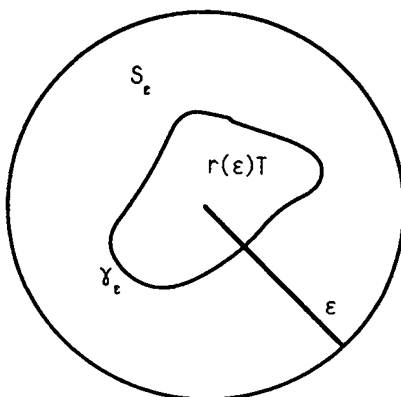


Figure 2.1.

smooth, then  $\Phi$  is a smooth function of its argument. Let us denote by  $\rho^{N-1} J(\theta)$  the Jacobian determinant of the standard transformation from Cartesian coordinates to spherical coordinates, and let  $\chi(\rho, \theta)$  be the function  $\varphi(x)$  written in spherical coordinates. We have :

$$(2.20a) \quad \|\varphi\|_{0, \gamma_\epsilon}^2 = r(\epsilon)^{N-1} \int_Q |\chi(r(\epsilon) \Phi(\theta), \theta)|^2 J(\theta) F(\theta) d\theta$$

where

$$(2.20b) \quad F(\theta) = \prod_{i=1}^{N-1} \sqrt{\Phi(\theta)^2 + (\partial\Phi/\partial\theta_i)^2}.$$

On the other hand, the region  $\bar{S}_\epsilon$  in spherical coordinates is represented by :

$$S_\epsilon = \{(\rho, \theta) \mid r(\epsilon) \Phi(\theta) \leq \rho \leq \epsilon, \theta \in Q\},$$

and for all  $(\rho, \theta)$  in  $S_\epsilon$ , we have :

$$\chi(r(\epsilon) \Phi(\theta), \theta) = \chi(\rho, \theta) - \int_{r(\epsilon)\Phi}^{\rho} (\partial\chi/\partial t)(t, \theta) dt$$

which implies that :

$$(2.21) \quad |\chi(r(\epsilon) \Phi, \theta)|^2 \leq 2|\chi(\rho, \theta)|^2 + 2 \left| \int_{r(\epsilon)\Phi}^{\rho} (\partial\chi/\partial t)(t, \theta) dt \right|^2.$$

By using the fact that  $t^{(N-1)/2} t^{-(N-1)/2}$  is equal to 1, it follows from Cauchy-Schwarz inequality :

$$(2.22) \quad \left| \int_{r(\varepsilon)\Phi}^{\rho} (\partial\chi/\partial t)(t, \theta) dt \right|^2 \leq \left( \int_{r(\varepsilon)\Phi}^{\rho} t^{-(N-1)} dt \right) \times \left( \int_{r(\varepsilon)\Phi}^{\rho} t^{N-1} |(\partial\chi/\partial t)(t, \theta)|^2 dt \right).$$

Besides that, let  $b_1 \leq b_2$  be defined by :

$$(2.23a) \quad b_1 = \min_{\theta \in Q} \Phi(\theta)$$

$$(2.23b) \quad b_2 = \max_{\theta \in Q} \Phi(\theta).$$

Thereby, since  $r(\varepsilon)\Phi(\theta) \geq b_1 r(\varepsilon)$  for all  $\theta$  in  $Q$ , and  $\rho$  is less or equal to  $\varepsilon$ , it follows from (2.21), (2.22) that :

$$(2.24a) \quad |\chi(r(\varepsilon)\Phi, \theta)|^2 \leq 2|\chi(\rho, \theta)|^2 + 2\tau_{1\varepsilon} \int_{r(\varepsilon)\Phi}^{\varepsilon} t^{N-1} |(\partial\chi/\partial t)(t, \theta)|^2 dt$$

where

$$(2.24b) \quad \tau_{1\varepsilon} = \int_{b_1 r(\varepsilon)}^{\varepsilon} t^{-(N-1)} dt.$$

We multiply (2.24a) by  $\rho^{N-1} J(\theta) F(\theta)$ , and we integrate in  $S_\varepsilon$ . We obtain :

$$(2.25a) \quad \left\{ \begin{aligned} & \int_Q \int_{r(\varepsilon)\Phi}^{\varepsilon} |\chi(r(\varepsilon)\Phi, \theta)|^2 \rho^{N-1} JF d\rho d\theta \\ & \leq 2 \int_Q \int_{r(\varepsilon)\Phi}^{\varepsilon} |\chi(\rho, \theta)|^2 \rho^{N-1} JF d\rho d\theta + \\ & + 2\tau_{1\varepsilon} \tau_{2\varepsilon} \int_Q \int_{r(\varepsilon)\Phi}^{\varepsilon} t^{N-1} |(\partial\chi/\partial t)(t, \theta)|^2 JF dt d\theta \end{aligned} \right.$$

where

$$(2.25b) \quad \tau_{2\varepsilon} = \int_{b_1 r(\varepsilon)}^{\varepsilon} \rho^{N-1} d\rho.$$

Since  $r(\varepsilon)\Phi(\theta) \leq b_2 r(\varepsilon)$  for all  $\theta$  in  $Q$ , it follows from (2.20a) that the left-hand side of (2.25a) verifies :

$$(2.26a) \quad \int_Q \int_{r(\varepsilon)\Phi}^{\varepsilon} |\chi(r(\varepsilon)\Phi, \theta)|^2 \rho^{N-1} JF d\rho d\theta \geq r(\varepsilon)^{-(N-1)} \tau_{3\varepsilon} \|\varphi\|_{0, \gamma_\varepsilon}^2$$

where

$$(2.26b) \quad \tau_{3\varepsilon} = \int_{b_2 r(\varepsilon)}^\varepsilon \rho^{N-1} d\rho .$$

On the other hand, since

$$|(\partial\chi/\partial t)(t, \theta)|^2 \leq N |\nabla_x \varphi(t, \theta)|^2$$

for all  $(t, \theta) \in S_\varepsilon$ , then the second term in the right-hand side of (2.25a) can be estimated as follows :

(2.27)

$$\int_Q \int_{r(\varepsilon)\Phi}^\varepsilon t^{N-1} |(\partial\chi/\partial t)(t, \theta)|^2 JF dt d\theta \leq N \left( \max_{\theta \in Q} F(\theta) \right) \|\nabla\varphi\|_{0, S_\varepsilon}^2 .$$

Therefore, since the first term in the right-hand side of (2.25a) is lower or equal to  $2 \left( \max_{\theta \in Q} F(\theta) \right) \|\varphi\|_{0, S_\varepsilon}^2$ , it follows combining (2.25a) with (2.26a),

(2.27) that :

$$(2.28) \quad \|\varphi\|_{0, \gamma_\varepsilon}^2 \leq C_0 \tau_{3\varepsilon}^{-1} r(\varepsilon)^{N-1} \left\{ \|\varphi\|_{0, S_\varepsilon}^2 + \tau_{1\varepsilon} \tau_{2\varepsilon} \|\nabla\varphi\|_{0, S_\varepsilon}^2 \right\}$$

where  $C_0 = 2N \max_{\theta \in Q} F(\theta)$ .

But using (2.25b), (2.26b), we have :

$$\begin{aligned} \tau_{2\varepsilon} &\leq C\varepsilon^N \\ \tau_{3\varepsilon}^{-1} &\leq C\varepsilon^{-N} . \end{aligned}$$

Therefore, (2.28) implies that :

$$(2.29) \quad \|\varphi\|_{0, \gamma_\varepsilon}^2 \leq C \left\{ r(\varepsilon)^{N-1} \varepsilon^{-N} \|\varphi\|_{0, S_\varepsilon}^2 + r(\varepsilon)^{N-1} \tau_{1\varepsilon} \|\nabla\varphi\|_{0, S_\varepsilon}^2 \right\} .$$

On the other hand, using (2.24b) we have :

$$\begin{aligned} C_1 r(\varepsilon)^{-(N-2)} &\leq \tau_{1\varepsilon} \leq C_2 r(\varepsilon)^{-(N-2)} && \text{if } N \geq 3 \\ C_1 \log(\varepsilon/r(\varepsilon)) &\leq \tau_{1\varepsilon} \leq C_2 \log(\varepsilon/r(\varepsilon)) && \text{if } N = 2 \end{aligned}$$

where  $C_1 \leq C_2$  are two constants, independent of  $\varepsilon$ . To complete the proof of (2.17), (2.18), and (2.19), it suffices to remark that : (i) if  $r(\cdot)$  verifies (1.12) or (1.13), then

$$r(\varepsilon)^{N-1} \varepsilon^{-N} \geq Cr(\varepsilon)^{N-1} \tau_{1\varepsilon}$$

so (2.29) implies that (2.17) holds. (ii) If  $r(\cdot)$  verifies (1.14a), and  $N \geq 3$ , or it verifies (1.14b), and  $N = 2$ , then

$$r(\varepsilon)^{N-1} \varepsilon^{-N} \leq Cr(\varepsilon)^{N-1} \tau_{1\varepsilon}$$

so (2.29) implies that (2.18), (2.19) hold. This completes the proof of (2.17)-(2.19) for any smooth function  $\varphi$  in  $H^1(S_\varepsilon)$ . Using standard density arguments it follows that (2.17)-(2.19) hold for any function  $\varphi$  in  $H^1(S_\varepsilon)$ . This completes the proof of Lemma 2.1, and Theorem 1.1. ■

**2.2. Proof of Theorem 1.2**

We shall divide the proof of Theorem 1.2 into two parts. First, we prove the theorem assuming that  $r(\cdot)$  verifies (1.12) or (1.13). Next, we prove (1.25) and (1.26).

(a) *Proof of (1.23)* : By using (1.15) and (1.22b), it is an easy matter to see that the sequence  $\{r(\varepsilon)^{-(N-1)} \varepsilon^N P_\varepsilon v_\varepsilon\}$  remains bounded in  $H_0^1(\Omega)$ , as  $\varepsilon \rightarrow 0$ . We can therefore extract from this sequence a subsequence, still denoted by  $\{r(\varepsilon)^{-(N-1)} \varepsilon^N P_\varepsilon v_\varepsilon\}$ , weakly convergent in  $H_0^1(\Omega)$ . That is,

$$(2.30) \quad r(\varepsilon)^{-(N-1)} \varepsilon^N P_\varepsilon v_\varepsilon \rightharpoonup v \text{ weakly in } H_0^1(\Omega), \text{ as } \varepsilon \rightarrow 0.$$

Our goal in what follows is to prove that  $v$  is the (unique) solution of problem (1.24). Let  $\varphi$  be any given function in  $C_0^\infty(\Omega)$ , and let us take  $\varphi$  as test function in (1.11b). We have :

$$(2.31) \quad \int_{\Omega_\varepsilon} \nabla v_\varepsilon \cdot \nabla \varphi \, dx = \bar{g} \int_{\partial T_\varepsilon} \varphi \, ds.$$

Multiplying this identity by  $r(\varepsilon)^{-(N-1)} \varepsilon^N$ , and introducing the characteristic function  $\chi_{\Omega_\varepsilon}$  of  $\Omega_\varepsilon$ , (2.31) can be rewritten as :

$$(2.32) \quad \int_{\Omega} \chi_{\Omega_\varepsilon} \nabla(r(\varepsilon)^{-(N-1)} \varepsilon^N P_\varepsilon v_\varepsilon) \cdot \nabla \varphi \, dx = r(\varepsilon)^{-(N-1)} \varepsilon^N \bar{g} \int_{\partial T_\varepsilon} \varphi \, ds.$$

Since  $r(\cdot)$  verifies (1.1), it is easy to check that the sequence  $\{\chi_{\Omega_\varepsilon}\}$  satisfies :

$$(2.33) \quad \chi_{\Omega_\varepsilon} \rightarrow 1 \text{ strongly in } L^2(\Omega), \text{ as } \varepsilon \rightarrow 0.$$

Therefore, combining (2.30) with (2.33), we can pass to the limit in the left-hand side of (2.32). We have :

$$(2.34) \quad \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \chi_{\Omega_\varepsilon} \nabla(r(\varepsilon)^{-(N-1)} \varepsilon^N P_\varepsilon v_\varepsilon) \cdot \nabla \varphi \, dx = \int_{\Omega} \nabla v \cdot \nabla \varphi \, dx.$$

The next step of the proof consists in passing to the limit in the right-hand side of (2.32). To do that, let us introduce the sequence  $\{\nu_\varepsilon\}$  of positive Radon measures defined in  $C_0^0(\Omega)$  by :

$$(2.35) \quad \langle \nu_\varepsilon, \psi \rangle = r(\varepsilon)^{-(N-1)} \varepsilon^N \int_{\partial T_\varepsilon} \psi \, ds \quad \forall \psi \in C_0^0(\Omega).$$

By using (2.5), it follows that :

$$|\langle \nu_\varepsilon, \psi \rangle| \leq C_2 \|\psi\|_{C_0^0(\Omega)}$$

i.e., the sequence  $\{\nu_\varepsilon\}$  remains bounded, as  $\varepsilon \rightarrow 0$ , in the space of Radon's measures on  $\Omega$ . We can therefore extract from  $\{\nu_\varepsilon\}$  a subsequence, still denoted by  $\{\nu_\varepsilon\}$ , weakly \* convergent to a (positive) measure  $\nu$  on the space of Radon measures on  $\Omega$ , i.e.,

$$(2.36) \quad \forall \psi \in C_0^0(\Omega), \langle \nu_\varepsilon, \psi \rangle \rightarrow \langle \nu, \psi \rangle, \text{ as } \varepsilon \rightarrow 0.$$

In order to identify  $\nu$ , let us begin remarking that  $\nu$  can be identified, using the Riesz Representation Theorem (cf. e.g. W. Rudin [9, Th. 2.14]), with a (positive) measure  $\tilde{\nu}$  on  $\Omega$ , such that :

$$(2.37) \quad \langle \nu, \psi \rangle = \int_{\Omega} \psi \, d\tilde{\nu} \quad \forall \psi \in C_0^0(\Omega)$$

where  $\tilde{\nu}$  is (uniquely) defined by :

$$\tilde{\nu}(A) = \sup \{ \langle \nu, \psi \rangle \mid \psi \in C_0^0(A), 0 \leq \psi \leq 1 \}$$

for all open subset  $A$  of  $\Omega$ .

Let  $A$  be any open subset of  $\Omega$ . Since

$$(2.38a) \quad \varepsilon^N N_\varepsilon(A) \rightarrow |A|, \text{ as } \varepsilon \rightarrow 0$$

where  $N_\varepsilon(A)$  is the number of holes having a non empty intersection with  $A$ , it follows that :

$$(2.38b) \quad \lim_{\varepsilon \rightarrow 0} r(\varepsilon)^{-(N-1)} \varepsilon^N \int_{A \cap \partial T_\varepsilon} ds = |A| |\partial T|.$$

Therefore, using the definition of  $\tilde{\nu}$  it is an easy matter to prove that :

$$\tilde{\nu}(A) = |\partial T| |A|$$

which implies that  $\tilde{v}$  is  $|\partial T|$  times the restriction to  $\Omega$  of the  $\mathbb{R}^N$  – Lebesgue measure, since  $A$  is an arbitrary open subset of  $\Omega$ . Hence, we can pass to the limit in the right-hand side of (2.32).

Using (2.35), (2.36), and (2.37), we have :

$$(2.39) \quad \lim_{\varepsilon \rightarrow 0} \bar{g} r(\varepsilon)^{-(N-1)} \varepsilon^N \int_{\partial T_\varepsilon} \varphi \, ds = \bar{g} \int_{\Omega} \varphi \, d\tilde{v} = \bar{g} |\partial T| \int_{\Omega} \varphi \, dx$$

for all  $\varphi$  in  $C_0^\infty(\Omega)$ . Combining (2.34) with (2.39), we conclude that  $v$  is a solution of problem (1.24). Since this problem has a unique solution, it follows that the whole sequence  $\{r(\varepsilon)^{-(N-1)} \varepsilon^N P_\varepsilon v_\varepsilon\}$  in (2.30) weakly converge to  $v$  in  $H_0^1(\Omega)$ . This completes the proof of (1.23).

*Proof of (1.25) and (1.26) :* We begin the proof by assuming that (1.14a) holds. As in the proof of (1.23), by using (1.16) it follows that the sequence  $\{(r(\varepsilon)/\varepsilon)^{-N/2} P_\varepsilon v_\varepsilon\}$  remains bounded in  $H_0^1(\Omega)$ , as  $\varepsilon \rightarrow 0$ . We can therefore extract from this sequence a subsequence, that we shall still denote by  $\{(r(\varepsilon)/\varepsilon)^{-N/2} P_\varepsilon v_\varepsilon\}$ , such that :

$$(2.40) \quad (r(\varepsilon)/\varepsilon)^{-N/2} P_\varepsilon v_\varepsilon \rightharpoonup v \text{ weakly in } H_0^1(\Omega), \text{ as } \varepsilon \rightarrow 0.$$

Following the same arguments of the proof of (1.25), it can be easily checked that for all  $\varphi$  in  $C_0^\infty(\Omega)$ , we have :

$$(2.41a) \quad \int_{\Omega} \chi_{\Omega_\varepsilon} \nabla((r(\varepsilon)/\varepsilon)^{-N/2} P_\varepsilon v_\varepsilon) \cdot \nabla \varphi \, dx = \bar{g} (r(\varepsilon)/\varepsilon)^{-N/2} \int_{\partial T_\varepsilon} \varphi \, ds$$

and

$$(2.41b) \quad \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \chi_{\Omega_\varepsilon} \nabla((r(\varepsilon)/\varepsilon)^{-N/2} P_\varepsilon v_\varepsilon) \cdot \nabla \varphi \, dx = \int_{\Omega} \nabla v \cdot \nabla \varphi \, dx.$$

Besides that, by using (2.5), the right-hand side of (2.41a) can be estimated as follows :

$$(2.42) \quad |\bar{g}| (r(\varepsilon)/\varepsilon)^{-N/2} \left| \int_{\partial T_\varepsilon} \varphi \, ds \right| \leq C r(\varepsilon)^{(N-2)/2} \varepsilon^{-N/2} \|\varphi\|_{C_0^\infty(\Omega)}.$$

Since  $r(\cdot)$  verifies (1.14a), the right-hand side of this inequality goes to zero as  $\varepsilon \rightarrow 0$ .

Therefore, using (2.41b) and passing to the limit in (2.41a), we obtain :

$$\int_{\Omega} \nabla v \cdot \nabla \varphi \, dx = 0 \quad \forall \varphi \in C_0^\infty(\Omega)$$

which clearly implies  $v = 0$ . This proves that  $v = 0$  is the unique (weak) cluster point of the sequence  $\{(r(\varepsilon)/\varepsilon)^{-N/2} P_\varepsilon v_\varepsilon\}$ . This completes the proof of (1.25). The proof of (1.26) is very similar and we shall omit it. Theorem 1.2 is therefore proved. ■

**2.3. A correcting term for the solution of problem (1.6)**

In this section we study a corrector for the sequence  $\{v_\varepsilon\}$ , solutions of problem (1.6). In what follows we use the following notations :

$$\begin{aligned} G_\varepsilon &= \cup \{T(\varepsilon, \mathbf{k}) \mid \mathbf{k} \in \mathbb{Z}^N\} \\ \Gamma_\varepsilon &= \partial G_\varepsilon = \cup \{\partial T(\varepsilon, \vec{\mathbf{k}}) \mid \mathbf{k} \in \mathbb{Z}^N\} \\ Y_\varepsilon &= ]-\varepsilon/2, \varepsilon/2[^N, Y_\varepsilon^* = Y_\varepsilon \setminus r(\varepsilon) \bar{T}. \end{aligned}$$

Let us introduce the sequence  $\{\xi_\varepsilon\}$  of functions defined in  $G_\varepsilon$  up to an additive constant, as follows :

If  $r(\cdot)$  verifies (1.12) or (1.13), then

$$\begin{aligned} (2.43a) \quad & -\Delta \xi_\varepsilon = \rho_\varepsilon && \text{in } \mathbb{R}^N \setminus \bar{G}_\varepsilon \\ (2.43b) \quad & \partial \xi_\varepsilon / \partial n = \bar{g} r(\varepsilon)^{-(N-1)} \varepsilon^N && \text{on } \Gamma_\varepsilon \\ (2.43c) \quad & \xi_\varepsilon \text{ is } \varepsilon\text{-periodic in all its variables,} \end{aligned}$$

where  $\rho_\varepsilon$  is defined by :

$$(2.44) \quad \rho_\varepsilon = -\bar{g} |\partial T| \varepsilon^N / (\varepsilon^N - r(\varepsilon)^N |T|).$$

On the other hand, if  $r(\cdot)$  verifies (1.14a), and  $N \geq 3$ , then

$$\begin{aligned} (2.45a) \quad & -\Delta \xi_\varepsilon = \rho_\varepsilon && \text{in } \mathbb{R}^N \setminus \bar{G}_\varepsilon \\ (2.45b) \quad & \partial \xi_\varepsilon / \partial n = \bar{g} (r(\varepsilon)/\varepsilon)^{-N/2} && \text{on } \Gamma_\varepsilon \\ (2.45c) \quad & \xi_\varepsilon \text{ is } \varepsilon\text{-periodic in all its variables,} \end{aligned}$$

where  $\rho_\varepsilon$  is defined in this case by :

$$(2.46) \quad \rho_\varepsilon = -\bar{g} |\partial T| \varepsilon^{N/2} r(\varepsilon)^{(N-2)/2} / (\varepsilon^N - r(\varepsilon)^N |T|).$$

Finally, if  $r(\cdot)$  verifies (1.14b) and  $N = 2$ , then

$$\begin{aligned} (2.47a) \quad & -\Delta \xi_\varepsilon = \rho_\varepsilon && \text{in } \mathbb{R}^2 \setminus \bar{G}_\varepsilon \\ (2.47b) \quad & \partial \xi_\varepsilon / \partial n = \bar{g} (r(\varepsilon)/\varepsilon)^{-1} (\log(\varepsilon/r(\varepsilon)))^{-1/2} && \text{on } \Gamma_\varepsilon \\ (2.47c) \quad & \xi_\varepsilon \text{ is } \varepsilon\text{-periodic in all its variables,} \end{aligned}$$



where  $\rho_\varepsilon$  is defined by :

$$(2.48) \quad \rho_\varepsilon = -\bar{g} |\partial T| \varepsilon (\log (\varepsilon / r(\varepsilon)))^{-1} / (\varepsilon^2 - r^2(\varepsilon) |T|).$$

Besides that, let  $\{Q_\varepsilon\}$  be a family of linear extension operators verifying the following conditions :

$$(2.49a) \quad (Q_\varepsilon \varphi)(x) = \varphi(x) \quad \forall x \in G_\varepsilon$$

$$(2.49b) \quad \text{If } \varphi \text{ is } \varepsilon\text{-periodic, then } Q_\varepsilon \varphi \text{ is } \varepsilon\text{-periodic.}$$

$$(2.49c) \quad \|\nabla Q_\varepsilon \varphi\|_{0, \omega} \leq C \|\nabla \varphi\|_{0, \omega \setminus \bar{G}_\varepsilon} \quad \forall \varphi \in H^1_{loc}(\mathbb{R}^N / \bar{G}_\varepsilon), \quad \forall \omega \subset \subset \mathbb{R}^N$$

where  $C$  is independent of  $\varepsilon$  and  $\omega$  (for the existence of at least one such family, we refer to the appendix A). By using the operator  $Q_\varepsilon$ , we shall fix the indeterminated constant occurring in the definition of  $\xi_\varepsilon$  by imposing that for all  $\varepsilon$ ,

$$(2.50) \quad \int_{Y_\varepsilon} Q_\varepsilon \xi_\varepsilon dx = 0.$$

**THEOREM 2.2 :** *Assume that  $r(\cdot)$  verifies (1.1) and that (1.10) holds. Let  $\Omega'$  be any open (bounded) subset of  $\Omega$  such that  $\bar{\Omega}' \subset \Omega$ . Then we have :*

*If  $r(\cdot)$  verifies (1.12) or (1.13), then there exists a (rest) function  $\alpha_\varepsilon$  in  $H^1(\Omega_\varepsilon)$ , such that :*

$$(2.51a) \quad r(\varepsilon)^{-(N-1)} \varepsilon^N v_\varepsilon = v + \xi_\varepsilon + \alpha_\varepsilon \quad \text{in } \Omega_\varepsilon$$

$$(2.51b) \quad \|\nabla \alpha_\varepsilon\|_{0, \Omega' \cap \Omega_\varepsilon} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0$$

where  $v$  is the solution of the (limit) problem (1.24) and  $\xi_\varepsilon$  is given by (2.43), (2.50).

*If  $r(\cdot)$  verifies (1.14a), and  $N \geq 3$ , then there exists a (rest) function  $\alpha_\varepsilon$  in  $H^1(\Omega_\varepsilon)$ , such that :*

$$(2.52a) \quad (r(\varepsilon)/\varepsilon)^{-N/2} v_\varepsilon = \xi_\varepsilon + \alpha_\varepsilon \quad \text{in } \Omega_\varepsilon$$

$$(2.52b) \quad \|\nabla \alpha_\varepsilon\|_{0, \Omega' \cap \Omega_\varepsilon} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0$$

where  $\xi_\varepsilon$  is given by (2.45), (2.50).

*If  $r(\cdot)$  verifies (1.14b), and  $N = 2$ , then there exists a (rest) function  $\alpha_\varepsilon$  in  $H^1(\Omega_\varepsilon)$ , such that :*

$$(2.53a) \quad (r(\varepsilon)/\varepsilon)^{-1} (\log (\varepsilon / r(\varepsilon)))^{-1/2} v_\varepsilon = \xi_\varepsilon + \alpha_\varepsilon \quad \text{in } \Omega_\varepsilon$$

$$(2.53b) \quad \|\nabla \alpha_\varepsilon\|_{0, \Omega' \cap \Omega_\varepsilon} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0$$

where  $\xi_\varepsilon$  is given by (2.47), (2.50). ■

*Proof:* We prove the theorem in case the function  $r(\cdot)$  verifies (1.12) or (1.13). In the other cases, the proof is very similar, and we shall omit it.

We begin the proof of (2.51) by studying the limit behaviour of the sequence  $\{\xi_\varepsilon\}$ , defined by (2.43), (2.50). By (2.17) and similar arguments to those used in the proof of the upper bounds in Theorem 1.1 (cf. Section 2.1(a)), it can be easily checked that :

$$(2.54a) \quad \|\nabla \xi_\varepsilon\|_{0, Y_\varepsilon^*}^2 \leq C \varepsilon^{N/2} \|\xi_\varepsilon\|_{1, Y_\varepsilon^*}$$

where  $C$  is independent of  $\varepsilon$ . Besides that, since  $Q_\varepsilon \xi_\varepsilon$  verifies (2.50), the following Poincaré's inequality holds :

$$(2.54b) \quad \|Q_\varepsilon \xi_\varepsilon\|_{0, Y_\varepsilon} \leq C \varepsilon \|\nabla Q_\varepsilon \xi_\varepsilon\|_{0, Y_\varepsilon}$$

Therefore, combining (2.49) with (2.54) we deduce that there exists a constant  $C$ , independent of  $\varepsilon$ , such that :

$$\|Q_\varepsilon \xi_\varepsilon\|_{1, Y_\varepsilon}^2 \leq C \varepsilon^N$$

which proves that the sequence  $\{Q_\varepsilon \xi_\varepsilon\}$  remains bounded in  $H^1_{loc}(\mathbb{R}^N)$ , as  $\varepsilon \rightarrow 0$ . Hence, from (2.50), Lemma B1 (cf. appendix B) shows that :

$$(2.54c) \quad Q_\varepsilon \xi_\varepsilon \rightharpoonup 0 \text{ weakly in } H^1_{loc}(\mathbb{R}^N), \text{ as } \varepsilon \rightarrow 0.$$

We pass now to prove (2.51). First, let us define  $\alpha_\varepsilon$  in  $\Omega_\varepsilon$  by :

$$\alpha_\varepsilon = r(\varepsilon)^{-(N-1)} \varepsilon^N v_\varepsilon - \xi_\varepsilon - v.$$

According to this definition, the proof of (2.51) reduces to prove (2.51b), or equivalently, to prove that :

$$(2.55) \quad \lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} \varphi^2 |\nabla \alpha_\varepsilon|^2 dx = 0,$$

for all  $\varphi$  in  $C^\infty_0(\Omega)$ . In order to prove (2.55), let  $\varphi$  be any given function in  $C^\infty_0(\Omega)$ . Using the definition of  $\alpha_\varepsilon$ , we decompose the left-hand side of (2.55), as follows :

$$(2.56) \quad \int_{\Omega_\varepsilon} \varphi^2 |\nabla \alpha_\varepsilon|^2 dx = A_\varepsilon + B_\varepsilon + C_\varepsilon - 2 D_\varepsilon - 2 E_\varepsilon + 2 F_\varepsilon$$

where

$$(2.57a) \quad A_\varepsilon = \int_{\Omega_\varepsilon} \varphi^2 |\nabla(r(\varepsilon)^{-(N-1)} \varepsilon^N v_\varepsilon)|^2 dx$$

$$(2.57b) \quad B_\varepsilon = \int_{\Omega_\varepsilon} \varphi^2 |\nabla \xi_\varepsilon|^2 dx$$

$$(2.57c) \quad C_\varepsilon = \int_{\Omega_\varepsilon} \varphi^2 |\nabla v|^2 dx$$

$$(2.57d) \quad D_\varepsilon = \int_{\Omega_\varepsilon} \varphi^2 \nabla(r(\varepsilon)^{-(N-1)} \varepsilon^N v_\varepsilon) \cdot \nabla \xi_\varepsilon dx$$

$$(2.57e) \quad E_\varepsilon = \int_{\Omega_\varepsilon} \varphi^2 \nabla(r(\varepsilon)^{-(N-1)} \varepsilon^N v_\varepsilon) \cdot \nabla v dx$$

$$(2.57f) \quad F_\varepsilon = \int_{\Omega_\varepsilon} \varphi^2 \nabla \xi_\varepsilon \cdot \nabla v dx.$$

Since  $r(\cdot)$  verifies (1.1), we have :

$$(2.58) \quad \chi_{\Omega_\varepsilon} \rightharpoonup 1 \text{ weakly } * \text{ in } L^\infty(\Omega), \text{ as } \varepsilon \rightarrow 0.$$

We can therefore pass to the limit in (2.57c). We obtain :

$$(2.59) \quad \lim_{\varepsilon \rightarrow 0} C_\varepsilon = \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \chi_{\Omega_\varepsilon} \varphi^2 |\nabla v|^2 dx = \int_{\Omega} \varphi^2 |\nabla v|^2 dx.$$

On the other hand, by using (1.23), (2.33), (2.54c), we deduce :

(2.60)

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} (E_\varepsilon - F_\varepsilon) &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \varphi^2 \nabla(r(\varepsilon)^{-(N-1)} \varepsilon^N P_\varepsilon v_\varepsilon - Q_\varepsilon \xi_\varepsilon) \cdot \chi_{\Omega_\varepsilon} \nabla v dx \\ &= \int_{\Omega} \varphi^2 |\nabla v|^2 dx. \end{aligned}$$

Next, taking  $\varphi^2(r(\varepsilon)^{-(N-1)} \varepsilon^N)^2 v_\varepsilon$  as test function in (1.11b), we have :

$$(2.61) \quad \begin{aligned} A_\varepsilon &= - \int_{\Omega_\varepsilon} \nabla(r(\varepsilon)^{-(N-1)} \varepsilon^N v_\varepsilon) \cdot \nabla(\varphi^2)(r(\varepsilon)^{-(N-1)} \varepsilon^N v_\varepsilon) dx \\ &\quad + \bar{g}(r(\varepsilon)^{-(N-1)} \varepsilon^N)^2 \int_{\partial T_\varepsilon} \varphi^2 v_\varepsilon ds. \end{aligned}$$

Besides that, multiplying (2.43a) by  $r(\varepsilon)^{-(N-1)} \varepsilon^N \varphi^2 v_\varepsilon$ , and integrating by parts in  $\Omega_\varepsilon$ , we obtain :

$$(2.62) \quad D_\varepsilon = - \int_{\Omega_\varepsilon} \nabla \xi_\varepsilon \cdot \nabla(\varphi^2)(r(\varepsilon)^{-(N-1)} \varepsilon^N v_\varepsilon) dx + \bar{g} \int_{\partial T_\varepsilon} \varphi^2 (r(\varepsilon)^{-(N-1)} \varepsilon^N)^2 v_\varepsilon ds + \rho_\varepsilon \int_{\Omega_\varepsilon} \varphi^2 r(\varepsilon)^{-(N-1)} \varepsilon^N v_\varepsilon dx .$$

Combining (2.61) with (2.62), and using (1.23), (2.33), (2.54c), we have :

$$(2.63) \quad \lim_{\varepsilon \rightarrow 0} (A_\varepsilon - D_\varepsilon) = - \int_{\Omega} \nabla v \cdot \nabla(\varphi^2) v dx + \bar{g} |\partial T| \int_{\Omega} \varphi^2 v dx .$$

On the other hand, multiplying (2.43a) by  $\varphi^2 \xi_\varepsilon$ , and integrating by parts in  $\Omega_\varepsilon$ , we deduce :

$$(2.64) \quad B_\varepsilon = - \int_{\Omega_\varepsilon} \nabla \xi_\varepsilon \cdot \nabla(\varphi^2) \xi_\varepsilon dx + \bar{g} r(\varepsilon)^{-(N-1)} \varepsilon^N \int_{\partial T_\varepsilon} \varphi^2 \xi_\varepsilon ds + \rho_\varepsilon \int_{\Omega_\varepsilon} \varphi^2 \xi_\varepsilon dx .$$

Besides that, taking  $r(\varepsilon)^{-(N-1)} \varepsilon^N \varphi^2 \xi_\varepsilon$  as test function in (1.11b), we have :

$$(2.65) \quad D_\varepsilon = - \int_{\Omega_\varepsilon} \nabla(r(\varepsilon)^{-(N-1)} \varepsilon^N v_\varepsilon) \cdot \nabla(\varphi^2) \xi_\varepsilon dx + \bar{g} r(\varepsilon)^{-(N-1)} \varepsilon^N \int_{\partial T_\varepsilon} \varphi^2 \xi_\varepsilon ds .$$

Combining (2.64) with (2.65), and using (1.23), (2.33), (2.54c), it follows that :

$$(2.66) \quad \lim_{\varepsilon \rightarrow 0} (B_\varepsilon - D_\varepsilon) = 0 .$$

From (2.56), (2.59), (2.60), (2.63), (2.66), we deduce :

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} \varphi^2 |\nabla \alpha_\varepsilon|^2 dx = - \int_{\Omega} \varphi^2 |\nabla v|^2 dx - \int_{\Omega} \nabla v \cdot \nabla(\varphi^2) v dx + \bar{g} |\partial T| \int_{\Omega} \varphi^2 v dx ,$$

which using Green's formula and the fact that  $v$  is a solution of problem (1.24), implies (2.55). This completes the proof of Theorem 2.2. ■

§ 3. PROOF OF THE RESULTS IN THE CASE OF A NEUMANN DATA WITH A ZERO MEAN-VALUE

In this section we shall prove Theorems 1.3 and 1.4 stated in Section 1.5.

3.1. A priori estimates. Proof of Theorem 1.3

We shall divide the proof of this theorem into two parts :

(a) *Proof of the upper estimate in Theorem 1.3*

In order to prove the upper estimate we shall use the following result :

LEMMA 3.1 : *Assume that  $r(\cdot)$  verifies (1.1). Let  $H_\varepsilon$  be defined by :*

$$(3.1) \quad H_\varepsilon = \left\{ \Phi \in H^1(Y_\varepsilon \setminus r(\varepsilon) \bar{T}) \mid \int_{\partial r(\varepsilon) T} \phi \, ds = 0 \right\}$$

where  $Y_\varepsilon = ]-\varepsilon/2, \varepsilon/2[{}^N$ . Then there exists a constant  $C$ , independent of  $\varepsilon$ , such that :

$$(3.2) \quad \|\Phi\|_{0, \partial r(\varepsilon) T}^2 \leq Cr(\varepsilon) \|\nabla \Phi\|_{0, Y_\varepsilon \setminus r(\varepsilon) \bar{T}}^2 \quad \forall \Phi \in H_\varepsilon. \quad \blacksquare$$

Taking  $\varphi = t_\varepsilon$  in (1.30b), it follows that :

$$(3.3) \quad \|\nabla t_\varepsilon\|_{0, \Omega_\varepsilon}^2 = \int_{\partial T_\varepsilon} g_\varepsilon^0 t_\varepsilon \, ds.$$

Let  $\mathbf{k}$  in  $\mathbb{Z}^N$  be any integer vector such that  $T(\varepsilon, \bar{\mathbf{k}})$  is strictly contained in  $\Omega$ . Since  $g_\varepsilon^0$  satisfies (1.28), (1.29), we have :

$$(3.4) \quad \int_{\partial T(\varepsilon, \mathbf{k})} g_\varepsilon^0 t_\varepsilon \, ds = \int_{\partial T(\varepsilon, \mathbf{k})} g_\varepsilon^0 t_\varepsilon^0 \, ds$$

where

$$(3.5) \quad t_\varepsilon^0 = t_\varepsilon - (1/|r(\varepsilon) \partial T|) \int_{\partial T(\varepsilon, \mathbf{k})} t_\varepsilon \, ds \quad \text{on } Y^*(\varepsilon, \mathbf{k}).$$

Therefore, using Lemma 3.1 (cf. (3.2)), and Cauchy-Schwarz inequality, we deduce that :

$$(3.6) \quad \left| \int_{\partial T(\varepsilon, \mathbf{k})} g_\varepsilon^0 t_\varepsilon^0 \, ds \right| \leq Cr(\varepsilon)^{1/2} \|g_\varepsilon^0\|_{0, \partial T(\varepsilon, \mathbf{k})} \|\nabla t_\varepsilon^0\|_{0, Y_{\varepsilon \mathbf{k}}^*}$$

where  $Y^*(\epsilon, \mathbf{k}) = \epsilon \mathbf{k} + (Y_\epsilon \setminus r(\epsilon) T)$ . Since  $g_\epsilon^0$  is defined by (1.28), we have :

$$(3.7) \quad \|g_\epsilon^0\|_{0, \partial T(\epsilon, \mathbf{k})}^2 = r(\epsilon)^{N-1} \|g^0\|_{0, \partial T}^2 .$$

Combining (3.4) with (3.6), (3.7), and using the fact that  $\nabla t_\epsilon^0 = \nabla t_\epsilon$ , we deduce :

$$(3.8) \quad \left| \int_{\partial T(\epsilon, \mathbf{k})} g_\epsilon^0 t_\epsilon ds \right|^2 \leq Cr(\epsilon)^N \|\nabla t_\epsilon\|_{0, Y^*(\epsilon, \mathbf{k})}^2 .$$

Since

$$\sum_{\mathbf{k}} \left| \int_{\partial T(\epsilon, \mathbf{k})} g_\epsilon^0 t_\epsilon ds \right|^2 \geq C \epsilon^N \left| \int_{\partial T_\epsilon} g_\epsilon^0 t_\epsilon ds \right|^2$$

where the sum stands for all  $\mathbf{k}$  in  $\mathbb{Z}^N$  such that  $T(\epsilon, \mathbf{k}) \subset \Omega$ , it follows from (3.8) that :

$$(3.9) \quad \left| \int_{\partial T_\epsilon} g_\epsilon^0 t_\epsilon ds \right| \leq C (r(\epsilon)/\epsilon)^{N/2} \|\nabla t_\epsilon\|_{0, \Omega_\epsilon} .$$

Therefore, the upper bound in (1.31) follows immediately from (3.3) and (3.9).

(b) *Proof of the lower estimate in Theorem 1.3*

Let us define the function  $\omega$  by :

$$(3.10a) \quad -\Delta \omega = 0 \quad \text{in } 2T \setminus \bar{T}$$

$$(3.10b) \quad \omega = 0 \quad \text{on } \partial(2T)$$

$$(3.10c) \quad \omega = h \quad \text{on } \partial T$$

where  $h$  is any function in  $H^{1/2}(\partial T)$  verifying the following condition :

$$(3.11) \quad \int_{\partial T} g^0 h ds \neq 0 .$$

Next, we define  $\omega_\epsilon(\cdot)$  in  $\Omega_\epsilon$  as follows :

$$(3.12a) \quad \omega_\epsilon(x) = \omega((x - \epsilon \mathbf{k})/r(\epsilon))$$

$$(3.12b) \quad \begin{aligned} &\text{if } x \in (2T(\epsilon, \bar{\mathbf{k}}) \setminus \bar{T}(\epsilon, \bar{\mathbf{k}})) ; 2\bar{T}(\epsilon, \bar{\mathbf{k}}) \subset \Omega \\ &\omega_\epsilon(x) = 0 \quad \text{otherwise .} \end{aligned}$$

Taking  $\omega_\varepsilon$  as test function in (1.30b), we have :

$$(3.13) \quad \int_{\Omega_\varepsilon} \nabla t_\varepsilon \cdot \nabla \omega_\varepsilon \, dx = \int_{\partial T_\varepsilon} g_\varepsilon^0 \omega_\varepsilon \, ds .$$

Using (1.28), (3.10c) and (3.12), it follows that :

$$(3.14) \quad \left| \int_{\partial T_\varepsilon} g_\varepsilon^0 \omega_\varepsilon \, ds \right| \geq Cr(\varepsilon)^{N-1} \varepsilon^{-N} \left| \int_{\partial T} g^0 h \, ds \right|$$

where  $C$  is a strictly positive constant, which is independent of  $\varepsilon$ .

On the other hand, using the fact that  $\omega_\varepsilon$  is  $\varepsilon$ -periodic, we have :

$$(3.15) \quad \|\nabla \omega_\varepsilon\|_{0, \Omega_\varepsilon}^2 \leq Cr(\varepsilon)^{N-2} \varepsilon^{-N} \|\nabla \omega\|_{0, (2T \setminus \bar{T})}^2 .$$

Combining (3.13), (3.14) with (3.15), and using Cauchy-Schwarz inequality, we obtain :

$$\|\nabla t_\varepsilon\|_{0, \Omega_\varepsilon} \geq C (r(\varepsilon)/\varepsilon)^{N/2} (1/\|\nabla \omega\|_{0, (2T \setminus \bar{T})}) \left| \int_{\partial T} g^0 h \, ds \right|$$

which proves the lower estimate in (1.31), with  $m = \left( C \left| \int_{\partial T} g^0 h \, ds \right| \right) / \|\nabla \omega\|_{0, (2T \setminus \bar{T})}$ , since  $h$  verifies (3.11).

To conclude the proof of Theorem 1.3, let us prove Lemma 3.1.

*Proof of Lemma 3.1 :* Let  $\Phi$  be a given function in  $H_\varepsilon$ . We have :

$$(3.16a) \quad \|\Phi\|_{0, \partial r(\varepsilon)T}^2 = r(\varepsilon)^{N-1} \int_{\partial T} |\Phi(r(\varepsilon)y)|^2 \, ds(y)$$

and

$$(3.16b) \quad \begin{aligned} \|\nabla \Phi\|_{0, Y_\varepsilon \setminus r(\varepsilon)\bar{T}}^2 &\geq \|\nabla \Phi\|_{0, r(\varepsilon)(2T \setminus \bar{T})}^2 = \\ &= r(\varepsilon)^{N-2} \int_{2T \setminus \bar{T}} |\nabla_y \Phi(r(\varepsilon)y)|^2 \, dy . \end{aligned}$$

By using a generalized Poincaré's inequality in  $(2T \setminus \bar{T})$ , it follows that there exists a constant  $C = C(T)$ , such that :

$$(3.17) \quad \int_{\partial T} |\Phi(r(\varepsilon)y)|^2 \, ds(y) \leq C \int_{2T \setminus \bar{T}} |\nabla_y \Phi(r(\varepsilon)y)|^2 \, dy .$$

Therefore, combining (3.16) with (3.17), we obtain :

$$\|\Phi\|_{0, \partial r(\varepsilon)T}^2 \leq Cr(\varepsilon) \|\nabla\Phi\|_{0, Y_\varepsilon \setminus r(\varepsilon)T}^2$$

which proves (3.2). This completes the proof of Lemma 3.1 and of Theorem 1.3. ■

**3.2. Proof of Theorem 1.4**

In the first part of this proof we shall follow the same pattern that in the proof of Theorem 1.2. First, by using (1.31), it follows that we can extract from the sequence  $\{(r(\varepsilon)/\varepsilon)^{-N/2} P_\varepsilon t_\varepsilon\}$  a subsequence, that we still denote by  $\{(r(\varepsilon)/\varepsilon)^{-N/2} P_\varepsilon t_\varepsilon\}$ , such that :

$$(3.18) \quad (r(\varepsilon)/\varepsilon)^{-N/2} P_\varepsilon t_\varepsilon \rightharpoonup t \text{ weakly in } H_0^1(\Omega), \text{ as } \varepsilon \rightarrow 0.$$

Let  $\varphi$  be any given function in  $C_0^\infty(\Omega)$ , and let us take  $(r(\varepsilon)/\varepsilon)^{-N/2} \varphi$  as test function in (1.30b). We have :

$$(3.19) \quad \int_{\Omega_\varepsilon} \nabla((r(\varepsilon)/\varepsilon)^{-N/2} t_\varepsilon) \cdot \nabla\varphi \, dx = (r(\varepsilon)/\varepsilon)^{-N/2} \int_{\partial T_\varepsilon} g_\varepsilon^0 \varphi \, ds.$$

As in the proof of Theorem 1.2, by using (2.33), (3.18), we deduce :

$$(3.20) \quad \lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} \nabla((r(\varepsilon)/\varepsilon)^{-N/2} t_\varepsilon) \cdot \nabla\varphi \, dx = \int_\Omega \nabla t \cdot \nabla\varphi \, dx.$$

The next step of the proof consists in passing to the limit in the right-hand side term in (3.19). To do that, let  $\mathbf{N}$  in  $H^1(T)^N$  be any function verifying the following properties :

$$(3.21a) \quad \operatorname{div} \mathbf{N} = 0 \quad \text{in } T$$

$$(3.21b) \quad \mathbf{N} \cdot \vec{n} = -g^0 \quad \text{on } \partial T.$$

We remark that the existence of at least one function  $\mathbf{N}$  with these properties is ensured by the fact that  $g^0$  verifies (1.29), and it belongs to  $L^2(\partial T)$ .

Next, we define  $N_\varepsilon(\cdot)$  in  $\cup \{T(\varepsilon, \mathbf{k}) \mid \mathbf{k} \in \mathbb{Z}^N\}$  as follows :

$$(3.22) \quad N_\varepsilon(x) = \mathbf{N}((x - \varepsilon\mathbf{k})/r(\varepsilon)) \quad \text{if } x \in T(\varepsilon, \mathbf{k}).$$

By using (3.21), (3.22), and Green's formula, we have :

$$\int_{\partial T_\varepsilon} g_\varepsilon^0 \varphi \, ds = \int_{T_\varepsilon} \nabla\varphi \cdot \mathbf{N}_\varepsilon \, dx.$$



Therefore, the right-hand side term in (3.19) can be estimated as follows :

$$(3.23) \quad (r(\varepsilon)/\varepsilon)^{-N/2} \left| \int_{\partial T_\varepsilon} g_\varepsilon^0 \varphi \, ds \right| \leq (r(\varepsilon)/\varepsilon)^{-N/2} \|\nabla \varphi\|_{0, T_\varepsilon} \|\mathbf{N}_\varepsilon\|_{0, T_\varepsilon}.$$

An explicit computation using (3.22) shows that :

$$(3.24) \quad \|\mathbf{N}_\varepsilon\|_{0, T_\varepsilon} \leq C (r(\varepsilon)/\varepsilon)^{N/2} \|\mathbf{N}\|_{0, T}.$$

Combining (3.23) with (3.24), we have :

$$(3.25) \quad (r(\varepsilon)/\varepsilon)^{-N/2} \left| \int_{\partial T_\varepsilon} g_\varepsilon^0 \varphi \, ds \right| \leq C \|\nabla \varphi\|_{0, T_\varepsilon}.$$

By using (2.58), we deduce that the right-hand side of (3.25) goes to zero, as  $\varepsilon \rightarrow 0$ , which proves that the right-hand side in (3.19) goes to zero as  $\varepsilon \rightarrow 0$ . Together with (3.20), this completes the proof of Theorem 1.4. ■

### 3.3. A correcting term for the solution of problem (1.7)

In this section we shall use the same notations as in Section 2.3 concerning the regions  $G_\varepsilon$ ,  $\Gamma_\varepsilon$ ,  $Y_\varepsilon$ ,  $Y_\varepsilon^*$  and the  $\{Q_\varepsilon\}$ .

Let us introduce the sequence  $\{\xi_\varepsilon\}$  of functions defined in  $G_\varepsilon$  as follows :

$$(3.26a) \quad -\Delta \xi_\varepsilon = 0 \quad \text{in } \mathbb{R}^N \setminus \bar{G}_\varepsilon$$

$$(3.26b) \quad \partial \xi_\varepsilon / \partial n = (r(\varepsilon)/\varepsilon)^{-N/2} g_\varepsilon^0 \quad \text{on } \Gamma_\varepsilon$$

$$(3.26c) \quad \xi_\varepsilon \text{ is } \varepsilon\text{-periodic in all its variables}$$

$$(3.26d) \quad \int_{Y_\varepsilon} Q_\varepsilon \xi_\varepsilon \, dx = 0.$$

**THEOREM 3.2 :** *Assume that  $r(\cdot)$  verifies (1.1) and that  $g^0$  verifies (1.29). Let  $\Omega'$  be any open (bounded) subset of  $\Omega$  such that  $\bar{\Omega}' \subset \Omega$ . Then there exists a (rest) function  $\alpha_\varepsilon$  in  $H^1(\Omega_\varepsilon)$ , such that :*

$$(3.27a) \quad (r(\varepsilon)/\varepsilon)^{-N/2} t_\varepsilon = \xi_\varepsilon + \alpha_\varepsilon \quad \text{in } \Omega_\varepsilon$$

$$(3.27b) \quad \|\nabla \alpha_\varepsilon\|_{0, \Omega' \cap \Omega_\varepsilon} \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0,$$

where  $\xi_\varepsilon$  is given by (3.26). ■

*Proof:* The proof of this theorem follows exactly the same steps as the proof of Theorem 2.2. For brevity, we shall therefore omit it. ■

## 4. PROOFS OF THE RESULTS FOR THE CASE OF A HOMOGENEOUS NEUMANN DATA

In this section we shall prove Theorem 1.5 stated in Section 1.6. As it has been already mentioned, a proof of this theorem can be found in H. Attouch [1, Th. 1.1]. Here, we give an alternative proof.

*Proof of Theorem 1.5.* The first step of the proof consists in proving that the sequence  $\{P_\varepsilon z_\varepsilon\}$  remains bounded in  $H_0^1(\Omega)$ , as  $\varepsilon \rightarrow 0$ . In order to prove that, let us take  $z_\varepsilon$  as test function in the variational formulation of problem (1.8). We have :

$$\int_{\Omega_\varepsilon} |\nabla z_\varepsilon|^2 dx = \int_{\Omega_\varepsilon} f z_\varepsilon dx$$

which using Cauchy-Schwarz inequality implies that :

$$(4.1) \quad \|\nabla z_\varepsilon\|_{0, \Omega_\varepsilon}^2 \leq C \|z_\varepsilon\|_{1, \Omega_\varepsilon}$$

with  $C = \|f\|_{0, \Omega}$ . Therefore, since the family  $\{P_\varepsilon\}$  verifies (1.22), it follows from Poincaré's inequality that  $\{P_\varepsilon z_\varepsilon\}$  lies bounded in  $H_0^1(\Omega)$ , as  $\varepsilon \rightarrow 0$ . We can therefore extract from this sequence a subsequence, still denoted by  $\{P_\varepsilon z_\varepsilon\}$ , such that :

$$(4.2) \quad P_\varepsilon z_\varepsilon \rightharpoonup z \text{ weakly in } H_0^1(\Omega), \text{ as } \varepsilon \rightarrow 0.$$

Our next goal is to pass to the limit in the variational formulation of problem (1.8) (cf. (1.33)). By using (2.33), (4.2), we can pass to the limit at both sides of (1.33b). We obtain :

$$\int_{\Omega} \nabla z \cdot \nabla \varphi dx = \int_{\Omega} f \varphi dx \quad \forall \varphi \in H_0^1(\Omega)$$

which implies that  $z$  is a (weak) solution of problem (1.35). Since this problem admits a unique solution, it follows that in (4.2) the whole sequence  $\{P_\varepsilon z_\varepsilon\}$  weakly converges to  $z$  in  $H_0^1(\Omega)$  as  $\varepsilon \rightarrow 0$ . This completes the proof of (1.34a).

To prove (1.34b) it suffices to remark that taking  $z$  as test function in (1.33b), it can be easily checked that :

$$\int_{\Omega_\varepsilon} |\nabla(z_\varepsilon - z)|^2 dx = \int_{\Omega_\varepsilon} |\nabla z|^2 dx + \int_{\Omega_\varepsilon} f z_\varepsilon dx - 2 \int_{\Omega_\varepsilon} f z dx.$$

Therefore, using (2.58) and (4.2), we can pass to the limit in each one of the terms in the right-hand side of this expression. We obtain :

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} |\nabla(z_\varepsilon - z)|^2 dx = \int_{\Omega} |\nabla z|^2 dx - \int_{\Omega} f z dx .$$

Since  $z$  is the solution of problem (1.35), the right-hand side of this identity is zero. This proves (1.34b), and it completes the proof of Theorem 1.5. ■

5. PROOF OF THEOREM 1.6

We shall prove the theorem in case the function  $r(\cdot)$  verifies (1.12) or (1.13). In the other cases, the proof is very similar, so we shall omit it.

Let us therefore assume that  $r(\cdot)$  verifies (1.12) or (1.13). The proof of (1.49) follows step by step the proof of Theorem 2.2. We shall limit ourselves to prove the following result :

$$(5.1) \quad Q_\varepsilon \theta_\varepsilon \rightharpoonup 0 \text{ weakly in } H^1_{loc}(\mathbb{R}^N), \text{ as } \varepsilon \rightarrow 0 ,$$

where  $\{Q_\varepsilon\}$  is any family of linear extension-operators verifying (2.49). Indeed, this is the only different step between both proofs.

We begin the proof of (5.1) by remarking that using the explicit expression giving  $\theta_\varepsilon$  (cf. (1.47)), a brief computation shows that :

$$(5.2) \quad \partial\theta_\varepsilon/\partial n = 0 \text{ on } \partial B(\varepsilon, \vec{k}) .$$

Therefore, from (1.41) it follows that  $\theta_\varepsilon$  verifies :

$$(5.3a) \quad -\Delta\theta_\varepsilon = \tilde{\mu}_\varepsilon \quad \text{in } \mathbb{R}^N \setminus \bar{G}_\varepsilon$$

$$(5.3b) \quad \partial\theta_\varepsilon/\partial n = \bar{g}r(\varepsilon)^{-(N-1)} \varepsilon^N \quad \text{on } \Gamma_\varepsilon$$

$$(5.3c) \quad \theta_\varepsilon \text{ is } \varepsilon\text{-periodic in all its variables ,}$$

where

$$(5.4) \quad \tilde{\mu}_\varepsilon = \begin{cases} \mu_\varepsilon & \text{in } \cup \{B(\varepsilon, \mathbf{k}) \setminus \bar{T}(\varepsilon, \mathbf{k}) \mid \mathbf{k} \in \mathbb{Z}^N\} \\ 0 & \text{in } \cup \{Y(\varepsilon, \mathbf{k}) \setminus \bar{B}(\varepsilon, \mathbf{k}) \mid \mathbf{k} \in \mathbb{Z}^N\} \end{cases}$$

and  $\mu_\varepsilon$  is defined by (1.42).

Multiplying (5.3a) by  $\theta_\varepsilon$ , and integrating by parts in  $Y_\varepsilon^* = Y_\varepsilon \setminus r(\varepsilon) T$ , we have :

$$\|\nabla\theta_\varepsilon\|_{0, Y_\varepsilon^*}^2 = \tilde{\mu}_\varepsilon \int_{Y_\varepsilon^*} \theta_\varepsilon dx + \bar{g}r(\varepsilon)^{-(N-1)} \varepsilon^N \int_{\partial T(\varepsilon, 0)} \theta_\varepsilon dx$$

which using Cauchy-Schwarz inequality and (2.17), implies that :

$$\|\nabla\theta_\varepsilon\|_{0, Y_\varepsilon^*}^2 = C\varepsilon^{N/2} \left\{ |\tilde{\mu}_\varepsilon| \|\theta_\varepsilon\|_{0, Y_\varepsilon^*} + |\bar{g}| \|\theta_\varepsilon\|_{1, Y_\varepsilon^*} \right\}.$$

Since  $\{|\tilde{\mu}_\varepsilon|\}$  remains bounded as  $\varepsilon \rightarrow 0$ , it follows that :

$$\|\nabla\theta_\varepsilon\|_{0, Y_\varepsilon^*}^2 \leq C\varepsilon^{N/2} \|\theta_\varepsilon\|_{1, Y_\varepsilon^*}$$

which using (2.49) implies that :

$$(5.5a) \quad \|\nabla Q_\varepsilon \theta_\varepsilon\|_{0, Y_\varepsilon}^2 \leq C\varepsilon^{N/2} \|Q_\varepsilon \theta_\varepsilon\|_{1, Y_\varepsilon}.$$

Since  $Q_\varepsilon \theta_\varepsilon$  vanishes on the external boundary of  $Y_\varepsilon$ , by Poincaré's inequality, we have :

$$(5.5b) \quad \|Q_\varepsilon \theta_\varepsilon\|_{0, Y_\varepsilon} \leq C\varepsilon \|\nabla Q_\varepsilon \theta_\varepsilon\|_{0, Y_\varepsilon}.$$

Combining (5.5a) with (5.5b), we obtain :

$$\|Q_\varepsilon \theta_\varepsilon\|_{1, Y_\varepsilon}^2 \leq C\varepsilon^N,$$

which clearly implies that the sequence  $\{Q_\varepsilon \theta_\varepsilon\}$  remains bounded in  $H_{loc}^1(\mathbb{R}^N)$  as  $\varepsilon \rightarrow 0$ , because  $Q_\varepsilon \theta_\varepsilon$  is  $\varepsilon$ -periodic. Therefore, up to a subsequence, we have :

$$(5.6) \quad Q_\varepsilon \theta_\varepsilon \rightharpoonup \theta \text{ weakly in } H_{loc}^1(\mathbb{R}^N), \text{ as } \varepsilon \rightarrow 0.$$

To identify  $\theta$ , let us remark that  $Q_\varepsilon \theta_\varepsilon = 0$  in the squares  $R(\varepsilon, \vec{k})$  of side  $(\sqrt{N} - 1)\varepsilon / \sqrt{N}$  centered at the edges of  $Y(\varepsilon, \vec{k})$  (see fig. 1.4). Therefore, we have :

$$(5.7) \quad \chi_{R_\varepsilon} Q_\varepsilon \theta_\varepsilon = 0$$

where  $\chi_{R_\varepsilon}$  is the characteristic function of  $\cup \{R(\varepsilon, \mathbf{k}) \mid \mathbf{k} \in \mathbb{Z}^N\}$ . Since  $\chi_{R_\varepsilon}$  converge to  $[(\sqrt{N} - 1) / \sqrt{N}]^N$  weakly in  $L_{loc}^2(\mathbb{R}^N)$ , by passing to the limit in (5.7) we deduce that  $\theta = 0$ , which proves (5.1).

Once (5.1) has been established, the proof of Theorem 1.6 to be completed follows step by step the proof of Theorem 2.2. ■

APPENDIX A

In this appendix we prove the following lemma concerning the existence of the family  $\{P_\varepsilon\}$  of extension-operators that we have systematically used throughout the paper.

LEMMA A.1 : Assume that  $r(\cdot)$  verifies (1.1). Then there exists a family  $\{P_\varepsilon\}$  of linear continuous extension-operators,  $P_\varepsilon \in \mathcal{L}(V_\varepsilon, H_0^1(\Omega))$ , verifying the following conditions :

$$(A.1a) \quad (P_\varepsilon \varphi)(x) = \varphi(x) \quad \forall x \in \Omega_\varepsilon$$

$$(A.1b) \quad \int_{\Omega} |\nabla P_\varepsilon \varphi|^2 dx \leq C \int_{\Omega_\varepsilon} |\nabla \varphi|^2 dx \quad \forall \varphi \in V_\varepsilon$$

where  $C$  is a constant independent of  $\varepsilon$ . ■

*Proof:* We begin the proof by remarking that in order to prove the existence of the family  $\{P_\varepsilon\}$ , it suffices to prove that there exists a family  $\{Q_\varepsilon\}$ ,  $Q_\varepsilon \in \mathcal{L}(H^1(Y_\varepsilon^*), H^1(Y_\varepsilon))$ , verifying :

$$(A.2a) \quad (Q_\varepsilon \varphi)(x) = \varphi(x) \quad \forall x \in Y_\varepsilon^*$$

$$(A.2b) \quad \int_{Y_\varepsilon} |\nabla Q_\varepsilon \varphi|^2 dx \leq C \int_{Y_\varepsilon^*} |\nabla \varphi|^2 dx \quad \forall \varphi \in H^1(Y_\varepsilon^*)$$

where  $C$  is independent of  $\varepsilon$ ,  $Y_\varepsilon = ]-\varepsilon/2, \varepsilon/2[^N$ , and  $Y_\varepsilon^* = Y_\varepsilon \setminus r(\varepsilon) \bar{T}$ .

We construct  $Q_\varepsilon$  as follows : (i) Let us denote  $\tilde{S}$ ,  $S$  the domains defined by :

$$\begin{aligned} \tilde{S} &= 2T \\ S &= \tilde{S} - \bar{T}. \end{aligned}$$

From D. Cioranescu & J. Saint Jean Paulin [5], we know that there exists a linear continuous extension operator  $Q \in \mathcal{L}(H^1(S), H^1(\tilde{S}))$  verifying :

$$(A.3a) \quad Q\psi(y) = \psi(y) \quad \forall y \in S$$

$$(A.3b) \quad \int_{\tilde{S}} |\nabla Q\psi|^2 dy \leq C_1 \int_S |\nabla \psi|^2 dy \quad \forall \psi \in H^1(S)$$

where  $C_1$  only depends on  $T$ . (ii) We define  $Q_\varepsilon$  by :

$$(A.4) \quad \forall \varphi \in H^1(Y_\varepsilon^*), \quad (Q_\varepsilon \varphi)(x) = \begin{cases} \varphi(x) & \text{if } x \in Y_\varepsilon^* \\ (Q\psi) & \text{if } x \in r(\varepsilon) \bar{T} \end{cases}$$

where  $\psi(y) = \varphi(x/r(\varepsilon))$ ,  $y = x/r(\varepsilon)$ .

By using (A.4), it follows that  $Q_\varepsilon \varphi$  satisfies (A.2a). To prove that this extension-operator verifies (A.2b), it suffices to remark that the following identity holds :

$$(A.5) \quad \int_{r(\varepsilon)T} |\nabla_x Q_\varepsilon \varphi|^2 dx = r(\varepsilon)^{N-2} \int_T |\nabla_y Q\psi|^2 dy.$$

Effectively, from (A.3b) and (A.5), it follows that :

$$\int_{r(\varepsilon)T} |\nabla_x Q_\varepsilon \varphi|^2 dx \leq C_1 r(\varepsilon)^{N-2} \int_S |\nabla_y \psi|^2 dy = C_1 \int_{r(\varepsilon)S} |\nabla_x \varphi|^2 dx$$

which implies

$$\int_{Y_\varepsilon} |\nabla Q_\varepsilon \varphi|^2 dx \leq (1 + C_1) \int_{Y_\varepsilon^*} |\nabla \varphi|^2 dx$$

which proves (A.3b) with  $C = (1 + C_1)$ . This completes the proof of Lemma A.1. ■

**APPENDIX B**

In this appendix we prove the following lemma which characterizes the weak limits of sequences of  $\varepsilon$ -periodic functions.

LEMMA B.1 : Let  $\{f_\varepsilon\}$ ,  $f$  be given in  $L^2_{loc}(\mathbb{R}^N)$ . Assume that  $\{f_\varepsilon\}$ ,  $f$  verify the following conditions :

(B.1a)  $\forall \varepsilon$ ,  $f_\varepsilon$  is  $\varepsilon$ -periodic in all its variables .

(B.1b)  $f_\varepsilon \rightharpoonup f$  weakly in  $L^2_{loc}(\mathbb{R}^N)$ , as  $\varepsilon \rightarrow 0$  .

(B.1c)  $m_\varepsilon(f_\varepsilon) = (1/\varepsilon^N) \int_{Y_\varepsilon} f_\varepsilon dx \rightarrow a \in \mathbb{R}$ , as  $\varepsilon \rightarrow 0$  .

Then  $f \equiv a$ . ■

*Proof:* Without loss of generality, assume that  $a = 0$ . Let  $P$  be an open interval of  $\mathbb{R}^N$ . We set :

$$P_\varepsilon^0 = \cup \{Y(\varepsilon, \mathbf{k}) \mid Y(\varepsilon, \mathbf{k}) \subset P\}$$

$$\bar{P}_\varepsilon = P \setminus P_\varepsilon^0$$

Since  $P = P_\varepsilon^0 \cup \bar{P}_\varepsilon$ , we have :

(B.2) 
$$\int_P f_\varepsilon dx = \int_{P_\varepsilon^0} f_\varepsilon dx + \int_{\bar{P}_\varepsilon} f_\varepsilon dx .$$

Let us denote by  $N(P_\varepsilon^0)$  the number of periods  $Y(\varepsilon, \mathbf{k})$  contained in  $P_\varepsilon^0$ . Since  $f_\varepsilon$  is  $\varepsilon$ -periodic, we have :

(B.3a) 
$$\int_{P_\varepsilon^0} f_\varepsilon dx = \varepsilon^N m_\varepsilon(f_\varepsilon) N_\varepsilon(P_\varepsilon^0) .$$

On the other hand, by using Cauchy-Schwarz inequality, we have :

$$(B.3b) \quad \left| \int_{\bar{P}_\varepsilon} f_\varepsilon dx \right| \leq |\bar{P}_\varepsilon|^{1/2} \|f_\varepsilon\|_{0, \bar{P}_\varepsilon} \leq |\bar{P}_\varepsilon|^{1/2} \|f_\varepsilon\|_{0, P}.$$

Since

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^N N_\varepsilon(P_\varepsilon^0) = |P|$$

and

$$\lim_{\varepsilon \rightarrow 0} |\bar{P}_\varepsilon| = 0,$$

then using (B.1b), (B.1c) (with  $a = 0$ ), and (B.3), we can pass to the limit in (B.2). We obtain :

$$\int_P f dx = 0.$$

Since  $P$  is arbitrary, by passing to the Lebesgue points we conclude that  $f \equiv 0$ . This completes the proof of Lemma B.1. ■

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